# Introductions to Tan

## Introduction to the trigonometric functions

### General

The six trigonometric functions sine $\sin(z)$, cosine $\cos(z)$, tangent $\tan(z)$, cotangent $\cot(z)$, cosecant $\csc(z)$, and secant $\sec(z)$ are well known and among the most frequently used elementary functions. The most popular functions $\sin(z)$, $\cos(z)$, $\tan(z)$, and $\cot(z)$ are taught worldwide in high school programs because of their natural appearance in problems involving angle measurement and their wide applications in the quantitative sciences.

The trigonometric functions share many common properties.

### Definitions of trigonometric functions

All trigonometric functions can be defined as simple rational functions of the exponential function of $i z$:

$$\sin(z) = \frac{e^{i z} - e^{-i z}}{2 i}$$

$$\cos(z) = \frac{e^{i z} + e^{-i z}}{2}$$

$$\tan(z) = -\frac{i \left(e^{i z} - e^{-i z}\right)}{e^{i z} + e^{-i z}}$$

$$\cot(z) = \frac{i \left(e^{i z} + e^{-i z}\right)}{e^{i z} - e^{-i z}}$$

$$\csc(z) = \frac{2 i}{e^{i z} - e^{-i z}}$$

$$\sec(z) = \frac{2}{e^{i z} + e^{-i z}}.$$

The functions $\tan(z)$, $\cot(z)$, $\csc(z)$, and $\sec(z)$ can also be defined through the functions $\sin(z)$ and $\cos(z)$ using the following formulas:

$$\tan(z) = \frac{\sin(z)}{\cos(z)}$$

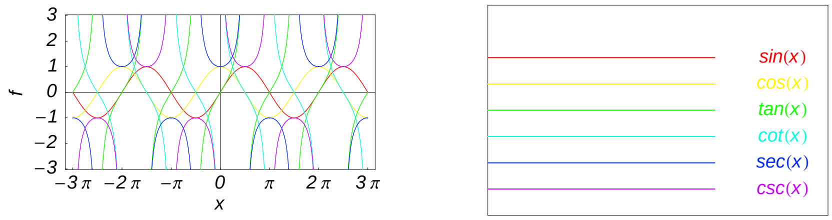
$$\cot(z) = \frac{\cos(z)}{\sin(z)}$$

$$\csc(z) = \frac{1}{\sin(z)}$$

$$\sec(z) = \frac{1}{\cos(z)}.$$

### A quick look at the trigonometric functions

Here is a quick look at the graphics for the six trigonometric functions along the real axis.




## Connections within the group of trigonometric functions and with other function groups

### Representations through more general functions

The trigonometric functions are particular cases of more general functions. Among these more general functions, four different classes of special functions are particularly relevant: Bessel, Jacobi, Mathieu, and hypergeometric functions.

For example, $\sin(z)$ and $\cos(z)$ have the following representations through Bessel, Mathieu, and hypergeometric functions:

$$\sin(z) = \sqrt{\frac{\pi z}{2}}\, J_{1/2}(z) \quad \sin(z) = -i\sqrt{\frac{\pi i z}{2}}\, I_{1/2}(i z) \quad \sin(z) = \sqrt{\frac{\pi z}{2}}\, Y_{-1/2}(z) \quad \sin(z) = \frac{i}{\sqrt{2\pi}}\left(\sqrt{i z}\, K_{1/2}(i z) - \sqrt{-i z}\, K_{1/2}(-i z)\right)$$

$$\cos(z) = \sqrt{\frac{\pi z}{2}}\, J_{-1/2}(z) \quad \cos(z) = \sqrt{\frac{\pi i z}{2}}\, I_{-1/2}(i z) \quad \cos(z) = -\sqrt{\frac{\pi z}{2}}\, Y_{1/2}(z) \quad \cos(z) = \sqrt{\frac{i z}{2\pi}}\, K_{1/2}(i z) + \sqrt{-\frac{i z}{2\pi}}\, K_{1/2}(-i z)$$

$$\sin(z) = \mathrm{Se}(1, 0, z) \quad \cos(z) = \mathrm{Ce}(1, 0, z)$$

$$\sin(z) = z\, {}_0F_1\left(; \tfrac{3}{2}; -\tfrac{z^2}{4}\right) \quad \cos(z) = {}_0F_1\left(; \tfrac{1}{2}; -\tfrac{z^2}{4}\right).$$

On the other hand, all trigonometric functions can be represented as degenerate cases of the corresponding doubly periodic Jacobi elliptic functions when their second parameter is equal to 0 or 1:

$$\begin{array}{ll}
\sin(z) = \mathrm{sd}(z \mid 0) = \mathrm{sn}(z \mid 0) & \sin(z) = -i\,\mathrm{sc}(i z \mid 1) = -i\,\mathrm{sd}(i z \mid 1) \\
\cos(z) = \mathrm{cd}(z \mid 0) = \mathrm{cn}(z \mid 0) & \cos(z) = \mathrm{nc}(i z \mid 1) = \mathrm{nd}(i z \mid 1) \\
\tan(z) = \mathrm{sc}(z \mid 0) & \tan(z) = -i\,\mathrm{sn}(i z \mid 1) \\
\cot(z) = \mathrm{cs}(z \mid 0) & \cot(z) = i\,\mathrm{ns}(i z \mid 1) \\
\csc(z) = \mathrm{ds}(z \mid 0) = \mathrm{ns}(z \mid 0) & \csc(z) = i\,\mathrm{cs}(i z \mid 1) = i\,\mathrm{ds}(i z \mid 1) \\
\sec(z) = \mathrm{dc}(z \mid 0) = \mathrm{nc}(z \mid 0) & \sec(z) = \mathrm{cn}(i z \mid 1) = \mathrm{dn}(i z \mid 1).
\end{array}$$

### Representations through related equivalent functions

Each of the six trigonometric functions can be represented through the corresponding hyperbolic function:

$$\begin{array}{ll}
\sin(z) = -i\sinh(i z) & \sin(i z) = i\sinh(z) \\
\cos(z) = \cosh(i z) & \cos(i z) = \cosh(z) \\
\tan(z) = -i\tanh(i z) & \tan(i z) = i\tanh(z) \\
\cot(z) = i\coth(i z) & \cot(i z) = -i\coth(z) \\
\csc(z) = i\,\mathrm{csch}(i z) & \csc(i z) = -i\,\mathrm{csch}(z) \\
\sec(z) = \mathrm{sech}(i z) & \sec(i z) = \mathrm{sech}(z).
\end{array}$$

### Relations to inverse functions

Each of the six trigonometric functions is connected with its corresponding inverse trigonometric function by two formulas. One is a simple formula, and the other is much more complicated because of the multivalued nature of the inverse function:

$$\sin(\sin^{-1}(z)) \equiv z \quad \sin^{-1}(\sin(z)) \equiv z \;/;\; -\frac{\pi}{2} < \operatorname{Re}(z) < \frac{\pi}{2} \bigvee \operatorname{Re}(z) \equiv -\frac{\pi}{2} \bigwedge \operatorname{Im}(z) \geq 0 \bigvee \operatorname{Re}(z) \equiv \frac{\pi}{2} \bigwedge \operatorname{Im}(z) \leq 0$$

$$\cos(\cos^{-1}(z)) \equiv z \quad \cos^{-1}(\cos(z)) \equiv z \;/;\; 0 < \operatorname{Re}(z) < \pi \lor \operatorname{Re}(z) \equiv 0 \land \operatorname{Im}(z) \geq 0 \lor \operatorname{Re}(z) \equiv \pi \land \operatorname{Im}(z) \leq 0$$

$$\tan(\tan^{-1}(z)) \equiv z \quad \tan^{-1}(\tan(z)) \equiv z \;/;\; |\operatorname{Re}(z)| < \frac{\pi}{2} \bigvee \operatorname{Re}(z) \equiv -\frac{\pi}{2} \bigwedge \operatorname{Im}(z) < 0 \bigvee \operatorname{Re}(z) \equiv \frac{\pi}{2} \bigwedge \operatorname{Im}(z) > 0$$

$$\cot(\cot^{-1}(z)) \equiv z \quad \cot^{-1}(\cot(z)) \equiv z \;/;\; |\operatorname{Re}(z)| < \frac{\pi}{2} \bigvee \operatorname{Re}(z) \equiv -\frac{\pi}{2} \bigwedge \operatorname{Im}(z) < 0 \bigvee \operatorname{Re}(z) \equiv \frac{\pi}{2} \bigwedge \operatorname{Im}(z) \geq 0$$

$$\csc(\csc^{-1}(z)) \equiv z \quad \csc^{-1}(\csc(z)) \equiv z \;/;\; |\operatorname{Re}(z)| < \frac{\pi}{2} \bigvee \operatorname{Re}(z) \equiv -\frac{\pi}{2} \bigwedge \operatorname{Im}(z) \leq 0 \bigvee \operatorname{Re}(z) \equiv \frac{\pi}{2} \bigwedge \operatorname{Im}(z) \geq 0$$

$$\sec(\sec^{-1}(z)) \equiv z \quad \sec^{-1}(\sec(z)) \equiv z \;/;\; 0 < \operatorname{Re}(z) < \pi \lor \operatorname{Re}(z) \equiv 0 \land \operatorname{Im}(z) \geq 0 \lor \operatorname{Re}(z) \equiv \pi \land \operatorname{Im}(z) \leq 0.$$

### Representations through other trigonometric functions

Each of the six trigonometric functions can be represented by any other trigonometric function as a rational function of that function with linear arguments. For example, the sine function can be representative as a group-defining function because the other five functions can be expressed as follows:

$$\cos(z) \equiv \sin\left(\frac{\pi}{2} - z\right) \qquad \cos^2(z) \equiv 1 - \sin^2(z)$$

$$\tan(z) \equiv \frac{\sin(z)}{\cos(z)} \equiv \frac{\sin(z)}{\sin\left(\frac{\pi}{2}-z\right)} \qquad \tan^2(z) \equiv \frac{\sin^2(z)}{1-\sin^2(z)}$$

$$\cot(z) \equiv \frac{\cos(z)}{\sin(z)} \equiv \frac{\sin\left(\frac{\pi}{2}-z\right)}{\sin(z)} \qquad \cot^2(z) \equiv \frac{1-\sin^2(z)}{\sin^2(z)}$$

$$\csc(z) \equiv \frac{1}{\sin(z)} \qquad \csc^2(z) \equiv \frac{1}{\sin^2(z)}$$

$$\sec(z) \equiv \frac{1}{\cos(z)} \equiv \frac{1}{\sin\left(\frac{\pi}{2}-z\right)} \qquad \sec^2(z) \equiv \frac{1}{1-\sin^2(z)}.$$

All six trigonometric functions can be transformed into any other trigonometric function of this group if the argument $z$ is replaced by $p\,\pi/2 + q\,z$ with $q^2 \equiv 1 \bigwedge p \in \mathbb{Z}$:

$$\sin(-z - 2\pi) \equiv -\sin(z) \qquad \sin(z - 2\pi) \equiv \sin(z)$$

$$\sin\left(-z - \frac{3\pi}{2}\right) \equiv \cos(z) \qquad \sin\left(z - \frac{3\pi}{2}\right) \equiv \cos(z)$$

$$\sin(-z - \pi) \equiv \sin(z) \qquad \sin(z - \pi) \equiv -\sin(z)$$

$$\sin\left(-z - \frac{\pi}{2}\right) \equiv -\cos(z) \qquad \sin\left(z - \frac{\pi}{2}\right) \equiv -\cos(z)$$

$$\sin\left(z + \frac{\pi}{2}\right) \equiv \cos(z) \qquad \sin\left(\frac{\pi}{2} - z\right) \equiv \cos(z)$$

$$\sin(z + \pi) \equiv -\sin(z) \qquad \sin(\pi - z) \equiv \sin(z)$$

$$\sin\left(z + \frac{3\pi}{2}\right) \equiv -\cos(z) \qquad \sin\left(\frac{3\pi}{2} - z\right) \equiv -\cos(z)$$

$$\sin(z + 2\pi) \equiv \sin(z) \qquad \sin(2\pi - z) \equiv -\sin(z)$$

$\cos(-z-2\pi) == \cos(z) \quad \cos(z-2\pi) == \cos(z)$
$\cos\left(-z-\frac{3\pi}{2}\right) == \sin(z) \quad \cos\left(z-\frac{3\pi}{2}\right) == -\sin(z)$
$\cos(-z-\pi) == -\cos(z) \quad \cos(z-\pi) == -\cos(z)$
$\cos\left(-z-\frac{\pi}{2}\right) == -\sin(z) \quad \cos\left(z-\frac{\pi}{2}\right) == \sin(z)$
$\cos\left(z+\frac{\pi}{2}\right) == -\sin(z) \quad \cos\left(\frac{\pi}{2}-z\right) == \sin(z)$
$\cos(z+\pi) == -\cos(z) \quad \cos(\pi-z) == -\cos(z)$
$\cos\left(z+\frac{3\pi}{2}\right) == \sin(z) \quad \cos\left(\frac{3\pi}{2}-z\right) == -\sin(z)$
$\cos(z+2\pi) == \cos(z) \quad \cos(2\pi-z) == \cos(z)$

$\tan(-z-\pi) == -\tan(z) \quad \tan(z-\pi) == \tan(z)$
$\tan\left(-z-\frac{\pi}{2}\right) == \cot(z) \quad \tan\left(z-\frac{\pi}{2}\right) == -\cot(z)$
$\tan\left(z+\frac{\pi}{2}\right) == -\cot(z) \quad \tan\left(\frac{\pi}{2}-z\right) == \cot(z)$
$\tan(z+\pi) == \tan(z) \quad \tan(\pi-z) == -\tan(z)$

$\cot(-z-\pi) == -\cot(z) \quad \cot(z-\pi) == \cot(z)$
$\cot\left(-z-\frac{\pi}{2}\right) == \tan(z) \quad \cot\left(z-\frac{\pi}{2}\right) == -\tan(z)$
$\cot\left(z+\frac{\pi}{2}\right) == -\tan(z) \quad \cot\left(\frac{\pi}{2}-z\right) == \tan(z)$
$\cot(z+\pi) == \cot(z) \quad \cot(\pi-z) == -\cot(z)$

$\csc(-z-2\pi) == -\csc(z) \quad \csc(z-2\pi) == \csc(z)$
$\csc\left(-z-\frac{3\pi}{2}\right) == \sec(z) \quad \csc\left(z-\frac{3\pi}{2}\right) == \sec(z)$
$\csc(-z-\pi) == \csc(z) \quad \csc(z-\pi) == -\csc(z)$
$\csc\left(-z-\frac{\pi}{2}\right) == -\sec(z) \quad \csc\left(z-\frac{\pi}{2}\right) == -\sec(z)$
$\csc\left(z+\frac{\pi}{2}\right) == \sec(z) \quad \csc\left(\frac{\pi}{2}-z\right) == \sec(z)$
$\csc(z+\pi) == -\csc(z) \quad \csc(\pi-z) == \csc(z)$
$\csc\left(z+\frac{3\pi}{2}\right) == -\sec(z) \quad \csc\left(\frac{3\pi}{2}-z\right) == -\sec(z)$
$\csc(z+2\pi) == \csc(z) \quad \csc(2\pi-z) == -\csc(z)$

$\sec(-z-2\pi) == \sec(z) \quad \sec(z-2\pi) == \sec(z)$
$\sec\left(-z-\frac{3\pi}{2}\right) == \csc(z) \quad \sec\left(z-\frac{3\pi}{2}\right) == -\csc(z)$
$\sec(-z-\pi) == -\sec(z) \quad \sec(z-\pi) == -\sec(z)$
$\sec\left(-z-\frac{\pi}{2}\right) == -\csc(z) \quad \sec\left(z-\frac{\pi}{2}\right) == \csc(z)$
$\sec\left(z+\frac{\pi}{2}\right) == -\csc(z) \quad \sec\left(\frac{\pi}{2}-z\right) == \csc(z)$
$\sec(z+\pi) == -\sec(z) \quad \sec(\pi-z) == -\sec(z)$
$\sec\left(z+\frac{3\pi}{2}\right) == \csc(z) \quad \sec\left(\frac{3\pi}{2}-z\right) == -\csc(z)$
$\sec(z+2\pi) == \sec(z) \quad \sec(2\pi-z) == \sec(z).$

## The best-known properties and formulas for trigonometric functions

### Real values for real arguments

For real values of argument $z$, the values of all the trigonometric functions are real (or infinity).

In the points $z = 2\pi n / m$ /; $n \in \mathbb{Z} \bigwedge m \in \mathbb{Z}$, the values of trigonometric functions are algebraic. In several cases they can even be rational numbers or integers (like $\sin(\pi/2) = 1$ or $\sin(\pi/6) = 1/2$). The values of trigonometric functions can be expressed using only square roots if $n \in \mathbb{Z}$ and $m$ is a product of a power of 2 and distinct Fermat primes {3, 5, 17, 257, …}.

### Simple values at zero

All trigonometric functions have rather simple values for arguments $z == 0$ and $z == \pi/2$:

$\sin(0) == 0 \quad \sin\left(\frac{\pi}{2}\right) == 1$

$\cos(0) == 1 \quad \cos\left(\frac{\pi}{2}\right) == 0$

$\tan(0) == 0 \quad \tan\left(\frac{\pi}{2}\right) == \tilde{\infty}$

$\cot(0) == \tilde{\infty} \quad \cot\left(\frac{\pi}{2}\right) == 0$

$\csc(0) == \tilde{\infty} \quad \csc\left(\frac{\pi}{2}\right) == 1$

$\sec(0) == 1 \quad \sec\left(\frac{\pi}{2}\right) == \tilde{\infty}.$

### Analyticity

All trigonometric functions are defined for all complex values of $z$, and they are analytical functions of $z$ over the whole complex $z$-plane and do not have branch cuts or branch points. The two functions $\sin(z)$ and $\cos(z)$ are entire functions with an essential singular point at $z = \tilde{\infty}$. All other trigonometric functions are meromorphic functions with simple poles at points $z == \pi k$ /; $k \in \mathbb{Z}$ for $\csc(z)$ and $\cot(z)$, and at points $z == \pi/2 + \pi k$ /; $k \in \mathbb{Z}$ for $\sec(z)$ and $\tan(z)$.

### Periodicity

All trigonometric functions are periodic functions with a real period ($2\pi$ or $\pi$):

$\sin(z) == \sin(z + 2\pi) \quad \sin(z + 2\pi k) == \sin(z) \;/;\; k \in \mathbb{Z}$

$\cos(z) == \cos(z + 2\pi) \quad \cos(z + 2\pi k) == \cos(z) \;/;\; k \in \mathbb{Z}$

$\tan(z) == \tan(z + \pi) \quad \tan(z + \pi k) == \tan(z) \;/;\; k \in \mathbb{Z}$

$\cot(z) == \cot(z + \pi) \quad \cot(z + \pi k) == \cot(z) \;/;\; k \in \mathbb{Z}$

$\csc(z) == \csc(z + 2\pi) \quad \csc(z + 2\pi k) == \csc(z) \;/;\; k \in \mathbb{Z}$

$\sec(z) == \sec(z + 2\pi) \quad \sec(z + 2\pi k) == \sec(z) \;/;\; k \in \mathbb{Z}.$

### Parity and symmetry

All trigonometric functions have parity (either odd or even) and mirror symmetry:

$\sin(-z) == -\sin(z) \quad \sin(\bar{z}) == \overline{\sin(z)}$

$\cos(-z) == \cos(z) \quad \cos(\bar{z}) == \overline{\cos(z)}$

$\tan(-z) == -\tan(z) \quad \tan(\bar{z}) == \overline{\tan(z)}$

$\cot(-z) == -\cot(z) \quad \cot(\bar{z}) == \overline{\cot(z)}$

$\csc(-z) == -\csc(z) \quad \csc(\bar{z}) == \overline{\csc(z)}$

$\sec(-z) == \sec(z) \quad \sec(\bar{z}) == \overline{\sec(z)}.$

### Simple representations of derivatives

The derivatives of all trigonometric functions have simple representations that can be expressed through other trigonometric functions:

$$\frac{\partial\sin(z)}{\partial z} == \cos(z) \qquad \frac{\partial\cos(z)}{\partial z} == -\sin(z) \qquad \frac{\partial\tan(z)}{\partial z} == \sec^2(z)$$
$$\frac{\partial\cot(z)}{\partial z} == -\csc^2(z) \qquad \frac{\partial\csc(z)}{\partial z} == -\cot(z)\csc(z) \qquad \frac{\partial\sec(z)}{\partial z} == \sec(z)\tan(z).$$

### Simple differential equations

The solutions of the simplest second-order linear ordinary differential equation with constant coefficients can be represented through $\sin(z)$ and $\cos(z)$:

$$w''(z) + w(z) == 0 \;/;\; w(z) == \cos(z) \bigwedge w(0) == 1 \bigwedge w'(0) == 0$$
$$w''(z) + w(z) == 0 \;/;\; w(z) == \sin(z) \bigwedge w(0) == 0 \bigwedge w'(0) == 1$$
$$w''(z) + w(z) == 0 \;/;\; w(z) == c_1\cos(z) + c_2\sin(z).$$

All six trigonometric functions satisfy first-order nonlinear differential equations:

$$w'(z) - \sqrt{1-(w(z))^2} == 0 \;/;\; w(z) == \sin(z) \bigwedge w(0) == 0 \bigwedge |\mathrm{Re}(z)| < \frac{\pi}{2}$$
$$w'(z) - \sqrt{1-(w(z))^2} == 0 \;/;\; w(z) == \cos(z) \bigwedge w(0) == 1 \bigwedge |\mathrm{Re}(z)| < \frac{\pi}{2}$$
$$w'(z) - w(z)^2 - 1 == 0 \;/;\; w(z) == \tan(z) \wedge w(0) == 0$$
$$w'(z) + w(z)^2 + 1 == 0 \;/;\; w(z) == \cot(z) \bigwedge w\left(\frac{\pi}{2}\right) == 0$$
$$w'(z)^2 - w(z)^4 + w(z)^2 == 0 \;/;\; w(z) == \csc(z)$$
$$w'(z)^2 - w(z)^4 + w(z)^2 == 0 \;/;\; w(z) == \sec(z).$$

## Applications of trigonometric functions

### Triangle theorems

The prime application of the trigonometric functions are triangle theorems. In a triangle, $a$, $b$, and $c$ represent the lengths of the sides opposite to the angles, $\Delta$ the area , $R$ the circumradius, and $r$ the inradius. Then the following identities hold:

$$\alpha + \beta + \gamma == \pi$$
$$\frac{\sin(\alpha)}{a} == \frac{\sin(\beta)}{b} == \frac{\sin(\gamma)}{c}$$

$$\sin(\alpha)\sin(\beta)\sin(\gamma) == \frac{\Delta}{2R^2} \qquad \sin(\alpha) == \frac{2\Delta}{b\,c}$$
$$\cos(\alpha) == \frac{b^2+c^2-a^2}{2\,b\,c} \qquad \cot(\alpha) == \frac{b^2+c^2-a^2}{4\Delta}$$
$$\sin\left(\frac{\alpha}{2}\right)\sin\left(\frac{\beta}{2}\right)\sin\left(\frac{\gamma}{2}\right) == \frac{r}{4R} \qquad \cos(\alpha)+\cos(\beta)+\cos(\gamma) == 1 + \frac{r}{R}$$

$$\cot(\alpha) + \cot(\beta) + \cot(\gamma) == \frac{a^2 + b^2 + c^2}{4\,\Delta}$$

$$\tan(\alpha) + \tan(\beta) + \tan(\gamma) == \tan(\alpha)\tan(\beta)\tan(\gamma)$$

$$\cot(\alpha)\cot(\beta) + \cot(\alpha)\cot(\gamma) + \cot(\beta)\cot(\gamma) == 1$$

$$\cos^2(\alpha) + \cos^2(\beta) + \cos^2(\gamma) == 1 - 2\cos(\alpha)\cos(\beta)\cos(\gamma)$$

$$\frac{\tan\left(\frac{\alpha}{2}\right)\tan\left(\frac{\beta}{2}\right)}{\tan\left(\frac{\alpha}{2}\right) + \tan\left(\frac{\beta}{2}\right)} == \frac{r}{c}.$$

For a right-angle triangle the following relations hold:

$$\sin(\alpha) == \frac{a}{c} \text{ /; } \gamma == \frac{\pi}{2} \quad \cos(\alpha) == \frac{b}{c} \text{ /; } \gamma == \frac{\pi}{2}$$

$$\tan(\alpha) == \frac{a}{b} \text{ /; } \gamma == \frac{\pi}{2} \quad \cot(\alpha) == \frac{b}{a} \text{ /; } \gamma == \frac{\pi}{2}$$

$$\csc(\alpha) == \frac{c}{a} \text{ /; } \gamma == \frac{\pi}{2} \quad \sec(\alpha) == \frac{c}{b} \text{ /; } \gamma == \frac{\pi}{2}.$$

### Other applications

Because the trigonometric functions appear virtually everywhere in quantitative sciences, it is impossible to list their numerous applications in teaching, science, engineering, and art.

# Introduction to the Tangent Function

## Defining the tangent function

The tangent function is an old mathematical function. It was mentioned in 1583 by T. Fincke who introduced the word "tangens" in Latin. E. Gunter (1624) used the notation "tan", and J. H. Lambert (1770) discovered the continued fraction representation of this function.

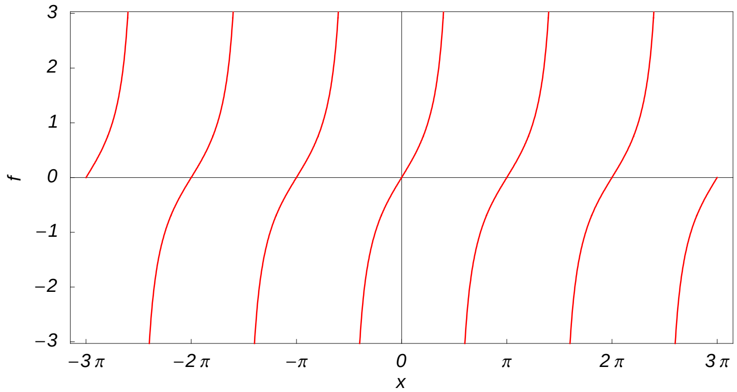
The classical definition of the tangent function for real arguments is: "the tangent of an angle $\alpha$ in a right-angle triangle is the ratio of the length of the opposite leg to the length of the adjacent leg." This description of $\tan(\alpha)$ is valid for $0 < \alpha < \pi/2$ when the triangle is nondegenerate. This approach to the tangent can be expanded to arbitrary real values of $\alpha$ if consideration is given to the arbitrary point $\{x, y\}$ in the $x$,$y$-Cartesian plane and $\tan(\alpha)$ is defined as the ratio $y/x$, assuming that $\alpha$ is the value of the angle between the positive direction of the $x$-axis and the direction from the origin to the point $\{x, y\}$.

Comparing this definition with definitions of the sine and cosine functions shows that the following formula can also be used as a definition of the tangent function:

$$\tan(z) == \frac{\sin(z)}{\cos(z)}.$$

## A quick look at the tangent function

Here is a graphic of the tangent function $f(x) = \tan(x)$ for real values of its argument $x$.




## Representation through more general functions

The tangent function $\tan(z)$ can be represented using more general mathematical functions. As the ratio of the sine and cosine functions that are particular cases of the generalized hypergeometric, Bessel, Struve, and Mathieu functions, the tangent function can also be represented as ratios of those special functions. But these representations are not very useful. It is more useful to write the tangent function as particular cases of one special function. That can be done using doubly periodic Jacobi elliptic functions that degenerate into the tangent function when their second parameter is equal to 0 or 1:

$$\tan(z) = \operatorname{sc}(z \mid 0) = \operatorname{cs}\left(\frac{\pi}{2} - z \,\middle|\, 0\right) = -i \operatorname{sn}(i z \mid 1) = i \operatorname{ns}\left(\frac{\pi i}{2} - i z \,\middle|\, 1\right).$$

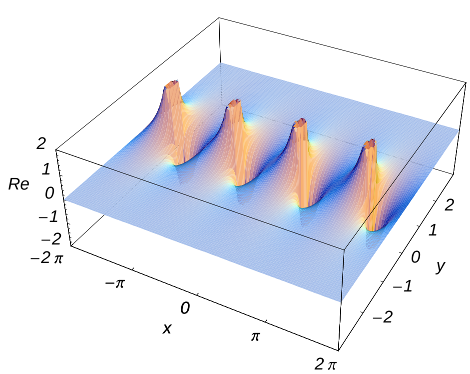
## Definition of the tangent function for a complex argument

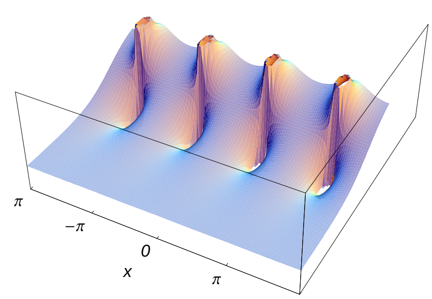
In the complex $z$-plane, the function $\tan(z)$ is defined using $\sin(z)$ and $\cos(z)$ or the exponential function $e^w$ in the points $i z$ and $-i z$ through the formula:

$$\tan(z) = \frac{\sin(z)}{\cos(z)} = -\frac{i\left(e^{i z} - e^{-i z}\right)}{e^{i z} + e^{-i z}}.$$

In the points $z = \pi/2 + \pi k \,/;\, k \in \mathbb{Z}$, where $\cos(z)$ has zeros, the denominator of the last formula equals zero and $\tan(z)$ has singularities (poles of the first order).

Here are two graphics showing the real and imaginary parts of the tangent function over the complex plane.



# The best-known properties and formulas for the tangent function

## Values in points

Students usually learn the following basic table of tangent function values for special points of the circle:

$\tan(0) = 0 \quad \tan\left(\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}} \quad \tan\left(\frac{\pi}{4}\right) = 1 \quad \tan\left(\frac{\pi}{3}\right) = \sqrt{3}$

$\tan\left(\frac{\pi}{2}\right) = \tilde{\infty} \quad \tan\left(\frac{2\pi}{3}\right) = -\sqrt{3} \quad \tan\left(\frac{3\pi}{4}\right) = -1 \quad \tan\left(\frac{5\pi}{6}\right) = -\frac{1}{\sqrt{3}}$

$\tan(\pi) = 0$

$\tan(\pi m) = 0 \;/;\; m \in \mathbb{Z} \quad \tan\left(\pi\left(\frac{1}{2} + m\right)\right) = \tilde{\infty} \;/;\; m \in \mathbb{Z}.$

## General characteristics

For real values of argument $z$, the values of $\tan(z)$ are real.

In the points $z = \pi n / m \;/;\; n \in \mathbb{Z} \land m \in \mathbb{Z}$, the values of $\tan(z)$ are algebraic. In several cases they can be integers $-1$, 0, or 1:

$\tan\left(-\frac{\pi}{4}\right) = -1 \quad \tan(0) = 0 \quad \tan\left(\frac{\pi}{4}\right) = 1.$

The values of $\tan\left(\frac{n\pi}{m}\right)$ can be expressed using only square roots if $n \in \mathbb{Z}$ and $m$ is a product of a power of 2 and distinct Fermat primes $\{3, 5, 17, 257, \ldots\}$.

The function $\tan(z)$ is an analytical function of $z$ that is defined over the whole complex $z$-plane and does not have branch cuts and branch points. It has an infinite set of singular points:

(a) $z = \pi / 2 + \pi k \;/;\; k \in \mathbb{Z}$ are the simple poles with residues –1.
(b) $z = \tilde{\infty}$ is an essential singular point.

It is a periodic function with the real period $\pi$:

$\tan(z + \pi) = \tan(z)$

$\tan(z) = \tan(z + \pi k) \;/;\; k \in \mathbb{Z}.$

The function $\tan(z)$ is an odd function with mirror symmetry:

$\tan(-z) = -\tan(z) \quad \tan(\bar{z}) = \overline{\tan(z)}.$

## Differentiation

The first derivative of $\tan(z)$ has simple representations using either the $\cos(z)$ function or the $\sec(z)$ function:

$$\frac{\partial \tan(z)}{\partial z} = \frac{1}{\cos^2(z)} = \sec^2(z).$$

The $n^{\text{th}}$ derivative of $\tan(z)$ has much more complicated representations than symbolic $n^{\text{th}}$ derivatives for $\sin(z)$ and $\cos(z)$:

$$\frac{\partial^n \tan(z)}{\partial z^n} = \delta_n \tan(z) + \delta_{n-1} \sec^2(z) + n \sum_{k=0}^{n-1} \sum_{j=0}^{k-1} \frac{(-1)^k (k-j)^{n-1} 2^{n-2k} \cos^{-2k-2}(z)}{k+1} \binom{n-1}{k} \binom{2k}{j} \sin\left(\frac{\pi n}{2} + 2(k-j) z\right) /; n \in \mathbb{N},$$

where $\delta_n$ is the Kronecker delta symbol: $\delta_0 = 1$ and $\delta_n = 0 /; n \neq 0$.

### Ordinary differential equation

The function $\tan(z)$ satisfies the following first-order nonlinear differential equation:

$$w'(z) - w(z)^2 - 1 = 0 /; w(z) = \tan(z) \land w(0) = 0.$$

### Series representation

The function $\tan(z)$ has a simple series expansion at the origin that converges for all finite values $z$ with $|z| < \frac{\pi}{2}$:

$$\tan(z) = z + \frac{z^3}{3} + \frac{2 z^5}{15} + \ldots = \sum_{k=1}^{\infty} \frac{(-1)^{k-1} 2^{2k} \left(2^{2k} - 1\right) B_{2k}}{(2k)!} z^{2k-1},$$

where $B_{2k}$ are the Bernoulli numbers.

### Integral representation

The function $\tan(z)$ has a well-known integral representation through the following definite integral along the positive part of the real axis:

$$\tan(z) = \frac{2}{\pi} \int_0^{\infty} \frac{t^{\frac{2z}{\pi}} - 1}{t^2 - 1} \, dt /; 0 < \operatorname{Re}(z) < \frac{\pi}{2}.$$

### Continued fraction representations

The function $\tan(z)$ has the following simple continued fraction representations:

$$\tan(z) = \cfrac{z}{1 - \cfrac{z^2}{3 - \cfrac{z^2}{5 - \cfrac{z^2}{7 - \cfrac{z^2}{9 - \cfrac{z^2}{11 - \ldots}}}}}} /; \frac{z}{\pi} - \frac{1}{2} \notin \mathbb{Z}$$

$$\tan(z) = \cfrac{1}{\cfrac{1}{z} - \cfrac{1}{\cfrac{3}{z} - \cfrac{1}{\cfrac{5}{z} - \cfrac{1}{\cfrac{7}{z} - \cfrac{1}{\cfrac{9}{z} - \cfrac{1}{\cfrac{11}{z} - \ldots}}}}}} \;/;\; \frac{z}{\pi} - \frac{1}{2} \notin \mathbb{Z}.$$

### Indefinite integration

Indefinite integrals of expressions involving the tangent function can sometimes be expressed using elementary functions. However, special functions are frequently needed to express the results even when the integrands have a simple form (if they can be evaluated in closed form). Here are some examples:

$$\int \tan(z)\, dz = -\log(\cos(z))$$

$$\int \sqrt{\tan(z)}\, dz = \frac{1}{2\sqrt{2}} \left(2\tan^{-1}\left(\sqrt{2}\, \tan^{\frac{1}{2}}(z) + 1\right) - 2\tan^{-1}\left(1 - \sqrt{2}\, \tan^{\frac{1}{2}}(z)\right) + \log\left(-\tan(z) + \sqrt{2}\, \tan^{\frac{1}{2}}(z) - 1\right) - \log\left(\tan(z) + \sqrt{2}\, \tan^{\frac{1}{2}}(z) + 1\right)\right)$$

$$\int \tan^{\nu}(a\, z)\, dz = \frac{\tan^{\nu+1}(a\, z)}{a\, (\nu+1)}\, {}_2F_1\left(\frac{\nu+1}{2}, 1; \frac{\nu+1}{2} + 1; -\tan^2(a\, z)\right).$$

### Definite integration

Definite integrals that contain the tangent function are sometimes simple. For example, the famous Catalan constant $C$ can be defined as the value of the following integral:

$$\int_0^{\frac{\pi}{4}} \log(\tan(t))\, dt = -C.$$

This constant also appears in the following integral:

$$\int_0^{\frac{\pi}{4}} t \tan(t)\, dt = \frac{1}{16}\left(8\, C + i\, \pi^2 - 4\, \pi \log(1 + i)\right).$$

Some special functions can be used to evaluate more complicated definite integrals. For example, the generalized hypergeometric and polygamma functions are needed to express the following integral:

$$\int_0^{\frac{\pi}{4}} \sin^{\alpha-1}(t) \tan(t)\, dt = \frac{1}{8}\left((\alpha - 1)\, {}_3F_2\left(1, 1, \frac{3}{2} - \frac{\alpha}{2}; 2, 2; \frac{1}{2}\right) + \log(16) - 4\, \psi\left(\frac{\alpha+1}{2}\right) - 4\, \gamma\right) /;\; \mathrm{Re}(\alpha) > -1.$$

### Finite summation

The following finite sums that contain the tangent function can be expressed using cotangent functions:

$$\sum_{k=0}^{n} \frac{1}{2^k} \tan\left(\frac{z}{2^k}\right) = \frac{1}{2^n} \cot\left(\frac{z}{2^n}\right) - 2\cot(2z)$$

$$\sum_{k=0}^{n-1} \tan^2\left(\frac{\pi k}{n} + z\right) = \cot^2\left(z n + \frac{\pi n}{2}\right) n^2 + n^2 - n \;/;\; n \in \mathbb{N}^+.$$

Other finite sums that contain the tangent function can be expressed using polynomial functions:

$$\sum_{k=0}^{n-1} (-1)^k \tan\left(\frac{(2k+1)\pi}{4n}\right) = (-1)^{n-1} n \;/;\; n \in \mathbb{N}^+$$

$$\sum_{k=1}^{n} \tan^4\left(\frac{k\pi}{2n+1}\right) = \frac{1}{3} n (2n+1) \left(4n^2 + 6n - 1\right) \;/;\; n \in \mathbb{N}^+$$

$$\sum_{k=1}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \tan^2\left(\frac{k\pi}{n}\right) = \frac{1}{6} (n-1) (-(-1)^n (n+1) + 2n - 1) \;/;\; n \in \mathbb{N}^+.$$

#### Infinite summation

The evaluation limit of the first formula from the previous subsubsection for $n \to \infty$ gives the following value for the corresponding infinite sum from the tangent:

$$\sum_{k=1}^{\infty} \frac{1}{2^k} \tan\left(\frac{z}{2^k}\right) = \frac{1}{z} - \cot(z).$$

Other infinite sums that contain the tangent can also be expressed using elementary functions:

$$\sum_{k=1}^{\infty} \frac{1}{2^{2k}} \tan^2\left(\frac{z}{2^k}\right) = \csc^2(z) - \frac{1}{3} - \frac{1}{z^2}.$$

#### Finite products

The following finite product from the tangent has a very simple value:

$$\prod_{k=1}^{n-1} \tan\left(\frac{k\pi}{n}\right) = (-1)^{(n-1)/2} n \;/;\; 2n+1 \in \mathbb{N}^+.$$

#### Addition formulas

The tangent of a sum can be represented by the rule: "the tangent of a sum is equal to the sum of tangents divided by one minus the product of tangents." A similar rule is valid for the tangent of the difference:

$$\tan(a+b) = \frac{\tan(a) + \tan(b)}{1 - \tan(a)\tan(b)}$$

$$\tan(a-b) = \frac{\tan(a) - \tan(b)}{\tan(a)\tan(b) + 1}.$$

#### Multiple arguments

In the case of multiple arguments $z$, $2z$, $3z$, …, the function $\tan(z)$ can be represented as the ratio of the finite sums including powers of tangents:

$$\tan(2z) = \frac{2\tan(z)}{1-\tan^2(z)}$$

$$\tan(3z) = \frac{3\tan(z)-\tan^3(z)}{1-3\tan^2(z)}$$

$$\tan(nz) = \frac{1}{\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \binom{n}{2k} \tan^{2k}(z)} \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^k \binom{n}{2k+1} \tan^{2k+1}(z) \;/;\; n \in \mathbb{N}^+.$$

### Half-angle formulas

The tangent of the half-angle can be represented using two trigonometric functions by the following simple formulas:

$$\tan\left(\frac{z}{2}\right) = \csc(z) - \cot(z)$$

$$\tan\left(\frac{z}{2}\right) = \frac{\sin(z)}{\cos(z)+1}.$$

The sine function in the last formula can be replaced by the cosine function. But it leads to a more complicated representation that is valid in some vertical strips:

$$\tan\left(\frac{z}{2}\right) = \sqrt{\frac{1-\cos(z)}{1+\cos(z)}} \;/;\; 0 < \mathrm{Re}(z) < \pi \lor \mathrm{Re}(z) = 0 \land \mathrm{Im}(z) \geq 0 \lor \mathrm{Re}(z) = \pi \land \mathrm{Im}(z) > 0.$$

To make this formula correct for all complex $z$, a complicated prefactor is needed:

$$\tan\left(\frac{z}{2}\right) = c(z)\sqrt{\frac{1-\cos(z)}{1+\cos(z)}} \;/;\; c(z) = (-1)^{\left\lfloor \frac{\mathrm{Re}(z)}{\pi} - \frac{1}{2} \right\rceil} \left(1 - \left(1 + (-1)^{\left\lfloor \frac{\mathrm{Re}(z)}{\pi} \right\rfloor + \left\lfloor -\frac{\mathrm{Re}(z)}{\pi} \right\rfloor}\right) \theta(-\mathrm{Im}(z))\right),$$

where $c(z)$ contains the unit step, real part, imaginary part, the floor, and the round functions.

### Sums of two direct functions

The sum of two tangent functions can be described by the rule: "the sum of tangents is equal to the sine of the sum multiplied by the secants." A similar rule is valid for the difference of two tangents:

$$\tan(a) + \tan(b) = \sec(a)\sec(b)\sin(a+b)$$
$$\tan(a) - \tan(b) = \sec(a)\sec(b)\sin(a-b).$$

### Products involving the direct function

The product of two tangent functions and the product of the tangent and cotangent have the following representations:

$$\tan(a)\tan(b) = \frac{\cos(a-b)-\cos(a+b)}{\cos(a-b)+\cos(a+b)}$$

$$\tan(a)\cot(b) = \frac{\sin(a-b)+\sin(a+b)}{\sin(a+b)-\sin(a-b)}.$$

**Inequalities**

The most famous inequality for the tangent function is the following:

$$\tan(x) \geq x\ /;\ 0 \leq x < \frac{\pi}{2} \bigwedge x \in \mathbb{R}.$$

**Relations with its inverse function**

There are simple relations between the function $\tan(z)$ and its inverse function $\tan^{-1}(z)$:

$$\tan\left(\tan^{-1}(z)\right) = z \quad \tan^{-1}(\tan(z)) = z\ /;\ |\mathrm{Re}(z)| < \frac{\pi}{2} \bigvee \mathrm{Re}(z) = -\frac{\pi}{2} \bigwedge \mathrm{Im}(z) < 0 \bigvee \mathrm{Re}(z) = \frac{\pi}{2} \bigwedge \mathrm{Im}(z) > 0.$$

The second formula is valid at least in the vertical strip $-\frac{\pi}{2} < \mathrm{Re}(z) < \frac{\pi}{2}$. Outside of this strip a much more complicated relation (that contain the unit step, real part, and the floor functions) holds:

$$\tan^{-1}(\tan(z)) = z - \pi \left\lfloor \frac{\mathrm{Re}(z)}{\pi} + \frac{1}{2} \right\rfloor + \frac{1}{2}\left(1 + (-1)^{\left\lfloor \frac{\mathrm{Re}(z)}{\pi} - \frac{1}{2} \right\rfloor + \left\lfloor \frac{1}{2} - \frac{\mathrm{Re}(z)}{\pi} \right\rfloor}\right) \pi\, \theta(\mathrm{Im}(z))\ /;\ \frac{z}{\pi} - \frac{1}{2} \notin \mathbb{Z}.$$

**Representations through other trigonometric functions**

Tangent and cotangent functions are connected by a very simple formula that contains the linear function in the argument:

$$\tan(z) = \cot\left(\frac{\pi}{2} - z\right).$$

The tangent function can also be represented using other trigonometric functions by the following formulas:

$$\tan(z) = \frac{\sin(z)}{\sin\left(\frac{\pi}{2}-z\right)} \quad \tan(z) = \frac{\cos\left(\frac{\pi}{2}-z\right)}{\cos(z)}$$

$$\tan(z) = \frac{\csc\left(\frac{\pi}{2}-z\right)}{\csc(z)} \quad \tan(z) = \frac{\sec(z)}{\sec\left(\frac{\pi}{2}-z\right)}.$$

**Representations through hyperbolic functions**

The tangent function has representations using the hyperbolic functions:

$$\tan(z) = \frac{\sinh(i z)}{\sinh\left(\frac{i\pi}{2}-i z\right)} \quad \tan(z) = \frac{\cosh\left(\frac{i\pi}{2}-i z\right)}{\cosh(i z)} \quad \tan(z) = -i \tanh(i z) \quad \tan(i z) = i \tanh(z)$$

$$\tan(z) = i \coth\left(\frac{\pi i}{2} - i z\right) \quad \tan(z) = \frac{\mathrm{csch}\left(\frac{\pi i}{2}-i z\right)}{\mathrm{csch}(i z)} \quad \tan(z) = \frac{\mathrm{sech}(i z)}{\mathrm{sech}\left(\frac{\pi i}{2}-i z\right)}.$$

**Applications**

The tangent function is used throughout mathematics, the exact sciences, and engineering.

# Introduction to the Trigonometric Functions in *Mathematica*

## Overview

The following shows how the six trigonometric functions are realized in *Mathematica*. Examples of evaluating *Mathematica* functions applied to various numeric and exact expressions that involve the trigonometric functions or return them are shown. These involve numeric and symbolic calculations and plots.

## Notations

### *Mathematica* forms of notations

All six trigonometric functions are represented as built-in functions in *Mathematica*. Following *Mathematica*'s general naming convention, the `StandardForm` function names are simply capitalized versions of the traditional mathematics names. Here is a list `trigFunctions` of the six trigonometric functions in `StandardForm`.

```
trigFunctions = {Sin[z], Cos[z], Tan[z], Cot[z], Sec[z], Cos[z]}
```

```
{Sin[z], Cos[z], Tan[z], Cot[z], Sec[z], Cos[z]}
```

Here is a list `trigFunctions` of the six trigonometric functions in `TraditionalForm`.

```
trigFunctions // TraditionalForm
```

{sin(z), cos(z), tan(z), cot(z), sec(z), cos(z)}

### Additional forms of notations

*Mathematica* also knows the most popular forms of notations for the trigonometric functions that are used in other programming languages. Here are three examples: `CForm`, `TeXForm`, and `FortranForm`.

```
trigFunctions /. {z → 2 π z} // (CForm /@ #) &
```

```
{Sin(2*Pi*z), Cos(2*Pi*z), Tan(2*Pi*z),
 Cot(2*Pi*z), Sec(2*Pi*z), Cos(2*Pi*z)}
```

```
trigFunctions /. {z → 2 π z} // (TeXForm /@ #) &
```

```
{\sin(2\, \pi \, z), \cos(2\, \pi \, z), \tan(2\, \pi \, z), \cot
  (2\, \pi \, z), \sec(2\, \pi \, z), \cos(2\, \pi \, z)}
```

```
trigFunctions /. {z → 2 π z} // (FortranForm /@ #) &
```

```
{Sin(2*Pi*z), Cos(2*Pi*z), Tan(2*Pi*z),
 Cot(2*Pi*z), Sec(2*Pi*z), Cos(2*Pi*z)}
```

## Automatic evaluations and transformations

### Evaluation for exact, machine-number, and high-precision arguments

For a simple exact argument, *Mathematica* returns exact results. For instance, for the argument $\pi/6$, the `Sin` function evaluates to $1/2$.

**Sin**$\left[\frac{\pi}{6}\right]$

$\frac{1}{2}$

**{Sin[z], Cos[z], Tan[z], Cot[z], Csc[z], Sec[z]} /. z →** $\frac{\pi}{6}$

$\left\{\frac{1}{2}, \frac{\sqrt{3}}{2}, \frac{1}{\sqrt{3}}, \sqrt{3}, 2, \frac{2}{\sqrt{3}}\right\}$

For a generic machine-number argument (a numerical argument with a decimal point and not too many digits), a machine number is returned.

```
Cos[3.]
```

```
-0.989992
```

```
{Sin[z], Cos[z], Tan[z], Cot[z], Csc[z], Sec[z]} /. z → 2.
```

```
{0.909297, -0.416147, -2.18504, -0.457658, 1.09975, -2.403}
```

The next inputs calculate 100-digit approximations of the six trigonometric functions at $z = 1$.

```
N[Tan[1], 40]
```

```
1.557407724654902230506974807458360173087
```

```
Cot[1] // N[#, 50] &
```

```
0.64209261593433070300641998659426562023027811391817
```

```
N[{Sin[z], Cos[z], Tan[z], Cot[z], Csc[z], Sec[z]} /. z → 1, 100]
```

```
{0.8414709848078965066525023216302989996225630607983710656727517099919104043912396689
  486397435430526959,
 0.5403023058681397174009366074429766037323104206179222276700972553811003947744717645
  179518560871830893,
 1.5574077246549022305069748074583601730872507723815200383839466056988613971517272895
  55099965202242984,
 0.6420926159343307030064199865942656202302781139181713791011622804262768568391646721
  984829197601968047,
 1.1883951057781212162615994523745510035278298340979626252652536663591843673571904879
  13663568030853023,
 1.8508157176809256179117532413986501934703966550940092988351582778588154112615967059
  21841413287306671}
```

Within a second, it is possible to calculate thousands of digits for the trigonometric functions. The next input calculates 10000 digits for sin(1), cos(1), tan(1), cot(1), sec(1), and csc(1) and analyzes the frequency of the occurrence of the digit *k* in the resulting decimal number.

```
Map[Function[w, {First[#], Length[#]} & /@ Split[Sort[First[RealDigits[w]]]]],
 N[{Sin[z], Cos[z], Tan[z], Cot[z], Csc[z], Sec[z]} /. z → 1, 10 000]]
```

```
{{{0, 983}, {1, 1069}, {2, 1019}, {3, 983}, {4, 972}, {5, 994},
  {6, 994}, {7, 988}, {8, 988}, {9, 1010}}, {{0, 998}, {1, 1034}, {2, 982},
  {3, 1015}, {4, 1013}, {5, 963}, {6, 1034}, {7, 966}, {8, 991}, {9, 1004}},
 {{0, 1024}, {1, 1025}, {2, 1000}, {3, 969}, {4, 1026}, {5, 944}, {6, 999},
  {7, 1001}, {8, 1008}, {9, 1004}}, {{0, 1006}, {1, 1030}, {2, 986},
  {3, 954}, {4, 1003}, {5, 1034}, {6, 999}, {7, 998}, {8, 1009}, {9, 981}},
 {{0, 1031}, {1, 976}, {2, 1045}, {3, 917}, {4, 1001}, {5, 996}, {6, 964},
  {7, 1012}, {8, 982}, {9, 1076}}, {{0, 978}, {1, 1034}, {2, 1016},
  {3, 974}, {4, 987}, {5, 1067}, {6, 943}, {7, 1006}, {8, 1027}, {9, 968}}}
```

Here are 50-digit approximations to the six trigonometric functions at the complex argument $z = 3 + 5\,i$.

```
N[Csc[3 + 5 ⅈ], 100]
```

```
0.001901970423701089996670017296320805840459252512171274310801719695392870034046820 2
  96847410109982878354 +
 0.013341591397996678721837322466473194390132347157253190972075437462485814431570118
   67262664488519840339 ⅈ
```

```
N[{Sin[z], Cos[z], Tan[z], Cot[z], Csc[z], Sec[z]} /. z → 3 + 5 ⅈ, 50]
```

```
{10.472508533940392276673322536853503271126419950388 -
  73.460621695673676366791192505081750407213922814475 ⅈ,
 -73.467292212645262467746454594833950830814859165299 -
  10.471557674805574377394464224329537808548330651734 ⅈ,
 -0.000025368676207676032417806136707426288195560702602478 +
  0.99991282015135380828209263013972954140566020462086 ⅈ,
 -0.000025373100044545977383763346789469656754050037355986 -
  1.0000871868058967743285316881045218577131612831891 ⅈ,
 0.0019019704237010899966700172963208058404592525121713 +
  0.013341591397996678721837322466473194390132347157253 ⅈ,
 -0.013340476530549737487361100811100839468470481725038 +
  0.0019014661516951513089519270013254277867588978133499 ⅈ}
```

*Mathematica* always evaluates mathematical functions with machine precision, if the arguments are machine numbers. In this case, only six digits after the decimal point are shown in the results. The remaining digits are suppressed, but can be displayed using the function `InputForm`.

```
{Sin[2.], N[Sin[2]], N[Sin[2], 16], N[Sin[2], 5], N[Sin[2], 20]}
```

```
{0.909297, 0.909297, 0.909297, 0.909297, 0.90929742682568169540}
```

```
% // InputForm
```

```
{0.9092974268256817, 0.9092974268256817, 0.9092974268256817, 0.9092974268256817,
 0.909297426825681695396019865911745`20}
```

```
Precision[%%]
```

```
16
```

### Simplification of the argument

*Mathematica* uses symmetries and periodicities of all the trigonometric functions to simplify expressions. Here are some examples.

```
Sin[-z]
```

```
-Sin[z]
```

```
Sin[z + π]
```

```
-Sin[z]
```

```
Sin[z + 2 π]
```

```
Sin[z]
```

```
Sin[z + 34 π]
```

```
Sin[z]
```

```
{Sin[-z], Cos[-z], Tan[-z], Cot[-z], Csc[-z], Sec[-z]}
```

```
{-Sin[z], Cos[z], -Tan[z], -Cot[z], -Csc[z], Sec[z]}
```

```
{Sin[z + π], Cos[z + π], Tan[z + π], Cot[z + π], Csc[z + π], Sec[z + π]}
```

```
{-Sin[z], -Cos[z], Tan[z], Cot[z], -Csc[z], -Sec[z]}
```

```
{Sin[z + 2 π], Cos[z + 2 π], Tan[z + 2 π], Cot[z + 2 π], Csc[z + 2 π], Sec[z + 2 π]}
```

```
{Sin[z], Cos[z], Tan[z], Cot[z], Csc[z], Sec[z]}
```

```
{Sin[z + 342 π], Cos[z + 342 π], Tan[z + 342 π], Cot[z + 342 π], Csc[z + 342 π], Sec[z + 342 π]}
```

```
{Sin[z], Cos[z], Tan[z], Cot[z], Csc[z], Sec[z]}
```

*Mathematica* automatically simplifies the composition of the direct and the inverse trigonometric functions into the argument.

```
{Sin[ArcSin[z]], Cos[ArcCos[z]], Tan[ArcTan[z]],
 Cot[ArcCot[z]], Csc[ArcCsc[z]], Sec[ArcSec[z]]}
```

```
{z, z, z, z, z, z}
```

*Mathematica* also automatically simplifies the composition of the direct and any of the inverse trigonometric functions into algebraic functions of the argument.

```
{Sin[ArcSin[z]], Sin[ArcCos[z]], Sin[ArcTan[z]],
 Sin[ArcCot[z]], Sin[ArcCsc[z]], Sin[ArcSec[z]]}
```

$$\left\{z,\ \sqrt{1-z^2},\ \frac{z}{\sqrt{1+z^2}},\ \frac{1}{\sqrt{1+\frac{1}{z^2}}\ z},\ \frac{1}{z},\ \sqrt{1-\frac{1}{z^2}}\right\}$$

```
{Cos[ArcSin[z]], Cos[ArcCos[z]], Cos[ArcTan[z]],
 Cos[ArcCot[z]], Cos[ArcCsc[z]], Cos[ArcSec[z]]}
```

$$\left\{\sqrt{1-z^2},\ z,\ \frac{1}{\sqrt{1+z^2}},\ \frac{1}{\sqrt{1+\frac{1}{z^2}}},\ \sqrt{1-\frac{1}{z^2}},\ \frac{1}{z}\right\}$$

```
{Tan[ArcSin[z]], Tan[ArcCos[z]], Tan[ArcTan[z]],
 Tan[ArcCot[z]], Tan[ArcCsc[z]], Tan[ArcSec[z]]}
```

$$\left\{\frac{z}{\sqrt{1-z^2}},\ \frac{\sqrt{1-z^2}}{z},\ z,\ \frac{1}{z},\ \frac{1}{\sqrt{1-\frac{1}{z^2}}\ z},\ \sqrt{1-\frac{1}{z^2}}\ z\right\}$$

```
{Cot[ArcSin[z]], Cot[ArcCos[z]], Cot[ArcTan[z]],
 Cot[ArcCot[z]], Cot[ArcCsc[z]], Cot[ArcSec[z]]}
```

$$\left\{\frac{\sqrt{1-z^2}}{z},\ \frac{z}{\sqrt{1-z^2}},\ \frac{1}{z},\ z,\ \sqrt{1-\frac{1}{z^2}}\ z,\ \frac{1}{\sqrt{1-\frac{1}{z^2}}\ z}\right\}$$

```
{Csc[ArcSin[z]], Csc[ArcCos[z]], Csc[ArcTan[z]],
 Csc[ArcCot[z]], Csc[ArcCsc[z]], Csc[ArcSec[z]]}
```

$$\left\{\frac{1}{z},\ \frac{1}{\sqrt{1-z^2}},\ \frac{\sqrt{1+z^2}}{z},\ \sqrt{1+\frac{1}{z^2}}\ z,\ z,\ \frac{1}{\sqrt{1-\frac{1}{z^2}}}\right\}$$

```
{Sec[ArcSin[z]], Sec[ArcCos[z]], Sec[ArcTan[z]],
 Sec[ArcCot[z]], Sec[ArcCsc[z]], Sec[ArcSec[z]]}
```

$$\left\{\frac{1}{\sqrt{1-z^2}},\ \frac{1}{z},\ \sqrt{1+z^2},\ \sqrt{1+\frac{1}{z^2}},\ \frac{1}{\sqrt{1-\frac{1}{z^2}}},\ z\right\}$$

In cases where the argument has the structure $\pi k/2+z$ or $\pi k/2-z$, and $\pi k/2+i z$ or $\pi k/2-i z$ with integer $k$, trigonometric functions can be automatically transformed into other trigonometric or hyperbolic functions. Here are some examples.

$$\mathbf{Tan}\left[\frac{\pi}{2}-\mathbf{z}\right]$$

```
Cot[z]
```

```
Csc[ⅈ z]
```

```
-ⅈ Csch[z]
```

$$\left\{\text{Sin}\left[\frac{\pi}{2}-z\right],\ \text{Cos}\left[\frac{\pi}{2}-z\right],\ \text{Tan}\left[\frac{\pi}{2}-z\right],\ \text{Cot}\left[\frac{\pi}{2}-z\right],\ \text{Csc}\left[\frac{\pi}{2}-z\right],\ \text{Sec}\left[\frac{\pi}{2}-z\right]\right\}$$

```
{Cos[z], Sin[z], Cot[z], Tan[z], Sec[z], Csc[z]}
```

```
{Sin[ⅈ z], Cos[ⅈ z], Tan[ⅈ z], Cot[ⅈ z], Csc[ⅈ z], Sec[ⅈ z]}
```

```
{ⅈ Sinh[z], Cosh[z], ⅈ Tanh[z], -ⅈ Coth[z], -ⅈ Csch[z], Sech[z]}
```

### Simplification of simple expressions containing trigonometric functions

Sometimes simple arithmetic operations containing trigonometric functions can automatically produce other trigonometric functions.

```
1 / Sec[z]
```

```
Cos[z]
```

```
{1/Sin[z], 1/Cos[z], 1/Tan[z], 1/Cot[z], 1/Csc[z], 1/Sec[z],
 Sin[z]/Cos[z], Cos[z]/Sin[z], Sin[z]/Sin[π/2 - z], Cos[z]/Sin[z]^2}
```

```
{Csc[z], Sec[z], Cot[z], Tan[z], Sin[z], Cos[z], Tan[z], Cot[z], Tan[z], Cot[z] Csc[z]}
```

### Trigonometric functions arising as special cases from more general functions

All trigonometric functions can be treated as particular cases of some more advanced special functions. For example, sin($z$) and cos($z$) are sometimes the results of auto-simplifications from Bessel, Mathieu, Jacobi, hypergeometric, and Meijer functions (for appropriate values of their parameters).

$$\text{BesselJ}\left[\frac{1}{2},\ z\right]$$

$$\frac{\sqrt{\frac{2}{\pi}}\ \text{Sin}[z]}{\sqrt{z}}$$

```
MathieuC[1, 0, z]
```

```
Cos[z]
```

```
JacobiSC[z, 0]
```

```
Tan[z]
```

In[14]:= $$\left\{\text{BesselJ}\left[\frac{1}{2},\ z\right],\ \text{MathieuS}[1,\ 0,\ z],\ \text{JacobiSN}[z,\ 0],\ \text{HypergeometricPFQ}\left[\{\},\ \left\{\frac{3}{2}\right\},\ -\frac{z^2}{4}\right],\ \text{MeijerG}\left[\{\{\},\ \{\}\},\ \left\{\left\{\frac{1}{2}\right\},\ \{0\}\right\},\ \frac{z^2}{4}\right]\right\}$$

Out[14]= $\left\{\frac{\sqrt{\frac{2}{\pi}}\,\text{Sin}[z]}{\sqrt{z}},\ \text{Sin}[z],\ \text{Sin}[z],\ \frac{\text{Sin}\left[\sqrt{z^2}\right]}{\sqrt{z^2}},\ \frac{\sqrt{z^2}\,\text{Sin}[z]}{\sqrt{\pi}\,z}\right\}$

In[15]:= $\left\{\text{BesselJ}\left[-\frac{1}{2}, z\right], \text{MathieuC}[1, 0, z], \text{JacobiCD}[z, 0], \text{Hypergeometric0F1}\left[\frac{1}{2}, -\frac{z^2}{4}\right], \text{MeijerG}\left[\{\{\}, \{\}\}, \left\{\{0\}, \left\{\frac{1}{2}\right\}\right\}, \frac{z^2}{4}\right]\right\}$

Out[15]= $\left\{\frac{\sqrt{\frac{2}{\pi}}\,\text{Cos}[z]}{\sqrt{z}},\ \text{Cos}[z],\ \text{Cos}[z],\ \text{Cos}\left[\sqrt{z^2}\right],\ \frac{\text{Cos}[z]}{\sqrt{\pi}}\right\}$

In[16]:= `{JacobiSC[z, 0], JacobiCS[z, 0], JacobiDS[z, 0], JacobiDC[z, 0]}`

Out[16]= `{Tan[z], Cot[z], Csc[z], Sec[z]}`

## Equivalence transformations carried out by specialized *Mathematica* functions

### General remarks

Almost everybody prefers using $\sin(z)/2$ instead of $\cos(\pi/2 - z)\sin(\pi/6)$. *Mathematica* automatically transforms the second expression into the first one. The automatic application of transformation rules to mathematical expressions can give overly complicated results. Compact expressions like $\sin(2\,z)\sin(\pi/16)$ should not be automatically expanded into the more complicated expression $\sin(z)\cos(z)\left(2-\left(2+2^{1/2}\right)^{1/2}\right)^{1/2}$. *Mathematica* has special commands that produce these types of expansions. Some of them are demonstrated in the next section.

### TrigExpand

The function `TrigExpand` expands out trigonometric and hyperbolic functions. In more detail, it splits up sums and integer multiples that appear in the arguments of trigonometric and hyperbolic functions, and then expands out the products of the trigonometric and hyperbolic functions into sums of powers, using the trigonometric and hyperbolic identities where possible. Here are some examples.

```
TrigExpand[Sin[x - y]]
```

```
Cos[y] Sin[x] - Cos[x] Sin[y]
```

```
Cos[4 z] // TrigExpand
```

$\text{Cos}[z]^4 - 6\,\text{Cos}[z]^2\,\text{Sin}[z]^2 + \text{Sin}[z]^4$

```
TrigExpand[{{Sin[x + y], Sin[3 z]},
           {Cos[x + y], Cos[3 z]},
           {Tan[x + y], Tan[3 z]},
           {Cot[x + y], Cot[3 z]},
           {Csc[x + y], Csc[3 z]},
           {Sec[x + y], Sec[3 z]}}]
```

$$\left\{\{\cos(y)\sin(x)+\cos(x)\sin(y),\ 3\cos^2(z)\sin(z)-\sin^3(z)\},\ \{\cos(x)\cos(y)-\sin(x)\sin(y),\ \cos^3(z)-3\cos(z)\sin^2(z)\},\right.$$
$$\left\{\frac{\cos(y)\sin(x)}{\cos(x)\cos(y)-\sin(x)\sin(y)}+\frac{\cos(x)\sin(y)}{\cos(x)\cos(y)-\sin(x)\sin(y)},\ \frac{3\cos^2(z)\sin(z)}{\cos^3(z)-3\cos(z)\sin^2(z)}-\frac{\sin^3(z)}{\cos^3(z)-3\cos(z)\sin^2(z)}\right\},$$
$$\left\{\frac{\cos(x)\cos(y)}{\cos(y)\sin(x)+\cos(x)\sin(y)}-\frac{\sin(x)\sin(y)}{\cos(y)\sin(x)+\cos(x)\sin(y)},\ \frac{\cos^3(z)}{3\cos^2(z)\sin(z)-\sin^3(z)}-\frac{3\cos(z)\sin^2(z)}{3\cos^2(z)\sin(z)-\sin^3(z)}\right\},$$
$$\left\{\frac{1}{\cos(y)\sin(x)+\cos(x)\sin(y)},\ \frac{1}{3\cos^2(z)\sin(z)-\sin^3(z)}\right\},$$
$$\left.\left\{\frac{1}{\cos(x)\cos(y)-\sin(x)\sin(y)},\ \frac{1}{\cos^3(z)-3\cos(z)\sin^2(z)}\right\}\right\}$$

```
TableForm[(# == TrigExpand[#]) & /@
  Flatten[{{Sin[x + y], Sin[3 z]}, {Cos[x + y], Cos[3 z]}, {Tan[x + y], Tan[3 z]},
   {Cot[x + y], Cot[3 z]}, {Csc[x + y], Csc[3 z]}, {Sec[x + y], Sec[3 z]}}]]
```

$$\sin(x+y)=\cos(y)\sin(x)+\cos(x)\sin(y)$$
$$\sin(3z)=3\cos^2(z)\sin(z)-\sin^3(z)$$
$$\cos(x+y)=\cos(x)\cos(y)-\sin(x)\sin(y)$$
$$\cos(3z)=\cos^3(z)-3\cos(z)\sin^2(z)$$
$$\tan(x+y)=\frac{\cos(y)\sin(x)}{\cos(x)\cos(y)-\sin(x)\sin(y)}+\frac{\cos(x)\sin(y)}{\cos(x)\cos(y)-\sin(x)\sin(y)}$$
$$\tan(3z)=\frac{3\cos^2(z)\sin(z)}{\cos^3(z)-3\cos(z)\sin^2(z)}-\frac{\sin^3(z)}{\cos^3(z)-3\cos(z)\sin^2(z)}$$
$$\cot(x+y)=\frac{\cos(x)\cos(y)}{\cos(y)\sin(x)+\cos(x)\sin(y)}-\frac{\sin(x)\sin(y)}{\cos(y)\sin(x)+\cos(x)\sin(y)}$$
$$\cot(3z)=\frac{\cos^3(z)}{3\cos^2(z)\sin(z)-\sin^3(z)}-\frac{3\cos(z)\sin^2(z)}{3\cos^2(z)\sin(z)-\sin^3(z)}$$
$$\csc(x+y)=\frac{1}{\cos(y)\sin(x)+\cos(x)\sin(y)}$$
$$\csc(3z)=\frac{1}{3\cos^2(z)\sin(z)-\sin^3(z)}$$
$$\sec(x+y)=\frac{1}{\cos(x)\cos(y)-\sin(x)\sin(y)}$$
$$\sec(3z)=\frac{1}{\cos^3(z)-3\cos(z)\sin^2(z)}$$

### TrigFactor

The function `TrigFactor` factors trigonometric and hyperbolic functions. In more detail, it splits up sums and integer multiples that appear in the arguments of trigonometric and hyperbolic functions, and then factors the resulting polynomials in the trigonometric and hyperbolic functions, using the corresponding identities where possible. Here are some examples.

```
TrigFactor[Sin[x] + Cos[y]]
```

$\left(\text{Cos}\left[\frac{x}{2}-\frac{y}{2}\right]+\text{Sin}\left[\frac{x}{2}-\frac{y}{2}\right]\right)\left(\text{Cos}\left[\frac{x}{2}+\frac{y}{2}\right]+\text{Sin}\left[\frac{x}{2}+\frac{y}{2}\right]\right)$

```
Tan[x] - Cot[y] // TrigFactor
```

-Cos[x + y] Csc[y] Sec[x]

```
TrigFactor[{Sin[x] + Sin[y],
            Cos[x] + Cos[y],
            Tan[x] + Tan[y],
            Cot[x] + Cot[y],
            Csc[x] + Csc[y],
            Sec[x] + Sec[y]}]
```

$\left\{2\,\text{Cos}\left[\frac{x}{2}-\frac{y}{2}\right]\text{Sin}\left[\frac{x}{2}+\frac{y}{2}\right],\ 2\,\text{Cos}\left[\frac{x}{2}-\frac{y}{2}\right]\text{Cos}\left[\frac{x}{2}+\frac{y}{2}\right],\ \text{Sec}[x]\,\text{Sec}[y]\,\text{Sin}[x+y],\ \text{Csc}[x]\,\text{Csc}[y]\,\text{Sin}[x+y],\ \frac{1}{2}\text{Cos}\left[\frac{x}{2}-\frac{y}{2}\right]\text{Csc}\left[\frac{x}{2}\right]\text{Csc}\left[\frac{y}{2}\right]\text{Sec}\left[\frac{x}{2}\right]\text{Sec}\left[\frac{y}{2}\right]\text{Sin}\left[\frac{x}{2}+\frac{y}{2}\right],\ \frac{2\,\text{Cos}\left[\frac{x}{2}-\frac{y}{2}\right]\text{Cos}\left[\frac{x}{2}+\frac{y}{2}\right]}{\left(\text{Cos}\left[\frac{x}{2}\right]-\text{Sin}\left[\frac{x}{2}\right]\right)\left(\text{Cos}\left[\frac{x}{2}\right]+\text{Sin}\left[\frac{x}{2}\right]\right)\left(\text{Cos}\left[\frac{y}{2}\right]-\text{Sin}\left[\frac{y}{2}\right]\right)\left(\text{Cos}\left[\frac{y}{2}\right]+\text{Sin}\left[\frac{y}{2}\right]\right)}\right\}$

## TrigReduce

The function `TrigReduce` rewrites products and powers of trigonometric and hyperbolic functions in terms of those functions with combined arguments. In more detail, it typically yields a linear expression involving trigonometric and hyperbolic functions with more complicated arguments. `TrigReduce` is approximately inverse to `TrigExpand` and `TrigFactor`. Here are some examples.

```
TrigReduce[Sin[z]^3]
```

$\frac{1}{4}(3\,\text{Sin}[z]-\text{Sin}[3\,z])$

```
Sin[x] Cos[y] // TrigReduce
```

$\frac{1}{2}(\text{Sin}[x-y]+\text{Sin}[x+y])$

```
TrigReduce[{Sin[z]^2, Cos[z]^2, Tan[z]^2, Cot[z]^2, Csc[z]^2, Sec[z]^2}]
```

$\left\{\frac{1}{2}(1-\text{Cos}[2\,z]),\ \frac{1}{2}(1+\text{Cos}[2\,z]),\ \frac{1-\text{Cos}[2\,z]}{1+\text{Cos}[2\,z]},\ \frac{-1-\text{Cos}[2\,z]}{-1+\text{Cos}[2\,z]},\ -\frac{2}{-1+\text{Cos}[2\,z]},\ \frac{2}{1+\text{Cos}[2\,z]}\right\}$

```
TrigReduce[TrigExpand[{{Sin[x + y], Sin[3 z], Sin[x] Sin[y]},
                       {Cos[x + y], Cos[3 z], Cos[x] Cos[y]},
                       {Tan[x + y], Tan[3 z], Tan[x] Tan[y]},
                       {Cot[x + y], Cot[3 z], Cot[x] Cot[y]},
                       {Csc[x + y], Csc[3 z], Csc[x] Csc[y]},
                       {Sec[x + y], Sec[3 z], Sec[x] Sec[y]}}]]
```

$$\left\{\left\{\text{Sin}[x+y],\ \text{Sin}[3z],\ \frac{1}{2}(\text{Cos}[x-y]-\text{Cos}[x+y])\right\},\ \left\{\text{Cos}[x+y],\ \text{Cos}[3z],\ \frac{1}{2}(\text{Cos}[x-y]+\text{Cos}[x+y])\right\},\ \left\{\text{Tan}[x+y],\ \text{Tan}[3z],\ \frac{\text{Cos}[x-y]-\text{Cos}[x+y]}{\text{Cos}[x-y]+\text{Cos}[x+y]}\right\},\ \left\{\text{Cot}[x+y],\ \text{Cot}[3z],\ \frac{\text{Cos}[x-y]+\text{Cos}[x+y]}{\text{Cos}[x-y]-\text{Cos}[x+y]}\right\},\ \left\{\text{Csc}[x+y],\ \text{Csc}[3z],\ \frac{2}{\text{Cos}[x-y]-\text{Cos}[x+y]}\right\},\ \left\{\text{Sec}[x+y],\ \text{Sec}[3z],\ \frac{2}{\text{Cos}[x-y]+\text{Cos}[x+y]}\right\}\right\}$$

```
TrigReduce[TrigFactor[{Sin[x] + Sin[y], Cos[x] + Cos[y],
   Tan[x] + Tan[y], Cot[x] + Cot[y], Csc[x] + Csc[y], Sec[x] + Sec[y]}]]
```

$$\left\{\text{Sin}[x]+\text{Sin}[y],\ \text{Cos}[x]+\text{Cos}[y],\ \frac{2\,\text{Sin}[x+y]}{\text{Cos}[x-y]+\text{Cos}[x+y]},\ \frac{2\,\text{Sin}[x+y]}{\text{Cos}[x-y]-\text{Cos}[x+y]},\ \frac{2\,(\text{Sin}[x]+\text{Sin}[y])}{\text{Cos}[x-y]-\text{Cos}[x+y]},\ \frac{2\,(\text{Cos}[x]+\text{Cos}[y])}{\text{Cos}[x-y]+\text{Cos}[x+y]}\right\}$$

### TrigToExp

The function `TrigToExp` converts direct and inverse trigonometric and hyperbolic functions to exponential or logarithmic functions. It tries, where possible, to give results that do not involve explicit complex numbers. Here are some examples.

```
TrigToExp[Sin[2 z]]
```

$$\frac{1}{2}\,i\,e^{-2iz}-\frac{1}{2}\,i\,e^{2iz}$$

```
Sin[z] Tan[2 z] // TrigToExp
```

$$-\frac{\left(e^{-iz}-e^{iz}\right)\left(e^{-2iz}-e^{2iz}\right)}{2\left(e^{-2iz}+e^{2iz}\right)}$$

```
TrigToExp[{Sin[z], Cos[z], Tan[z], Cot[z], Csc[z], Sec[z]}]
```

$$\left\{\frac{1}{2}\,i\,e^{-iz}-\frac{1}{2}\,i\,e^{iz},\ \frac{e^{-iz}}{2}+\frac{e^{iz}}{2},\ \frac{i\left(e^{-iz}-e^{iz}\right)}{e^{-iz}+e^{iz}},\ -\frac{i\left(e^{-iz}+e^{iz}\right)}{e^{-iz}-e^{iz}},\ -\frac{2\,i}{e^{-iz}-e^{iz}},\ \frac{2}{e^{-iz}+e^{iz}}\right\}$$

### ExpToTrig

The function `ExpToTrig` converts exponentials to trigonometric or hyperbolic functions. It tries, where possible, to give results that do not involve explicit complex numbers. It is approximately inverse to `TrigToExp`. Here are some examples.

$$\text{ExpToTrig}\left[e^{i\,x\,\beta}\right]$$

$\text{Cos}[x\,\beta] + i\,\text{Sin}[x\,\beta]$

$$\frac{e^{i x \alpha} - e^{i x \beta}}{e^{i x \gamma} + e^{i x \delta}} \text{ // ExpToTrig}$$

$$\frac{\text{Cos}[x\,\alpha] - \text{Cos}[x\,\beta] + i\,\text{Sin}[x\,\alpha] - i\,\text{Sin}[x\,\beta]}{\text{Cos}[x\,\gamma] + \text{Cos}[x\,\delta] + i\,\text{Sin}[x\,\gamma] + i\,\text{Sin}[x\,\delta]}$$

```
ExpToTrig[TrigToExp[{Sin[z], Cos[z], Tan[z], Cot[z], Csc[z], Sec[z]}]]
```

```
{Sin[z], Cos[z], Tan[z], Cot[z], Csc[z], Sec[z]}
```

$$\text{ExpToTrig}\left[\left\{\alpha\, e^{-i x \beta} + \alpha\, e^{i x \beta},\ \alpha\, e^{-i x \beta} + \gamma\, e^{i x \beta}\right\}\right]$$

$$\{2\,\alpha\,\text{Cos}[x\,\beta],\ \alpha\,\text{Cos}[x\,\beta] + \gamma\,\text{Cos}[x\,\beta] - i\,\alpha\,\text{Sin}[x\,\beta] + i\,\gamma\,\text{Sin}[x\,\beta]\}$$

## ComplexExpand

The function `ComplexExpand` expands expressions assuming that all the occurring variables are real. The value option `TargetFunctions` is a list of functions from the set `{Re, Im, Abs, Arg, Conjugate, Sign}`. `ComplexExpand` tries to give results in terms of the specified functions. Here are some examples

```
ComplexExpand[Sin[x + i y] Cos[x - i y]]
```

$$\text{Cos}[x]\,\text{Cosh}[y]^2\,\text{Sin}[x] - \text{Cos}[x]\,\text{Sin}[x]\,\text{Sinh}[y]^2 + i\left(\text{Cos}[x]^2\,\text{Cosh}[y]\,\text{Sinh}[y] + \text{Cosh}[y]\,\text{Sin}[x]^2\,\text{Sinh}[y]\right)$$

```
Csc[x + i y] Sec[x - i y] // ComplexExpand
```

$$-\frac{4\,\text{Cos}[x]\,\text{Cosh}[y]^2\,\text{Sin}[x]}{(\text{Cos}[2x] - \text{Cosh}[2y])\,(\text{Cos}[2x] + \text{Cosh}[2y])} + \frac{4\,\text{Cos}[x]\,\text{Sin}[x]\,\text{Sinh}[y]^2}{(\text{Cos}[2x] - \text{Cosh}[2y])\,(\text{Cos}[2x] + \text{Cosh}[2y])} + i\left(\frac{4\,\text{Cos}[x]^2\,\text{Cosh}[y]\,\text{Sinh}[y]}{(\text{Cos}[2x] - \text{Cosh}[2y])\,(\text{Cos}[2x] + \text{Cosh}[2y])} + \frac{4\,\text{Cosh}[y]\,\text{Sin}[x]^2\,\text{Sinh}[y]}{(\text{Cos}[2x] - \text{Cosh}[2y])\,(\text{Cos}[2x] + \text{Cosh}[2y])}\right)$$

In[17]:= `li1 = {Sin[x + i y], Cos[x + i y], Tan[x + i y], Cot[x + i y], Csc[x + i y], Sec[x + i y]}`

Out[17]= `{Sin[x + i y], Cos[x + i y], Tan[x + i y], Cot[x + i y], Csc[x + i y], Sec[x + i y]}`

In[18]:= `ComplexExpand[li1]`

Out[18]=
$$\left\{\text{Cosh}[y]\,\text{Sin}[x] + i\,\text{Cos}[x]\,\text{Sinh}[y],\ \text{Cos}[x]\,\text{Cosh}[y] - i\,\text{Sin}[x]\,\text{Sinh}[y],\ \frac{\text{Sin}[2x]}{\text{Cos}[2x] + \text{Cosh}[2y]} + \frac{i\,\text{Sinh}[2y]}{\text{Cos}[2x] + \text{Cosh}[2y]},\ -\frac{\text{Sin}[2x]}{\text{Cos}[2x] - \text{Cosh}[2y]} + \frac{i\,\text{Sinh}[2y]}{\text{Cos}[2x] - \text{Cosh}[2y]},\ -\frac{2\,\text{Cosh}[y]\,\text{Sin}[x]}{\text{Cos}[2x] - \text{Cosh}[2y]} + \frac{2\,i\,\text{Cos}[x]\,\text{Sinh}[y]}{\text{Cos}[2x] - \text{Cosh}[2y]},\ \frac{2\,\text{Cos}[x]\,\text{Cosh}[y]}{\text{Cos}[2x] + \text{Cosh}[2y]} + \frac{2\,i\,\text{Sin}[x]\,\text{Sinh}[y]}{\text{Cos}[2x] + \text{Cosh}[2y]}\right\}$$

In[19]:= `ComplexExpand[Re[#] & /@ li1, TargetFunctions → {Re, Im}]`

Out[19]= $$\left\{\cosh(y)\sin(x),\ \cos(x)\cosh(y),\ \frac{\sin(2x)}{\cos(2x)+\cosh(2y)},\ -\frac{\sin(2x)}{\cos(2x)-\cosh(2y)},\ -\frac{2\cosh(y)\sin(x)}{\cos(2x)-\cosh(2y)},\ \frac{2\cos(x)\cosh(y)}{\cos(2x)+\cosh(2y)}\right\}$$

In[20]:=
```
ComplexExpand[Im[#] & /@ li1, TargetFunctions → {Re, Im}]
```

Out[20]= $$\left\{\cos(x)\sinh(y),\ -\sin(x)\sinh(y),\ \frac{\sinh(2y)}{\cos(2x)+\cosh(2y)},\ \frac{\sinh(2y)}{\cos(2x)-\cosh(2y)},\ \frac{2\cos(x)\sinh(y)}{\cos(2x)-\cosh(2y)},\ \frac{2\sin(x)\sinh(y)}{\cos(2x)+\cosh(2y)}\right\}$$

In[21]:=
```
ComplexExpand[Abs[#] & /@ li1, TargetFunctions → {Re, Im}]
```

Out[21]= $$\left\{\sqrt{\cosh(y)^2\sin(x)^2+\cos(x)^2\sinh(y)^2},\ \sqrt{\cos(x)^2\cosh(y)^2+\sin(x)^2\sinh(y)^2},\ \sqrt{\frac{\sin(2x)^2}{(\cos(2x)+\cosh(2y))^2}+\frac{\sinh(2y)^2}{(\cos(2x)+\cosh(2y))^2}},\ \sqrt{\frac{\sin(2x)^2}{(\cos(2x)-\cosh(2y))^2}+\frac{\sinh(2y)^2}{(\cos(2x)-\cosh(2y))^2}},\ \sqrt{\frac{4\cosh(y)^2\sin(x)^2}{(\cos(2x)-\cosh(2y))^2}+\frac{4\cos(x)^2\sinh(y)^2}{(\cos(2x)-\cosh(2y))^2}},\ \sqrt{\frac{4\cos(x)^2\cosh(y)^2}{(\cos(2x)+\cosh(2y))^2}+\frac{4\sin(x)^2\sinh(y)^2}{(\cos(2x)+\cosh(2y))^2}}\right\}$$

In[22]:=
```
% // Simplify[#, {x, y} ∈ Reals] &
```

Out[22]= $$\left\{\frac{\sqrt{-\cos(2x)+\cosh(2y)}}{\sqrt{2}},\ \frac{\sqrt{\cos(2x)+\cosh(2y)}}{\sqrt{2}},\ \frac{\sqrt{\sin(2x)^2+\sinh(2y)^2}}{\cos(2x)+\cosh(2y)},\ \sqrt{-\frac{\cos(2x)+\cosh(2y)}{\cos(2x)-\cosh(2y)}},\ \frac{\sqrt{2}}{\sqrt{-\cos(2x)+\cosh(2y)}},\ \frac{\sqrt{2}}{\sqrt{\cos(2x)+\cosh(2y)}}\right\}$$

In[23]:=
```
ComplexExpand[Arg[#] & /@ li1, TargetFunctions → {Re, Im}]
```

Out[23]=
$$\left\{\text{ArcTan}[\text{Cosh}[y]\,\text{Sin}[x],\ \text{Cos}[x]\,\text{Sinh}[y]],\ \text{ArcTan}[\text{Cos}[x]\,\text{Cosh}[y],\ -\text{Sin}[x]\,\text{Sinh}[y]],\ \text{ArcTan}\left[\frac{\text{Sin}[2x]}{\text{Cos}[2x]+\text{Cosh}[2y]},\ \frac{\text{Sinh}[2y]}{\text{Cos}[2x]+\text{Cosh}[2y]}\right],\ \text{ArcTan}\left[-\frac{\text{Sin}[2x]}{\text{Cos}[2x]-\text{Cosh}[2y]},\ \frac{\text{Sinh}[2y]}{\text{Cos}[2x]-\text{Cosh}[2y]}\right],\ \text{ArcTan}\left[-\frac{2\,\text{Cosh}[y]\,\text{Sin}[x]}{\text{Cos}[2x]-\text{Cosh}[2y]},\ \frac{2\,\text{Cos}[x]\,\text{Sinh}[y]}{\text{Cos}[2x]-\text{Cosh}[2y]}\right],\ \text{ArcTan}\left[\frac{2\,\text{Cos}[x]\,\text{Cosh}[y]}{\text{Cos}[2x]+\text{Cosh}[2y]},\ \frac{2\,\text{Sin}[x]\,\text{Sinh}[y]}{\text{Cos}[2x]+\text{Cosh}[2y]}\right]\right\}$$

In[24]:=
```
ComplexExpand[Conjugate[#] & /@ li1, TargetFunctions → {Re, Im}] // Simplify
```

Out[24]=
$$\left\{\text{Cosh}[y]\,\text{Sin}[x]-i\,\text{Cos}[x]\,\text{Sinh}[y],\ \text{Cos}[x]\,\text{Cosh}[y]+i\,\text{Sin}[x]\,\text{Sinh}[y],\ \frac{\text{Sin}[2x]-i\,\text{Sinh}[2y]}{\text{Cos}[2x]+\text{Cosh}[2y]},\ -\frac{\text{Sin}[2x]+i\,\text{Sinh}[2y]}{\text{Cos}[2x]-\text{Cosh}[2y]},\ \frac{1}{\text{Cosh}[y]\,\text{Sin}[x]-i\,\text{Cos}[x]\,\text{Sinh}[y]},\ \frac{1}{\text{Cos}[x]\,\text{Cosh}[y]+i\,\text{Sin}[x]\,\text{Sinh}[y]}\right\}$$

### Simplify

The function `Simplify` performs a sequence of algebraic transformations on its argument, and returns the simplest form it finds. Here are two examples.

```
Simplify[Sin[2 z] / Sin[z]]
```

2 Cos[z]

```
Sin[2 z] / Cos[z] // Simplify
```

2 Sin[z]

Here is a large collection of trigonometric identities. All are written as one large logical conjunction.

$$\text{Simplify}[\#]\ \&\ /@ \Bigg(\text{Cos}[z]^2+\text{Sin}[z]^2 == 1 \bigwedge$$
$$\text{Sin}[z]^2 == \frac{1-\text{Cos}[2z]}{2} \bigwedge \text{Cos}[z]^2 == \frac{1+\text{Cos}[2z]}{2} \bigwedge$$
$$\text{Tan}[z]^2 == \frac{1-\text{Cos}[2z]}{1+\text{Cos}[2z]} \bigwedge \text{Cot}[z]^2 == \frac{1+\text{Cos}[2z]}{1-\text{Cos}[2z]} \bigwedge$$
$$\text{Sin}[2z] == 2\,\text{Sin}[z]\,\text{Cos}[z] \bigwedge \text{Cos}[2z] == \text{Cos}[z]^2-\text{Sin}[z]^2 == 2\,\text{Cos}[z]^2-1 \bigwedge$$
$$\text{Sin}[a+b] == \text{Sin}[a]\,\text{Cos}[b]+\text{Cos}[a]\,\text{Sin}[b] \bigwedge \text{Sin}[a-b] == \text{Sin}[a]\,\text{Cos}[b]-\text{Cos}[a]\,\text{Sin}[b] \bigwedge$$
$$\text{Cos}[a+b] == \text{Cos}[a]\,\text{Cos}[b]-\text{Sin}[a]\,\text{Sin}[b] \bigwedge \text{Cos}[a-b] == \text{Cos}[a]\,\text{Cos}[b]+\text{Sin}[a]\,\text{Sin}[b] \bigwedge$$
$$\text{Sin}[a]+\text{Sin}[b] == 2\,\text{Sin}\Big[\frac{a+b}{2}\Big]\,\text{Cos}\Big[\frac{a-b}{2}\Big] \bigwedge \text{Sin}[a]-\text{Sin}[b] == 2\,\text{Cos}\Big[\frac{a+b}{2}\Big]\,\text{Sin}\Big[\frac{a-b}{2}\Big] \bigwedge$$
$$\text{Cos}[a]+\text{Cos}[b] == 2\,\text{Cos}\Big[\frac{a+b}{2}\Big]\,\text{Cos}\Big[\frac{a-b}{2}\Big] \bigwedge \text{Cos}[a]-\text{Cos}[b] == 2\,\text{Sin}\Big[\frac{a+b}{2}\Big]\,\text{Sin}\Big[\frac{b-a}{2}\Big] \bigwedge$$
$$\text{Tan}[a]+\text{Tan}[b] == \frac{\text{Sin}[a+b]}{\text{Cos}[a]\,\text{Cos}[b]} \bigwedge \text{Tan}[a]-\text{Tan}[b] == \frac{\text{Sin}[a-b]}{\text{Cos}[a]\,\text{Cos}[b]} \bigwedge$$
$$A\,\text{Sin}[z]+B\,\text{Cos}[z] == A\sqrt{1+\frac{B^2}{A^2}}\ \text{Sin}\Big[z+\text{ArcTan}\Big[\frac{B}{A}\Big]\Big] \bigwedge$$
$$\text{Sin}[a]\,\text{Sin}[b] == \frac{\text{Cos}[a-b]-\text{Cos}[a+b]}{2} \bigwedge$$
$$\text{Cos}[a]\,\text{Cos}[b] == \frac{\text{Cos}[a-b]+\text{Cos}[a+b]}{2} \bigwedge \text{Sin}[a]\,\text{Cos}[b] == \frac{\text{Sin}[a+b]+\text{Sin}[a-b]}{2} \bigwedge$$
$$\text{Sin}\Big[\frac{z}{2}\Big]^2 == \frac{1-\text{Cos}[z]}{2} \bigwedge \text{Cos}\Big[\frac{z}{2}\Big]^2 == \frac{1+\text{Cos}[z]}{2} \bigwedge$$
$$\text{Tan}\Big[\frac{z}{2}\Big] == \frac{1-\text{Cos}[z]}{\text{Sin}[z]} == \frac{\text{Sin}[z]}{1+\text{Cos}[z]} \bigwedge \text{Cot}\Big[\frac{z}{2}\Big] == \frac{\text{Sin}[z]}{1-\text{Cos}[z]} == \frac{1+\text{Cos}[z]}{\text{Sin}[z]}\Bigg)$$

```
True
```

The function `Simplify` has the `Assumption` option. For example, *Mathematica* knows that $-1 \le \sin(x) \le 1$ for all real $x$, and uses the periodicity of trigonometric functions for the symbolic integer coefficient $k$ of $k\,\pi$.

```
Simplify[Abs[Sin[x]] ≤ 1, x ∈ Reals]
```

```
True
```

```
Abs[Sin[x]] ≤ 1 // Simplify[#, x ∈ Reals] &
```

```
True
```

```
Simplify[{Sin[z + 2 k π], Cos[z + 2 k π], Tan[z + k π],
  Cot[z + k π], Csc[z + 2 k π], Sec[z + 2 k π]}, k ∈ Integers]
```

```
{Sin[z], Cos[z], Tan[z], Cot[z], Csc[z], Sec[z]}
```

```
Simplify[{Sin[z + k π] / Sin[z], Cos[z + k π] / Cos[z], Tan[z + k π] / Tan[z],
  Cot[z + k π] / Cot[z], Csc[z + k π] / Csc[z], Sec[z + k π] / Sec[z]}, k ∈ Integers]
```

$\{(-1)^k, (-1)^k, 1, 1, (-1)^k, (-1)^k\}$

*Mathematica* also knows that the composition of inverse and direct trigonometric functions produces the value of the inner argument under the appropriate restriction. Here are some examples.

```
Simplify[{ArcSin[Sin[z]], ArcTan[Tan[z]], ArcCot[Cot[z]], ArcCsc[Csc[z]]},
 -π / 2 < Re[z] < π / 2]
```

{z, z, z, z}

```
Simplify[{ArcCos[Cos[z]], ArcSec[Sec[z]]}, 0 < Re[z] < π]
```

{z, z}

### FunctionExpand (and Together)

While the trigonometric functions auto-evaluate for simple fractions of $\pi$, for more complicated cases they stay as trigonometric functions to avoid the build up of large expressions. Using the function `FunctionExpand`, such expressions can be transformed into explicit radicals.

$\mathbf{Cos}\left[\frac{\pi}{32}\right]$

$\cos\left[\frac{\pi}{32}\right]$

$\mathbf{FunctionExpand}\left[\mathbf{Cos}\left[\frac{\pi}{32}\right]\right]$

$$\frac{1}{2}\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}$$

$\mathbf{Cot}\left[\frac{\pi}{24}\right]$ // **FunctionExpand**

$$\frac{\frac{\sqrt{2-\sqrt{2}}}{4}+\frac{1}{4}\sqrt{3\left(2+\sqrt{2}\right)}}{-\frac{1}{4}\sqrt{3\left(2-\sqrt{2}\right)}+\frac{\sqrt{2+\sqrt{2}}}{4}}$$

$\left\{\mathbf{Sin}\left[\frac{\pi}{16}\right], \mathbf{Cos}\left[\frac{\pi}{16}\right], \mathbf{Tan}\left[\frac{\pi}{16}\right], \mathbf{Cot}\left[\frac{\pi}{16}\right], \mathbf{Csc}\left[\frac{\pi}{16}\right], \mathbf{Sec}\left[\frac{\pi}{16}\right]\right\}$

$\left\{\mathrm{Sin}\left[\frac{\pi}{16}\right], \mathrm{Cos}\left[\frac{\pi}{16}\right], \mathrm{Tan}\left[\frac{\pi}{16}\right], \mathrm{Cot}\left[\frac{\pi}{16}\right], \mathrm{Csc}\left[\frac{\pi}{16}\right], \mathrm{Sec}\left[\frac{\pi}{16}\right]\right\}$

```
FunctionExpand[%]
```

$$\left\{\frac{1}{2}\sqrt{2-\sqrt{2+\sqrt{2}}},\ \frac{1}{2}\sqrt{2+\sqrt{2+\sqrt{2}}},\ \sqrt{\frac{2-\sqrt{2+\sqrt{2}}}{2+\sqrt{2+\sqrt{2}}}},\ \sqrt{\frac{2+\sqrt{2+\sqrt{2}}}{2-\sqrt{2+\sqrt{2}}}},\ \frac{2}{\sqrt{2-\sqrt{2+\sqrt{2}}}},\ \frac{2}{\sqrt{2+\sqrt{2+\sqrt{2}}}}\right\}$$

$$\left\{\mathbf{Sin}\left[\frac{\pi}{60}\right],\ \mathbf{Cos}\left[\frac{\pi}{60}\right],\ \mathbf{Tan}\left[\frac{\pi}{60}\right],\ \mathbf{Cot}\left[\frac{\pi}{60}\right],\ \mathbf{Csc}\left[\frac{\pi}{60}\right],\ \mathbf{Sec}\left[\frac{\pi}{60}\right]\right\}$$

$$\left\{\text{Sin}\left[\frac{\pi}{60}\right],\ \text{Cos}\left[\frac{\pi}{60}\right],\ \text{Tan}\left[\frac{\pi}{60}\right],\ \text{Cot}\left[\frac{\pi}{60}\right],\ \text{Csc}\left[\frac{\pi}{60}\right],\ \text{Sec}\left[\frac{\pi}{60}\right]\right\}$$

**Together[FunctionExpand[%]]**

$$\left\{\frac{1}{16}\left(-\sqrt{2}-\sqrt{6}+\sqrt{10}+\sqrt{30}+2\sqrt{5+\sqrt{5}}-2\sqrt{3\left(5+\sqrt{5}\right)}\right),\right.$$
$$\frac{1}{16}\left(\sqrt{2}-\sqrt{6}-\sqrt{10}+\sqrt{30}+2\sqrt{5+\sqrt{5}}+2\sqrt{3\left(5+\sqrt{5}\right)}\right),$$
$$\frac{-1-\sqrt{3}+\sqrt{5}+\sqrt{15}+\sqrt{2\left(5+\sqrt{5}\right)}-\sqrt{6\left(5+\sqrt{5}\right)}}{1-\sqrt{3}-\sqrt{5}+\sqrt{15}+\sqrt{2\left(5+\sqrt{5}\right)}+\sqrt{6\left(5+\sqrt{5}\right)}},$$
$$\frac{-1+\sqrt{3}+\sqrt{5}-\sqrt{15}-\sqrt{2\left(5+\sqrt{5}\right)}-\sqrt{6\left(5+\sqrt{5}\right)}}{1+\sqrt{3}-\sqrt{5}-\sqrt{15}-\sqrt{2\left(5+\sqrt{5}\right)}+\sqrt{6\left(5+\sqrt{5}\right)}},$$
$$\frac{16}{-\sqrt{2}-\sqrt{6}+\sqrt{10}+\sqrt{30}+2\sqrt{5+\sqrt{5}}-2\sqrt{3\left(5+\sqrt{5}\right)}},$$
$$\left.\frac{16}{\sqrt{2}-\sqrt{6}-\sqrt{10}+\sqrt{30}+2\sqrt{5+\sqrt{5}}+2\sqrt{3\left(5+\sqrt{5}\right)}}\right\}$$

If the denominator contains squares of integers other than 2, the results always contain complex numbers (meaning that the imaginary number $i = \sqrt{-1}$ appears unavoidably).

$$\left\{\mathbf{Sin}\left[\frac{\pi}{9}\right],\ \mathbf{Cos}\left[\frac{\pi}{9}\right],\ \mathbf{Tan}\left[\frac{\pi}{9}\right],\ \mathbf{Cot}\left[\frac{\pi}{9}\right],\ \mathbf{Csc}\left[\frac{\pi}{9}\right],\ \mathbf{Sec}\left[\frac{\pi}{9}\right]\right\}$$

$$\left\{\text{Sin}\left[\frac{\pi}{9}\right],\ \text{Cos}\left[\frac{\pi}{9}\right],\ \text{Tan}\left[\frac{\pi}{9}\right],\ \text{Cot}\left[\frac{\pi}{9}\right],\ \text{Csc}\left[\frac{\pi}{9}\right],\ \text{Sec}\left[\frac{\pi}{9}\right]\right\}$$

**FunctionExpand[%] // Together**

$$\Big\{\frac{1}{8}\Big(-i\,2^{2/3}\left(-1-i\sqrt{3}\right)^{1/3}+2^{2/3}\sqrt{3}\left(-1-i\sqrt{3}\right)^{1/3}+i\,2^{2/3}\left(-1+i\sqrt{3}\right)^{1/3}+2^{2/3}\sqrt{3}\left(-1+i\sqrt{3}\right)^{1/3}\Big),$$

$$\frac{1}{8}\Big(2^{2/3}\left(-1-i\sqrt{3}\right)^{1/3}+i\,2^{2/3}\sqrt{3}\left(-1-i\sqrt{3}\right)^{1/3}+2^{2/3}\left(-1+i\sqrt{3}\right)^{1/3}-i\,2^{2/3}\sqrt{3}\left(-1+i\sqrt{3}\right)^{1/3}\Big),$$

$$\frac{-\left(-1-i\sqrt{3}\right)^{1/3}-i\sqrt{3}\left(-1-i\sqrt{3}\right)^{1/3}+\left(-1+i\sqrt{3}\right)^{1/3}-i\sqrt{3}\left(-1+i\sqrt{3}\right)^{1/3}}{-i\left(-1-i\sqrt{3}\right)^{1/3}+\sqrt{3}\left(-1-i\sqrt{3}\right)^{1/3}-i\left(-1+i\sqrt{3}\right)^{1/3}-\sqrt{3}\left(-1+i\sqrt{3}\right)^{1/3}},$$

$$\frac{\left(-1-i\sqrt{3}\right)^{1/3}+i\sqrt{3}\left(-1-i\sqrt{3}\right)^{1/3}+\left(-1+i\sqrt{3}\right)^{1/3}-i\sqrt{3}\left(-1+i\sqrt{3}\right)^{1/3}}{-i\left(-1-i\sqrt{3}\right)^{1/3}+\sqrt{3}\left(-1-i\sqrt{3}\right)^{1/3}+i\left(-1+i\sqrt{3}\right)^{1/3}+\sqrt{3}\left(-1+i\sqrt{3}\right)^{1/3}},$$

$$8\Big/\Big(-i\,2^{2/3}\left(-1-i\sqrt{3}\right)^{1/3}+2^{2/3}\sqrt{3}\left(-1-i\sqrt{3}\right)^{1/3}+i\,2^{2/3}\left(-1+i\sqrt{3}\right)^{1/3}+2^{2/3}\sqrt{3}\left(-1+i\sqrt{3}\right)^{1/3}\Big),$$

$$-(8\,i)\Big/\Big(-i\,2^{2/3}\left(-1-i\sqrt{3}\right)^{1/3}+2^{2/3}\sqrt{3}\left(-1-i\sqrt{3}\right)^{1/3}-i\,2^{2/3}\left(-1+i\sqrt{3}\right)^{1/3}-2^{2/3}\sqrt{3}\left(-1+i\sqrt{3}\right)^{1/3}\Big)\Big\}$$

Here the function `RootReduce` is used to express the previous algebraic numbers as numbered roots of polynomial equations.

**RootReduce[Simplify[%]]**

$$\{\text{Root}[-3+36\,\#1^2-96\,\#1^4+64\,\#1^6\,\&,\,4],\ \text{Root}[-1-6\,\#1+8\,\#1^3\,\&,\,3],$$
$$\text{Root}[-3+27\,\#1^2-33\,\#1^4+\#1^6\,\&,\,4],\ \text{Root}[-1+33\,\#1^2-27\,\#1^4+3\,\#1^6\,\&,\,6],$$
$$\text{Root}[-64+96\,\#1^2-36\,\#1^4+3\,\#1^6\,\&,\,6],\ \text{Root}[-8+6\,\#1^2+\#1^3\,\&,\,3]\}$$

The function `FunctionExpand` also reduces trigonometric expressions with compound arguments or compositions, including hyperbolic functions, to simpler ones. Here are some examples.

**FunctionExpand** $\left[\mathbf{Cot}\left[\sqrt{-z^2}\right]\right]$

$$-\frac{\sqrt{-z}\ \text{Coth}[z]}{\sqrt{z}}$$

$\mathbf{Tan}\left[\sqrt{i\,z^2}\right]$ **// FunctionExpand**

$$-\frac{(-1)^{3/4}\sqrt{-(-1)^{3/4}z}\ \sqrt{(-1)^{3/4}z}\ \text{Tan}\left[(-1)^{1/4}z\right]}{z}$$

**{Sin[$\sqrt{z^2}$], Cos[$\sqrt{z^2}$], Tan[$\sqrt{z^2}$], Cot[$\sqrt{z^2}$], Csc[$\sqrt{z^2}$], Sec[$\sqrt{z^2}$]} // FunctionExpand**

$$\left\{\frac{\sqrt{-i z}\ \sqrt{i z}\ \text{Sin}[z]}{z}, \text{Cos}[z], \frac{\sqrt{-i z}\ \sqrt{i z}\ \text{Tan}[z]}{z}, \frac{\sqrt{-i z}\ \sqrt{i z}\ \text{Cot}[z]}{z}, \frac{\sqrt{-i z}\ \sqrt{i z}\ \text{Csc}[z]}{z}, \text{Sec}[z]\right\}$$

Applying `Simplify` to the last expression gives a more compact result.

**Simplify[%]**

$$\left\{\frac{\sqrt{z^2}\ \text{Sin}[z]}{z}, \text{Cos}[z], \frac{\sqrt{z^2}\ \text{Tan}[z]}{z}, \frac{\sqrt{z^2}\ \text{Cot}[z]}{z}, \frac{\sqrt{z^2}\ \text{Csc}[z]}{z}, \text{Sec}[z]\right\}$$

Here are some similar examples.

**Sin[2 ArcTan[z]] // FunctionExpand**

$$\frac{2z}{1+z^2}$$

**Cos[$\frac{\text{ArcCot}[z]}{2}$] // FunctionExpand**

$$\frac{\sqrt{1+\frac{\sqrt{-z}\ \sqrt{z}}{\sqrt{-1-z^2}}}}{\sqrt{2}}$$

**{Sin[2 ArcSin[z]], Cos[2 ArcCos[z]], Tan[2 ArcTan[z]], Cot[2 ArcCot[z]], Csc[2 ArcCsc[z]], Sec[2 ArcSec[z]]} // FunctionExpand**

$$\left\{2\sqrt{1-z}\ z\sqrt{1+z}, -1+2z^2, -\frac{2z}{(-1+z)(1+z)}, \frac{1}{2}\left(1+\frac{1}{z^2}\right)z\left(\frac{1}{-1-z^2}-\frac{z^2}{-1-z^2}\right), \frac{\sqrt{-i z}\ \sqrt{i z}\ z}{2\sqrt{(-1+z)(1+z)}}, \frac{z^2}{2-z^2}\right\}$$

**{Sin[$\frac{\text{ArcSin}[z]}{2}$], Cos[$\frac{\text{ArcCos}[z]}{2}$], Tan[$\frac{\text{ArcTan}[z]}{2}$], Cot[$\frac{\text{ArcCot}[z]}{2}$], Csc[$\frac{\text{ArcCsc}[z]}{2}$], Sec[$\frac{\text{ArcSec}[z]}{2}$]} // FunctionExpand**

$$\left\{\frac{z\sqrt{1-\sqrt{1-z}\ \sqrt{1+z}}}{\sqrt{2}\ \sqrt{-i z}\ \sqrt{i z}},\ \frac{\sqrt{1+z}}{\sqrt{2}},\ \frac{z}{1+\sqrt{i\ (-i+z)}\ \sqrt{-i\ (i+z)}},\right.$$

$$\left. z\left(1+\frac{\sqrt{-1-z^2}}{\sqrt{-z}\ \sqrt{z}}\right),\ \frac{\sqrt{2}\ \sqrt{-\frac{i}{z}}\ \sqrt{\frac{i}{z}}\ z}{\sqrt{1-\frac{\sqrt{(-1+z)\ (1+z)}}{\sqrt{-i z}\ \sqrt{i z}}}},\ \frac{\sqrt{2}\ \sqrt{-z}}{\sqrt{-1-z}}\right\}$$

**Simplify[%]**

$$\left\{\frac{z\sqrt{1-\sqrt{1-z^2}}}{\sqrt{2}\ \sqrt{z^2}},\ \frac{\sqrt{1+z}}{\sqrt{2}},\ \frac{z}{1+\sqrt{1+z^2}},\ z+\frac{\sqrt{z}\ \sqrt{-1-z^2}}{\sqrt{-z}},\ \frac{\sqrt{2}\ \sqrt{\frac{1}{z^2}}\ z}{\sqrt{1-\frac{\sqrt{z^2}\ \sqrt{-1+z^2}}{z^2}}},\ \frac{\sqrt{2}}{\sqrt{1+\frac{1}{z}}}\right\}$$

### FullSimplify

The function `FullSimplify` tries a wider range of transformations than `Simplify` and returns the simplest form it finds. Here are some examples that contrast the results of applying these functions to the same expressions.

$$\mathbf{Cos}\left[\frac{1}{2}\ i\ \mathbf{Log}[1-i\ z]-\frac{1}{2}\ i\ \mathbf{Log}[1+i\ z]\right]\ //\ \mathbf{Simplify}$$

$$\text{Cosh}\left[\frac{1}{2}\ (\text{Log}[1-i\ z]-\text{Log}[1+i\ z])\right]$$

$$\mathbf{Cos}\left[\frac{1}{2}\ i\ \mathbf{Log}[1-i\ z]-\frac{1}{2}\ i\ \mathbf{Log}[1+i\ z]\right]\ //\ \mathbf{FullSimplify}$$

$$\frac{1}{\sqrt{1+z^2}}$$

$$\left\{\mathbf{Sin}\left[-i\ \mathbf{Log}\left[i\ z+\sqrt{1-z^2}\right]\right],\ \mathbf{Cos}\left[-i\ \mathbf{Log}\left[i\ z+\sqrt{1-z^2}\right]\right],\right.$$
$$\mathbf{Tan}\left[-i\ \mathbf{Log}\left[i\ z+\sqrt{1-z^2}\right]\right],\ \mathbf{Cot}\left[-i\ \mathbf{Log}\left[i\ z+\sqrt{1-z^2}\right]\right],$$
$$\left.\mathbf{Csc}\left[-i\ \mathbf{Log}\left[i\ z+\sqrt{1-z^2}\right]\right],\ \mathbf{Sec}\left[-i\ \mathbf{Log}\left[i\ z+\sqrt{1-z^2}\right]\right]\right\}\ //\ \mathbf{Simplify}$$

$$\left\{z,\ \frac{1-z^2+i\ z\sqrt{1-z^2}}{i\ z+\sqrt{1-z^2}},\ \frac{z\left(z-i\sqrt{1-z^2}\right)}{-i+i\ z^2+z\sqrt{1-z^2}},\ \frac{1-z^2+i\ z\sqrt{1-z^2}}{i\ z^2+z\sqrt{1-z^2}},\ \frac{1}{z},\ \frac{2\left(i\ z+\sqrt{1-z^2}\right)}{1+\left(i\ z+\sqrt{1-z^2}\right)^2}\right\}$$

$$\left\{\mathbf{Sin}\left[-i\ \mathbf{Log}\left[i\ z+\sqrt{1-z^2}\right]\right],\ \mathbf{Cos}\left[-i\ \mathbf{Log}\left[i\ z+\sqrt{1-z^2}\right]\right],\right.$$
$$\mathbf{Tan}\left[-i\ \mathbf{Log}\left[i\ z+\sqrt{1-z^2}\right]\right],\ \mathbf{Cot}\left[-i\ \mathbf{Log}\left[i\ z+\sqrt{1-z^2}\right]\right],$$
$$\left.\mathbf{Csc}\left[-i\ \mathbf{Log}\left[i\ z+\sqrt{1-z^2}\right]\right],\ \mathbf{Sec}\left[-i\ \mathbf{Log}\left[i\ z+\sqrt{1-z^2}\right]\right]\right\}\ //\ \mathbf{FullSimplify}$$

$$\left\{z, \sqrt{1-z^2}, \frac{z}{\sqrt{1-z^2}}, \frac{\sqrt{1-z^2}}{z}, \frac{1}{z}, \frac{1}{\sqrt{1-z^2}}\right\}$$

## Operations carried out by specialized *Mathematica* functions

### Series expansions

Calculating the series expansion of trigonometric functions to hundreds of terms can be done in seconds. Here are some examples.

```
Series[Sin[z], {z, 0, 5}]
```

$$z - \frac{z^3}{6} + \frac{z^5}{120} + O[z]^6$$

```
Normal[%]
```

$$z - \frac{z^3}{6} + \frac{z^5}{120}$$

```
Series[{Sin[z], Cos[z], Tan[z], Cot[z], Csc[z], Sec[z]}, {z, 0, 3}]
```

$$\left\{z - \frac{z^3}{6} + O[z]^4, 1 - \frac{z^2}{2} + O[z]^4, z + \frac{z^3}{3} + O[z]^4, \frac{1}{z} - \frac{z}{3} - \frac{z^3}{45} + O[z]^4, \frac{1}{z} + \frac{z}{6} + \frac{7 z^3}{360} + O[z]^4, 1 + \frac{z^2}{2} + O[z]^4\right\}$$

```
Series[Cot[z], {z, 0, 100}] // Timing
```

$$\left\{1.442\ \text{Second}, \frac{1}{z} - \frac{z}{3} - \frac{z^3}{45} - \frac{2 z^5}{945} - \frac{z^7}{4725} - \frac{2 z^9}{93\,555} - \frac{1382 z^{11}}{638\,512\,875} - \frac{4 z^{13}}{18\,243\,225} - \frac{3617 z^{15}}{162\,820\,783\,125} - \frac{87\,734 z^{17}}{38\,979\,295\,480\,125} - \frac{349\,222 z^{19}}{1\,531\,329\,465\,290\,625} - \frac{310\,732 z^{21}}{13\,447\,856\,940\,643\,125} - \frac{472\,728\,182 z^{23}}{201\,919\,571\,963\,756\,521\,875} - \frac{2\,631\,724 z^{25}}{11\,094\,481\,976\,030\,578\,125} - \frac{13\,571\,120\,588 z^{27}}{564\,653\,660\,170\,076\,273\,671\,875} - \frac{13\,785\,346\,041\,608 z^{29}}{5\,660\,878\,804\,669\,082\,674\,070\,015\,625} - \frac{7\,709\,321\,041\,217 z^{31}}{31\,245\,110\,285\,511\,170\,603\,633\,203\,125} - \frac{303\,257\,395\,102 z^{33}}{12\,130\,454\,581\,433\,748\,587\,292\,890\,625} - \frac{52\,630\,543\,106\,106\,954\,746 z^{35}}{20\,777\,977\,561\,866\,588\,586\,487\,628\,662\,044\,921\,875} - \frac{616\,840\,823\,966\,644 z^{37}}{2\,403\,467\,618\,492\,375\,776\,343\,276\,883\,984\,375} - \frac{522\,165\,436\,992\,898\,244\,102 z^{39}}{20\,080\,431\,172\,289\,638\,826\,798\,401\,128\,390\,556\,640\,625} - \frac{6\,080\,390\,575\,672\,283\,210\,764 z^{41}}{2\,307\,789\,189\,818\,960\,127\,712\,594\,427\,864\,667\,427\,734\,375} - \frac{10\,121\,188\,937\,927\,645\,176\,372 z^{43}}{37\,913\,679\,547\,025\,773\,526\,706\,908\,457\,776\,679\,169\,921\,875} -$$

$$\frac{207\,461\,256\,206\,578\,143\,748\,856\, z^{45}}{7\,670\,102\,214\,448\,301\,053\,033\,358\,480\,610\,212\,529\,462\,890\,625} -$$

$$\frac{11\,218\,806\,737\,995\,635\,372\,498\,255\,094\, z^{47}}{4\,093\,648\,603\,384\,274\,996\,519\,698\,921\,478\,879\,580\,162\,286\,669\,921\,875} -$$

$$\frac{79\,209\,152\,838\,572\,743\,713\,996\,404\, z^{49}}{285\,258\,771\,457\,546\,764\,463\,363\,635\,252\,374\,414\,183\,254\,365\,234\,375} -$$

$$\frac{246\,512\,528\,657\,073\,833\,030\,130\,766\,724\, z^{51}}{8\,761\,982\,491\,474\,419\,367\,550\,817\,114\,626\,909\,562\,924\,278\,968\,505\,859\,375} -$$

$$\frac{233\,199\,709\,079\,078\,899\,371\,344\,990\,501\,528\, z^{53}}{81\,807\,125\,729\,900\,063\,867\,074\,959\,072\,425\,603\,825\,198\,823\,017\,351\,806\,640\,625} -$$

$$\frac{1\,416\,795\,959\,607\,558\,144\,963\,094\,708\,378\,988\, z^{55}}{4\,905\,352\,087\,939\,496\,310\,826\,487\,207\,538\,302\,184\,255\,342\,959\,123\,162\,841\,796\,875} -$$

$$\frac{23\,305\,824\,372\,104\,839\,134\,357\,731\,308\,699\,592\, z^{57}}{796\,392\,368\,980\,577\,121\,745\,974\,726\,570\,063\,253\,238\,310\,542\,073\,919\,837\,646\,484\,375} -$$

$$\frac{9\,721\,865\,123\,870\,044\,576\,322\,439\,952\,638\,561\,968\,331\,928\, z^{59}}{3\,278\,777\,586\,273\,629\,598\,615\,520\,165\,380\,455\,583\,231\,003\,564\,645\,636\,125\,000\,418\,914\,794\,921\,875} -$$

$$\frac{6\,348\,689\,256\,302\,894\,731\,330\,601\,216\,724\,328\,336\, z^{61}}{21\,132\,271\,510\,899\,613\,925\,529\,439\,369\,536\,628\,424\,678\,570\,233\,931\,462\,891\,949\,462\,890\,625} -$$

$$\frac{106\,783\,830\,147\,866\,529\,886\,385\,444\,979\,142\,647\,942\,017\, z^{63}}{3\,508\,062\,732\,166\,890\,409\,707\,514\,582\,539\,928\,001\,638\,766\,051\,683\,792\,497\,378\,070\,587\,158\,203\,125} -$$

$$\left(267\,745\,458\,568\,424\,664\,373\,021\,714\,282\,169\,516\,771\,254\,382\, z^{65}\right) / 86\,812\,790\,293\,146\,213\,360\,651\,966\,604\,262\,937\,105\,495\,141\,563\,588\,806\,888\,204\,273\,501\,373\,291\,015\,625 -$$

$$\left(250\,471\,004\,320\,250\,327\,955\,196\,022\,920\,428\,000\,776\,938\, z^{67}\right) / 801\,528\,196\,428\,242\,695\,121\,010\,267\,455\,843\,804\,062\,822\,357\,897\,831\,858\,125\,102\,407\,684\,326\,171\,875 -$$

$$\left(172\,043\,582\,552\,384\,800\,434\,637\,321\,986\,040\,823\,829\,878\,646\,884\, z^{69}\right) / 5\,433\,748\,964\,547\,053\,581\,149\,916\,185\,708\,338\,218\,048\,392\,402\,830\,337\,634\,114\,958\,370\,880\,742\,156\,982\,421\,875 -$$

$$\left(11\,655\,909\,923\,339\,888\,220\,876\,554\,489\,282\,134\,730\,564\,976\,603\,688\,520\,858\, z^{71}\right) / 3\,633\,348\,205\,269\,879\,230\,856\,840\,004\,304\,821\,536\,968\,049\,780\,112\,803\,650\,817\,771\,432\,558\,560\,793\,458\,452\,606\,201\,171\,875 -$$

$$\left(3\,692\,153\,220\,456\,342\,488\,035\,683\,646\,645\,690\,290\,452\,790\,030\,604\, z^{73}\right) / 11\,359\,005\,221\,796\,317\,918\,049\,302\,062\,760\,294\,302\,183\,889\,391\,189\,419\,445\,133\,951\,612\,582\,060\,536\,346\,435\,546\,875 -$$

$$\left(5\,190\,545\,015\,986\,394\,254\,249\,936\,008\,544\,252\,611\,445\,319\,542\,919\,116\, z^{75}\right) / 157\,606\,197\,452\,423\,911\,112\,934\,066\,120\,799\,083\,442\,801\,465\,302\,753\,194\,801\,233\,578\,624\,576\,089\,941\,806\,793\,212\,890\,625 -$$

$$\left(255\,290\,071\,123\,323\,586\,643\,187\,098\,799\,718\,199\,072\,122\,692\,536\,861\,835\,992\, z^{77}\right) / 76\,505\,736\,228\,426\,953\,173\,738\,238\,352\,183\,101\,801\,688\,392\,812\,244\,485\,181\,277\,127\,930\,109\,049\,138\,257\,655\,704\,498\,291\,015\,625 -$$

$$\left(9\,207\,568\,598\,958\,915\,293\,871\,149\,938\,038\,093\,699\,588\,515\,745\,502\,577\,839\,313\,734\, z^{79}\right) / 27\,233\,582\,984\,369\,795\,892\,070\,228\,410\,001\,578\,355\,986\,013\,571\,390\,071\,723\,225\,259\,349\,721\,067\,988\,068\,852\,863\,296\,604\,156\,494\,140\,625 -$$

$$\left(163\,611\,136\,505\,867\,886\,519\,332\,147\,296\,221\,453\,678\,803\,514\,884\,902\,772\,183\,572\, z^{81}\right) / 4\,776\,089\,171\,877\,348\,057\,451\,105\,924\,101\,750\,653\,118\,402\,745\,283\,825\,543\,113\,171\,217\,116\,857\,704\,024\,700\,607\,798\,175\,811\,767\,578\,125 -$$

$$\left(8\,098\,304\,783\,741\,161\,440\,924\,524\,640\,446\,924\,039\,959\,669\,564\,792\,363\,509\,124\,335\,729\,908\, z^{83}\right) /$$

```
2 333 207 846 470 426 678 843 707 227 616 712 214 909 162 634 745 895 349 325 948 586 531 533 393 \
  530 725 143 500 144 033 328 342 437 744 140 625 -
(122 923 650 124 219 284 385 832 157 660 699 813 260 991 755 656 444 452 420 836 648 z^85) /
 349 538 086 043 843 717 584 559 187 055 386 621 548 470 304 913 596 772 372 737 435 524 697 231 \
  069 047 713 981 709 496 784 210 205 078 125 -
(476 882 359 517 824 548 362 004 154 188 840 670 307 545 554 753 464 961 562 516 323 845 108 z^87) /
 13 383 510 964 174 348 021 497 060 628 653 950 829 663 288 548 327 870 152 944 013 988 358 928 114 \
  528 962 242 087 062 453 152 690 410 614 013 671 875 -
(1 886 491 646 433 732 479 814 597 361 998 744 134 040 407 919 471 435 385 970 472 345 164 676 056
   z^89) /
 522 532 651 330 971 490 226 753 590 247 329 744 050 384 290 675 644 135 735 656 667 608 610 471 \
  400 391 047 234 539 824 350 830 981 313 610 076 904 296 875 -
(450 638 590 680 882 618 431 105 331 665 591 912 924 988 342 163 281 788 877 675 244 114 763 912
   z^91) /
 1 231 931 818 039 911 948 327 467 370 123 161 265 684 460 571 086 659 079 080 437 659 781 065 743 \
  269 173 212 919 832 661 978 537 311 246 395 111 083 984 375 -
(415 596 189 473 955 564 121 634 614 268 323 814 113 534 779 643 471 190 276 158 333 713 923 216
   z^93) /
 11 213 200 675 690 943 223 287 032 785 929 540 201 272 600 687 465 377 745 332 153 847 964 679 254 \
  692 602 138 023 498 144 562 090 675 557 613 372 802 734 375 -
(423 200 899 194 533 026 195 195 456 219 648 467 346 087 908 778 120 468 301 277 466 840 101 336 \
    699 974 518 z^95) /
 112 694 926 530 960 148 011 367 752 417 874 063 473 378 698 369 880 587 800 838 274 234 349 237 \
  591 647 453 413 782 021 538 312 594 164 677 406 144 702 434 539 794 921 875 -
(5 543 531 483 502 489 438 698 050 411 951 314 743 456 505 773 755 468 368 087 670 306 121 873 229 \
    244 z^97) /
 14 569 479 835 935 377 894 165 191 004 250 040 526 616 509 162 234 077 285 176 247 476 968 227 225 \
  810 918 346 966 001 491 701 692 846 112 140 419 483 184 814 453 125 -
(378 392 151 276 488 501 180 909 732 277 974 887 490 811 366 132 267 744 533 542 784 817 245 581 \
    660 788 990 844 z^99) /
 9 815 205 420 757 514 710 108 178 059 369 553 458 327 392 260 750 404 049 930 407 987 933 582 359 \
  080 767 225 644 716 670 683 512 153 512 547 802 166 033 089 160 919 189 453 125 + O[z]^101}
```

*Mathematica* comes with the add-on package `DiscreteMath`RSolve`` that allows finding the general terms of series for many functions. After loading this package, and using the package function `SeriesTerm`, the following $n^{th}$ term for odd trigonometric functions can be evaluated.

```
<< DiscreteMath`RSolve`
```

```
SeriesTerm[{Sin[z], Tan[z], Cot[z], Csc[z], Cos[z], Sec[z]}, {z, 0, n}]
```

$$\left\{\frac{i^{-1+n}\,\text{KroneckerDelta}[\text{Mod}[-1+n,2]]\,\text{UnitStep}[-1+n]}{\text{Gamma}[1+n]},\right.$$
$$\text{If}\left[\text{Odd}[n],\ \frac{i^{-1+n}\,2^{1+n}\,\left(-1+2^{1+n}\right)\text{BernoulliB}[1+n]}{(1+n)\,!},\ 0\right],\ \frac{i\,i^{n}\,2^{1+n}\,\text{BernoulliB}[1+n]}{(1+n)\,!},$$
$$\left.\frac{i\,i^{n}\,2^{1+n}\,\text{BernoulliB}\left[1+n,\frac{1}{2}\right]}{(1+n)\,!},\ \frac{i^{n}\,\text{KroneckerDelta}[\text{Mod}[n,2]]}{\text{Gamma}[1+n]},\ \frac{i^{n}\,\text{EulerE}[n]}{n\,!}\right\}$$

### Differentiation

*Mathematica* can evaluate derivatives of trigonometric functions of an arbitrary positive integer order.

**D[Sin[z], z]**

Cos[z]

**Sin[z] // D[#, z] &**

Cos[z]

$\partial_z$ **{Sin[z], Cos[z], Tan[z], Cot[z], Csc[z], Sec[z]}**

$\{\text{Cos}[z],\ -\text{Sin}[z],\ \text{Sec}[z]^2,\ -\text{Csc}[z]^2,\ -\text{Cot}[z]\,\text{Csc}[z],\ \text{Sec}[z]\,\text{Tan}[z]\}$

$\partial_{\{z,2\}}$ **{Sin[z], Cos[z], Tan[z], Cot[z], Csc[z], Sec[z]}**

$\{-\text{Sin}[z],\ -\text{Cos}[z],\ 2\,\text{Sec}[z]^2\,\text{Tan}[z],\ 2\,\text{Cot}[z]\,\text{Csc}[z]^2,$
$\text{Cot}[z]^2\,\text{Csc}[z]+\text{Csc}[z]^3,\ \text{Sec}[z]^3+\text{Sec}[z]\,\text{Tan}[z]^2\}$

**Table[D[{Sin[z], Cos[z], Tan[z], Cot[z], Csc[z], Sec[z]}, {z, n}], {n, 4}]**

$\{\{\text{Cos}[z],\ -\text{Sin}[z],\ \text{Sec}[z]^2,\ -\text{Csc}[z]^2,\ -\text{Cot}[z]\,\text{Csc}[z],\ \text{Sec}[z]\,\text{Tan}[z]\},\ \{-\text{Sin}[z],\ -\text{Cos}[z],$
$2\,\text{Sec}[z]^2\,\text{Tan}[z],\ 2\,\text{Cot}[z]\,\text{Csc}[z]^2,\ \text{Cot}[z]^2\,\text{Csc}[z]+\text{Csc}[z]^3,\ \text{Sec}[z]^3+\text{Sec}[z]\,\text{Tan}[z]^2\},$
$\{-\text{Cos}[z],\ \text{Sin}[z],\ 2\,\text{Sec}[z]^4+4\,\text{Sec}[z]^2\,\text{Tan}[z]^2,\ -4\,\text{Cot}[z]^2\,\text{Csc}[z]^2-2\,\text{Csc}[z]^4,$
$-\text{Cot}[z]^3\,\text{Csc}[z]-5\,\text{Cot}[z]\,\text{Csc}[z]^3,\ 5\,\text{Sec}[z]^3\,\text{Tan}[z]+\text{Sec}[z]\,\text{Tan}[z]^3\},$
$\{\text{Sin}[z],\ \text{Cos}[z],\ 16\,\text{Sec}[z]^4\,\text{Tan}[z]+8\,\text{Sec}[z]^2\,\text{Tan}[z]^3,$
$8\,\text{Cot}[z]^3\,\text{Csc}[z]^2+16\,\text{Cot}[z]\,\text{Csc}[z]^4,\ \text{Cot}[z]^4\,\text{Csc}[z]+18\,\text{Cot}[z]^2\,\text{Csc}[z]^3+5\,\text{Csc}[z]^5,$
$5\,\text{Sec}[z]^5+18\,\text{Sec}[z]^3\,\text{Tan}[z]^2+\text{Sec}[z]\,\text{Tan}[z]^4\}\}$

### Finite summation

*Mathematica* can calculate finite sums that contain trigonometric functions. Here are two examples.

**Sum[Sin[a k], {k, 0, n}]**

$$\frac{1}{2}\left(\text{Cos}\left[\frac{a}{2}\right]-\text{Cos}\left[\frac{a}{2}+a\,n\right]\right)\text{Csc}\left[\frac{a}{2}\right]$$

$$\sum_{k=0}^{n}(-1)^k\,\textbf{Sin}[a\,k]$$

$$\frac{1}{2}\,\text{Sec}\left[\frac{a}{2}\right]\left(-\text{Sin}\left[\frac{a}{2}\right]+\text{Sin}\left[\frac{a}{2}+a\,n+n\,\pi\right]\right)$$

### Infinite summation

*Mathematica* can calculate infinite sums that contain trigonometric functions. Here are some examples.

$$\sum_{k=1}^{\infty} z^k\,\text{Sin}[k\,x]$$

$$\frac{i\left(-1+e^{2\,i\,x}\right)z}{2\left(e^{i\,x}-z\right)\left(-1+e^{i\,x}\,z\right)}$$

$$\sum_{k=1}^{\infty}\frac{\text{Sin}[k\,x]}{k\,!}$$

$$\frac{1}{2}\,i\left(e^{e^{-i\,x}}-e^{e^{i\,x}}\right)$$

$$\sum_{k=1}^{\infty}\frac{\text{Cos}[k\,x]}{k}$$

$$\frac{1}{2}\left(-\text{Log}\left[1-e^{-i\,x}\right]-\text{Log}\left[1-e^{i\,x}\right]\right)$$

### Finite products

*Mathematica* can calculate some finite symbolic products that contain the trigonometric functions. Here are two examples.

$$\text{Product}\left[\text{Sin}\left[\frac{\pi\,k}{n}\right],\{k,1,n-1\}\right]$$

$$2^{1-n}\,n$$

$$\prod_{k=1}^{n-1}\text{Cos}\left[z+\frac{\pi\,k}{n}\right]$$

$$-(-1)^n\,2^{1-n}\,\text{Sec}[z]\,\text{Sin}\left[\frac{1}{2}\,n\,(\pi-2\,z)\right]$$

### Infinite products

*Mathematica* can calculate infinite products that contain trigonometric functions. Here are some examples.

In[2]:= $$\prod_{k=1}^{\infty}\text{Exp}\left[z^k\,\text{Sin}[k\,x]\right]$$

Out[2]= $$e^{\frac{i\left(-1+e^{2\,i\,x}\right)z}{2\left(z+e^{2\,i\,x}\,z-e^{i\,x}\left(1+z^2\right)\right)}}$$

In[3]:= $\prod_{k=1}^{\infty} \text{Exp}\left[\frac{\text{Cos}[k\,x]}{k\,!}\right]$

Out[3]= $e^{\frac{1}{2}\left(-2+e^{e^{-i\,x}}+e^{e^{i\,x}}\right)}$

### Indefinite integration

*Mathematica* can calculate a huge number of doable indefinite integrals that contain trigonometric functions. Here are some examples.

$$\int \text{Sin}[7\,z]\,dz$$

$$-\frac{1}{7}\,\text{Cos}[7\,z]$$

$$\int \{\{\text{Sin}[z],\ \text{Sin}[z]^a\},\ \{\text{Cos}[z],\ \text{Cos}[z]^a\},\ \{\text{Tan}[z],\ \text{Tan}[z]^a\},\ \{\text{Cot}[z],\ \text{Cot}[z]^a\},\ \{\text{Csc}[z],\ \text{Csc}[z]^a\},\ \{\text{Sec}[z],\ \text{Sec}[z]^a\}\}\,dz$$

$$\Big\{\Big\{-\text{Cos}[z],\ -\text{Cos}[z]\ \text{Hypergeometric2F1}\Big[\frac{1}{2},\ \frac{1-a}{2},\ \frac{3}{2},\ \text{Cos}[z]^2\Big]\ \text{Sin}[z]^{1+a}\ \left(\text{Sin}[z]^2\right)^{\frac{1}{2}(-1-a)}\Big\},$$

$$\Big\{\text{Sin}[z],\ -\frac{\text{Cos}[z]^{1+a}\ \text{Hypergeometric2F1}\Big[\frac{1+a}{2},\ \frac{1}{2},\ \frac{3+a}{2},\ \text{Cos}[z]^2\Big]\ \text{Sin}[z]}{(1+a)\ \sqrt{\text{Sin}[z]^2}}\Big\},$$

$$\Big\{-\text{Log}[\text{Cos}[z]],\ \frac{\text{Hypergeometric2F1}\Big[\frac{1+a}{2},\ 1,\ 1+\frac{1+a}{2},\ -\text{Tan}[z]^2\Big]\ \text{Tan}[z]^{1+a}}{1+a}\Big\},$$

$$\Big\{\text{Log}[\text{Sin}[z]],\ -\frac{\text{Cot}[z]^{1+a}\ \text{Hypergeometric2F1}\Big[\frac{1+a}{2},\ 1,\ 1+\frac{1+a}{2},\ -\text{Cot}[z]^2\Big]}{1+a}\Big\},$$

$$\Big\{-\text{Log}\Big[\text{Cos}\Big[\frac{z}{2}\Big]\Big]+\text{Log}\Big[\text{Sin}\Big[\frac{z}{2}\Big]\Big],$$

$$-\text{Cos}[z]\ \text{Csc}[z]^{-1+a}\ \text{Hypergeometric2F1}\Big[\frac{1}{2},\ \frac{1+a}{2},\ \frac{3}{2},\ \text{Cos}[z]^2\Big]\ \left(\text{Sin}[z]^2\right)^{\frac{1}{2}(-1+a)}\Big\},$$

$$\Big\{-\text{Log}\Big[\text{Cos}\Big[\frac{z}{2}\Big]-\text{Sin}\Big[\frac{z}{2}\Big]\Big]+\text{Log}\Big[\text{Cos}\Big[\frac{z}{2}\Big]+\text{Sin}\Big[\frac{z}{2}\Big]\Big],$$

$$-\frac{\text{Hypergeometric2F1}\Big[\frac{1-a}{2},\ \frac{1}{2},\ \frac{3-a}{2},\ \text{Cos}[z]^2\Big]\ \text{Sec}[z]^{-1+a}\ \text{Sin}[z]}{(1-a)\ \sqrt{\text{Sin}[z]^2}}\Big\}\Big\}$$

### Definite integration

*Mathematica* can calculate wide classes of definite integrals that contain trigonometric functions. Here are some examples.

$$\int_0^{\pi/2} \sqrt[3]{\text{Sin}[z]}\ dz$$

$$\frac{\sqrt{\pi}\ \text{Gamma}\left[\frac{2}{3}\right]}{2\ \text{Gamma}\left[\frac{7}{6}\right]}$$

$$\int_0^{\pi/2}\left\{\sqrt{\text{Sin}[z]},\ \sqrt{\text{Cos}[z]},\ \sqrt{\text{Tan}[z]},\ \sqrt{\text{Cot}[z]},\ \sqrt{\text{Csc}[z]},\ \sqrt{\text{Sec}[z]}\right\} dz$$

$$\left\{2\ \text{EllipticE}\left[\frac{\pi}{4}, 2\right], 2\ \text{EllipticE}\left[\frac{\pi}{4}, 2\right], \frac{\pi}{\sqrt{2}}, \frac{\pi}{\sqrt{2}}, \frac{2\sqrt{\pi}\ \text{Gamma}\left[\frac{5}{4}\right]}{\text{Gamma}\left[\frac{3}{4}\right]}, \frac{2\sqrt{\pi}\ \text{Gamma}\left[\frac{5}{4}\right]}{\text{Gamma}\left[\frac{3}{4}\right]}\right\}$$

$$\int_0^{\frac{\pi}{2}}\{\{\text{Sin}[z], \text{Sin}[z]^a\}, \{\text{Cos}[z], \text{Cos}[z]^a\}, \{\text{Tan}[z], \text{Tan}[z]^a\}, \{\text{Cot}[z], \text{Cot}[z]^a\}, \{\text{Csc}[z], \text{Csc}[z]^a\}, \{\text{Sec}[z], \text{Sec}[z]^a\}\} dz$$

$$\left\{\left\{1, \frac{\sqrt{\pi}\ \text{Gamma}\left[\frac{1+a}{2}\right]}{a\ \text{Gamma}\left[\frac{a}{2}\right]}\right\}, \left\{1, \frac{\sqrt{\pi}\ \text{Gamma}\left[\frac{1+a}{2}\right]}{a\ \text{Gamma}\left[\frac{a}{2}\right]}\right\},\right.$$

$$\left\{\int_0^{\frac{\pi}{2}}\text{Tan}[z]\,dz, \text{If}\left[\text{Re}[a] < 1, \frac{1}{2}\pi\,\text{Sec}\left[\frac{a\pi}{2}\right], \int_0^{\frac{\pi}{2}}\text{Tan}[z]^a\,dz\right]\right\},$$

$$\left\{\int_0^{\frac{\pi}{2}}\text{Cot}[z]\,dz, \text{If}\left[\text{Re}[a] < 1, \frac{1}{2}\pi\,\text{Sec}\left[\frac{a\pi}{2}\right], \int_0^{\frac{\pi}{2}}\text{Cot}[z]^a\,dz\right]\right\},$$

$$\left.\left\{\int_0^{\frac{\pi}{2}}\text{Csc}[z]\,dz, \frac{\sqrt{\pi}\ \text{Gamma}\left[\frac{1}{2}-\frac{a}{2}\right]}{2\ \text{Gamma}\left[1-\frac{a}{2}\right]}\right\}, \left\{\int_0^{\frac{\pi}{2}}\text{Sec}[z]\,dz, \frac{\sqrt{\pi}\ \text{Gamma}\left[\frac{1}{2}-\frac{a}{2}\right]}{2\ \text{Gamma}\left[1-\frac{a}{2}\right]}\right\}\right\}$$

### Limit operation

*Mathematica* can calculate limits that contain trigonometric functions.

$$\text{Limit}\left[\frac{\text{Sin}[z]}{z} + \text{Cos}[z]^3, z \to 0\right]$$

2

$$\text{Limit}\left[\left(\frac{\text{Tan}[x]}{x}\right)^{\frac{1}{x^2}}, x \to 0\right]$$

$e^{1/3}$

### Solving equations

The next input solves equations that contain trigonometric functions. The message indicates that the multivalued functions are used to express the result and that some solutions might be absent.

$$\text{Solve}\left[\text{Tan}[z]^2 + 3\,\text{Sin}\left[z + \text{Pi}/6\right] == 4, z\right]$$

Solve::ifun : Inverse functions are being used by Solve, so some solutions may not be found.

```
{{z → ArcCos[Root[4 - 40 #1^2 + 12 #1^3 + 73 #1^4 - 60 #1^5 + 36 #1^6 &, 1]]},
 {z → -ArcCos[Root[4 - 40 #1^2 + 12 #1^3 + 73 #1^4 - 60 #1^5 + 36 #1^6 &, 2]]},
 {z → -ArcCos[Root[4 - 40 #1^2 + 12 #1^3 + 73 #1^4 - 60 #1^5 + 36 #1^6 &, 3]]},
 {z → ArcCos[Root[4 - 40 #1^2 + 12 #1^3 + 73 #1^4 - 60 #1^5 + 36 #1^6 &, 4]]},
 {z → ArcCos[Root[4 - 40 #1^2 + 12 #1^3 + 73 #1^4 - 60 #1^5 + 36 #1^6 &, 5]]},
 {z → ArcCos[Root[4 - 40 #1^2 + 12 #1^3 + 73 #1^4 - 60 #1^5 + 36 #1^6 &, 6]]}}
```

Complete solutions can be obtained by using the function `Reduce`.

```
Reduce[Sin[x] == a, x] // TraditionalForm
```

```
// InputForm =
 C[1] ∈ Integers && (x == Pi - ArcSin[a] + 2 * Pi * C[1] || x == ArcSin[a] + 2 * Pi * C[1])
```

```
Reduce[Cos[x] == a, x] // TraditionalForm
```

```
// InputForm = C[1] ∈ Integers && (x == -ArcCos[a] + 2 * Pi * C[1] || x == ArcCos[a] + 2 * Pi * C[1])
```

```
Reduce[Tan[x] == a, x] // TraditionalForm
```

```
// InputForm = C[1] ∈ Integers && 1 + a^2 ≠ 0 && x == ArcTan[a] + Pi * C[1]
```

```
Reduce[Cot[x] == a, x] // TraditionalForm
```

```
// InputForm = C[1] ∈ Integers && 1 + a^2 ≠ 0 && x == ArcCot[a] + Pi * C[1]
```

```
Reduce[Csc[x] == a, x] // TraditionalForm
```

$$c_1 \in \mathbb{Z} \bigwedge a \neq 0 \bigwedge \left( x == -\sin^{-1}\left(\frac{1}{a}\right) + 2\pi c_1 + \pi \bigvee x == \sin^{-1}\left(\frac{1}{a}\right) + 2\pi c_1 \right)$$

```
Reduce[Sec[x] == a, x] // TraditionalForm
```

```
// InputForm = C[1] ∈ Integers && a ≠ 0 &&
  (x == -ArcCos[a^(-1)] + 2 * Pi * C[1] || x == ArcCos[a^(-1)] + 2 * Pi * C[1])
```

### Solving differential equations

Here are differential equations whose linear-independent solutions are trigonometric functions. The solutions of the simplest second-order linear ordinary differential equation with constant coefficients can be represented through $\sin(z)$ and $\cos(z)$.

```
DSolve[w''[z] + w[z] == 0, w[z], z]
```

```
{{w[z] → C[1] Cos[z] + C[2] Sin[z]}}
```

```
dsol1 = DSolve[2 w[z] + 3 w''[z] + w^(4)[z] == 0, w[z], z]
```

```
{{w[z] → C[3] Cos[z] + C[1] Cos[√2 z] + C[4] Sin[z] + C[2] Sin[√2 z]}}
```

In the last input, the differential equation was solved for $w(z)$. If the argument is suppressed, the result is returned as a pure function (in the sense of the $\lambda$-calculus).

```
dsol2 = DSolve[2 w[z] + 3 w''[z] + w^(4)[z] == 0, w, z]
```

$\{\{w \to \text{Function}[\{z\}, C[3]\,\text{Cos}[z] + C[1]\,\text{Cos}[\sqrt{2}\,z] + C[4]\,\text{Sin}[z] + C[2]\,\text{Sin}[\sqrt{2}\,z]]\}\}$

The advantage of such a pure function is that it can be used for different arguments, derivatives, and more.

```
w'[ζ] /. dsol1
```

$\{w'[\zeta]\}$

```
w'[ζ] /. dsol2
```

$\{C[4]\,\text{Cos}[\zeta] + \sqrt{2}\,C[2]\,\text{Cos}[\sqrt{2}\,\zeta] - C[3]\,\text{Sin}[\zeta] - \sqrt{2}\,C[1]\,\text{Sin}[\sqrt{2}\,\zeta]\}$

All trigonometric functions satisfy first-order nonlinear differential equations. In carrying out the algorithm to solve the nonlinear differential equation, *Mathematica* has to solve a transcendental equation. In doing so, the generically multivariate inverse of a function is encountered, and a message is issued that a solution branch is potentially missed.

$\text{DSolve}\left[\left\{w'[z] == \sqrt{1 - w[z]^2},\ w[0] == 0\right\},\ w[z],\ z\right]$

```
Solve::ifun : Inverse functions are being used by Solve, so some solutions may not be found.
```

$\{\{w[z] \to \text{Sin}[z]\}\}$

$\text{DSolve}\left[\left\{w'[z] == \sqrt{1 - w[z]^2},\ w[0] == 1\right\},\ w[z],\ z\right]$

```
Solve::ifun : Inverse functions are being used by Solve, so some solutions may not be found.
```

$\{\{w[z] \to \text{Cos}[z]\}\}$

$\text{DSolve}\left[\left\{w'[z] - w[z]^2 - 1 == 0,\ w[0] == 0\right\},\ w[z],\ z\right]$

```
Solve::ifun : Inverse functions are being used by Solve, so some solutions may not be found.
```

$\{\{w[z] \to \text{Tan}[z]\}\}$

$\text{DSolve}\left[\left\{w'[z] + w[z]^2 + 1 == 0,\ w\left[\frac{\pi}{2}\right] == 0\right\},\ w[z],\ z\right]$

```
Solve::ifun : Inverse functions are being used by Solve, so some solutions may not be found.
```

$\{\{w[z] \to \text{Cot}[z]\}\}$

$\text{DSolve}\left[\left\{w'[z] == \sqrt{w[z]^4 - w[z]^2},\ 1/w[0] == 0\right\},\ w[z],\ z\right]\ //\ \text{Simplify}[\#,\ 0 < z < \text{Pi}/2]\ \&$

```
Solve::verif : Potential solution {C[1] → Indeterminate} (possibly
   discarded by verifier) should be checked by hand. May require use of limits.

Solve::ifun : Inverse functions are being used by Solve, so some solutions may not be found.

Solve::verif : Potential solution {C[1] → Indeterminate} (possibly
   discarded by verifier) should be checked by hand. May require use of limits.

Solve::ifun : Inverse functions are being used by Solve, so some solutions may not be found.
```

```
{{w[z] → -Csc[z]}, {w[z] → Csc[z]}}
```

$$\text{DSolve}\left[\left\{w'[z] == \sqrt{w[z]^4 - w[z]^2},\ 1 \Big/ w\left[\frac{\pi}{2}\right] == 0\right\},\ w[z],\ z\right] \text{ // Simplify}\left[\#,\ 0 < z < \text{Pi}/2\right] \&$$

```
Solve::verif : Potential solution {C[1] → Indeterminate} (possibly
   discarded by verifier) should be checked by hand. May require use of limits.

Solve::ifun : Inverse functions are being used by Solve, so some solutions may not be found.

Solve::verif : Potential solution {C[1] → Indeterminate} (possibly
   discarded by verifier) should be checked by hand. May require use of limits.

Solve::ifun : Inverse functions are being used by Solve, so some solutions may not be found.
```

```
{{w[z] → -Sec[z]}, {w[z] → Sec[z]}}
```

## Integral transforms

*Mathematica* supports the main integral transforms like direct and inverse Fourier, Laplace, and Z transforms that can give results that contain classical or generalized functions. Here are some transforms of trigonometric functions.

**LaplaceTransform[Sin[t], t, s]**

$$\frac{1}{1+s^2}$$

**FourierTransform[Sin[t], t, s]**

$$i\sqrt{\frac{\pi}{2}}\ \text{DiracDelta}[-1+s] - i\sqrt{\frac{\pi}{2}}\ \text{DiracDelta}[1+s]$$

**FourierSinTransform[Sin[t], t, s]**

$$\sqrt{\frac{\pi}{2}}\ \text{DiracDelta}[-1+s] - \sqrt{\frac{\pi}{2}}\ \text{DiracDelta}[1+s]$$

**FourierCosTransform[Sin[t], t, s]**

$$-\frac{1}{\sqrt{2\pi}\ (-1+s)} + \frac{1}{\sqrt{2\pi}\ (1+s)}$$

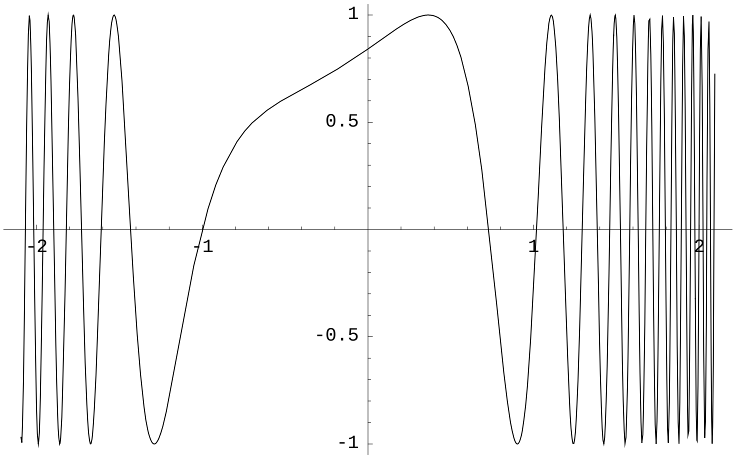
**ZTransform[Sin[π t], t, s]**

0

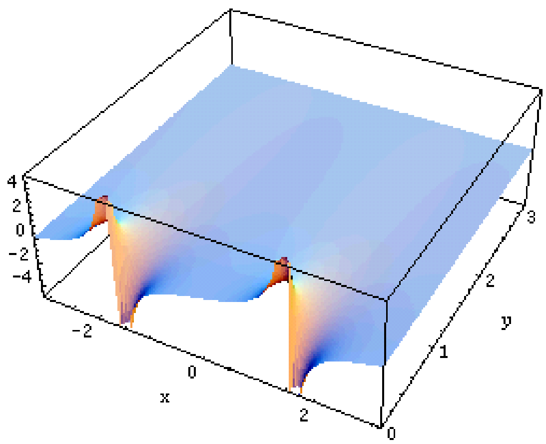
## Plotting

*Mathematica* has built-in functions for 2D and 3D graphics. Here are some examples.

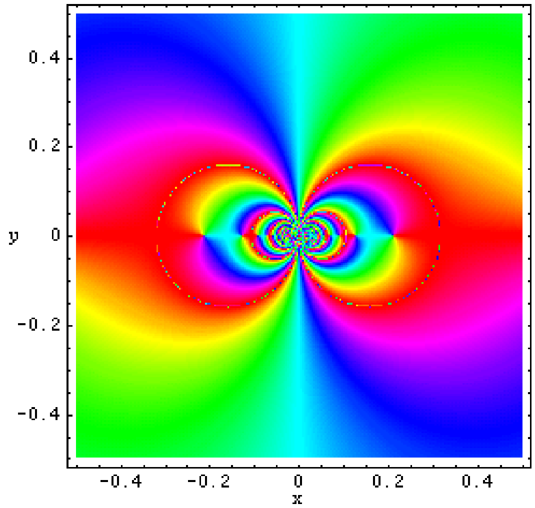
$$\text{Plot}\left[\text{Sin}\left[\sum_{k=0}^{5} z^k\right],\ \left\{z,\ -\frac{2\pi}{3},\ \frac{2\pi}{3}\right\}\right];$$




```
Plot3D[Re[Tan[x + ⅈ y]], {x, -π, π}, {y, 0, π},
  PlotPoints → 240, PlotRange → {-5, 5},
  ClipFill → None, Mesh → False, AxesLabel → {"x", "y", None}];
```




```
ContourPlot[Arg[Sec[1/(x + ⅈ y)]], {x, -1/2, 1/2}, {y, -1/2, 1/2},
  PlotPoints → 400, PlotRange → {-π, π}, FrameLabel → {"x", "y", None, None},
  ColorFunction → Hue, ContourLines → False, Contours → 200];
```




# Introduction to the Tangent Function in *Mathematica*

## Overview

The following shows how the tangent function is realized in *Mathematica*. Examples of evaluating *Mathematica* functions applied to various numeric and exact expressions that involve the tangent function or return it are shown. These involve numeric and symbolic calculations and plots.

## Notations

### *Mathematica* forms of notations

Following *Mathematica*'s general naming convention, function names in `StandardForm` are just the capitalized versions of their traditional mathematics names. This shows the tangent function in `StandardForm`.

```
Tan[z]
```

```
Tan[z]
```

This shows the tangent function in `TraditionalForm`.

```
% // TraditionalForm
```

tan($z$)

### Additional forms of notations

*Mathematica* also knows the most popular forms of notations for the tangent function that are used in other programming languages. Here are three examples: `CForm`, `TeXForm`, and `FortranForm`.

```
In[139]:= {CForm[Tan[2 π z]], TeXForm[Tan[2 π z]], FortranForm[Tan[2 π z]]}

Out[139]= {Tan(2*Pi*z), \tan(2\, \pi \, z), Tan(2*Pi*z)}
```

## Automatic evaluations and transformations

### Evaluation for exact, machine-number, and high-precision arguments

For the exact argument $z = \pi / 4$, *Mathematica* returns an exact result.

```
Tan[π/4]
```

```
1
```

```
Tan[z] /. z → π/4
```

```
1
```

For a machine-number argument (a numerical argument with a decimal point and not too many digits), a machine number is also returned.

```
Tan[4.]
```

```
1.15782
```

```
Tan[z] /. z → 2.
```

```
-2.18504
```

The next inputs calculate 100-digit approximations at $z = 1$ and $z = 2$.

```
N[Tan[z] /. z → 1, 100]
```

```
0.8414709848078965066525023216302989996225630607983710656727517099919104043912396689
 486397435430526959
```

```
N[Tan[2], 100]
```

```
-2.185039863261518991643306102313682543432017746227663164562955869966773747209194182
  319743542104728548
```

```
Tan[2] // N[#, 100] &
```

```
-2.185039863261518991643306102313682543432017746227663164562955869966773747209194182
  319743542104728548
```

Within a second, it is possible to calculate thousands of digits for the tangent function. The next input calculates 10000 digits for tan(1) and analyzes the frequency of the digit $k$ in the resulting decimal number.

```
Map[Function[w, {First[#], Length[#]} & /@ Split[Sort[First[RealDigits[w]]]]],
 N[{Tan[z]} /. z → 1, 10 000]]
```

```
{{{0, 1024}, {1, 1025}, {2, 1000}, {3, 969},
  {4, 1026}, {5, 944}, {6, 999}, {7, 1001}, {8, 1008}, {9, 1004}}}
```

Here is a 50-digit approximation of the tangent function at the complex argument $z = 3 - 2 i$.

```
N[Tan[3 - 2 ⅈ], 50]
```

```
-0.00988437503832249372031403430350121097961813353467 04 -
 0.965385879022133124278480269394560685879729650005 76 ⅈ
```

```
{N[Tan[z] /. z → 3 - 2 ⅈ, 50], Tan[3 - 2 ⅈ] // N[#, 50] &}
```

```
{-0.00988437503832249372031403430350121097961813353467 04 -
  0.965385879022133124278480269394560685879729650005 76 ⅈ,
 -0.00988437503832249372031403430350121097961813353467 04 -
  0.965385879022133124278480269394560685879729650005 76 ⅈ}
```

*Mathematica* automatically evaluates mathematical functions with machine precision, if the arguments of the function are machine-number elements. In this case, only six digits after the decimal point are shown in the results. The remaining digits are suppressed, but can be displayed using the function `InputForm`.

```
{Tan[3.], N[Tan[3]], N[Tan[3], 16], N[Tan[3], 5], N[Tan[3], 20]}
```

```
{-0.142547, -0.142547, -0.142547, -0.142547, -0.14254654307427780530}
```

```
% // InputForm
```

```
{-0.1425465430742778, -0.1425465430742778, -0.1425465430742778, -0.1425465430742778,
 -0.142546543074277805295635410533911874 52`20}
```

### Simplification of the argument

*Mathematica* knows symmetry and periodicity of the tangent function. Here are some examples.

```
Tan[-3]
```

```
-Tan[3]
```

```
{Tan[-z], Tan[z + π], Tan[z + 2 π], Tan[-z + 21 π]}
```

```
{-Tan[z], Tan[z], Tan[z], -Tan[z]}
```

*Mathematica* automatically simplifies the composition of the direct and the inverse tangent functions into its argument.

```
Tan[ArcTan[z]]
```

```
z
```

*Mathematica* also automatically simplifies the composition of the direct and any of the inverse trigonometric functions into algebraic functions of the argument.

```
{Tan[ArcSin[z]], Tan[ArcCos[z]], Tan[ArcTan[z]],
 Tan[ArcCot[z]], Tan[ArcCsc[z]], Tan[ArcSec[z]]}
```

$$\left\{\frac{z}{\sqrt{1-z^2}}, \frac{\sqrt{1-z^2}}{z}, z, \frac{1}{z}, \frac{1}{\sqrt{1-\frac{1}{z^2}}\, z}, \sqrt{1-\frac{1}{z^2}}\, z\right\}$$

If the argument has the structure $\pi k/2+z$ or $\pi k/2-z$, and $\pi k/2+i z$ or $\pi k/2-i z$ with integer $k$, the tangent function can be automatically transformed into trigonometric or hyperbolic tangent or cotangent functions.

$\text{Tan}\left[\frac{\pi}{2} - 4\right]$

Cot[4]

$\left\{\text{Tan}\left[\frac{\pi}{2} - z\right], \text{Tan}\left[\frac{\pi}{2} + z\right], \text{Tan}\left[-\frac{\pi}{2} - z\right], \text{Tan}\left[-\frac{\pi}{2} + z\right], \text{Tan}[\pi - z], \text{Tan}[\pi + z]\right\}$

{Cot[z], -Cot[z], Cot[z], -Cot[z], -Tan[z], Tan[z]}

Tan[ⅈ 5]

ⅈ Tanh[5]

$\left\{\text{Tan}[i z], \text{Tan}\left[\frac{\pi}{2} - i z\right], \text{Tan}\left[\frac{\pi}{2} + i z\right], \text{Tan}[\pi - i z], \text{Tan}[\pi + i z]\right\}$

{ⅈ Tanh[z], -ⅈ Coth[z], ⅈ Coth[z], -ⅈ Tanh[z], ⅈ Tanh[z]}

### Simplification of simple expressions containing the tangent function

Sometimes simple arithmetic operations containing the tangent function can automatically produce other trigonometric functions.

1 / Tan[4]

Cot[4]

```
{1 / Tan[z], 1 / Tan[π / 2 - z], Tan[π / 2 - z] / Tan[z],
 Tan[z] / Tan[π / 2 - z], 1 / Tan[π / 2 - z], Tan[π / 2 - z] / Tan[z] ^2}
```

$\left\{\text{Cot}[z], \text{Tan}[z], \text{Cot}[z]^2, \text{Tan}[z]^2, \text{Tan}[z], \text{Cot}[z]^3\right\}$

### The tangent function arising as special cases from more general functions

The tangent function can be treated as a particular case of some more general special functions. For example, tan($z$) can appear automatically from Bessel, Mathieu, Jacobi, hypergeometric, and Meijer functions for appropriate values of their parameters.

$\left\{\text{BesselJ}\left[\frac{1}{2}, z\right] \Big/ \text{BesselJ}\left[-\frac{1}{2}, z\right], \frac{\text{MathieuS}[1, 0, z]}{\text{MathieuC}[1, 0, z]}, \text{JacobiSC}[z, 0],\right.$
$\text{JacobiCS}\left[\frac{\pi}{2} - z, 0\right], -i\,\text{JacobiSN}[i z, 1], i\,\text{JacobiNS}\left[\frac{\pi i}{2} - i z, 1\right],$
$\text{HypergeometricPFQ}\left[\{\}, \left\{\frac{3}{2}\right\}, -\frac{z^2}{4}\right] \Big/ \text{HypergeometricPFQ}\left[\{\}, \left\{\frac{1}{2}\right\}, -\frac{z^2}{4}\right],$
$\left.\text{MeijerG}\left[\{\{\}, \{\}\}, \left\{\left\{\frac{1}{2}\right\}, \{0\}\right\}, \frac{z^2}{4}\right] \Big/ \text{MeijerG}\left[\{\{\}, \{\}\}, \left\{\left\{-\frac{1}{2}\right\}, \{0\}\right\}, \frac{z^2}{4}\right]\right\}$

$\left\{\text{Tan}[z], \text{Tan}[z], \text{Tan}[z], \text{Tan}[z], \text{Tan}[z], \text{Tan}[z], \frac{\text{Tan}\left[\sqrt{z^2}\right]}{\sqrt{z^2}}, \frac{1}{2} z\,\text{Tan}[z]\right\}$

## Equivalence transformations carried out by specialized *Mathematica* functions

### General remarks

Almost everybody prefers using $1-\tan(z)$ instead of $\tan(\pi - z)+\tan(\pi/4)$. *Mathematica* automatically transforms the second expression into the first one. The automatic application of transformation rules to mathematical expressions can give overly complicated results. Compact expressions like $\tan(\pi/16)$ should not be automatically expanded into the more complicated expression $\left(\left(2-\left(2+2^{1/2}\right)^{1/2}\right)\Big/\left(2+\left(2+2^{1/2}\right)\right)\right)^{1/2}$. *Mathematica* has special functions that produce such expansions. Some are demonstrated in the next section.

### TrigExpand

The function `TrigExpand` expands out trigonometric and hyperbolic functions. In more detail, it splits up sums and integer multiples that appear in the arguments of trigonometric and hyperbolic functions, and then expands out the products of trigonometric and hyperbolic functions into sums of powers, using trigonometric and hyperbolic identities where possible. Here are some examples.

```
TrigExpand[Tan[x - y]]
```

$$\frac{\text{Cos}[y]\,\text{Sin}[x]}{\text{Cos}[x]\,\text{Cos}[y]+\text{Sin}[x]\,\text{Sin}[y]}-\frac{\text{Cos}[x]\,\text{Sin}[y]}{\text{Cos}[x]\,\text{Cos}[y]+\text{Sin}[x]\,\text{Sin}[y]}$$

```
Tan[4 z] // TrigExpand
```

$$\frac{4\,\text{Cos}[z]^3\,\text{Sin}[z]}{\text{Cos}[z]^4-6\,\text{Cos}[z]^2\,\text{Sin}[z]^2+\text{Sin}[z]^4}-\frac{4\,\text{Cos}[z]\,\text{Sin}[z]^3}{\text{Cos}[z]^4-6\,\text{Cos}[z]^2\,\text{Sin}[z]^2+\text{Sin}[z]^4}$$

```
Tan[2 z]^2 // TrigExpand
```

$$\frac{1}{2\left(\text{Cos}[z]^2-\text{Sin}[z]^2\right)^2}-\frac{\text{Cos}[z]^4}{2\left(\text{Cos}[z]^2-\text{Sin}[z]^2\right)^2}+\frac{3\,\text{Cos}[z]^2\,\text{Sin}[z]^2}{\left(\text{Cos}[z]^2-\text{Sin}[z]^2\right)^2}-\frac{\text{Sin}[z]^4}{2\left(\text{Cos}[z]^2-\text{Sin}[z]^2\right)^2}$$

```
TrigExpand[{Tan[x + y + z], Tan[3 z]}]
```

$$\Big\{\frac{\text{Cos}[y]\,\text{Cos}[z]\,\text{Sin}[x]}{\text{Cos}[x]\,\text{Cos}[y]\,\text{Cos}[z]-\text{Cos}[z]\,\text{Sin}[x]\,\text{Sin}[y]-\text{Cos}[y]\,\text{Sin}[x]\,\text{Sin}[z]-\text{Cos}[x]\,\text{Sin}[y]\,\text{Sin}[z]}+\frac{\text{Cos}[x]\,\text{Cos}[z]\,\text{Sin}[y]}{\text{Cos}[x]\,\text{Cos}[y]\,\text{Cos}[z]-\text{Cos}[z]\,\text{Sin}[x]\,\text{Sin}[y]-\text{Cos}[y]\,\text{Sin}[x]\,\text{Sin}[z]-\text{Cos}[x]\,\text{Sin}[y]\,\text{Sin}[z]}+\frac{\text{Cos}[x]\,\text{Cos}[y]\,\text{Sin}[z]}{\text{Cos}[x]\,\text{Cos}[y]\,\text{Cos}[z]-\text{Cos}[z]\,\text{Sin}[x]\,\text{Sin}[y]-\text{Cos}[y]\,\text{Sin}[x]\,\text{Sin}[z]-\text{Cos}[x]\,\text{Sin}[y]\,\text{Sin}[z]}-\frac{\text{Sin}[x]\,\text{Sin}[y]\,\text{Sin}[z]}{\text{Cos}[x]\,\text{Cos}[y]\,\text{Cos}[z]-\text{Cos}[z]\,\text{Sin}[x]\,\text{Sin}[y]-\text{Cos}[y]\,\text{Sin}[x]\,\text{Sin}[z]-\text{Cos}[x]\,\text{Sin}[y]\,\text{Sin}[z]},\ \frac{3\,\text{Cos}[z]^2\,\text{Sin}[z]}{\text{Cos}[z]^3-3\,\text{Cos}[z]\,\text{Sin}[z]^2}-\frac{\text{Sin}[z]^3}{\text{Cos}[z]^3-3\,\text{Cos}[z]\,\text{Sin}[z]^2}\Big\}$$

### TrigFactor

The function `TrigFactor` factors trigonometric and hyperbolic functions. In more detail, it splits up sums and integer multiples that appear in the arguments of trigonometric and hyperbolic functions, and then factors the resulting polynomials into trigonometric and hyperbolic functions, using trigonometric and hyperbolic identities where possible. Here are some examples.

```
TrigFactor[Tan[x] + Tan[y]]
```

```
Sec[x] Sec[y] Sin[x + y]
```

```
Tan[x] - Cot[y] // TrigFactor
```

```
-Cos[x + y] Csc[y] Sec[x]
```

## TrigReduce

The function `TrigReduce` rewrites products and powers of trigonometric and hyperbolic functions in terms of trigonometric and hyperbolic functions with combined arguments. In more detail, it typically yields a linear expression involving trigonometric and hyperbolic functions with more complicated arguments. `TrigReduce` is approximately opposite to `TrigExpand` and `TrigFactor`. Here are some examples.

```
TrigReduce[Tan[x] Tan[y]]
```

$$\frac{\text{Cos}[x-y]-\text{Cos}[x+y]}{\text{Cos}[x-y]+\text{Cos}[x+y]}$$

```
Tan[x] Cot[y] // TrigReduce
```

$$\frac{-\text{Sin}[x-y]-\text{Sin}[x+y]}{\text{Sin}[x-y]-\text{Sin}[x+y]}$$

```
Table[TrigReduce[Tan[z]^n], {n, 2, 5}]
```

$$\left\{\frac{1-\text{Cos}[2\,z]}{1+\text{Cos}[2\,z]},\ \frac{3\,\text{Sin}[z]-\text{Sin}[3\,z]}{3\,\text{Cos}[z]+\text{Cos}[3\,z]},\ \frac{3-4\,\text{Cos}[2\,z]+\text{Cos}[4\,z]}{3+4\,\text{Cos}[2\,z]+\text{Cos}[4\,z]},\ \frac{10\,\text{Sin}[z]-5\,\text{Sin}[3\,z]+\text{Sin}[5\,z]}{10\,\text{Cos}[z]+5\,\text{Cos}[3\,z]+\text{Cos}[5\,z]}\right\}$$

```
TrigReduce[TrigExpand[{Tan[x + y + z], Tan[3 z], Tan[x] Tan[y]}]]
```

$$\left\{\text{Tan}[x+y+z],\ \text{Tan}[3\,z],\ \frac{\text{Cos}[x-y]-\text{Cos}[x+y]}{\text{Cos}[x-y]+\text{Cos}[x+y]}\right\}$$

```
TrigFactor[Tan[x] + Tan[y]] // TrigReduce
```

$$\frac{2\,\text{Sin}[x+y]}{\text{Cos}[x-y]+\text{Cos}[x+y]}$$

## TrigToExp

The function `TrigToExp` converts trigonometric and hyperbolic functions to exponentials. It tries, where possible, to give results that do not involve explicit complex numbers. Here are some examples.

```
TrigToExp[Tan[z]]
```

$$\frac{i\left(e^{-i z}-e^{i z}\right)}{e^{-i z}+e^{i z}}$$

```
Tan[a z] + Tan[b z] // TrigToExp
```

$$\frac{i\, e^{-i a z}}{e^{-i a z}+e^{i a z}}-\frac{i\, e^{i a z}}{e^{-i a z}+e^{i a z}}+\frac{i\, e^{-i b z}}{e^{-i b z}+e^{i b z}}-\frac{i\, e^{i b z}}{e^{-i b z}+e^{i b z}}$$

### ExpToTrig

The function `ExpToTrig` converts exponentials to trigonometric and hyperbolic functions. It is approximately inverse to `TrigToExp`. Here are some examples.

```
ExpToTrig[TrigToExp[Tan[z]]]
```

```
Tan[z]
```

$$\left\{\alpha\, e^{-i x \beta}+\alpha\, e^{i x \beta} \big/ \left(\alpha\, e^{-i x \beta}+\gamma\, e^{i x \beta}\right)\right\} \text{ // ExpToTrig}$$

$$\left\{\alpha \operatorname{Cos}[x\beta]-i\,\alpha \operatorname{Sin}[x\beta]+\frac{\alpha\,(\operatorname{Cos}[x\beta]+i \operatorname{Sin}[x\beta])}{\alpha \operatorname{Cos}[x\beta]+\gamma \operatorname{Cos}[x\beta]-i\,\alpha \operatorname{Sin}[x\beta]+i\,\gamma \operatorname{Sin}[x\beta]}\right\}$$

### ComplexExpand

The function `ComplexExpand` expands expressions assuming that all the variables are real. The value option `TargetFunctions` is a list of functions from the set `{Re, Im, Abs, Arg, Conjugate, Sign}`. `ComplexExpand` tries to give results in terms of the functions specified. Here are some examples.

```
ComplexExpand[Tan[x + i y]]
```

$$\frac{\operatorname{Sin}[2x]}{\operatorname{Cos}[2x]+\operatorname{Cosh}[2y]}+\frac{i \operatorname{Sinh}[2y]}{\operatorname{Cos}[2x]+\operatorname{Cosh}[2y]}$$

```
Tan[x + i y] + Tan[x - i y] // ComplexExpand
```

$$\frac{2 \operatorname{Sin}[2x]}{\operatorname{Cos}[2x]+\operatorname{Cosh}[2y]}$$

```
ComplexExpand[Re[Tan[x + i y]], TargetFunctions → {Re, Im}]
```

$$\frac{\operatorname{Sin}[2x]}{\operatorname{Cos}[2x]+\operatorname{Cosh}[2y]}$$

```
ComplexExpand[Im[Tan[x + i y]], TargetFunctions → {Re, Im}]
```

$$\frac{\operatorname{Sinh}[2y]}{\operatorname{Cos}[2x]+\operatorname{Cosh}[2y]}$$

```
ComplexExpand[Abs[Tan[x + i y]], TargetFunctions → {Re, Im}]
```

$$\sqrt{\frac{\operatorname{Sin}[2x]^2}{(\operatorname{Cos}[2x]+\operatorname{Cosh}[2y])^2}+\frac{\operatorname{Sinh}[2y]^2}{(\operatorname{Cos}[2x]+\operatorname{Cosh}[2y])^2}}$$

```
ComplexExpand[Abs[Tan[x + i y]], TargetFunctions → {Re, Im}] //
 Simplify[#, {x, y} ∈ Reals] &
```

$$\frac{\sqrt{\text{Sin}[2x]^2 + \text{Sinh}[2y]^2}}{\text{Cos}[2x] + \text{Cosh}[2y]}$$

```
ComplexExpand[Re[Tan[x + ⅈ y]] + Im[Tan[x + ⅈ y]], TargetFunctions → {Re, Im}]
```

$$\frac{\text{Sin}[2x]}{\text{Cos}[2x] + \text{Cosh}[2y]} + \frac{\text{Sinh}[2y]}{\text{Cos}[2x] + \text{Cosh}[2y]}$$

```
ComplexExpand[Arg[Tan[x + ⅈ y]], TargetFunctions → {Re, Im}]
```

$$\text{ArcTan}\left[\frac{\text{Sin}[2x]}{\text{Cos}[2x] + \text{Cosh}[2y]}, \frac{\text{Sinh}[2y]}{\text{Cos}[2x] + \text{Cosh}[2y]}\right]$$

```
ComplexExpand[Arg[Tan[x + ⅈ y]], TargetFunctions → {Re, Im}] //
 Simplify[#, {x, y} ∈ Reals] &
```

```
ArcTan[Sin[2 x], Sinh[2 y]]
```

```
ComplexExpand[Conjugate[Tan[x + ⅈ y]], TargetFunctions → {Re, Im}] // Simplify
```

$$\frac{\text{Sin}[2x] - i\,\text{Sinh}[2y]}{\text{Cos}[2x] + \text{Cosh}[2y]}$$

### Simplify

The function `Simplify` performs a sequence of algebraic transformations on the expression, and returns the simplest form it finds. Here are some examples.

$$\frac{\text{Tan}[z_1] + \text{Tan}[z_2] + \text{Tan}[z_3] - \text{Tan}[z_1]\,\text{Tan}[z_2]\,\text{Tan}[z_3]}{1 - \text{Tan}[z_1]\,\text{Tan}[z_2] - \text{Tan}[z_1]\,\text{Tan}[z_3] - \text{Tan}[z_2]\,\text{Tan}[z_3]} \text{ // Simplify}$$

$$\text{Tan}[z_1 + z_2 + z_3]$$

$$\text{Simplify}\left[\text{Tan}\left[z - \frac{\pi}{3}\right]\text{Tan}\left[\frac{\pi}{3} + z\right] + \text{Tan}\left[z - \frac{\pi}{3}\right]\text{Tan}[z] + \text{Tan}[z]\,\text{Tan}\left[\frac{\pi}{3} + z\right]\right]$$

$$-3$$

Here is a collection of trigonometric identities. All are written as one large logical conjunction.

$$\text{Simplify}[\#]\ \&\ /@ \left(\text{Tan}[2z]\left(1 - \text{Tan}[z]^2\right) == 2\,\text{Tan}[z] \bigwedge \right.$$
$$\text{Tan}[z]^2 == \frac{1 - \text{Cos}[2z]}{1 + \text{Cos}[2z]} \bigwedge \text{Tan}[z]^3 == \frac{3\,\text{Sin}[z] - \text{Sin}[3z]}{3\,\text{Cos}[z] + \text{Cos}[3z]} \bigwedge$$
$$\text{Tan}[a] + \text{Tan}[b] == \frac{\text{Sin}[a+b]}{\text{Cos}[a]\,\text{Cos}[b]} \bigwedge \text{Tan}[a] - \text{Tan}[b] == \frac{\text{Sin}[a-b]}{\text{Cos}[a]\,\text{Cos}[b]} \bigwedge$$
$$\text{Tan}\left[\frac{z}{2}\right] == \frac{1 - \text{Cos}[z]}{\text{Sin}[z]} == \frac{\text{Sin}[z]}{1 + \text{Cos}[z]} \bigwedge$$
$$\left. \text{Tan}[a]^2 - \text{Tan}[b]^2 == \text{Sec}[a]^2\,\text{Sec}[b]^2\,\text{Sin}[a-b]\,\text{Sin}[a+b]\right)$$

```
True
```

The function `Simplify` has the `Assumption` option. For example, *Mathematica* treats the periodicity of trigonometric functions for the symbolic integer coefficient $k$ of $k\pi$.

```
Simplify[{Tan[z + 2 k π], Tan[z + k π] / Tan[z]}, k ∈ Integers]
```

```
{Tan[z], 1}
```

*Mathematica* also knows that the composition of the inverse and the direct trigonometric functions produces the value of the internal argument under the corresponding restriction.

```
ArcTan[Tan[z]]
```

```
ArcTan[Tan[z]]
```

```
Simplify[ArcTan[Tan[z]], -π / 2 < Re[z] < π / 2]
```

```
z
```

### FunctionExpand (and Together)

While the tangent function auto-evaluates for simple fractions of $\pi$, for more complicated cases it stays as a tangent function to avoid the build up of large expressions. Using the function `FunctionExpand`, the tangent function can sometimes be transformed into explicit radicals. Here are some examples.

$$\left\{\tan\left[\frac{\pi}{16}\right], \tan\left[\frac{\pi}{60}\right]\right\}$$

$$\left\{\tan\left[\frac{\pi}{16}\right], \tan\left[\frac{\pi}{60}\right]\right\}$$

```
FunctionExpand[%]
```

$$\left\{\sqrt{\frac{2-\sqrt{2+\sqrt{2}}}{2+\sqrt{2+\sqrt{2}}}}, \frac{-\frac{-\frac{1}{8}\sqrt{3}\left(-1+\sqrt{5}\right)-\frac{1}{4}\sqrt{\frac{1}{2}\left(5+\sqrt{5}\right)}}{\sqrt{2}}+\frac{\frac{1}{8}\left(-1+\sqrt{5}\right)-\frac{1}{4}\sqrt{\frac{3}{2}\left(5+\sqrt{5}\right)}}{\sqrt{2}}}{-\frac{-\frac{1}{8}\sqrt{3}\left(-1+\sqrt{5}\right)-\frac{1}{4}\sqrt{\frac{1}{2}\left(5+\sqrt{5}\right)}}{\sqrt{2}}-\frac{\frac{1}{8}\left(-1+\sqrt{5}\right)-\frac{1}{4}\sqrt{\frac{3}{2}\left(5+\sqrt{5}\right)}}{\sqrt{2}}}\right\}$$

```
Together[%]
```

$$\left\{\sqrt{\frac{2-\sqrt{2+\sqrt{2}}}{2+\sqrt{2+\sqrt{2}}}}, \frac{-1-\sqrt{3}+\sqrt{5}+\sqrt{15}+\sqrt{2\left(5+\sqrt{5}\right)}-\sqrt{6\left(5+\sqrt{5}\right)}}{1-\sqrt{3}-\sqrt{5}+\sqrt{15}+\sqrt{2\left(5+\sqrt{5}\right)}+\sqrt{6\left(5+\sqrt{5}\right)}}\right\}$$

If the denominator contains squares of integers other than 2, the results always contain complex numbers (meaning that the imaginary number $i = \sqrt{-1}$ appears unavoidably).

$$\left\{\tan\left[\frac{\pi}{9}\right]\right\}$$

$\{\text{Tan}[\frac{\pi}{9}]\}$

```
FunctionExpand[%] // Together
```

$$\left\{\frac{-\left(-1-i\sqrt{3}\right)^{1/3}-i\sqrt{3}\left(-1-i\sqrt{3}\right)^{1/3}+\left(-1+i\sqrt{3}\right)^{1/3}-i\sqrt{3}\left(-1+i\sqrt{3}\right)^{1/3}}{-i\left(-1-i\sqrt{3}\right)^{1/3}+\sqrt{3}\left(-1-i\sqrt{3}\right)^{1/3}-i\left(-1+i\sqrt{3}\right)^{1/3}-\sqrt{3}\left(-1+i\sqrt{3}\right)^{1/3}}\right\}$$

Here the function `RootReduce` is used to express the previous algebraic numbers as roots of polynomial equations.

```
RootReduce[Simplify[%]]
```

$\{\text{Root}[-3+27\,\#1^2-33\,\#1^4+\#1^6\,\&,\,4]\}$

The function `FunctionExpand` also reduces trigonometric expressions with compound arguments or compositions, including inverse trigonometric functions, to simpler ones. Here are some examples.

$\{\text{Tan}[\sqrt{z^2}],\ \text{Tan}[\frac{\text{ArcTan}[z]}{2}],\ \text{Tan}[2\,\text{ArcTan}[z]],\ \text{Tan}[3\,\text{ArcSin}[z]]\}$ // FunctionExpand

$$\left\{\frac{\sqrt{-i z}\,\sqrt{i z}\,\text{Tan}[z]}{z},\ \frac{z}{1+\sqrt{i\,(-i+z)}\,\sqrt{-i\,(i+z)}},\ -\frac{2z}{(-1+z)\,(1+z)},\ \frac{-z^3+3z\left(1-z^2\right)}{-3\sqrt{1-z}\,z^2\sqrt{1+z}+(1-z)^{3/2}\,(1+z)^{3/2}}\right\}$$

Applying `Simplify` to the previous expression gives a more compact result.

```
Simplify[%]
```

$$\left\{\frac{\sqrt{z^2}\,\text{Tan}[z]}{z},\ \frac{z}{1+\sqrt{1+z^2}},\ -\frac{2z}{-1+z^2},\ \frac{z\left(-3+4z^2\right)}{\sqrt{1-z^2}\left(-1+4z^2\right)}\right\}$$

### FullSimplify

The function `FullSimplify` tries a wider range of transformations than `Simplify` and returns the simplest form it finds. Here are some examples that contrast the results of applying the functions `Simplify` and `FullSimplify` to the same expressions.

$$\text{set1} = \left\{\text{Tan}\left[-i\,\text{Log}\left[i z+\sqrt{1-z^2}\right]\right],\ \text{Tan}\left[\frac{\pi}{2}+i\,\text{Log}\left[i z+\sqrt{1-z^2}\right]\right],\ \text{Tan}\left[\frac{1}{2}\,i\,\text{Log}[1-i z]-\frac{1}{2}\,i\,\text{Log}[1+i z]\right],\ \text{Tan}\left[\frac{1}{2}\,i\,\text{Log}\left[1-\frac{i}{z}\right]-\frac{1}{2}\,i\,\text{Log}\left[1+\frac{i}{z}\right]\right],\ \text{Tan}\left[-i\,\text{Log}\left[\sqrt{1-\frac{1}{z^2}}+\frac{i}{z}\right]\right],\ \text{Tan}\left[\frac{\pi}{2}+i\,\text{Log}\left[\sqrt{1-\frac{1}{z^2}}+\frac{i}{z}\right]\right]\right\}$$

$$\left\{-\frac{i\left(-1+\left(i z+\sqrt{1-z^2}\right)^2\right)}{1+\left(i z+\sqrt{1-z^2}\right)^2}, \frac{i\left(1+\left(i z+\sqrt{1-z^2}\right)^2\right)}{-1+\left(i z+\sqrt{1-z^2}\right)^2}, i\,\mathrm{Tanh}\left[\frac{1}{2}\mathrm{Log}[1-i z]-\frac{1}{2}\mathrm{Log}[1+i z]\right],\right.$$

$$\left. i\,\mathrm{Tanh}\left[\frac{1}{2}\mathrm{Log}\left[1-\frac{i}{z}\right]-\frac{1}{2}\mathrm{Log}\left[1+\frac{i}{z}\right]\right], -\frac{i\left(-1+\left(\sqrt{1-\frac{1}{z^2}}+\frac{i}{z}\right)^2\right)}{1+\left(\sqrt{1-\frac{1}{z^2}}+\frac{i}{z}\right)^2}, \frac{i\left(1+\left(\sqrt{1-\frac{1}{z^2}}+\frac{i}{z}\right)^2\right)}{-1+\left(\sqrt{1-\frac{1}{z^2}}+\frac{i}{z}\right)^2}\right\}$$

```
set1 // Simplify
```

$$\left\{\frac{z\left(z-i\sqrt{1-z^2}\right)}{-i+i z^2+z\sqrt{1-z^2}}, \frac{1-z^2+i z\sqrt{1-z^2}}{i z^2+z\sqrt{1-z^2}}, i\,\mathrm{Tanh}\left[\frac{1}{2}\left(\mathrm{Log}[1-i z]-\mathrm{Log}[1+i z]\right)\right],\right.$$

$$\left. i\,\mathrm{Tanh}\left[\frac{1}{2}\left(\mathrm{Log}\left[\frac{-i+z}{z}\right]-\mathrm{Log}\left[\frac{i+z}{z}\right]\right)\right], \frac{1-i\sqrt{1-\frac{1}{z^2}}\,z}{i+\sqrt{1-\frac{1}{z^2}}\,z-i z^2}, \frac{-1+i\sqrt{1-\frac{1}{z^2}}\,z+z^2}{i+\sqrt{1-\frac{1}{z^2}}\,z}\right\}$$

```
set1 // FullSimplify
```

$$\left\{\frac{z}{\sqrt{1-z^2}}, \frac{\sqrt{1-z^2}}{z}, z, \frac{1}{z}, \frac{1}{\sqrt{1-\frac{1}{z^2}}\,z}, \sqrt{1-\frac{1}{z^2}}\,z\right\}$$

## Operations carried out by specialized *Mathematica* functions

### Series expansions

Calculating the series expansion of a tangent function to hundreds of terms can be done in seconds.

```
Series[Tan[z], {z, 0, 3}]
```

$$z+\frac{z^3}{3}+O[z]^4$$

```
Normal[%]
```

$$z+\frac{z^3}{3}$$

```
Series[Tan[z], {z, 0, 100}] // Timing
```

$$\Big\{0.45\ \text{Second},\ z + \frac{z^3}{3} + \frac{2 z^5}{15} + \frac{17 z^7}{315} + \frac{62 z^9}{2835} + \frac{1382 z^{11}}{155\,925} + \frac{21\,844 z^{13}}{6\,081\,075} +$$

$$\frac{929\,569 z^{15}}{638\,512\,875} + \frac{6\,404\,582 z^{17}}{10\,854\,718\,875} + \frac{443\,861\,162 z^{19}}{1\,856\,156\,927\,625} + \frac{18\,888\,466\,084 z^{21}}{194\,896\,477\,400\,625} +$$

$$\frac{113\,927\,491\,862 z^{23}}{2\,900\,518\,163\,668\,125} + \frac{58\,870\,668\,456\,604 z^{25}}{3\,698\,160\,658\,676\,859\,375} + \frac{8\,374\,643\,517\,010\,684 z^{27}}{1\,298\,054\,391\,195\,577\,640\,625} +$$

$$\frac{689\,005\,380\,505\,609\,448 z^{29}}{263\,505\,041\,412\,702\,261\,046\,875} + \frac{129\,848\,163\,681\,107\,301\,953 z^{31}}{122\,529\,844\,256\,906\,551\,386\,796\,875} +$$

$$\frac{1\,736\,640\,792\,209\,901\,647\,222 z^{33}}{4\,043\,484\,860\,477\,916\,195\,764\,296\,875} + \frac{418\,781\,231\,495\,293\,038\,913\,922 z^{35}}{2\,405\,873\,491\,984\,360\,136\,479\,756\,640\,625} +$$

$$\frac{56\,518\,638\,202\,982\,204\,522\,669\,764 z^{37}}{801\,155\,872\,830\,791\,925\,447\,758\,961\,328\,125} + \frac{32\,207\,686\,319\,158\,956\,594\,455\,462 z^{39}}{1\,126\,482\,925\,555\,250\,126\,673\,224\,649\,609\,375} +$$

$$\frac{1\,410\,211\,493\,828\,985\,228\,276\,049\,834\,684 z^{41}}{121\,699\,582\,862\,361\,447\,435\,141\,825\,020\,548\,828\,125} +$$

$$\frac{516\,098\,083\,439\,704\,913\,515\,348\,955\,653\,804 z^{43}}{109\,894\,723\,324\,712\,387\,033\,933\,067\,993\,555\,591\,796\,875} +$$

$$\frac{103\,537\,504\,005\,512\,749\,467\,288\,942\,622\,106\,408 z^{45}}{54\,397\,888\,045\,732\,631\,581\,796\,868\,656\,810\,017\,939\,453\,125} +$$

$$\frac{45\,361\,105\,584\,983\,995\,647\,044\,252\,937\,847\,808\,918 z^{47}}{58\,804\,116\,977\,436\,974\,739\,922\,415\,018\,011\,629\,392\,548\,828\,125} +$$

$$\frac{87\,176\,517\,890\,549\,500\,795\,745\,183\,943\,750\,553\,204 z^{49}}{278\,845\,328\,893\,007\,589\,895\,761\,129\,278\,958\,371\,635\,634\,765\,625} +$$

$$\frac{1\,396\,470\,103\,398\,938\,597\,044\,980\,843\,456\,514\,101\,088\,564 z^{51}}{11\,021\,361\,624\,496\,124\,990\,629\,958\,634\,750\,829\,638\,898\,464\,111\,328\,125} +$$

$$\frac{389\,951\,962\,465\,960\,362\,323\,362\,101\,491\,789\,115\,193\,414\,088 z^{53}}{7\,593\,718\,159\,277\,830\,118\,544\,041\,499\,343\,321\,621\,201\,041\,772\,705\,078\,125} +$$

$$\frac{321\,055\,735\,622\,680\,218\,266\,276\,441\,690\,024\,211\,623\,548\,948 z^{55}}{15\,426\,363\,155\,304\,483\,893\,348\,702\,635\,464\,887\,287\,939\,188\,826\,904\,296\,875} +$$

$$\frac{37\,951\,675\,284\,166\,717\,133\,668\,639\,194\,471\,627\,545\,621\,910\,540\,728 z^{57}}{4\,499\,391\,915\,144\,503\,512\,689\,122\,748\,983\,408\,210\,385\,935\,265\,954\,349\,365\,234\,375} +$$

$$\frac{641\,885\,182\,338\,872\,430\,017\,276\,041\,951\,405\,742\,741\,480\,681\,339\,128 z^{59}}{187\,767\,306\,507\,615\,744\,151\,489\,976\,183\,185\,645\,072\,447\,200\,976\,777\,847\,900\,390\,625} +$$

$$\frac{9\,759\,387\,159\,544\,076\,997\,817\,707\,959\,584\,835\,600\,439\,155\,088\,202\,173\,136 z^{61}}{7\,044\,090\,503\,633\,204\,641\,843\,146\,456\,512\,209\,474\,892\,856\,744\,643\,820\,963\,983\,154\,296\,875} +$$

$$\frac{7\,724\,760\,729\,208\,487\,305\,545\,342\,963\,324\,697\,288\,405\,380\,586\,579\,904\,269\,441 z^{63}}{13\,757\,108\,753\,595\,648\,665\,519\,665\,029\,568\,345\,104\,465\,749\,222\,289\,382\,342\,659\,100\,341\,796\,875} +$$

$$\frac{203\,497\,294\,113\,685\,566\,585\,581\,532\,155\,905\,318\,177\,648\,366\,860\,283\,186\,794\,902 z^{65}}{894\,212\,068\,983\,717\,163\,258\,778\,226\,921\,942\,431\,790\,273\,699\,448\,809\,852\,272\,841\,522\,216\,796\,875} +$$

$$\left(4\,928\,399\,482\,219\,367\,719\,407\,723\,316\,603\,414\,444\,925\,848\,984\,020\,131\,054\,275\,466 z^{67}\right) \big/$$
$$53\,435\,213\,095\,216\,179\,674\,734\,017\,830\,389\,586\,937\,521\,490\,526\,522\,123\,875\,006\,827\,178\,955\,078\,125 +$$

$$\left(65\,033\,291\,600\,604\,926\,267\,730\,204\,296\,537\,787\,351\,363\,679\,128\,255\,646\,162\,810\,228 z^{69}\right) \big/$$
$$1\,739\,791\,210\,461\,723\,491\,420\,203\,381\,737\,988\,344\,092\,550\,795\,455\,053\,787\,171\,171\,272\,613\,525\,390 \cdots$$

625 +

(5 135 746 785 881 293 900 665 825 337 251 063 912 099 812 018 333 491 820 410 850 621 042 $z^{71}$) /

339 003 946 094 735 963 173 345 929 589 646 769 396 958 527 805 814 290 898 437 688 022 862 701
416 015 625 +

(23 247 600 823 869 669 181 617 874 661 621 842 533 234 313 612 312 895 049 759 227 683 259 644
$z^{73}$) /

3 786 335 073 932 105 972 683 100 687 586 764 767 394 629 797 063 139 815 044 650 537 527 353 512
115 478 515 625 +

(26 145 766 198 741 584 025 528 698 683 516 199 629 583 197 662 307 446 174 936 102 767 991 445 644
$z^{75}$) /

10 507 079 830 161 594 074 195 604 408 053 272 229 520 097 686 850 212 986 748 905 241 638 405 996
120 452 880 859 375 +

(15 502 650 114 137 077 692 879 282 322 945 671 281 504 464 245 814 336 939 030 473 090 718 667 221
928 $z^{77}$) /

15 371 857 791 526 412 130 548 169 248 981 937 271 787 902 915 861 861 599 613 648 368 516 987 972
324 222 564 697 265 625 +

(624 447 880 395 344 915 327 575 701 011 165 339 822 237 764 093 445 803 722 686 391 628 220 033
123 558 $z^{79}$) /

1 527 764 317 925 576 637 878 029 337 293 978 991 431 565 447 863 561 148 013 214 536 238 736 772
346 159 023 284 912 109 375 +

(3 177 409 273 870 478 888 667 047 675 148 588 648 707 554 958 779 304 377 142 474 306 979 107 045
033 591 884 $z^{81}$) /

19 181 081 011 555 614 688 558 658 329 725 906 237 423 304 197 927 010 213 305 908 502 477 340 176
806 026 537 342 071 533 203 125 +

(4 382 231 878 630 427 838 203 781 834 402 677 719 332 895 725 070 795 917 855 918 399 166 899 403
692 359 715 604 $z^{83}$) /

65 273 218 682 323 756 785 165 114 296 057 258 925 951 504 185 545 615 755 880 006 633 930 388 621
670 908 306 575 069 427 490 234 375 +

(3 170 252 255 497 465 850 441 721 151 634 611 336 636 763 918 620 746 439 504 134 202 802 452 839
465 833 038 132 808 $z^{85}$) /

116 512 695 347 947 905 861 519 729 018 462 207 182 823 434 971 198 924 124 245 811 841 565 743
689 682 571 327 236 498 928 070 068 359 375 +

(282 743 645 351 878 196 427 175 381 372 737 723 603 024 007 898 834 815 645 760 821 235 876 739
738 607 970 302 830 604 $z^{87}$) /

25 639 646 664 510 183 283 996 782 721 062 771 592 408 380 601 603 245 596 988 446 005 841 026 302
535 441 137 364 220 146 465 301 513 671 875 +

(26 125 334 033 648 605 299 760 215 770 950 877 021 992 145 130 248 999 739 712 305 337 867 298 482
114 198 536 057 875 576 $z^{89}$) /

5 845 488 211 471 821 649 254 225 408 584 215 172 773 324 360 992 915 294 118 886 395 550 852 065
110 922 559 577 434 464 350 986 480 712 890 625 +

(3 165 183 288 800 001 305 552 844 563 646 295 861 445 487 620 103 190 581 092 424 979 876 576 485
235 244 926 077 451 042 788 728 $z^{91}$) /

1 747 421 018 496 329 004 719 811 872 515 122 362 672 993 717 853 417 133 447 429 304 653 993 962
083 933 635 347 280 371 600 762 143 611 907 958 984 375 +

(2 743 910 203 329 295 441 771 249 659 819 135 452 881 820 248 213 784 360 868 316 092 063 767 619
714 125 477 901 042 867 863 808 976 $z^{93}$) /

3 737 733 558 563 647 741 095 677 595 309 846 733 757 533 562 488 459 248 444 051 282 654 893 084

897 534 046 007 832 714 854 030 225 185 871 124 267 578 125 +
(4 965 369 860 827 668 851 290 623 237 994 135 971 062 634 507 133 697 075 181 739 578 164 128 821 ⋱
938 437 552 932 557 133 865 324 691 926 z^95) /
16 688 980 338 986 687 163 992 200 463 058 465 666 227 387 356 510 970 544 302 688 977 054 097 624 ⋱
067 489 515 424 973 071 823 244 955 454 914 569 854 736 328 125 +
(13 618 722 892 337 243 626 196 029 509 843 989 171 050 070 079 708 515 121 451 737 330 953 176 267 ⋱
769 526 553 238 319 205 989 401 876 548 z^97) /
112 941 704 154 537 813 133 063 496 156 977 058 345 864 412 110 341 684 381 211 220 751 691 683 ⋱
921 014 870 906 713 189 858 152 657 721 799 538 135 528 564 453 125 +
(905 838 570 048 586 218 745 173 742 117 616 558 174 626 778 700 773 083 971 608 582 082 083 300 ⋱
800 057 692 087 180 696 588 351 163 326 044 z^99) /
18 535 679 696 858 777 383 843 519 947 971 924 100 345 314 960 922 504 303 800 151 196 111 426 769 ⋱
580 058 982 725 432 267 570 131 653 944 398 944 377 899 169 921 875 + O[z]^101}

*Mathematica* comes with the add-on package `DiscreteMath`RSolve`` that allows finding the general terms of series for many functions. After loading this package, and using the package function `SeriesTerm`, the following $n^{th}$ term of tan(*z*) can be evaluated.

In[14]:= `<< DiscreteMath`RSolve``

In[33]:= `SeriesTerm[Tan[z], {z, 0, n}] z^n`

Out[33]= $z^n \text{If}\left[\text{Odd}[n], \frac{i^{-1+n}\, 2^{1+n} \left(-1+2^{1+n}\right) \text{BernoulliB}[1+n]}{(1+n)!}, 0\right]$

The result can be checked by the following process.

In[34]:= `% /. {Odd[n_] :> Element[(n + 1) / 2, Integers]} /. {n → 2 k - 1}`

Out[34]= $z^{-1+2k} \text{If}\left[k \in \text{Integers}, \frac{i^{-1+(-1+2k)}\, 2^{1+(-1+2k)} \left(-1+2^{1+(-1+2k)}\right) \text{BernoulliB}[1+(-1+2k)]}{(1+(-1+2k))!}, 0\right]$

In[35]:= `Simplify[%, k ∈ Integers]`

Out[35]= $-\frac{(2 i)^{2k} \left(-1+4^k\right) z^{-1+2k} \text{BernoulliB}[2k]}{(2k)!}$

In[36]:= $\text{FunctionExpand}\left[\sum_{k=1}^{\infty} \text{Evaluate}[\%]\right]$

Out[36]= `Tan[z]`

### Differentiation

*Mathematica* can evaluate derivatives of the tangent function of an arbitrary positive integer order.

$\partial_z$ `Tan[z]`

`Sec[z]^2`

$\partial_{\{z,2\}}$ `Tan[z]`

$2\,\text{Sec}[z]^2\,\text{Tan}[z]$

```
Table[D[Tan[z], {z, n}], {n, 10}]
```

$\{\text{Sec}[z]^2,\ 2\,\text{Sec}[z]^2\,\text{Tan}[z],\ 2\,\text{Sec}[z]^4 + 4\,\text{Sec}[z]^2\,\text{Tan}[z]^2,$
$16\,\text{Sec}[z]^4\,\text{Tan}[z] + 8\,\text{Sec}[z]^2\,\text{Tan}[z]^3,\ 16\,\text{Sec}[z]^6 + 88\,\text{Sec}[z]^4\,\text{Tan}[z]^2 + 16\,\text{Sec}[z]^2\,\text{Tan}[z]^4,$
$272\,\text{Sec}[z]^6\,\text{Tan}[z] + 416\,\text{Sec}[z]^4\,\text{Tan}[z]^3 + 32\,\text{Sec}[z]^2\,\text{Tan}[z]^5,$
$272\,\text{Sec}[z]^8 + 2880\,\text{Sec}[z]^6\,\text{Tan}[z]^2 + 1824\,\text{Sec}[z]^4\,\text{Tan}[z]^4 + 64\,\text{Sec}[z]^2\,\text{Tan}[z]^6,$
$7936\,\text{Sec}[z]^8\,\text{Tan}[z] + 24\,576\,\text{Sec}[z]^6\,\text{Tan}[z]^3 + 7680\,\text{Sec}[z]^4\,\text{Tan}[z]^5 + 128\,\text{Sec}[z]^2\,\text{Tan}[z]^7,$
$7936\,\text{Sec}[z]^{10} + 137\,216\,\text{Sec}[z]^8\,\text{Tan}[z]^2 + 185\,856\,\text{Sec}[z]^6\,\text{Tan}[z]^4 + 31\,616\,\text{Sec}[z]^4\,\text{Tan}[z]^6 +$
$256\,\text{Sec}[z]^2\,\text{Tan}[z]^8,\ 353\,792\,\text{Sec}[z]^{10}\,\text{Tan}[z] + 1\,841\,152\,\text{Sec}[z]^8\,\text{Tan}[z]^3 +$
$1\,304\,832\,\text{Sec}[z]^6\,\text{Tan}[z]^5 + 128\,512\,\text{Sec}[z]^4\,\text{Tan}[z]^7 + 512\,\text{Sec}[z]^2\,\text{Tan}[z]^9\}$

### Finite products

*Mathematica* can calculate some finite symbolic products that contain the tangent function. Here is an example.

$$\prod_{k=1}^{n-1} \text{Tan}\left[\frac{k\,\pi}{n}\right]$$

$$-(-1)^{-n}\,n\,\text{Csc}\left[\frac{n\,\pi}{2}\right]$$

### Indefinite integration

*Mathematica* can calculate a huge number of doable indefinite integrals that contain the tangent function. Here are some examples.

$$\int \text{Tan}[z]\,dz$$

$$-\text{Log}[\text{Cos}[z]]$$

$$\int \text{Tan}[z]^a\,dz$$

$$\frac{\text{Hypergeometric2F1}\left[\frac{1+a}{2}, 1, 1+\frac{1+a}{2}, -\text{Tan}[z]^2\right]\text{Tan}[z]^{1+a}}{1+a}$$

### Definite integration

*Mathematica* can calculate wide classes of definite integrals that contain the tangent function. Here are some examples.

$$\int_0^{\pi/2} \sqrt{\text{Tan}[z]}\,dz$$

$$\frac{\pi}{\sqrt{2}}$$

$$\int_0^{\frac{\pi}{2}} \text{Tan}[z]^a\,dz$$

$$\text{If}\left[\text{Re}[a] < 1,\ \frac{1}{2}\pi\,\text{Sec}\left[\frac{a\,\pi}{2}\right],\ \int_0^{\frac{\pi}{2}}\text{Tan}[z]^a\,dz\right]$$

### Limit operation

*Mathematica* can calculate limits that contain the tangent function. Here are some examples.

$$\textbf{Limit}\left[\frac{\text{Tan}[3\,z]}{z},\ z\to 0\right]$$

3

$$\textbf{Limit}\left[\frac{\text{Tan}\left[\sqrt{z^2}\right]}{z},\ z\to 0,\ \text{Direction}\to 1\right]$$

$-1$

$$\textbf{Limit}\left[\frac{\text{Tan}\left[\sqrt{z^2}\right]}{z},\ z\to 0,\ \text{Direction}\to -1\right]$$

1

### Solving equations

The next inputs solve two equations that contain the tangent function. Because of the multivalued nature of the inverse tangent function, a printed message indicates that only some of the possible solutions are returned.

$$\textbf{Solve}\left[\text{Tan}[z]^2 + 3\,\text{Tan}[z + \text{Pi}/6] == 4,\ z\right]$$

Solve::ifun : Inverse functions are being used by Solve, so some solutions may not be found.

$$\Bigg\{\Bigg\{z\to -\text{ArcCos}\Bigg[-\sqrt{\Bigg(\frac{23}{102}+\frac{5\sqrt{3}}{68}+}$$

$$1363\Big/\left(204\left(81\,856+30\,987\sqrt{3}+3\,i\sqrt{3\left(6\,908\,256+20\,974\,464\sqrt{3}\right)}\right)^{1/3}\right)+$$

$$\left(64\sqrt{3}\right)\Big/\left(17\left(81\,856+30\,987\sqrt{3}+3\,i\sqrt{3\left(6\,908\,256+20\,974\,464\sqrt{3}\right)}\right)^{1/3}\right)+$$

$$\frac{1}{204}\left(81\,856+30\,987\sqrt{3}+3\,i\sqrt{3\left(6\,908\,256+20\,974\,464\sqrt{3}\right)}\right)^{1/3}\Bigg)}\Bigg]\Bigg\},\ \Bigg\{z\to\text{ArcCos}\Bigg[$$

$$\sqrt{\Bigg(\frac{23}{102}+\frac{5\sqrt{3}}{68}+1363\Big/\left(204\left(81\,856+30\,987\sqrt{3}+3\,i\sqrt{3\left(6\,908\,256+20\,974\,464\sqrt{3}\right)}\right)^{1/3}\right)+}$$

$$\left(64\sqrt{3}\right)\Big/\left(17\left(81\,856+30\,987\sqrt{3}+3\,i\sqrt{3\left(6\,908\,256+20\,974\,464\sqrt{3}\right)}\right)^{1/3}\right)+$$

$$\frac{1}{204}\left(81856+30987\sqrt{3}+3\,i\sqrt{3\left(6908256+20974464\sqrt{3}\right)}\right)^{1/3}\Bigg)\Bigg]\Bigg\},$$

$$\Bigg\{z\to \text{ArcCos}\Bigg[-\sqrt{\Bigg(\frac{23}{102}+\frac{5\sqrt{3}}{68}-\left(\frac{1363}{408}-\frac{96\,i}{17}\right)\Bigg/}$$

$$\left(81856+30987\sqrt{3}+3\,i\sqrt{3\left(6908256+20974464\sqrt{3}\right)}\right)^{1/3}-$$

$$\left(\frac{96}{17}-\frac{1363\,i}{136}\right)\Bigg/\left(\sqrt{3}\left(81856+30987\sqrt{3}+3\,i\sqrt{3\left(6908256+20974464\sqrt{3}\right)}\right)^{1/3}\right)-$$

$$\frac{1}{408}\left(81856+30987\sqrt{3}+3\,i\sqrt{3\left(6908256+20974464\sqrt{3}\right)}\right)^{1/3}-$$

$$\frac{1}{136\sqrt{3}}\left(i\left(81856+30987\sqrt{3}+3\,i\sqrt{3\left(6908256+20974464\sqrt{3}\right)}\right)^{1/3}\right)\Bigg)\Bigg]\Bigg\},$$

$$\Bigg\{z\to -\text{ArcCos}\Bigg[\sqrt{\Bigg(\frac{23}{102}+\frac{5\sqrt{3}}{68}-\left(\frac{1363}{408}-\frac{96\,i}{17}\right)\Bigg/}$$

$$\left(81856+30987\sqrt{3}+3\,i\sqrt{3\left(6908256+20974464\sqrt{3}\right)}\right)^{1/3}-$$

$$\left(\frac{96}{17}-\frac{1363\,i}{136}\right)\Bigg/\left(\sqrt{3}\left(81856+30987\sqrt{3}+3\,i\sqrt{3\left(6908256+20974464\sqrt{3}\right)}\right)^{1/3}\right)-$$

$$\frac{1}{408}\left(81856+30987\sqrt{3}+3\,i\sqrt{3\left(6908256+20974464\sqrt{3}\right)}\right)^{1/3}-$$

$$\frac{1}{136\sqrt{3}}\left(i\left(81856+30987\sqrt{3}+3\,i\sqrt{3\left(6908256+20974464\sqrt{3}\right)}\right)^{1/3}\right)\Bigg)\Bigg]\Bigg\},$$

$$\Bigg\{z\to -\text{ArcCos}\Bigg[-\sqrt{\Bigg(\frac{23}{102}+\frac{5\sqrt{3}}{68}-\left(\frac{1363}{408}+\frac{96\,i}{17}\right)\Bigg/}$$

$$\left(81856+30987\sqrt{3}+3\,i\sqrt{3\left(6908256+20974464\sqrt{3}\right)}\right)^{1/3}-$$

$$\left(\frac{96}{17}+\frac{1363\,i}{136}\right)\Bigg/\left(\sqrt{3}\left(81856+30987\sqrt{3}+3\,i\sqrt{3\left(6908256+20974464\sqrt{3}\right)}\right)^{1/3}\right)-$$

$$\frac{1}{408}\left(81\,856+30\,987\sqrt{3}+3\,i\sqrt{3\left(6\,908\,256+20\,974\,464\sqrt{3}\right)}\right)^{1/3}+$$

$$\frac{1}{136\sqrt{3}}\left(i\left(81\,856+30\,987\sqrt{3}+3\,i\sqrt{3\left(6\,908\,256+20\,974\,464\sqrt{3}\right)}\right)^{1/3}\right)\Bigg)\Bigg]\Bigg\},$$

$$\Bigg\{z\to\mathrm{ArcCos}\Bigg[\sqrt{\Bigg(\frac{23}{102}+\frac{5\sqrt{3}}{68}-\left(\frac{1363}{408}+\frac{96\,i}{17}\right)\Big/}$$

$$\left(81\,856+30\,987\sqrt{3}+3\,i\sqrt{3\left(6\,908\,256+20\,974\,464\sqrt{3}\right)}\right)^{1/3}-$$

$$\left(\frac{96}{17}+\frac{1363\,i}{136}\right)\Big/\left(\sqrt{3}\left(81\,856+30\,987\sqrt{3}+3\,i\sqrt{3\left(6\,908\,256+20\,974\,464\sqrt{3}\right)}\right)^{1/3}\right)-$$

$$\frac{1}{408}\left(81\,856+30\,987\sqrt{3}+3\,i\sqrt{3\left(6\,908\,256+20\,974\,464\sqrt{3}\right)}\right)^{1/3}+$$

$$\frac{1}{136\sqrt{3}}\left(i\left(81\,856+30\,987\sqrt{3}+3\,i\sqrt{3\left(6\,908\,256+20\,974\,464\sqrt{3}\right)}\right)^{1/3}\right)\Bigg)\Bigg]\Bigg\}\Bigg\}$$

```
Solve[Tan[x] == a, x]
```

Solve::ifun : Inverse functions are being used by Solve, so some solutions may not be found.

```
{{x → ArcTan[a]}}
```

A complete solution of the previous equation can be obtained using the function `Reduce`.

```
Reduce[Tan[x] == a, x] // InputForm
```

```
C[1] ∈ Integers && 1 + a^2 ≠ 0 && x == ArcTan[a] + Pi * C[1]
```

### Solving differential equations

Here is a nonlinear differential equation whose independent solutions include the tangent function.

In[6]:= `dsol1 = DSolve[w'[z] - w[z]^2 - 1 == 0, w[z], z]`

Out[6]= `{{w[z] → Tan[z + C[1]]}}`

In the previous input, the differential equation was solved for $w(z)$. If the argument is suppressed, the result is returned as a pure function (in the sense of the $\lambda$-calculus).

In[38]:= `dsol2 = DSolve[w'[z] - w[z]^2 - 1 == 0, w, z]`

Out[38]= `{{w → Function[{z}, Tan[z + C[1]]]}}`

The advantage of such a pure function is that it can be used for different arguments, derivatives, and more.

In[39]:= 
```
w'[ξ] /. dsol1
```
Out[39]= 
```
{w'[ζ]}
```
In[40]:= 
```
w'[ξ] /. dsol2
```
Out[40]= 
```
{Sec[ζ + C[1]]^2}
```

In carrying out the algorithm to solve the following nonlinear differential equation, *Mathematica* has to solve a transcendental equation. In doing so, the generically multivariate inverse of a function is encountered, and a message is issued that a solution branch is potentially missed.

In[41]:= 
```
DSolve[{w'[z] - w[z]^2 - 1 == 0, w[0] == 0}, w[z], z]
```
```
Solve::ifun : Inverse functions are being used by Solve, so some solutions may not be found.
```
Out[41]= 
```
{{w[z] → Tan[z]}}
```

### Plotting

*Mathematica* has built-in functions for 2D and 3D graphics. Here are some examples.

```
Plot[Tan[Sum[z^k, {k, 0, 5}]], {z, -2π/3, 2π/3}];
```
$$\text{Plot}\left[\text{Tan}\left[\sum_{k=0}^{5} z^k\right], \left\{z, -\frac{2\pi}{3}, \frac{2\pi}{3}\right\}\right];$$

```
Plot3D[Re[Tan[x + ⅈ y]], {x, -π, π}, {y, 0, π},
  PlotPoints → 240, PlotRange → {-5, 5},
  ClipFill → None, Mesh → False, AxesLabel → {"x", "y", None}];
```

$$\text{ContourPlot}\left[\text{Arg}\left[\text{Tan}\left[\frac{1}{x + i\, y}\right]\right], \left\{x, -\frac{1}{2}, \frac{1}{2}\right\}, \left\{y, -\frac{1}{2}, \frac{1}{2}\right\},\right.$$
```
  PlotPoints → 400, PlotRange → {-π, π}, FrameLabel → {"x", "y", None, None},
  ColorFunction → Hue, ContourLines → False, Contours → 200];
```

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