

Introductions to Sinh

Introduction to the hyperbolic functions

General

The six well-known hyperbolic functions are the hyperbolic sine $\sinh(z)$, hyperbolic cosine $\cosh(z)$, hyperbolic tangent $\tanh(z)$, hyperbolic cotangent $\coth(z)$, hyperbolic cosecant $\operatorname{csch}(z)$, and hyperbolic secant $\operatorname{sech}(z)$. They are among the most used elementary functions. The hyperbolic functions share many common properties and they have many properties and formulas that are similar to those of the trigonometric functions.

Definitions of the hyperbolic functions

All hyperbolic functions can be defined as simple rational functions of the exponential function of z :

$$\sinh(z) = \frac{e^z - e^{-z}}{2}$$

$$\cosh(z) = \frac{e^z + e^{-z}}{2}$$

$$\tanh(z) = \frac{e^z - e^{-z}}{e^z + e^{-z}}$$

$$\coth(z) = \frac{e^z + e^{-z}}{e^z - e^{-z}}$$

$$\operatorname{csch}(z) = \frac{2}{e^z - e^{-z}}$$

$$\operatorname{sech}(z) = \frac{2}{e^z + e^{-z}}.$$

The functions $\tanh(z)$, $\coth(z)$, $\operatorname{csch}(z)$, and $\operatorname{sech}(z)$ can also be defined through the functions $\sinh(z)$ and $\cosh(z)$ using the following formulas:

$$\tanh(z) = \frac{\sinh(z)}{\cosh(z)}$$

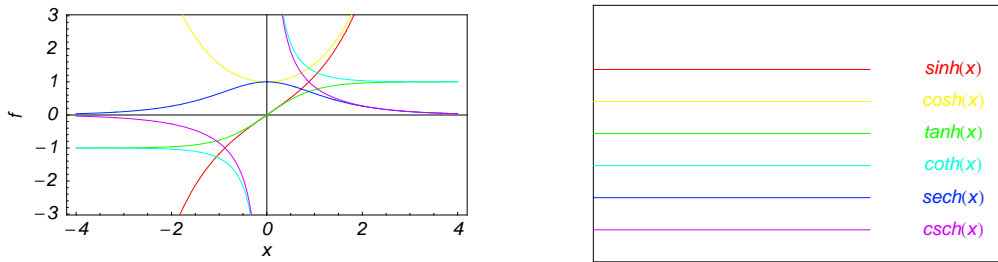
$$\coth(z) = \frac{\cosh(z)}{\sinh(z)}$$

$$\operatorname{csch}(z) = \frac{1}{\sinh(z)}$$

$$\operatorname{sech}(z) = \frac{1}{\cosh(z)}.$$

A quick look at the hyperbolic functions

Here is a quick look at the graphics of the six hyperbolic functions along the real axis.



Connections within the group of hyperbolic functions and with other function groups

Representations through more general functions

The hyperbolic functions are particular cases of more general functions. Among these more general functions, four classes of special functions are of special relevance: Bessel, Jacobi, Mathieu, and hypergeometric functions.

For example, $\sinh(z)$ and $\cosh(z)$ have the following representations through Bessel, Mathieu, and hypergeometric functions:

$$\sinh(z) = -i \sqrt{\frac{\pi iz}{2}} J_{1/2}(iz) \quad \sinh(z) = \sqrt{\frac{\pi z}{2}} I_{1/2}(z) \quad \sinh(z) = -i \sqrt{\frac{\pi iz}{2}} Y_{-1/2}(iz) \quad \sinh(z) = \frac{1}{\sqrt{2\pi}} (\sqrt{-z} K_{1/2}(-z) - \sqrt{z})$$

$$\cosh(z) = \sqrt{\frac{\pi iz}{2}} J_{-1/2}(iz) \quad \cosh(z) = \sqrt{\frac{\pi z}{2}} I_{-1/2}(z) \quad \cosh(z) = -\sqrt{\frac{\pi iz}{2}} Y_{1/2}(iz) \quad \cosh(z) = \frac{1}{\sqrt{2\pi}} (\sqrt{-z} K_{1/2}(-z) + \sqrt{z})$$

$$\sinh(z) = -i \operatorname{Se}(1, 0, iz) \quad \cosh(z) = \operatorname{Ce}(1, 0, iz)$$

$$\sinh(z) = {}_zF_1\left(\frac{3}{2}; \frac{z^2}{4}\right) \quad \cosh(z) = {}_0F_1\left(\frac{1}{2}; \frac{z^2}{4}\right).$$

All hyperbolic functions can be represented as degenerate cases of the corresponding doubly periodic Jacobi elliptic functions when their second parameter is equal to 0 or 1:

$$\begin{aligned} \sinh(z) &= -i \operatorname{sd}(iz | 0) = -i \operatorname{sn}(iz | 0) & \sinh(z) &= \operatorname{sc}(z | 1) = \operatorname{sd}(z | 1) \\ \cosh(z) &= \operatorname{cd}(iz | 0) = \operatorname{cn}(iz | 0) & \cosh(z) &= \operatorname{nc}(z | 1) = \operatorname{nd}(z | 1) \\ \tanh(z) &= -i \operatorname{sc}(iz | 0) & \tanh(z) &= \operatorname{sn}(z | 1) \\ \operatorname{coth}(z) &= i \operatorname{cs}(iz | 0) & \operatorname{coth}(z) &= \operatorname{ns}(z | 1) \\ \operatorname{csch}(z) &= i \operatorname{ds}(iz | 0) = i \operatorname{ns}(iz | 0) & \operatorname{csch}(z) &= \operatorname{cs}(z | 1) = \operatorname{ds}(z | 1) \\ \operatorname{sech}(z) &= \operatorname{dc}(iz | 0) = \operatorname{nc}(iz | 0) & \operatorname{sech}(z) &= \operatorname{cn}(z | 1) = \operatorname{dn}(z | 1). \end{aligned}$$

Representations through related equivalent functions

Each of the six hyperbolic functions can be represented through the corresponding trigonometric function:

$$\begin{aligned} \sinh(z) &= -i \sin(iz) & \sinh(iz) &= i \sin(z) \\ \cosh(z) &= \cos(iz) & \cosh(iz) &= \cos(z) \\ \tanh(z) &= -i \tan(iz) & \tanh(iz) &= i \tan(z) \\ \operatorname{coth}(z) &= i \cot(iz) & \operatorname{coth}(iz) &= -i \cot(z) \\ \operatorname{csch}(z) &= i \operatorname{csc}(iz) & \operatorname{csch}(iz) &= -i \operatorname{csc}(z) \\ \operatorname{sech}(z) &= \operatorname{sec}(iz) & \operatorname{sech}(iz) &= \operatorname{sec}(z). \end{aligned}$$

Relations to inverse functions

Each of the six hyperbolic functions is connected with a corresponding inverse hyperbolic function by two formulas. One direction can be expressed through a simple formula, but the other direction is much more complicated because of the multivalued nature of the inverse function:

$$\sinh(\sinh^{-1}(z)) = z \quad \sinh^{-1}(\sinh(z)) = z /; -\frac{\pi}{2} < \text{Im}(z) < \frac{\pi}{2} \bigvee \left(\text{Im}(z) = -\frac{\pi}{2} \wedge \text{Re}(z) \leq 0 \right) \bigvee \left(\text{Im}(z) = \frac{\pi}{2} \wedge \text{Re}(z) \geq 0 \right)$$

$$\cosh(\cosh^{-1}(z)) = z \quad \cosh^{-1}(\cosh(z)) = z /; \text{Re}(z) > 0 \wedge -\pi < \text{Im}(z) \leq \pi \vee \text{Re}(z) = 0 \wedge \text{Im}(z) \geq 0$$

$$\tanh(\tanh^{-1}(z)) = z \quad \tanh^{-1}(\tanh(z)) = z /; -\frac{\pi}{2} < \text{Im}(z) < \frac{\pi}{2} \bigvee \left(\text{Im}(z) = -\frac{\pi}{2} \wedge \text{Re}(z) > 0 \right) \bigvee \left(\text{Im}(z) = \frac{\pi}{2} \wedge \text{Re}(z) < 0 \right)$$

$$\coth(\coth^{-1}(z)) = z \quad \coth^{-1}(\coth(z)) = z /; -\frac{\pi}{2} < \text{Im}(z) < \frac{\pi}{2} \bigvee \left(\text{Im}(z) = -\frac{\pi}{2} \wedge \text{Re}(z) > 0 \right) \bigvee \left(\text{Im}(z) = \frac{\pi}{2} \wedge \text{Re}(z) \leq 0 \right)$$

$$\text{csch}(\text{csch}^{-1}(z)) = z \quad \text{csch}^{-1}(\text{csch}(z)) = z /; -\frac{\pi}{2} < \text{Im}(z) < \frac{\pi}{2} \bigvee \left(\text{Im}(z) = -\frac{\pi}{2} \wedge \text{Re}(z) \leq 0 \right) \bigvee \left(\text{Im}(z) = \frac{\pi}{2} \wedge \text{Re}(z) \geq 0 \right)$$

$$\text{sech}(\text{sech}^{-1}(z)) = z \quad \text{sech}^{-1}(\text{sech}(z)) = z /; -\pi < \text{Im}(z) \leq \pi \wedge \text{Re}(z) > 0 \vee \text{Re}(z) = 0 \wedge \text{Im}(z) \geq 0.$$

Representations through other hyperbolic functions

Each of the six hyperbolic functions can be represented through any other function as a rational function of that function with a linear argument. For example, the hyperbolic sine can be representative as a group-defining function because the other five functions can be expressed as:

$$\cosh(z) = -i \sinh\left(\frac{\pi i}{2} - z\right) \quad \cosh^2(z) = 1 + \sinh^2(z)$$

$$\tanh(z) = \frac{\sinh(z)}{\cosh(z)} = \frac{i \sinh(z)}{\sinh\left(\frac{\pi i}{2} - z\right)} \quad \tanh^2(z) = \frac{\sinh^2(z)}{1 + \sinh^2(z)}$$

$$\coth(z) = \frac{\cosh(z)}{\sinh(z)} = -\frac{i \sinh\left(\frac{\pi i}{2} - z\right)}{\sinh(z)} \quad \coth^2(z) = \frac{1 + \sinh^2(z)}{\sinh^2(z)}$$

$$\text{csch}(z) = \frac{1}{\sinh(z)} \quad \text{csch}^2(z) = \frac{1}{\sinh^2(z)}$$

$$\text{sech}(z) = \frac{1}{\cosh(z)} = \frac{i}{\sinh\left(\frac{\pi i}{2} - z\right)} \quad \text{sech}^2(z) = \frac{1}{1 + \sinh^2(z)}.$$

All six hyperbolic functions can be transformed into any other function of the group of hyperbolic functions if the argument z is replaced by $p \pi i / 2 + q z$ with $q^2 = 1 \wedge p \in \mathbb{Z}$:

$$\sinh(-z - 2 \pi i) = -\sinh(z) \quad \sinh(z - 2 \pi i) = \sinh(z)$$

$$\sinh\left(-z - \frac{3 \pi i}{2}\right) = i \cosh(z) \quad \sinh\left(z - \frac{3 \pi i}{2}\right) = i \cosh(z)$$

$$\sinh(-z - \pi i) = \sinh(z) \quad \sinh(z - \pi i) = -\sinh(z)$$

$$\sinh\left(-z - \frac{\pi i}{2}\right) = -i \cosh(z) \quad \sinh\left(z - \frac{\pi i}{2}\right) = -i \cosh(z)$$

$$\sinh\left(z + \frac{\pi i}{2}\right) = i \cosh(z) \quad \sinh\left(\frac{\pi i}{2} - z\right) = i \cosh(z)$$

$$\sinh(z + \pi i) = -\sinh(z) \quad \sinh(\pi i - z) = \sinh(z)$$

$$\sinh\left(z + \frac{3 \pi i}{2}\right) = -i \cosh(z) \quad \sinh\left(\frac{3 \pi i}{2} - z\right) = -i \cosh(z)$$

$$\sinh(z + 2 \pi i) = \sinh(z) \quad \sinh(2 \pi i - z) = -\sinh(z)$$

$$\begin{aligned} \cosh(-z - 2\pi i) &= \cosh(z) & \cosh(z - 2\pi i) &= \cosh(z) \\ \cosh\left(-z - \frac{3\pi i}{2}\right) &= -i \sinh(z) & \cosh\left(z - \frac{3\pi i}{2}\right) &= i \sinh(z) \\ \cosh(-z - \pi i) &= -\cosh(z) & \cosh(z - \pi i) &= -\cosh(z) \\ \cosh\left(-z - \frac{\pi i}{2}\right) &= i \sinh(z) & \cosh\left(z - \frac{\pi i}{2}\right) &= -i \sinh(z) \\ \cosh\left(z + \frac{\pi i}{2}\right) &= i \sinh(z) & \cosh\left(\frac{\pi i}{2} - z\right) &= -i \sinh(z) \\ \cosh(z + \pi i) &= -\cosh(z) & \cosh(\pi i - z) &= -\cosh(z) \\ \cosh\left(z + \frac{3\pi i}{2}\right) &= -i \sinh(z) & \cosh\left(\frac{3\pi i}{2} - z\right) &= i \sinh(z) \\ \cosh(z + 2\pi i) &= \cosh(z) & \cosh(2\pi i - z) &= \cosh(z) \end{aligned}$$

$$\begin{aligned} \tanh(-z - \pi i) &= -\tanh(z) & \tanh(z - \pi i) &= \tanh(z) \\ \tanh\left(-z - \frac{\pi i}{2}\right) &= -\coth(z) & \tanh\left(z - \frac{\pi i}{2}\right) &= \coth(z) \\ \tanh\left(z + \frac{\pi i}{2}\right) &= \coth(z) & \tanh\left(\frac{\pi i}{2} - z\right) &= -\coth(z) \\ \tanh(z + \pi i) &= \tanh(z) & \tanh(\pi i - z) &= -\tanh(z) \end{aligned}$$

$$\begin{aligned} \coth(-z - \pi i) &= -\coth(z) & \coth(z - \pi i) &= \coth(z) \\ \coth\left(-z - \frac{\pi i}{2}\right) &= -\tanh(z) & \coth\left(z - \frac{\pi i}{2}\right) &= \tanh(z) \\ \coth\left(z + \frac{\pi i}{2}\right) &= \tanh(z) & \coth\left(\frac{\pi i}{2} - z\right) &= -\tanh(z) \\ \coth(z + \pi i) &= \coth(z) & \coth(\pi i - z) &= -\coth(z) \end{aligned}$$

$$\begin{aligned} \operatorname{csch}(-z - 2\pi i) &= -\operatorname{csch}(z) & \operatorname{csch}(z - 2\pi i) &= \operatorname{csch}(z) \\ \operatorname{csch}\left(-z - \frac{3\pi i}{2}\right) &= -i \operatorname{sech}(z) & \operatorname{csch}\left(z - \frac{3\pi i}{2}\right) &= -i \operatorname{sech}(z) \\ \operatorname{csch}(-z - \pi i) &= \operatorname{csch}(z) & \operatorname{csch}(z - \pi i) &= -\operatorname{csch}(z) \\ \operatorname{csch}\left(-z - \frac{\pi i}{2}\right) &= i \operatorname{sech}(z) & \operatorname{csch}\left(z - \frac{\pi i}{2}\right) &= i \operatorname{sech}(z) \\ \operatorname{csch}\left(z + \frac{\pi i}{2}\right) &= -i \operatorname{sech}(z) & \operatorname{csch}\left(\frac{\pi i}{2} - z\right) &= -i \operatorname{sech}(z) \\ \operatorname{csch}(z + \pi i) &= -\operatorname{csch}(z) & \operatorname{csch}(\pi i - z) &= \operatorname{csch}(z) \\ \operatorname{csch}\left(z + \frac{3\pi i}{2}\right) &= i \operatorname{sech}(z) & \operatorname{csch}\left(\frac{3\pi i}{2} - z\right) &= i \operatorname{sech}(z) \\ \operatorname{csch}(z + 2\pi i) &= \operatorname{csch}(z) & \operatorname{csch}(2\pi i - z) &= -\operatorname{csch}(z) \end{aligned}$$

$$\begin{aligned} \operatorname{sech}(-z - 2\pi i) &= \operatorname{sech}(z) & \operatorname{sech}(z - 2\pi i) &= \operatorname{sech}(z) \\ \operatorname{sech}\left(-z - \frac{3\pi i}{2}\right) &= i \operatorname{csch}(z) & \operatorname{sech}\left(z - \frac{3\pi i}{2}\right) &= -i \operatorname{csch}(z) \\ \operatorname{sech}(-z - \pi i) &= -\operatorname{sech}(z) & \operatorname{sech}(z - \pi i) &= -\operatorname{sech}(z) \\ \operatorname{sech}\left(-z - \frac{\pi i}{2}\right) &= -i \operatorname{csch}(z) & \operatorname{sech}\left(z - \frac{\pi i}{2}\right) &= i \operatorname{csch}(z) \\ \operatorname{sech}\left(z + \frac{\pi i}{2}\right) &= -i \operatorname{csch}(z) & \operatorname{sech}\left(\frac{\pi i}{2} - z\right) &= i \operatorname{csch}(z) \\ \operatorname{sech}(z + \pi i) &= -\operatorname{sech}(z) & \operatorname{sech}(\pi i - z) &= -\operatorname{sech}(z) \\ \operatorname{sech}\left(z + \frac{3\pi i}{2}\right) &= i \operatorname{csch}(z) & \operatorname{sech}\left(\frac{3\pi i}{2} - z\right) &= -i \operatorname{csch}(z) \\ \operatorname{sech}(z + 2\pi i) &= \operatorname{sech}(z) & \operatorname{sech}(2\pi i - z) &= \operatorname{sech}(z). \end{aligned}$$

The best-known properties and formulas for hyperbolic functions

Real values for real arguments

For real values of argument z , the values of all the hyperbolic functions are real (or infinity).

In the points $z = 2\pi n i / m$; $n \in \mathbb{Z} \wedge m \in \mathbb{Z}$, the values of the hyperbolic functions are algebraic. In several cases, they can even be rational numbers, 1, or i (e.g. $\sinh(\pi i / 2) = i$, $\operatorname{sech}(0) = 1$, or $\cosh(\pi i / 3) = 1/2$). They can be expressed using only square roots if $n \in \mathbb{Z}$ and m is a product of a power of 2 and distinct Fermat primes $\{3, 5, 17, 257, \dots\}$.

Simple values at zero

All hyperbolic functions has rather simple values for arguments $z = 0$ and $z = \pi i / 2$:

$$\begin{aligned} \sinh(0) &= 0 & \sinh\left(\frac{\pi i}{2}\right) &= i \\ \cosh(0) &= 1 & \cosh\left(\frac{\pi i}{2}\right) &= 0 \\ \tanh(0) &= 0 & \tanh\left(\frac{\pi i}{2}\right) &= \infty \\ \operatorname{coth}(0) &= \infty & \operatorname{coth}\left(\frac{\pi i}{2}\right) &= 0 \\ \operatorname{csch}(0) &= \infty & \operatorname{csch}\left(\frac{\pi i}{2}\right) &= -i \\ \operatorname{sech}(0) &= 1 & \operatorname{sech}\left(\frac{\pi i}{2}\right) &= \infty. \end{aligned}$$

Analyticity

All hyperbolic functions are defined for all complex values of z , and they are analytical functions of z over the whole complex z -plane and do not have branch cuts or branch points. The two functions $\sinh(z)$ and $\cosh(z)$ are entire functions with an essential singular point at $z = \infty$. All other hyperbolic functions are meromorphic functions with simple poles at points $z = \pi k i$; $k \in \mathbb{Z}$ (for $\operatorname{csch}(z)$ and $\operatorname{coth}(z)$) and at points $z = \pi i / 2 + \pi k i$; $k \in \mathbb{Z}$ (for $\operatorname{sech}(z)$ and $\tanh(z)$).

Periodicity

All hyperbolic functions are periodic functions with a real period ($2\pi i$ or πi):

$$\begin{aligned} \sinh(z) &= \sinh(z + 2\pi i) & \sinh(z + 2\pi i k) &= \sinh(z) /; k \in \mathbb{Z} \\ \cosh(z) &= \cosh(z + 2\pi i) & \cosh(z + 2\pi i k) &= \cosh(z) /; k \in \mathbb{Z} \\ \tanh(z) &= \tanh(z + \pi i) & \tanh(z + \pi i k) &= \tanh(z) /; k \in \mathbb{Z} \\ \operatorname{coth}(z) &= \operatorname{coth}(z + \pi i) & \operatorname{coth}(z + \pi i k) &= \operatorname{coth}(z) /; k \in \mathbb{Z} \\ \operatorname{csch}(z) &= \operatorname{csch}(z + 2\pi i) & \operatorname{csch}(z + 2\pi i k) &= \operatorname{csch}(z) /; k \in \mathbb{Z} \\ \operatorname{sech}(z) &= \operatorname{sech}(z + 2\pi i) & \operatorname{sech}(z + 2\pi i k) &= \operatorname{sech}(z) /; k \in \mathbb{Z}. \end{aligned}$$

Parity and symmetry

All hyperbolic functions have parity (either odd or even) and mirror symmetry:

$$\begin{aligned} \sinh(-z) &= -\sinh(z) & \sinh(\bar{z}) &= \overline{\sinh(z)} \\ \cosh(-z) &= \cosh(z) & \cosh(\bar{z}) &= \overline{\cosh(z)} \\ \tanh(-z) &= -\tanh(z) & \tanh(\bar{z}) &= \overline{\tanh(z)} \\ \operatorname{coth}(-z) &= -\operatorname{coth}(z) & \operatorname{coth}(\bar{z}) &= \overline{\operatorname{coth}(z)} \\ \operatorname{csch}(-z) &= -\operatorname{csch}(z) & \operatorname{csch}(\bar{z}) &= \overline{\operatorname{csch}(z)} \\ \operatorname{sech}(-z) &= \operatorname{sech}(z) & \operatorname{sech}(\bar{z}) &= \overline{\operatorname{sech}(z)}. \end{aligned}$$

Simple representations of derivatives

The derivatives of all hyperbolic functions have simple representations that can be expressed through other hyperbolic functions:

$$\begin{aligned} \frac{\partial \sinh(z)}{\partial z} &= \cosh(z) & \frac{\partial \cosh(z)}{\partial z} &= \sinh(z) & \frac{\partial \tanh(z)}{\partial z} &= \operatorname{sech}^2(z) \\ \frac{\partial \coth(z)}{\partial z} &= -\operatorname{csch}^2(z) & \frac{\partial \operatorname{csch}(z)}{\partial z} &= -\coth(z) \operatorname{csch}(z) & \frac{\partial \operatorname{sech}(z)}{\partial z} &= -\operatorname{sech}(z) \tanh(z). \end{aligned}$$

Simple differential equations

The solutions of the simplest second-order linear ordinary differential equation with constant coefficients can be represented through $\sinh(z)$ and $\cosh(z)$. The other hyperbolic functions satisfy first-order nonlinear differential equations:

$$\begin{aligned} w''(z) - w(z) &= 0 ; w(z) = \cosh(z) \wedge w(0) = 1 \wedge w'(0) = 0 \\ w''(z) - w(z) &= 0 ; w(z) = \sinh(z) \wedge w(0) = 0 \wedge w'(0) = 1 \\ w''(z) - w(z) &= 0 ; w(z) = c_1 \cosh(z) + c_2 \sinh(z). \end{aligned}$$

All six hyperbolic functions satisfy first-order nonlinear differential equations:

$$\begin{aligned} w'(z) - \sqrt{1 + (w(z))^2} &= 0 ; w(z) = \sinh(z) \wedge w(0) = 0 \wedge |\operatorname{Im}(z)| < \frac{\pi}{2} \\ w'(z) - \sqrt{-1 + (w(z))^2} &= 0 ; w(z) = \cosh(z) \wedge w(0) = 1 \wedge |\operatorname{Im}(z)| < \frac{\pi}{2} \\ w'(z) + w(z)^2 - 1 &= 0 ; w(z) = \tanh(z) \wedge w(0) = 0 \\ w'(z) + w(z)^2 - 1 &= 0 ; w(z) = \coth(z) \wedge w\left(\frac{\pi i}{2}\right) = 0 \\ w'(z)^2 - w(z)^4 - w(z)^2 &= 0 ; w(z) = \operatorname{csch}(z) \\ w'(z)^2 + w(z)^4 - w(z)^2 &= 0 ; w(z) = \operatorname{sech}(z). \end{aligned}$$

Applications of hyperbolic functions

Trigonometric functions are intimately related to triangle geometry. Functions like sine and cosine are often introduced as edge lengths of right-angled triangles. Hyperbolic functions occur in the theory of triangles in hyperbolic spaces.

Lobachevsky (1829) and J. Bolyai (1832) independently recognized that Euclid's fifth postulate—saying that for a given line and a point not on the line, there is exactly one line parallel to the first—might be changed and still be a consistent geometry. In the hyperbolic geometry it is allowable for more than one line to be parallel to the first (meaning that the parallel lines will never meet the first, however far they are extended). Translated into triangles, this means that the sum of the three angles is always less than π .

A particularly nice representation of the hyperbolic geometry can be realized in the unit disk of complex numbers (the Poincaré disk model). In this model, points are complex numbers in the unit disk, and the lines are either arcs of circles that meet the boundary of the unit circle orthogonal or diameters of the unit circle.

The distance d between two points (meaning complex numbers) A and B in the Poincaré disk is:

$$d(A, B) = 2 \tanh^{-1} \left(\left| \frac{A - B}{1 - \bar{B}A} \right| \right).$$

The attractive feature of the Poincaré disk model is that the hyperbolic angles agree with the Euclidean angles. Formally, the angle α at a point A of two hyperbolic lines \overline{AB} and \overline{AC} is described by the formula:

$$\cos(\alpha) = \frac{\frac{-A+B}{1-A} \frac{-A+C}{1-A}}{\left| \frac{-A+B}{1-A} \right| \left| \frac{-A+C}{1-A} \right|}.$$

In the following, the values of the three angles of an hyperbolic triangle at the vertices A , B , and C are denoted through α , β , and γ . The hyperbolic length of the three edges opposite to the angles are denoted a , b , and c .

The cosine rule and the second cosine rule for hyperbolic triangles are:

$$\sinh(b) \sinh(c) \cos(\alpha) = \cosh(b) \cosh(c) - \cosh(a)$$

$$\sinh(a) \sinh(c) \cos(\beta) = \cosh(a) \cosh(c) - \cosh(b)$$

$$\sinh(a) \sinh(b) \cos(\gamma) = \cosh(a) \cosh(b) - \cosh(c)$$

$$\sin(\beta) \sin(\gamma) \cosh(a) = \cos(\beta) \cos(\gamma) + \cos(\alpha)$$

$$\sin(\alpha) \sin(\gamma) \cosh(b) = \cos(\alpha) \cos(\gamma) + \cos(\beta)$$

$$\sin(\alpha) \sin(\beta) \cosh(c) = \cos(\alpha) \cos(\beta) + \cos(\gamma).$$

The sine rule for hyperbolic triangles is:

$$\frac{\sin(\alpha)}{\sinh(a)} = \frac{\sin(\beta)}{\sinh(b)} = \frac{\sin(\gamma)}{\sinh(c)}.$$

For a right-angle triangle, the hyperbolic version of the Pythagorean theorem follows from the preceding formulas (the right angle is taken at vertex A):

$$\cosh(a) = \cosh(b) \cosh(c).$$

Using the series expansion $\cosh(x) \approx 1 + x^2/2$ at small scales the hyperbolic geometry is approximated by the familiar Euclidean geometry. The cosine formulas and the sine formulas for hyperbolic triangles with a right angle at vertex A become:

$$\cos(\beta) = \frac{\tanh(c)}{\tanh(a)}, \quad \sin(\beta) = \frac{\sinh(b)}{\sinh(a)}$$

$$\cos(\gamma) = \frac{\tanh(b)}{\tanh(a)}, \quad \sin(\gamma) = \frac{\sinh(c)}{\sinh(a)}.$$

The inscribed circle has the radius:

$$\rho = \sqrt{\tanh^{-1} \left(\frac{\cos^2(\alpha) + \cos^2(\beta) + \cos^2(\gamma) + 2 \cos(\alpha) \cos(\beta) \cos(\gamma) - 1}{2(1 + \cos(\alpha))(1 + \cos(\beta))(1 + \cos(\gamma))} \right)}.$$

The circumscribed circle has the radius:

$$\rho = \tanh^{-1} \left(\frac{4 \sinh\left(\frac{a}{2}\right) \sinh\left(\frac{b}{2}\right) \sinh\left(\frac{c}{2}\right)}{\sin(\gamma) \sinh(a) \sinh(b)} \right).$$

Other applications

As rational functions of the exponential function, the hyperbolic functions appear virtually everywhere in quantitative sciences. It is impossible to list their numerous applications in teaching, science, engineering, and art.

Introduction to the Hyperbolic Sine Function

Defining the hyperbolic sine function

The hyperbolic sine function is an old mathematical function. It was first used in the works of V. Riccati (1757), D. Foncenex (1759), and J. H. Lambert (1768).

The hyperbolic sine function is easily defined as the half difference of two exponential functions in the points z and $-z$:

$$\sinh(z) = \frac{e^z - e^{-z}}{2}.$$

After comparison with the famous Euler formula for sine ($\sin(z) = \frac{e^{iz} - e^{-iz}}{2i}$), it is easy to derive the following representation of the hyperbolic sine through the circular sine:

$$\sinh(z) = -i \sin(iz).$$

This formula allows the derivation of all the properties and formulas for the hyperbolic sine from the corresponding properties and formulas for the circular sine.

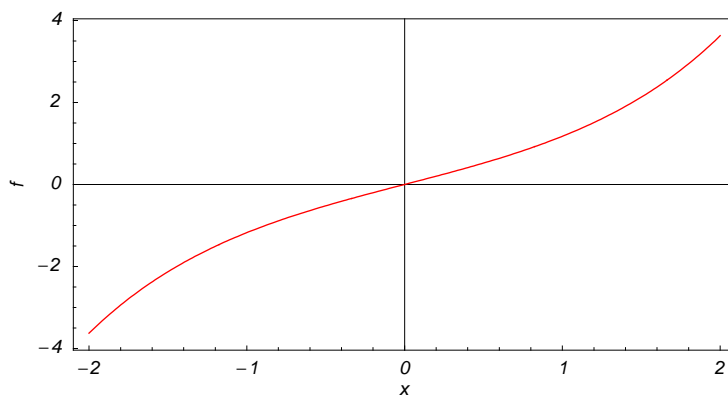
The following formula can sometimes be used as an equivalent definition of the hyperbolic sine function:

$$\sinh(z) = z + \frac{z^3}{6} + \frac{z^5}{120} + \dots = \sum_{k=0}^{\infty} \frac{z^{2k+1}}{(2k+1)!}.$$

This series converges for all finite numbers z .

A quick look at the hyperbolic sine function

Here is a graphic of the hyperbolic sine function $f(x) = \sinh(x)$ for real values of its argument x .



Representation through more general functions

The function $\sinh(z)$ is a particular case of more complicated mathematical functions. For example, it is a special case of the generalized hypergeometric function ${}_0F_1(; a; w)$ with the parameter $a = \frac{3}{2}$ at $w = \frac{z^2}{4}$, multiplied by z :

$$\sinh(z) = z {}_0F_1\left(\frac{3}{2}; \frac{z^2}{4}\right).$$

It is also a particular case of the modified Bessel function $I_\nu(z)$ with the parameter $\nu = \frac{1}{2}$, multiplied by $\sqrt{\frac{\pi z}{2}}$:

$$\sinh(z) = \sqrt{\frac{\pi z}{2}} I_{\frac{1}{2}}(z).$$

Other Bessel functions can also be expressed through the hyperbolic sine function for similar values of the parameter:

$$\sinh(z) = -i \sqrt{\frac{\pi iz}{2}} J_{\frac{1}{2}}(iz) \quad \sinh(z) = -i \sqrt{\frac{\pi iz}{2}} Y_{-\frac{1}{2}}(iz) \quad \sinh(z) = \frac{1}{\sqrt{2\pi}} \left(\sqrt{-z} K_{-\frac{1}{2}}(-z) - \sqrt{z} K_{-\frac{1}{2}}(z) \right).$$

Struve functions can also degenerate into the hyperbolic sine function for a similar parameter value:

$$\sinh(z) = -i \sqrt{\frac{\pi iz}{2}} \mathbf{H}_{-\frac{1}{2}}(iz) \quad \sinh(z) = \sqrt{\frac{\pi z}{2}} \mathbf{L}_{-\frac{1}{2}}(z).$$

But the function $\sinh(z)$ is also a degenerate case of the doubly periodic Jacobi elliptic functions when their second parameter is equal to 0 or 1:

$$\begin{aligned} \sinh(z) &= -i \operatorname{sd}(iz | 0) = -i \operatorname{sn}(iz | 0) \\ \sinh(z) &= \operatorname{sc}(z | 1) = \operatorname{sd}(z | 1). \end{aligned}$$

Finally, the function $\sinh(z)$ is the particular case of another class of functions—the Mathieu functions:

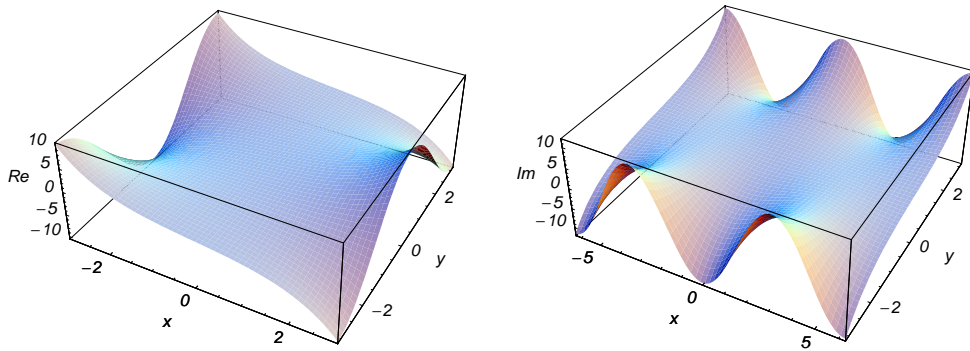
$$\sinh(z) = -i \operatorname{Se}(1, 0, iz).$$

Definition of the hyperbolic sine function for a complex argument

In the complex z -plane, the function $\sinh(z)$ is defined by the same formula used for real values:

$$\sinh(z) = \frac{e^z - e^{-z}}{2}.$$

Here are two graphics showing the real and imaginary parts of the hyperbolic sine function over the complex plane.



The best-known properties and formulas for the hyperbolic sine function

Values in points

The values of the hyperbolic sine function for special values of its argument can be easily derived from the corresponding values of the circular sine in special points of the circle:

$$\begin{aligned} \sinh(0) &= 0 & \sinh\left(\frac{\pi i}{6}\right) &= \frac{i}{2} & \sinh\left(\frac{\pi i}{4}\right) &= \frac{i}{\sqrt{2}} & \sinh\left(\frac{\pi i}{3}\right) &= \frac{\sqrt{3} i}{2} \\ \sinh\left(\frac{\pi i}{2}\right) &= i & \sinh\left(\frac{2\pi i}{3}\right) &= \frac{\sqrt{3} i}{2} & \sinh\left(\frac{3\pi i}{4}\right) &= \frac{i}{\sqrt{2}} & \sinh\left(\frac{5\pi i}{6}\right) &= \frac{i}{2} \\ \sinh(\pi i) &= 0 & \sinh\left(\frac{7\pi i}{6}\right) &= -\frac{i}{2} & \sinh\left(\frac{5\pi i}{4}\right) &= -\frac{i}{\sqrt{2}} & \sinh\left(\frac{4\pi i}{3}\right) &= -\frac{\sqrt{3} i}{2} \\ \sinh\left(\frac{3\pi i}{2}\right) &= -i & \sinh\left(\frac{5\pi i}{3}\right) &= -\frac{\sqrt{3} i}{2} & \sinh\left(\frac{7\pi i}{4}\right) &= -\frac{i}{\sqrt{2}} & \sinh\left(\frac{11\pi i}{6}\right) &= -\frac{i}{2} \end{aligned}$$

$$\sinh(2\pi i) = 0 \quad \sinh(\pi m i) = 0 /; m \in \mathbb{Z} \quad \sinh\left(\pi\left(\frac{1}{2} + m\right)i\right) = (-1)^m i /; m \in \mathbb{Z} .$$

The values at infinity can be expressed by the following formulas:

$$\sinh(\infty) = \infty \quad \sinh(-\infty) = -\infty .$$

General characteristics

For real values of argument z , the values of $\sinh(z)$ are real.

In the points $z = 2\pi n i / m /; n \in \mathbb{Z} \wedge m \in \mathbb{Z}$, the values of $\sinh(z)$ are algebraic. In several cases, they can even be rational numbers, 0, or 1, multiplied by i . Here are some examples:

$$\sinh(0) = 0 \quad \sinh\left(\frac{\pi i}{6}\right) = \frac{i}{2} \quad \sinh\left(\frac{\pi i}{2}\right) = i .$$

The values of $\sinh\left(\frac{n\pi i}{m}\right)$ can be expressed using only square roots if $n \in \mathbb{Z}$ and m is a product of a power of 2 and distinct Fermat primes $\{3, 5, 17, 257, \dots\}$.

The function $\sinh(z)$ is an entire analytical function of z that is defined over the whole complex z -plane and does not have branch cuts and branch points. It has an essential singular point at $z = \infty$. It is a periodic function with the period $2\pi i$:

$$\begin{aligned}\sinh(z + 2\pi i) &= \sinh(z) \\ \sinh(z) &= \sinh(z + 2\pi k i) \quad ; \quad k \in \mathbb{Z} \\ \sinh(z) &= (-1)^k \sinh(z + \pi k i) \quad ; \quad k \in \mathbb{Z}.\end{aligned}$$

The function $\sinh(z)$ is an odd function with mirror symmetry:

$$\sinh(-z) = -\sinh(z) \quad \sinh(\bar{z}) = \overline{\sinh(z)}.$$

Differentiation

The derivatives of $\sinh(z)$ have simple representations using either the $\sinh(z)$ function or the $\cosh(z)$ function:

$$\frac{\partial \sinh(z)}{\partial z} = \cosh(z) \quad \frac{\partial^n \sinh(z)}{\partial z^n} = (-i)^n \sinh\left(z + \frac{i\pi n}{2}\right) \quad ; \quad n \in \mathbb{N}^+.$$

Ordinary differential equation

The function $\sinh(z)$ satisfies the simplest possible linear differential equation with constant coefficients:

$$w''(z) - w(z) = 0 \quad ; \quad w(z) = \sinh(z) \wedge w(0) = 0 \wedge w'(0) = 1.$$

The complete solution of this equation can be represented as a linear combination of $\sinh(z)$ and $\cosh(z)$ with arbitrary constant coefficients c_1 and c_2 :

$$w''(z) - w(z) = 0 \quad ; \quad w(z) = c_1 \cosh(z) + c_2 \sinh(z).$$

The function $\sinh(z)$ also satisfies the first-order nonlinear differential equation:

$$w'(z) - \sqrt{1 + (w(z))^2} = 0 \quad ; \quad w(z) = \sinh(z) \wedge w(0) = 0 \wedge |\operatorname{Im}(z)| < \frac{\pi}{2}.$$

Series representation

The function $\sinh(z)$ has a simple series expansion at the origin that converges in the whole complex z -plane:

$$\sinh(z) = z + \frac{z^3}{3!} + \dots = \sum_{k=0}^{\infty} \frac{z^{2k+1}}{(2k+1)!}.$$

Product representation

The following infinite product representation for $\sinh(z)$ clearly illustrates that $\sinh(z) = 0$ at $z = \pi k i \wedge k \in \mathbb{Z}$:

$$\sinh(z) = z \prod_{k=1}^{\infty} \left(1 + \frac{z^2}{\pi^2 k^2}\right).$$

Indefinite integration

Indefinite integrals of expressions involving the hyperbolic sine function can sometimes be expressed using elementary functions. However, special functions are frequently needed to express the results even when the integrands have a simple form (if they can be evaluated in closed form). Here are some examples:

$$\int \sinh(z) dz = \cosh(z)$$

$$\int \sqrt{\sinh(z)} dz = \frac{2\sqrt{i \sinh(z)}}{\sinh^{\frac{1}{2}}(z)} E\left(\frac{1}{2}\left(\frac{\pi}{2} - iz\right) \middle| 2\right)$$

$$\int z^{\alpha-1} \sinh^{\nu}(az) dz = \frac{i^{\nu} 2^{-\nu} (1 - \nu \bmod 2)}{\alpha} z^{\alpha} \binom{\nu}{\frac{\nu}{2}} + i^{\nu} 2^{-\nu} z^{\alpha} \sum_{k=0}^{\lfloor \frac{\nu-1}{2} \rfloor} (-1)^{k-1} \binom{\nu}{k} (-a^2 (n-2k)^2 z^2)^{-\alpha}$$

$$\left(e^{\frac{\pi \nu i}{2}} \Gamma(\alpha, a(n-2k)z) (a(2k-n)z)^{\alpha} + e^{-\frac{1}{2}(\pi \nu i)} (a(n-2k)z)^{\alpha} \Gamma(\alpha, a(2k-n)z) \right) /; \alpha \neq 0 \wedge \nu \in \mathbb{N}^+$$

The last integral cannot be evaluated in closed form using the known classical special functions for arbitrary values of parameters α and ν .

Definite integration

Definite integrals that contain the hyperbolic sine are sometimes simple as shown in the following example:

$$\int_0^{\infty} \frac{1}{1 + \sinh(t)} dt = \frac{\log(3 + 2\sqrt{2})}{\sqrt{2}}$$

Some special functions can be used to evaluate more complicated definite integrals. For example, gamma and polygamma functions, are needed to express the following integrals:

$$\int_0^{\infty} \sinh^a(t) dt = \frac{\Gamma(-\frac{a}{2})\Gamma(\frac{a+1}{2})}{2\sqrt{\pi}} /; -1 < \text{Re}(a) < 0$$

$$\int_0^{\infty} t^2 \sinh^a(t) dt = \frac{1}{8\sqrt{\pi}} \left(\Gamma\left(-\frac{a}{2}\right)\Gamma\left(\frac{a+1}{2}\right) \left(\psi\left(-\frac{a}{2}\right)^2 - 2\psi\left(\frac{a}{2}+1\right)\psi\left(-\frac{a}{2}\right) + \psi\left(\frac{a}{2}+1\right)^2 + \psi^{(1)}\left(-\frac{a}{2}\right) - \psi^{(1)}\left(\frac{a}{2}+1\right) \right) \right) /;$$

$$-3 < \text{Re}(a) < 0.$$

Integral transforms

Numerous formulas for integral transforms from circular sine functions cannot be easily converted into corresponding formulas with the hyperbolic sine function because the hyperbolic sine grows exponentially at infinity. This holds for the Fourier cosine and sine transforms, and for Mellin, Hilbert, Hankel, and other transforms.

The exceptional case is the Laplace transform that itself includes the exponential function in the kernel:

$$\mathcal{L}_i[\sinh(t)](z) = \frac{1}{z^2 - 1}$$

Finite summation

The following finite sum of the hyperbolic sine can be expressed using the hyperbolic sine function:

$$\sum_{k=0}^n \sinh(xk) = \operatorname{csch}\left(\frac{x}{2}\right) \sinh\left(\frac{x(n+1)}{2}\right) \sinh\left(\frac{xn}{2}\right).$$

Infinite summation

The following infinite sum can be expressed using elementary functions:

$$\sum_{k=1}^{\infty} \frac{\sinh(kx)}{k!} = e^{\cosh(x)} \sinh(\sinh(x)).$$

Finite products

The following finite products of the hyperbolic sine can be expressed using elementary functions and formulas:

$$\prod_{k=1}^{n-1} \sinh\left(\frac{\pi i k}{n} + z\right) = \frac{i^{n-1} 2^{1-n} \sinh(nz)}{\sinh(z)} ; n \in \mathbb{N}^+$$

$$\prod_{k=1}^{n-1} \sinh\left(\frac{2i\pi k}{n} + z\right) = i^{-n} (-2)^{1-n} \left(\cos\left(\frac{n\pi}{2} - inz\right) - \cos\left(\frac{n\pi}{2}\right) \right) \operatorname{csch}(z) ; n \in \mathbb{N}^+.$$

Infinite products

The following infinite product can be expressed using the hyperbolic sine function:

$$\prod_{k=1}^{\infty} \left(1 + \frac{4}{3} \sinh^2\left(\frac{z}{3^k}\right) \right) = \frac{\sinh(z)}{z}.$$

Addition formulas

The hyperbolic sine of a sum can be represented by the rule: "the hyperbolic sine of a sum is equal to the product of the hyperbolic sine by the hyperbolic cosine plus the hyperbolic cosine by the hyperbolic sine." A similar rule is valid for the hyperbolic sine of the difference:

$$\begin{aligned} \sinh(a + b) &= \cosh(b) \sinh(a) + \cosh(a) \sinh(b) \\ \sinh(a - b) &= \cosh(b) \sinh(a) - \cosh(a) \sinh(b). \end{aligned}$$

Multiple arguments

In the case of multiple arguments $2z, 3z, \dots$, the function $\sinh(nz)$ can be represented as the finite sum of terms that include powers of the hyperbolic sine and cosine:

$$\sinh(2z) = 2 \sinh(z) \cosh(z)$$

$$\sinh(3z) = 4 \sinh^3(z) + 3 \sinh(z)$$

$$\sinh(nz) = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{2k+1} \sinh^{2k+1}(z) \cosh^{n-2k-1}(z) ; n \in \mathbb{N}^+.$$

The function $\sinh(nz)$ can also be represented as the finite product including the hyperbolic sine of the linear argument of z :

$$\sinh(nz) = i^{1-n} 2^{n-1} \prod_{k=0}^{n-1} \sinh\left(z + \frac{i\pi k}{n}\right); n \in \mathbb{N}^+.$$

Half-angle formulas

The hyperbolic sine of the half-angle can be represented by the following simple formula that is valid in half of the horizontal strip:

$$\sinh\left(\frac{z}{2}\right) = \sqrt{\frac{\cosh(z) - 1}{2}}; /;$$

$$\operatorname{Re}(z) > 0 \wedge -\pi < \operatorname{Im}(z) < \pi \vee \operatorname{Re}(z) < 0 \wedge \pi < \operatorname{Im}(z) < 3\pi \vee \operatorname{Re}(z) = 0 \wedge 0 \leq \operatorname{Im}(z) \leq 2\pi \vee \operatorname{Im}(z) = \pi.$$

To make this formula correct for all complex z , a complicated prefactor is needed:

$$\sinh\left(\frac{z}{2}\right) = c(z) \sqrt{\frac{\cosh(z) - 1}{2}}; /; c(z) = \frac{\sqrt{z^2}}{z} (-1)^{\lfloor \frac{\operatorname{Im}(z)}{2\pi} + \frac{1}{2} \rfloor} \left(1 - \left(1 + (-1)^{\lfloor \frac{\operatorname{Im}(z)}{2\pi} + \frac{1}{2} \rfloor + \lfloor -\frac{\operatorname{Im}(z)}{2\pi} - \frac{1}{2} \rfloor \right) \theta(\operatorname{Re}(z))\right); \frac{z - \pi i}{2\pi i} \notin \mathbb{Z},$$

where $c(z)$ contains the unit step, real part, imaginary part, and the floor functions.

Sums of two direct functions

The sum of two hyperbolic sine functions can be described by the rule: "the sum of hyperbolic sines is equal to the doubled hyperbolic cosine of the half-difference multiplied by the hyperbolic sine of the half-sum". A similar rule is valid for the difference of two hyperbolic sines:

$$\sinh(a) + \sinh(b) = 2 \cosh\left(\frac{a-b}{2}\right) \sinh\left(\frac{a+b}{2}\right)$$

$$\sinh(a) - \sinh(b) = 2 \cosh\left(\frac{a+b}{2}\right) \sinh\left(\frac{a-b}{2}\right).$$

Products involving the direct function

The product of two hyperbolic sine functions or the product of the hyperbolic sine and hyperbolic cosine have the following representations:

$$\sinh(a) \sinh(b) = \frac{1}{2} (\cosh(a+b) - \cosh(a-b))$$

$$\sinh(a) \cosh(b) = \frac{1}{2} (\sinh(a-b) + \sinh(a+b)).$$

Powers of the direct function

The integer powers of the hyperbolic sine functions can be expanded as finite sums of hyperbolic cosine (or sine) functions with multiple arguments. These sums include binomial coefficients:

$$\sinh^{2n}(z) = (-1)^n 2^{-2n} \binom{2n}{n} + 2^{1-2n} \sum_{k=0}^{n-1} (-1)^k \binom{2n}{k} \cosh(2(n-k)z); n \in \mathbb{N}^+$$

$$\sinh^{2n-1}(z) = 2^{2-2n} \sum_{k=0}^n (-1)^k \binom{2n-1}{k} \sinh((2n-2k-1)z); n \in \mathbb{N}^+.$$

These formulas can be combined into the following formula:

$$\sinh^n(z) = i^n 2^{-n} \binom{n}{\frac{n}{2}} (1 - n \bmod 2) - i^n 2^{1-n} \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^{k-1} \binom{n}{k} \cosh\left((n-2k)z - \frac{\pi i n}{2}\right); n \in \mathbb{N}^+.$$

Inequalities

The most famous inequality for the hyperbolic sine function can be described by the following formula:

$$\sinh(|z|) \geq |\sin(z)|.$$

Relations with its inverse function

There is a simple relation between the function $\sinh(z)$ and its inverse function $\sinh^{-1}(z)$:

$$\sinh(\sinh^{-1}(z)) = z \quad \sinh^{-1}(\sinh(z)) = z /; -\frac{\pi}{2} < \text{Im}(z) < \frac{\pi}{2} \bigvee (\text{Im}(z) = -\frac{\pi}{2} \wedge \text{Re}(z) \leq 0) \bigvee (\text{Im}(z) = \frac{\pi}{2} \wedge \text{Re}(z) \geq 0).$$

The second formula is valid at least in the horizontal strip $-\frac{\pi}{2} < \text{Re}(z) < \frac{\pi}{2}$. Outside this strip, a much more complicated relation (that contains the unit step, real part, imaginary part, and the floor functions) holds:

$$\begin{aligned} \sinh^{-1}(\sinh(z)) = & (-1)^{\lfloor -\frac{\text{Im}(z)-\frac{1}{2}}{\pi} \rfloor} \left(\left(1 + (-1)^{\lfloor \frac{\text{Im}(z)+\frac{1}{2}}{\pi} \rfloor + \lfloor -\frac{\text{Im}(z)-\frac{1}{2}}{\pi} \rfloor \right) \theta(-\text{Re}(z)) - 1 \right) \\ & \left(z + i\pi \left[-\frac{\text{Im}(z)}{\pi} - \frac{1}{2} \right] + \frac{\pi i}{2} \left(2 - \left(1 + (-1)^{\lfloor \frac{\text{Im}(z)+\frac{1}{2}}{\pi} \rfloor + \lfloor -\frac{\text{Im}(z)-\frac{1}{2}}{\pi} \rfloor \right) \theta(-\text{Re}(z)) \right) \right). \end{aligned}$$

Representations through other hyperbolic functions

Hyperbolic sine and cosine functions are connected by a very simple formula that contains the linear function in the argument:

$$\sinh(z) = i \cosh\left(\frac{\pi i}{2} - z\right).$$

Another famous formula, connecting $\sinh(z)$ and $\cosh(z)$, is expressed in the analog of the well-known Pythagorean theorem:

$$\sinh^2(z) = \cosh^2(z) - 1$$

$$\begin{aligned} \sinh(z) = \sqrt{\cosh^2(z) - 1} \quad /; \text{Re}(z) > 0 \wedge \frac{3\pi}{2} < \text{Im}(z) \bmod (2\pi) < 2\pi \bigvee \text{Re}(z) > 0 \wedge 0 \leq \text{Im}(z) \bmod (2\pi) < \frac{\pi}{2} \bigvee \\ \text{Re}(z) < 0 \wedge \frac{\pi}{2} < \text{Im}(z) \bmod (2\pi) < \frac{3\pi}{2} \bigvee \text{Re}(z) = 0 \wedge 0 \leq \text{Im}(z) \bmod (2\pi) \leq \pi \bigvee \text{Im}(z) \bmod (2\pi) = \frac{\pi}{2}. \end{aligned}$$

The last restriction on z can be removed, but the formula will get a complicated coefficient $c(z)$ that contains the unit step, real part, imaginary part, and the floor function and $|c(z)| = 1$:

$$\sinh(z) = c(z) \sqrt{\cosh^2(z) - 1} \quad /; c(z) = \frac{\sqrt{z^2}}{z} (-1)^{\lfloor \frac{\text{Im}(z)+\frac{1}{2}}{\pi} \rfloor} \left(1 - \left(1 + (-1)^{\lfloor \frac{\text{Im}(z)+\frac{1}{2}}{\pi} \rfloor + \lfloor -\frac{\text{Im}(z)-\frac{1}{2}}{\pi} \rfloor \right) \theta(\text{Re}(z)) \right) /; -\frac{2z - \pi i}{2\pi i} \notin \mathbb{N}^+.$$

The hyperbolic sine function can also be represented using other hyperbolic functions by the following formulas:

$$\sinh(z) = \frac{2 \tanh\left(\frac{z}{2}\right)}{1 - \tanh^2\left(\frac{z}{2}\right)} \quad \sinh(z) = \frac{2 \coth\left(\frac{z}{2}\right)}{\coth^2\left(\frac{z}{2}\right) - 1}$$

$$\sinh(z) = \frac{1}{\operatorname{csch}(z)} \quad \sinh(z) = \frac{i}{\operatorname{sech}\left(\frac{\pi i}{2} - z\right)}$$

Representations through hyperbolic functions

The hyperbolic sine function has representations using the other hyperbolic functions:

$$\sinh(z) = -i \sin(iz) \quad \sinh(iz) = i \sin(z) \quad \sinh(z) = -i \cos\left(\frac{\pi}{2} - iz\right)$$

$$\sinh(z) = -\frac{2i \tan\left(\frac{iz}{2}\right)}{\tan^2\left(\frac{iz}{2}\right) + 1} \quad \sinh(z) = -\frac{2i \cot\left(\frac{iz}{2}\right)}{\cot^2\left(\frac{iz}{2}\right) + 1} \quad \sinh(z) = -\frac{i}{\csc(iz)} \quad \sinh(z) = -\frac{i}{\sec\left(\frac{\pi}{2} - iz\right)}$$

Applications

The hyperbolic sine function is used throughout mathematics, the exact sciences, and engineering.

Introduction to the Hyperbolic Functions in *Mathematica*

Overview

The following shows how the six hyperbolic functions are realized in *Mathematica*. Examples of evaluating *Mathematica* functions applied to various numeric and exact expressions that involve the hyperbolic functions or return them are shown. These involve numeric and symbolic calculations and plots.

Notations

Mathematica forms of notations

All six hyperbolic functions are represented as built-in functions in *Mathematica*. Following *Mathematica*'s general naming convention, the `StandardForm` function names are simply capitalized versions of the traditional mathematics names. Here is a list `hypFunctions` of the six hyperbolic functions in `StandardForm`.

```
hypFunctions = {Sinh[z], Cosh[z], Tanh[z], Coth[z], Sech[z], Cosh[z]}
{Sinh[z], Cosh[z], Tanh[z], Coth[z], Sech[z], Cosh[z]}
```

Here is a list `hypFunctions` of the six trigonometric functions in `TraditionalForm`.

```
hypFunctions // TraditionalForm
{sinh(z), cosh(z), tanh(z), coth(z), sech(z), cosh(z)}
```

Additional forms of notations

Mathematica also knows the most popular forms of notations for the hyperbolic functions that are used in other programming languages. Here are three examples: `CForm`, `TeXForm`, and `FortranForm`.

```
hypFunctions /. {z -> 2 pi z} // CForm
```



```
List(Sinh(2*Pi*z),Cosh(2*Pi*z),Tanh(2*Pi*z),Coth(2*Pi*z),Sech(2*Pi*z),Cosh(2*Pi*z))
```

```
hypFunctions /. {z -> 2 Pi z} // TeXForm
```

```
\{ \sinh (2\,\pi \,z),\cosh (2\,\pi \,z),\tanh (2\,\pi \,z),\coth (2\,\pi \,z),
  \Mfunction{Sech}(2\,\pi \,z),\cosh (2\,\pi \,z)\}
```

```
hypFunctions /. {z -> 2 Pi z} // FortranForm
```

```
List(Sinh(2*Pi*z),Cosh(2*Pi*z),Tanh(2*Pi*z),Coth(2*Pi*z),Sech(2*Pi*z),Cosh(2*Pi*z))
```

Automatic evaluations and transformations

Evaluation for exact, machine-number, and high-precision arguments

For a simple exact argument, *Mathematica* returns an exact result. For instance, for the argument $\pi i/6$, the Sinh function evaluates to $i/2$.

```
Sinh[ $\frac{\pi i}{6}$ ]
```

$$\frac{i}{2}$$

```
{Sinh[z], Cosh[z], Tanh[z], Coth[z], Csch[z], Sech[z]} /. z ->  $\frac{\pi i}{6}$ 
```

$$\left\{ \frac{i}{2}, \frac{\sqrt{3}}{2}, \frac{i}{\sqrt{3}}, -i\sqrt{3}, -2i, \frac{2}{\sqrt{3}} \right\}$$

For a generic machine-number argument (a numerical argument with a decimal point and not too many digits), a machine number is returned.

```
Cosh[3.]
```

```
10.0677
```

```
{Sinh[z], Cosh[z], Tanh[z], Coth[z], Csch[z], Sech[z]} /. z -> 2.
```

```
{3.62686, 3.7622, 0.964028, 1.03731, 0.275721, 0.265802}
```

The next inputs calculate 100-digit approximations of the six hyperbolic functions at $z = 1$.

```
N[Tanh[1], 40]
```

```
0.7615941559557648881194582826047935904128
```

```
Coth[1] // N[#, 50] &
```

```
1.3130352854993313036361612469308478329120139412405
```

```
N[{Sinh[z], Cosh[z], Tanh[z], Coth[z], Csch[z], Sech[z]} /. z -> 1, 100]
```

```
{1.175201193643801456882381850595600815155717981334095870229565413013307567304323895\
607117452089623392,
1.543080634815243778477905620757061682601529112365863704737402214710769063049223698\
964264726435543036,
0.761594155955764888119458282604793590412768597257936551596810500121953244576638483\
4589475216736767144,
1.313035285499331303636161246930847832912013941240452655543152967567084270461874382\
674679241480856303,
0.850918128239321545133842763287175284181724660910339616990421151729003364321465103\
8997301773288938124,
0.648054273663885399574977353226150323108489312071942023037865337318717595646712830\
2808547853078928924}
```

Within a second, it is possible to calculate thousands of digits for the hyperbolic functions. The next input calculates 10000 digits for $\sinh(1)$, $\cosh(1)$, $\tanh(1)$, $\coth(1)$, $\operatorname{sech}(1)$, and $\operatorname{csch}(1)$ and analyzes the frequency of the occurrence of the digit k in the resulting decimal number.

```
Map[Function[w, {First[#], Length[#]} & /@ Split[Sort[First[RealDigits[w]]]],
N[{Sinh[z], Cosh[z], Tanh[z], Coth[z], Csch[z], Sech[z]} /. z -> 1, 10 000]]
{{{0, 980}, {1, 994}, {2, 996}, {3, 1014}, {4, 986}, {5, 1001},
{6, 1017}, {7, 1020}, {8, 981}, {9, 1011}}, {{0, 1015}, {1, 960}, {2, 997},
{3, 1037}, {4, 1070}, {5, 1018}, {6, 973}, {7, 997}, {8, 963}, {9, 970}},
{{0, 971}, {1, 1023}, {2, 1016}, {3, 970}, {4, 949}, {5, 1052}, {6, 981},
{7, 1056}, {8, 1010}, {9, 972}}, {{0, 975}, {1, 986}, {2, 1023},
{3, 1004}, {4, 1008}, {5, 977}, {6, 977}, {7, 1036}, {8, 1035}, {9, 979}},
{{0, 979}, {1, 1030}, {2, 987}, {3, 992}, {4, 1016}, {5, 1030}, {6, 1021},
{7, 969}, {8, 974}, {9, 1002}}, {{0, 1009}, {1, 971}, {2, 1018},
{3, 994}, {4, 1011}, {5, 1018}, {6, 958}, {7, 1019}, {8, 1016}, {9, 986}}}
```

Here are 50-digit approximations to the six hyperbolic functions at the complex argument $z = 3 + 5i$.

```
N[Csch[3 + 5 i], 100]
```

```
0.0280585164230800759963159842602743697051540123887285931631736730964453318082730911\
1484269546408531396 +
0.095323634674178402851915930706256451645442166878775479803879772793331583262276221\
38939784445056701747 i
```

```
N[{Sinh[z], Cosh[z], Tanh[z], Coth[z], Csch[z], Sech[z]} /. z -> 3 + 5 i, 50]
```

```
{2.8416922956063519438168753953062364359281841632360 -
 9.6541254768548391365515436340301659921919691213853 i,
2.8558150042273872913639018630946098374643609536732 -
9.6063834484325811198111562160434163877218590394033 i,
1.0041647106948152119205166259313184311852454735738 -
0.0027082358362240721322640353684331035927960259125751 i,
0.99584531857585412976042001587164841711026557204102 +
0.0026857984057585256446537711012814749378977439361108 i,
0.028058516423080075996315984260274369705154012388729 +
0.095323634674178402851915930706256451645442166878775 i,
0.028433530909971667358833684958646399417265586614624 +
0.095644640955286344684316595933099452259073530811833 i}
```

Mathematica always evaluates mathematical functions with machine precision, if the arguments are machine numbers. In this case, only six digits after the decimal point are shown in the results. The remaining digits are suppressed, but can be displayed using the function `InputForm`.

```
{Sinh[2.], N[Sinh[2]], N[Sinh[2], 16], N[Sinh[2], 5], N[Sinh[2], 20]}
{3.62686, 3.62686, 3.62686, 3.62686, 3.6268604078470187677}

% // InputForm
{3.6268604078470186, 3.6268604078470186, 3.6268604078470186, 3.6268604078470186,
3.62686040784701876766821398280126201644`20}
```

Precision[%%]

16

Simplification of the argument

Mathematica uses symmetries and periodicities of all the hyperbolic functions to simplify expressions. Here are some examples.

Sinh[-z]

-Sinh[z]

Sinh[z + πi]

-Sinh[z]

Sinh[z + 2 πi]

Sinh[z]

Sinh[z + 34 πi]

Sinh[z]

{Sinh[-z], Cosh[-z], Tanh[-z], Coth[-z], Csch[-z], Sech[-z]}

{-Sinh[z], Cosh[z], -Tanh[z], -Coth[z], -Csch[z], Sech[z]}

{Sinh[z + π i], Cosh[z + π i], Tanh[z + π i], Coth[z + π i], Csch[z + π i], Sech[z + π i]}

{-Sinh[z], -Cosh[z], Tanh[z], Coth[z], -Csch[z], -Sech[z]}

**{Sinh[z + 2 π i], Cosh[z + 2 π i], Tanh[z + 2 π i],
Coth[z + 2 π i], Csch[z + 2 π i], Sech[z + 2 π i]}**

{Sinh[z], Cosh[z], Tanh[z], Coth[z], Csch[z], Sech[z]}

**{Sinh[z + 342 π i], Cosh[z + 342 π i], Tanh[z + 342 π i],
Coth[z + 342 π i], Csch[z + 342 π i], Sech[z + 342 π i]}**

{Sinh[z], Cosh[z], Tanh[z], Coth[z], Csch[z], Sech[z]}

Mathematica automatically simplifies the composition of the direct and the inverse hyperbolic functions into the argument.

**{Sinh[ArcSinh[z]], Cosh[ArcCosh[z]], Tanh[ArcTanh[z]],
Coth[ArcCoth[z]], Csch[ArcCsch[z]], Sech[ArcSech[z]]}**

{z, z, z, z, z, z}

Mathematica also automatically simplifies the composition of the direct and any of the inverse hyperbolic functions into algebraic functions of the argument.

**{Sinh[ArcSinh[z]], Sinh[ArcCosh[z]], Sinh[ArcTanh[z]],
Sinh[ArcCoth[z]], Sinh[ArcCsch[z]], Sinh[ArcSech[z]]}**

$$\left\{ z, \sqrt{\frac{-1+z}{1+z}} (1+z), \frac{z}{\sqrt{1-z^2}}, \frac{1}{\sqrt{1-\frac{1}{z^2}} z}, \frac{1}{z}, \sqrt{\frac{1-z}{1+z}} (1+z) \right\}$$

**{Cosh[ArcSinh[z]], Cosh[ArcCosh[z]], Cosh[ArcTanh[z]],
Cosh[ArcCoth[z]], Cosh[ArcCsch[z]], Cosh[ArcSech[z]]}**

$$\left\{ \sqrt{1+z^2}, z, \frac{1}{\sqrt{1-z^2}}, \frac{1}{\sqrt{1-\frac{1}{z^2}}}, \sqrt{1+\frac{1}{z^2}}, \frac{1}{z} \right\}$$

**{Tanh[ArcSinh[z]], Tanh[ArcCosh[z]], Tanh[ArcTanh[z]],
Tanh[ArcCoth[z]], Tanh[ArcCsch[z]], Tanh[ArcSech[z]]}**

$$\left\{ \frac{z}{\sqrt{1+z^2}}, \frac{\sqrt{\frac{-1+z}{1+z}} (1+z)}{z}, z, \frac{1}{z}, \frac{1}{\sqrt{1+\frac{1}{z^2}} z}, \sqrt{\frac{1-z}{1+z}} (1+z) \right\}$$

**{Coth[ArcSinh[z]], Coth[ArcCosh[z]], Coth[ArcTanh[z]],
Coth[ArcCoth[z]], Coth[ArcCsch[z]], Coth[ArcSech[z]]}**

$$\left\{ \frac{\sqrt{1+z^2}}{z}, \frac{z}{\sqrt{\frac{-1+z}{1+z}}(1+z)}, \frac{1}{z}, z, \sqrt{1+\frac{1}{z^2}}z, \frac{1}{\sqrt{\frac{1-z}{1+z}}(1+z)} \right\}$$

$$\{\text{Csch}[\text{ArcSinh}[z]], \text{Csch}[\text{ArcCosh}[z]], \text{Csch}[\text{ArcTanh}[z]], \text{Csch}[\text{ArcCoth}[z]], \text{Csch}[\text{ArcCsch}[z]], \text{Csch}[\text{ArcSech}[z]]\}$$

$$\left\{ \frac{1}{z}, \frac{1}{\sqrt{\frac{-1+z}{1+z}}(1+z)}, \frac{\sqrt{1-z^2}}{z}, \sqrt{1-\frac{1}{z^2}}z, z, \frac{z}{\sqrt{\frac{1-z}{1+z}}(1+z)} \right\}$$

$$\{\text{Sech}[\text{ArcSinh}[z]], \text{Sech}[\text{ArcCosh}[z]], \text{Sech}[\text{ArcTanh}[z]], \text{Sech}[\text{ArcCoth}[z]], \text{Sech}[\text{ArcCsch}[z]], \text{Sech}[\text{ArcSech}[z]]\}$$

$$\left\{ \frac{1}{\sqrt{1+z^2}}, \frac{1}{z}, \sqrt{1-z^2}, \sqrt{1-\frac{1}{z^2}}, \frac{1}{\sqrt{1+\frac{1}{z^2}}}, z \right\}$$

In cases where the argument has the structure $\pi k/2 + z$ or $\pi k/2 - z$, $e^{\pi k/2 + iz}$ or $\pi k/2 - iz$ with integer k , trigonometric functions can be automatically transformed into other trigonometric or hyperbolic functions. Here are some examples.

$$\text{Tanh}\left[\frac{\pi i}{2} - z\right]$$

$$-\text{Coth}[z]$$

$$\text{Csch}[i z]$$

$$-i \text{Csc}[z]$$

$$\left\{ \text{Sinh}\left[\frac{\pi i}{2} - z\right], \text{Cosh}\left[\frac{\pi i}{2} - z\right], \text{Tanh}\left[\frac{\pi i}{2} - z\right], \text{Coth}\left[\frac{\pi i}{2} - z\right], \text{Csch}\left[\frac{\pi i}{2} - z\right], \text{Sech}\left[\frac{\pi i}{2} - z\right] \right\}$$

$$\{i \text{Cosh}[z], -i \text{Sinh}[z], -\text{Coth}[z], -\text{Tanh}[z], -i \text{Sech}[z], i \text{Csch}[z]\}$$

$$\{\text{Sinh}[i z], \text{Cosh}[i z], \text{Tanh}[i z], \text{Coth}[i z], \text{Csch}[i z], \text{Sech}[i z]\}$$

$$\{i \text{Sin}[z], \text{Cos}[z], i \text{Tan}[z], -i \text{Cot}[z], -i \text{Csc}[z], \text{Sec}[z]\}$$

Simplification of simple expressions containing hyperbolic functions

Sometimes simple arithmetic operations containing hyperbolic functions can automatically produce other hyperbolic functions.

$$1 / \text{Sech}[z]$$

$$\text{Cosh}[z]$$

$$\{1 / \text{Sinh}[z], 1 / \text{Cosh}[z], 1 / \text{Tanh}[z], 1 / \text{Coth}[z], 1 / \text{Csch}[z], 1 / \text{Sech}[z], \text{Sinh}[z] / \text{Cosh}[z], \text{Cosh}[z] / \text{Sinh}[z], \text{Sinh}[z] / \text{Sinh}[\pi i / 2 - z], \text{Cosh}[z] / \text{Sinh}[z]^2\}$$

{Csch[z], Sech[z], Coth[z], Tanh[z], Sinh[z],
Cosh[z], Tanh[z], Coth[z], -i Tanh[z], Coth[z] Csch[z]}

Hyperbolic functions as special cases of more general functions

All hyperbolic functions can be treated as particular cases of some more advanced special functions. For example, $\sinh(z)$ and $\cosh(z)$ are sometimes the results of auto-simplifications from Bessel, Mathieu, Jacobi, hypergeometric, and Meijer functions (for appropriate values of their parameters).

BesselI $\left[\frac{1}{2}, z\right]$

$$\frac{\sqrt{\frac{2}{\pi}} \operatorname{Sinh}[z]}{\sqrt{z}}$$

MathieuC $[1, 0, i z]$

Cosh $[z]$

JacobiSN $[z, 1]$

Tanh $[z]$

{**BesselI** $\left[\frac{1}{2}, z\right]$, **MathieuS** $[1, 0, i z]$, **JacobiSD** $[i z, 0]$,

HypergeometricPFQ $\left[\{\}, \left\{\frac{3}{2}\right\}, \frac{z^2}{4}\right]$, **MeijerG** $\left[\{\{\}, \{\}\}, \left\{\{0\}, \left\{-\frac{1}{2}\right\}\right\}, -\frac{z^2}{4}\right]$ }

$$\left\{\frac{\sqrt{\frac{2}{\pi}} \operatorname{Sinh}[z]}{\sqrt{z}}, i \operatorname{Sinh}[z], i \operatorname{Sinh}[z], \frac{\operatorname{Sinh}\left[\sqrt{z^2}\right]}{\sqrt{z^2}}, \frac{2 \operatorname{Sinh}[z]}{\sqrt{\pi} z}\right\}$$

{**BesselI** $\left[-\frac{1}{2}, z\right]$, **MathieuC** $[1, 0, i z]$, **JacobiCD** $[i z, 0]$,

Hypergeometric0F1 $\left[\frac{1}{2}, \frac{z^2}{4}\right]$, **MeijerG** $\left[\{\{\}, \{\}\}, \left\{\{0\}, \left\{\frac{1}{2}\right\}\right\}, -\frac{z^2}{4}\right]$ }

$$\left\{\frac{\sqrt{\frac{2}{\pi}} \operatorname{Cosh}[z]}{\sqrt{z}}, \operatorname{Cosh}[z], \operatorname{Cosh}[z], \operatorname{Cosh}\left[\sqrt{z^2}\right], \frac{\operatorname{Cosh}[z]}{\sqrt{\pi}}\right\}$$

{**JacobiSC** $[i z, 0]$, **JacobiNS** $[z, 1]$, **JacobiNS** $[i z, 0]$, **JacobiDC** $[i z, 0]$ }

{ $i \operatorname{Tanh}[z]$, $\operatorname{Coth}[z]$, $-i \operatorname{Csch}[z]$, $\operatorname{Sech}[z]$ }

Equivalence transformations carried out by specialized *Mathematica* functions

General remarks

Automatic evaluation and transformations can sometimes be inconvenient: They act in only one chosen direction and the result can be overly complicated. For example, the expression $i \cosh(z)/2$ is generally preferable to the more complicated $\sinh(\pi i/2 - z) \cosh(\pi i/3)$. *Mathematica* provides automatic transformation of the second expression into the first one. But compact expressions like $\sinh(2z) \cosh(\pi i/16)$ should not be automatically expanded into the more complicated expression $\sinh(z) \cosh(z) \left(2 + (2 + 2^{1/2})^{1/2}\right)^{1/2}$. *Mathematica* has special functions that produce these types of expansions. Some of them are demonstrated in the next section.

TrigExpand

The function `TrigExpand` expands out trigonometric and hyperbolic functions. In more detail, it splits up sums and integer multiples that appear in the arguments of trigonometric and hyperbolic functions, and then expands out the products of the trigonometric and hyperbolic functions into sums of powers, using the trigonometric and hyperbolic identities where possible. Here are some examples.

`TrigExpand[Sinh[x - y]]`

`Cosh[y] Sinh[x] - Cosh[x] Sinh[y]`

`Cosh[4 z] // TrigExpand`

`Cosh[z]^4 + 6 Cosh[z]^2 Sinh[z]^2 + Sinh[z]^4`

`TrigExpand[{{Sinh[x + y], Sinh[3 z]},
{Cosh[x + y], Cosh[3 z]},
{Tanh[x + y], Tanh[3 z]},
{Coth[x + y], Coth[3 z]},
{Csch[x + y], Csch[3 z]},
{Sech[x + y], Sech[3 z]}}}`

`{ {Cosh[y] Sinh[x] + Cosh[x] Sinh[y], 3 Cosh[z]^2 Sinh[z] + Sinh[z]^3},
{Cosh[x] Cosh[y] + Sinh[x] Sinh[y], Cosh[z]^3 + 3 Cosh[z] Sinh[z]^2},
{ Cosh[y] Sinh[x] / (Cosh[x] Cosh[y] + Sinh[x] Sinh[y]) + Cosh[x] Sinh[y] / (Cosh[x] Cosh[y] + Sinh[x] Sinh[y]),
3 Cosh[z]^2 Sinh[z] / (Cosh[z]^3 + 3 Cosh[z] Sinh[z]^2) + Sinh[z]^3 / (Cosh[z]^3 + 3 Cosh[z] Sinh[z]^2) },
{ Cosh[x] Cosh[y] / (Cosh[y] Sinh[x] + Cosh[x] Sinh[y]) + Sinh[x] Sinh[y] / (Cosh[y] Sinh[x] + Cosh[x] Sinh[y]),
Cosh[z]^3 / (3 Cosh[z]^2 Sinh[z] + Sinh[z]^3) + 3 Cosh[z] Sinh[z]^2 / (3 Cosh[z]^2 Sinh[z] + Sinh[z]^3) },
{ 1 / (Cosh[y] Sinh[x] + Cosh[x] Sinh[y]), 1 / (3 Cosh[z]^2 Sinh[z] + Sinh[z]^3) },
{ 1 / (Cosh[x] Cosh[y] + Sinh[x] Sinh[y]), 1 / (Cosh[z]^3 + 3 Cosh[z] Sinh[z]^2) } }`

`TableForm[(# == TrigExpand[#]) & /@`

`Flatten[{{Sinh[x + y], Sinh[3 z]}, {Cosh[x + y], Cosh[3 z]}, {Tanh[x + y], Tanh[3 z]},
{Coth[x + y], Coth[3 z]}, {Csch[x + y], Csch[3 z]}, {Sech[x + y], Sech[3 z]}}}`

$$\begin{aligned} \text{Sinh}[x + y] &== \text{Cosh}[y] \text{Sinh}[x] + \text{Cosh}[x] \text{Sinh}[y] \\ \text{Sinh}[3 z] &== 3 \text{Cosh}[z]^2 \text{Sinh}[z] + \text{Sinh}[z]^3 \\ \text{Cosh}[x + y] &== \text{Cosh}[x] \text{Cosh}[y] + \text{Sinh}[x] \text{Sinh}[y] \\ \text{Cosh}[3 z] &== \text{Cosh}[z]^3 + 3 \text{Cosh}[z] \text{Sinh}[z]^2 \\ \text{Tanh}[x + y] &== \frac{\text{Cosh}[y] \text{Sinh}[x]}{\text{Cosh}[x] \text{Cosh}[y] + \text{Sinh}[x] \text{Sinh}[y]} + \frac{\text{Cosh}[x] \text{Sinh}[y]}{\text{Cosh}[x] \text{Cosh}[y] + \text{Sinh}[x] \text{Sinh}[y]} \\ \text{Tanh}[3 z] &== \frac{3 \text{Cosh}[z]^2 \text{Sinh}[z]}{\text{Cosh}[z]^3 + 3 \text{Cosh}[z] \text{Sinh}[z]^2} + \frac{\text{Sinh}[z]^3}{\text{Cosh}[z]^3 + 3 \text{Cosh}[z] \text{Sinh}[z]^2} \\ \text{Coth}[x + y] &== \frac{\text{Cosh}[x] \text{Cosh}[y]}{\text{Cosh}[y] \text{Sinh}[x] + \text{Cosh}[x] \text{Sinh}[y]} + \frac{\text{Sinh}[x] \text{Sinh}[y]}{\text{Cosh}[y] \text{Sinh}[x] + \text{Cosh}[x] \text{Sinh}[y]} \\ \text{Coth}[3 z] &== \frac{\text{Cosh}[z]^3}{3 \text{Cosh}[z]^2 \text{Sinh}[z] + \text{Sinh}[z]^3} + \frac{3 \text{Cosh}[z] \text{Sinh}[z]^2}{3 \text{Cosh}[z]^2 \text{Sinh}[z] + \text{Sinh}[z]^3} \\ \text{Csch}[x + y] &== \frac{1}{\text{Cosh}[y] \text{Sinh}[x] + \text{Cosh}[x] \text{Sinh}[y]} \\ \text{Csch}[3 z] &== \frac{1}{3 \text{Cosh}[z]^2 \text{Sinh}[z] + \text{Sinh}[z]^3} \\ \text{Sech}[x + y] &== \frac{1}{\text{Cosh}[x] \text{Cosh}[y] + \text{Sinh}[x] \text{Sinh}[y]} \\ \text{Sech}[3 z] &== \frac{1}{\text{Cosh}[z]^3 + 3 \text{Cosh}[z] \text{Sinh}[z]^2} \end{aligned}$$

TrigFactor

The command `TrigFactor` factors trigonometric and hyperbolic functions. In more detail, it splits up sums and integer multiples that appear in the arguments of trigonometric and hyperbolic functions, and then factors the resulting polynomials in the trigonometric and hyperbolic functions, using the corresponding identities where possible. Here are some examples.

`TrigFactor[Sinh[x] + i Cosh[y]]`

$$\left(i \text{Cosh}\left[\frac{x}{2} - \frac{y}{2}\right] + \text{Sinh}\left[\frac{x}{2} - \frac{y}{2}\right] \right) \left(\text{Cosh}\left[\frac{x}{2} + \frac{y}{2}\right] - i \text{Sinh}\left[\frac{x}{2} + \frac{y}{2}\right] \right)$$

`Tanh[x] - Coth[y] // TrigFactor`

$$-\text{Cosh}[x - y] \text{Csch}[y] \text{Sech}[x]$$

`TrigFactor[{Sinh[x] + Sinh[y],
Cosh[x] + Cosh[y],
Tanh[x] + Tanh[y],
Coth[x] + Coth[y],
Csch[x] + Csch[y],
Sech[x] + Sech[y]}]`

$$\left\{ 2 \text{Cosh}\left[\frac{x}{2} - \frac{y}{2}\right] \text{Sinh}\left[\frac{x}{2} + \frac{y}{2}\right], 2 \text{Cosh}\left[\frac{x}{2} - \frac{y}{2}\right] \text{Cosh}\left[\frac{x}{2} + \frac{y}{2}\right], \text{Sech}[x] \text{Sech}[y] \text{Sinh}[x + y], \right. \\ \left. \text{Csch}[x] \text{Csch}[y] \text{Sinh}[x + y], \frac{1}{2} \text{Cosh}\left[\frac{x}{2} - \frac{y}{2}\right] \text{Csch}\left[\frac{x}{2}\right] \text{Csch}\left[\frac{y}{2}\right] \text{Sech}\left[\frac{x}{2}\right] \text{Sech}\left[\frac{y}{2}\right] \text{Sinh}\left[\frac{x}{2} + \frac{y}{2}\right], \right. \\ \left. \frac{2 \text{Cosh}\left[\frac{x}{2} - \frac{y}{2}\right] \text{Cosh}\left[\frac{x}{2} + \frac{y}{2}\right]}{\left(\text{Cosh}\left[\frac{x}{2}\right] - i \text{Sinh}\left[\frac{x}{2}\right] \right) \left(\text{Cosh}\left[\frac{x}{2}\right] + i \text{Sinh}\left[\frac{x}{2}\right] \right) \left(\text{Cosh}\left[\frac{y}{2}\right] - i \text{Sinh}\left[\frac{y}{2}\right] \right) \left(\text{Cosh}\left[\frac{y}{2}\right] + i \text{Sinh}\left[\frac{y}{2}\right] \right)} \right\}$$

TrigReduce

The command `TrigReduce` rewrites products and powers of trigonometric and hyperbolic functions in terms of those functions with combined arguments. In more detail, it typically yields a linear expression involving trigonometric and hyperbolic functions with more complicated arguments. `TrigReduce` is approximately opposite to `TrigExpand` and `TrigFactor`. Here are some examples.

TrigReduce[Sinh[z]^3]

$$\frac{1}{4} (-3 \operatorname{Sinh}[z] + \operatorname{Sinh}[3 z])$$

Sinh[x] Cosh[y] // TrigReduce

$$\frac{1}{2} (\operatorname{Sinh}[x - y] + \operatorname{Sinh}[x + y])$$

TrigReduce[{Sinh[z]^2, Cosh[z]^2, Tanh[z]^2, Coth[z]^2, Csch[z]^2, Sech[z]^2}]

$$\left\{ \frac{1}{2} (-1 + \operatorname{Cosh}[2 z]), \frac{1}{2} (1 + \operatorname{Cosh}[2 z]), \right. \\ \left. \frac{-1 + \operatorname{Cosh}[2 z]}{1 + \operatorname{Cosh}[2 z]}, \frac{1 + \operatorname{Cosh}[2 z]}{-1 + \operatorname{Cosh}[2 z]}, \frac{2}{-1 + \operatorname{Cosh}[2 z]}, \frac{2}{1 + \operatorname{Cosh}[2 z]} \right\}$$

**TrigReduce[TrigExpand[{{Sinh[x+y], Sinh[3 z], Sinh[x] Sinh[y]},
{Cosh[x+y], Cosh[3 z], Cosh[x] Cosh[y]},
{Tanh[x+y], Tanh[3 z], Tanh[x] Tanh[y]},
{Coth[x+y], Coth[3 z], Coth[x] Coth[y]},
{Csch[x+y], Csch[3 z], Csch[x] Csch[y]},
{Sech[x+y], Sech[3 z], Sech[x] Sech[y]}]}]]**

$$\left\{ \left\{ \operatorname{Sinh}[x + y], \operatorname{Sinh}[3 z], \frac{1}{2} (-\operatorname{Cosh}[x - y] + \operatorname{Cosh}[x + y]) \right\}, \right. \\ \left\{ \operatorname{Cosh}[x + y], \operatorname{Cosh}[3 z], \frac{1}{2} (\operatorname{Cosh}[x - y] + \operatorname{Cosh}[x + y]) \right\}, \\ \left\{ \operatorname{Tanh}[x + y], \operatorname{Tanh}[3 z], \frac{-\operatorname{Cosh}[x - y] + \operatorname{Cosh}[x + y]}{\operatorname{Cosh}[x - y] + \operatorname{Cosh}[x + y]} \right\}, \\ \left\{ \operatorname{Coth}[x + y], \operatorname{Coth}[3 z], \frac{-\operatorname{Cosh}[x - y] - \operatorname{Cosh}[x + y]}{\operatorname{Cosh}[x - y] - \operatorname{Cosh}[x + y]} \right\}, \\ \left\{ \operatorname{Csch}[x + y], \operatorname{Csch}[3 z], -\frac{2}{\operatorname{Cosh}[x - y] - \operatorname{Cosh}[x + y]} \right\}, \\ \left. \left\{ \operatorname{Sech}[x + y], \operatorname{Sech}[3 z], \frac{2}{\operatorname{Cosh}[x - y] + \operatorname{Cosh}[x + y]} \right\} \right\}$$

**TrigReduce[TrigFactor[{Sinh[x]+Sinh[y], Cosh[x]+Cosh[y],
Tanh[x]+Tanh[y], Coth[x]+Coth[y], Csch[x]+Csch[y], Sech[x]+Sech[y]}]]**

$$\left\{ \operatorname{Sinh}[x] + \operatorname{Sinh}[y], \operatorname{Cosh}[x] + \operatorname{Cosh}[y], \frac{2 \operatorname{Sinh}[x + y]}{\operatorname{Cosh}[x - y] + \operatorname{Cosh}[x + y]}, \right. \\ \left. -\frac{2 \operatorname{Sinh}[x + y]}{\operatorname{Cosh}[x - y] - \operatorname{Cosh}[x + y]}, -\frac{2 (\operatorname{Sinh}[x] + \operatorname{Sinh}[y])}{\operatorname{Cosh}[x - y] - \operatorname{Cosh}[x + y]}, \frac{2 (\operatorname{Cosh}[x] + \operatorname{Cosh}[y])}{\operatorname{Cosh}[x - y] + \operatorname{Cosh}[x + y]} \right\}$$

TrigToExp

The command `TrigToExp` converts direct and inverse trigonometric and hyperbolic functions to exponentials or logarithmic functions. It tries, where possible, to give results that do not involve explicit complex numbers. Here are some examples.

`TrigToExp[Sinh[2 z]]`

$$-\frac{1}{2} e^{-2z} + \frac{e^{2z}}{2}$$

`Sinh[z] Tanh[2 z] // TrigToExp`

$$\frac{(-e^{-z} + e^z) (-e^{-2z} + e^{2z})}{2 (e^{-2z} + e^{2z})}$$

`TrigToExp[{Sinh[z], Cosh[z], Tanh[z], Coth[z], Csch[z], Sech[z]}]`

$$\left\{ -\frac{e^{-z}}{2} + \frac{e^z}{2}, \frac{e^{-z}}{2} + \frac{e^z}{2}, \frac{-e^{-z} + e^z}{e^{-z} + e^z}, \frac{e^{-z} + e^z}{-e^{-z} + e^z}, \frac{2}{-e^{-z} + e^z}, \frac{2}{e^{-z} + e^z} \right\}$$

ExpToTrig

The command `ExpToTrig` converts exponentials to trigonometric or hyperbolic functions. It tries, where possible, to give results that do not involve explicit complex numbers. It is approximately opposite to `TrigToExp`. Here are some examples.

`ExpToTrig[e^{x\beta}]`

$$\text{Cosh}[x\beta] + \text{Sinh}[x\beta]$$

$$\frac{e^{x\alpha} - e^{x\beta}}{e^{x\gamma} + e^{x\delta}} // \text{ExpToTrig}$$

$$\frac{\text{Cosh}[x\alpha] - \text{Cosh}[x\beta] + \text{Sinh}[x\alpha] - \text{Sinh}[x\beta]}{\text{Cosh}[x\gamma] + \text{Cosh}[x\delta] + \text{Sinh}[x\gamma] + \text{Sinh}[x\delta]}$$

`ExpToTrig[TrigToExp[{Sinh[z], Cosh[z], Tanh[z], Coth[z], Csch[z], Sech[z]}]]`

$$\{\text{Sinh}[z], \text{Cosh}[z], \text{Tanh}[z], \text{Coth}[z], \text{Csch}[z], \text{Sech}[z]\}$$

`ExpToTrig[{α e^{-xβ} + α e^{xβ}, α e^{-xβ} + γ e^{ixβ}]`

$$\{2\alpha \text{Cosh}[x\beta], \gamma \text{Cos}[x\beta] + \alpha \text{Cosh}[x\beta] + i\gamma \text{Sin}[x\beta] - \alpha \text{Sinh}[x\beta]\}$$

ComplexExpand

The function `ComplexExpand` expands expressions assuming that all the occurring variables are real. The value option `TargetFunctions` is a list of functions from the set `{Re, Im, Abs, Arg, Conjugate, Sign}`. `ComplexExpand` tries to give results in terms of the specified functions. Here are some examples.

`ComplexExpand[Sinh[x + i y] Cosh[x - i y]]`

$$\text{Cos}[y]^2 \text{Cosh}[x] \text{Sinh}[x] + \text{Cosh}[x] \text{Sin}[y]^2 \text{Sinh}[x] + i (\text{Cos}[y] \text{Cosh}[x]^2 \text{Sin}[y] - \text{Cos}[y] \text{Sin}[y] \text{Sinh}[x]^2)$$

Csch[x + i y] Sech[x - i y] // ComplexExpand

$$-\frac{4 \cos [y]^2 \cosh [x] \sinh [x]}{(\cos [2 y]-\cosh [2 x])(\cos [2 y]+\cosh [2 x])}-\frac{4 \cosh [x] \sin [y]^2 \sinh [x]}{(\cos [2 y]-\cosh [2 x])(\cos [2 y]+\cosh [2 x])}+i\left(\frac{4 \cos [y] \cosh [x]^2 \sin [y]}{(\cos [2 y]-\cosh [2 x])(\cos [2 y]+\cosh [2 x])}-\frac{4 \cos [y] \sin [y] \sinh [x]^2}{(\cos [2 y]-\cosh [2 x])(\cos [2 y]+\cosh [2 x])}\right)$$

li1 = {Sinh[x + i y], Cosh[x + i y], Tanh[x + i y], Coth[x + i y], Csch[x + i y], Sech[x + i y]}

{Sinh[x + i y], Cosh[x + i y], Tanh[x + i y], Coth[x + i y], Csch[x + i y], Sech[x + i y]}

ComplexExpand[li1]

$$\left\{i \cosh [x] \sin [y]+\cos [y] \sinh [x], \cos [y] \cosh [x]+i \sin [y] \sinh [x], \frac{i \sin [2 y]}{\cos [2 y]+\cosh [2 x]}+\frac{\sinh [2 x]}{\cos [2 y]+\cosh [2 x]}, \frac{i \sin [2 y]}{\cos [2 y]-\cosh [2 x]}-\frac{\sinh [2 x]}{\cos [2 y]-\cosh [2 x]}, \frac{2 i \cosh [x] \sin [y]}{\cos [2 y]-\cosh [2 x]}-\frac{2 \cos [y] \sinh [x]}{\cos [2 y]-\cosh [2 x]}, \frac{2 \cos [y] \cosh [x]}{\cos [2 y]+\cosh [2 x]}-\frac{2 i \sin [y] \sinh [x]}{\cos [2 y]+\cosh [2 x]}\right\}$$

ComplexExpand[Re[#] & /@ li1, TargetFunctions -> {Re, Im}]

$$\left\{\cos [y] \sinh [x], \cos [y] \cosh [x], \frac{\sinh [2 x]}{\cos [2 y]+\cosh [2 x]}, -\frac{\sinh [2 x]}{\cos [2 y]-\cosh [2 x]}, -\frac{2 \cos [y] \sinh [x]}{\cos [2 y]-\cosh [2 x]}, \frac{2 \cos [y] \cosh [x]}{\cos [2 y]+\cosh [2 x]}\right\}$$

ComplexExpand[Im[#] & /@ li1, TargetFunctions -> {Re, Im}]

$$\left\{\cosh [x] \sin [y], \sin [y] \sinh [x], \frac{\sin [2 y]}{\cos [2 y]+\cosh [2 x]}, \frac{\sin [2 y]}{\cos [2 y]-\cosh [2 x]}, \frac{2 \cosh [x] \sin [y]}{\cos [2 y]-\cosh [2 x]}, -\frac{2 \sin [y] \sinh [x]}{\cos [2 y]+\cosh [2 x]}\right\}$$

ComplexExpand[Abs[#] & /@ li1, TargetFunctions -> {Re, Im}]

$$\left\{ \sqrt{\cosh[x]^2 \sin[y]^2 + \cos[y]^2 \sinh[x]^2}, \sqrt{\cos[y]^2 \cosh[x]^2 + \sin[y]^2 \sinh[x]^2}, \right.$$

$$\sqrt{\frac{\sin[2y]^2}{(\cos[2y] + \cosh[2x])^2} + \frac{\sinh[2x]^2}{(\cos[2y] + \cosh[2x])^2}},$$

$$\sqrt{\frac{\sin[2y]^2}{(\cos[2y] - \cosh[2x])^2} + \frac{\sinh[2x]^2}{(\cos[2y] - \cosh[2x])^2}},$$

$$\sqrt{\frac{4 \cosh[x]^2 \sin[y]^2}{(\cos[2y] - \cosh[2x])^2} + \frac{4 \cos[y]^2 \sinh[x]^2}{(\cos[2y] - \cosh[2x])^2}},$$

$$\left. \sqrt{\frac{4 \cos[y]^2 \cosh[x]^2}{(\cos[2y] + \cosh[2x])^2} + \frac{4 \sin[y]^2 \sinh[x]^2}{(\cos[2y] + \cosh[2x])^2}} \right\}$$

% // Simplify[#, {x, y} ∈ Reals] &

$$\left\{ \frac{\sqrt{-\cos[2y] + \cosh[2x]}}{\sqrt{2}}, \frac{\sqrt{\cos[2y] + \cosh[2x]}}{\sqrt{2}}, \frac{\sqrt{\sin[2y]^2 + \sinh[2x]^2}}{\cos[2y] + \cosh[2x]}, \right.$$

$$\left. \sqrt{-\frac{\cos[2y] + \cosh[2x]}{\cos[2y] - \cosh[2x]}}, \frac{\sqrt{2}}{\sqrt{-\cos[2y] + \cosh[2x]}}, \frac{\sqrt{2}}{\sqrt{\cos[2y] + \cosh[2x]}} \right\}$$

ComplexExpand[Arg[#] & /@ l11, TargetFunctions → {Re, Im}]

$$\left\{ \text{ArcTan}[\cos[y] \sinh[x], \cosh[x] \sin[y]], \text{ArcTan}[\cos[y] \cosh[x], \sin[y] \sinh[x]], \right.$$

$$\text{ArcTan}\left[\frac{\sinh[2x]}{\cos[2y] + \cosh[2x]}, \frac{\sin[2y]}{\cos[2y] + \cosh[2x]}\right],$$

$$\text{ArcTan}\left[-\frac{\sinh[2x]}{\cos[2y] - \cosh[2x]}, \frac{\sin[2y]}{\cos[2y] - \cosh[2x]}\right],$$

$$\text{ArcTan}\left[-\frac{2 \cos[y] \sinh[x]}{\cos[2y] - \cosh[2x]}, \frac{2 \cosh[x] \sin[y]}{\cos[2y] - \cosh[2x]}\right],$$

$$\left. \text{ArcTan}\left[\frac{2 \cos[y] \cosh[x]}{\cos[2y] + \cosh[2x]}, -\frac{2 \sin[y] \sinh[x]}{\cos[2y] + \cosh[2x]}\right] \right\}$$

% // Simplify[#, {x, y} ∈ Reals] &

$$\{\text{ArcTan}[\cos[y] \sinh[x], \cosh[x] \sin[y]], \text{ArcTan}[\cos[y] \cosh[x], \sin[y] \sinh[x]],$$

$$\text{ArcTan}[\sinh[2x], \sin[2y]], \text{ArcTan}[\cosh[x] \sinh[x], -\cos[y] \sin[y]],$$

$$\text{ArcTan}[\cos[y] \sinh[x], -\cosh[x] \sin[y]], \text{ArcTan}[\cos[y] \cosh[x], -\sin[y] \sinh[x]]\}$$

ComplexExpand[Conjugate[#] & /@ l11, TargetFunctions → {Re, Im}] // Simplify

$$\left\{ \begin{aligned} & -i \operatorname{Cosh}[x] \operatorname{Sin}[y] + \operatorname{Cos}[y] \operatorname{Sinh}[x], \operatorname{Cos}[y] \operatorname{Cosh}[x] - i \operatorname{Sin}[y] \operatorname{Sinh}[x], \\ & \frac{-i \operatorname{Sin}[2y] + \operatorname{Sinh}[2x]}{\operatorname{Cos}[2y] + \operatorname{Cosh}[2x]}, \frac{i \operatorname{Sin}[2y] + \operatorname{Sinh}[2x]}{\operatorname{Cos}[2y] - \operatorname{Cosh}[2x]}, \\ & \frac{1}{-i \operatorname{Cosh}[x] \operatorname{Sin}[y] + \operatorname{Cos}[y] \operatorname{Sinh}[x]}, \frac{1}{\operatorname{Cos}[y] \operatorname{Cosh}[x] - i \operatorname{Sin}[y] \operatorname{Sinh}[x]} \end{aligned} \right\}$$

Simplify

The command `Simplify` performs a sequence of algebraic transformations on an expression, and returns the simplest form it finds. Here are two examples.

```
Simplify[Sinh[2 z] / Sinh[z]]
```

```
2 Cosh[z]
```

```
Sinh[2 z] / Cosh[z] // Simplify
```

```
2 Sinh[z]
```

Here is a large collection of hyperbolic identities. All are written as one large logical conjunction.

$$\text{Simplify}[\#] \ \& \ /@ \left(\begin{aligned} & \text{Cosh}[z]^2 - \text{Sinh}[z]^2 == 1 \wedge \\ & \text{Sinh}[z]^2 == \frac{\text{Cosh}[2z] - 1}{2} \wedge \text{Cosh}[z]^2 == \frac{1 + \text{Cosh}[2z]}{2} \wedge \\ & \text{Tanh}[z]^2 == \frac{\text{Cosh}[2z] - 1}{\text{Cosh}[2z] + 1} \wedge \text{Coth}[z]^2 == \frac{\text{Cosh}[2z] + 1}{\text{Cosh}[2z] - 1} \wedge \\ & \text{Sinh}[2z] == 2 \text{Sinh}[z] \text{Cosh}[z] \wedge \text{Cosh}[2z] == \text{Cosh}[z]^2 + \text{Sinh}[z]^2 == 2 \text{Cosh}[z]^2 - 1 \wedge \\ & \text{Sinh}[a + b] == \text{Sinh}[a] \text{Cosh}[b] + \text{Cosh}[a] \text{Sinh}[b] \wedge \\ & \quad \text{Sinh}[a - b] == \text{Sinh}[a] \text{Cosh}[b] - \text{Cosh}[a] \text{Sinh}[b] \wedge \\ & \text{Cosh}[a + b] == \text{Cosh}[a] \text{Cosh}[b] + \text{Sinh}[a] \text{Sinh}[b] \wedge \\ & \quad \text{Cosh}[a - b] == \text{Cosh}[a] \text{Cosh}[b] - \text{Sinh}[a] \text{Sinh}[b] \wedge \\ & \text{Sinh}[a] + \text{Sinh}[b] == 2 \text{Sinh}\left[\frac{a+b}{2}\right] \text{Cosh}\left[\frac{a-b}{2}\right] \wedge \\ & \quad \text{Sinh}[a] - \text{Sinh}[b] == 2 \text{Cosh}\left[\frac{a+b}{2}\right] \text{Sinh}\left[\frac{a-b}{2}\right] \wedge \\ & \text{Cosh}[a] + \text{Cosh}[b] == 2 \text{Cosh}\left[\frac{a+b}{2}\right] \text{Cosh}\left[\frac{a-b}{2}\right] \wedge \\ & \quad \text{Cosh}[a] - \text{Cosh}[b] == -2 \text{Sinh}\left[\frac{a+b}{2}\right] \text{Sinh}\left[\frac{b-a}{2}\right] \wedge \\ & \text{Tanh}[a] + \text{Tanh}[b] == \frac{\text{Sinh}[a+b]}{\text{Cosh}[a] \text{Cosh}[b]} \wedge \text{Tanh}[a] - \text{Tanh}[b] == \frac{\text{Sinh}[a-b]}{\text{Cosh}[a] \text{Cosh}[b]} \wedge \\ & A \text{Sinh}[z] + B \text{Cosh}[z] == A \sqrt{1 - \frac{B^2}{A^2}} \text{Sinh}\left[z + \text{ArcTanh}\left[\frac{B}{A}\right]\right] \wedge \\ & \text{Sinh}[a] \text{Sinh}[b] == \frac{\text{Cosh}[a+b] - \text{Cosh}[a-b]}{2} \wedge \text{Cosh}[a] \text{Cosh}[b] == \\ & \quad \frac{\text{Cosh}[a-b] + \text{Cosh}[a+b]}{2} \wedge \text{Sinh}[a] \text{Cosh}[b] == \frac{\text{Sinh}[a+b] + \text{Sinh}[a-b]}{2} \wedge \\ & \text{Sinh}\left[\frac{z}{2}\right]^2 == \frac{\text{Cosh}[z] - 1}{2} \wedge \text{Cosh}\left[\frac{z}{2}\right]^2 == \frac{1 + \text{Cosh}[z]}{2} \wedge \\ & \text{Tanh}\left[\frac{z}{2}\right] == \frac{\text{Cosh}[z] - 1}{\text{Sinh}[z]} == \frac{\text{Sinh}[z]}{1 + \text{Cosh}[z]} \wedge \text{Coth}\left[\frac{z}{2}\right] == \frac{\text{Sinh}[z]}{\text{Cosh}[z] - 1} == \frac{1 + \text{Cosh}[z]}{\text{Sinh}[z]} \end{aligned} \right)$$

True

The command `Simplify` has the `Assumption` option. For example, *Mathematica* knows that $\sinh(x) > 0$ for all real positive x , and uses the periodicity of hyperbolic functions for the symbolic integer coefficient k of $k\pi i$.

`Simplify[Abs[Sinh[x]] > 0, x > 0]`

True

`Abs[Sinh[x]] > 0 // Simplify[#, x > 0] &`

True

```
Simplify[{Sinh[z + 2 k π i], Cosh[z + 2 k π i], Tanh[z + k π i],
  Coth[z + k π i], Csch[z + 2 k π i], Sech[z + 2 k π i]}, k ∈ Integers]
```

```
{Sinh[z], Cosh[z], Tanh[z], Coth[z], Csch[z], Sech[z]}
```

```
Simplify[{Sinh[z + k π i] / Sinh[z], Cosh[z + k π i] / Cosh[z], Tanh[z + k π i] / Tanh[z],
  Coth[z + k π i] / Coth[z], Csch[z + k π i] / Csch[z], Sech[z + k π i] / Sech[z]}, k ∈ Integers]
```

```
{(-1)k, (-1)k, 1, 1, (-1)k, (-1)k}
```

Mathematica also knows that the composition of inverse and direct hyperbolic functions produces the value of the inner argument under the appropriate restriction. Here are some examples.

```
Simplify[{ArcSinh[Sinh[z]], ArcTanh[Tanh[z]],
  ArcCoth[Coth[z]], ArcCsch[Csch[z]]}, -π / 2 < Im[z] < π / 2]
```

```
{z, z, z, z}
```

```
Simplify[{ArcCosh[Cosh[z]], ArcSech[Sech[z]]}, -π < Im[z] < π ∧ Re[z] > 0]
```

```
{z, z}
```

FunctionExpand (and Together)

While the hyperbolic functions auto-evaluate for simple fractions of πi , for more complicated cases they stay as hyperbolic functions to avoid the build up of large expressions. Using the function `FunctionExpand`, such expressions can be transformed into explicit radicals.

$$\text{Cosh}\left[\frac{\pi i}{32}\right]$$

$$\text{Cos}\left[\frac{\pi}{32}\right]$$

$$\text{FunctionExpand}\left[\text{Cosh}\left[\frac{\pi i}{32}\right]\right]$$

$$\frac{1}{2} \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2}}}}$$

$$\text{Coth}\left[\frac{\pi i}{24}\right] // \text{FunctionExpand}$$

$$\frac{i \left(\frac{\sqrt{2-\sqrt{2}}}{4} + \frac{1}{4} \sqrt{3(2+\sqrt{2})} \right)}{-\frac{1}{4} \sqrt{3(2-\sqrt{2})} + \frac{\sqrt{2+\sqrt{2}}}{4}}$$

$$\left\{ \operatorname{Sinh}\left[\frac{\pi i}{16}\right], \operatorname{Cosh}\left[\frac{\pi i}{16}\right], \operatorname{Tanh}\left[\frac{\pi i}{16}\right], \operatorname{Coth}\left[\frac{\pi i}{16}\right], \operatorname{Csch}\left[\frac{\pi i}{16}\right], \operatorname{Sech}\left[\frac{\pi i}{16}\right] \right\}$$

$$\left\{ i \operatorname{Sin}\left[\frac{\pi}{16}\right], \operatorname{Cos}\left[\frac{\pi}{16}\right], i \operatorname{Tan}\left[\frac{\pi}{16}\right], -i \operatorname{Cot}\left[\frac{\pi}{16}\right], -i \operatorname{Csc}\left[\frac{\pi}{16}\right], \operatorname{Sec}\left[\frac{\pi}{16}\right] \right\}$$

FunctionExpand[%]

$$\left\{ \frac{1}{2} i \sqrt{2 - \sqrt{2 + \sqrt{2}}}, \frac{1}{2} \sqrt{2 + \sqrt{2 + \sqrt{2}}}, i \sqrt{\frac{2 - \sqrt{2 + \sqrt{2}}}{2 + \sqrt{2 + \sqrt{2}}}}, \right.$$

$$\left. -i \sqrt{\frac{2 + \sqrt{2 + \sqrt{2}}}{2 - \sqrt{2 + \sqrt{2}}}}, -\frac{2i}{\sqrt{2 - \sqrt{2 + \sqrt{2}}}}, \frac{2}{\sqrt{2 + \sqrt{2 + \sqrt{2}}}} \right\}$$

$$\left\{ \operatorname{Sinh}\left[\frac{\pi i}{60}\right], \operatorname{Cosh}\left[\frac{\pi i}{60}\right], \operatorname{Tanh}\left[\frac{\pi i}{60}\right], \operatorname{Coth}\left[\frac{\pi i}{60}\right], \operatorname{Csch}\left[\frac{\pi i}{60}\right], \operatorname{Sech}\left[\frac{\pi i}{60}\right] \right\}$$

$$\left\{ i \operatorname{Sin}\left[\frac{\pi}{60}\right], \operatorname{Cos}\left[\frac{\pi}{60}\right], i \operatorname{Tan}\left[\frac{\pi}{60}\right], -i \operatorname{Cot}\left[\frac{\pi}{60}\right], -i \operatorname{Csc}\left[\frac{\pi}{60}\right], \operatorname{Sec}\left[\frac{\pi}{60}\right] \right\}$$

Together[FunctionExpand[%]]

$$\left\{ \frac{1}{16} i \left(-\sqrt{2} - \sqrt{6} + \sqrt{10} + \sqrt{30} + 2\sqrt{5 + \sqrt{5}} - 2\sqrt{3(5 + \sqrt{5})} \right), \right.$$

$$\frac{1}{16} \left(\sqrt{2} - \sqrt{6} - \sqrt{10} + \sqrt{30} + 2\sqrt{5 + \sqrt{5}} + 2\sqrt{3(5 + \sqrt{5})} \right),$$

$$- \frac{i \left(1 + \sqrt{3} - \sqrt{5} - \sqrt{15} - \sqrt{2(5 + \sqrt{5})} + \sqrt{6(5 + \sqrt{5})} \right)}{1 - \sqrt{3} - \sqrt{5} + \sqrt{15} + \sqrt{2(5 + \sqrt{5})} + \sqrt{6(5 + \sqrt{5})}},$$

$$\frac{i \left(1 - \sqrt{3} - \sqrt{5} + \sqrt{15} + \sqrt{2(5 + \sqrt{5})} + \sqrt{6(5 + \sqrt{5})} \right)}{1 + \sqrt{3} - \sqrt{5} - \sqrt{15} - \sqrt{2(5 + \sqrt{5})} + \sqrt{6(5 + \sqrt{5})}},$$

$$- \frac{16i}{-\sqrt{2} - \sqrt{6} + \sqrt{10} + \sqrt{30} + 2\sqrt{5 + \sqrt{5}} - 2\sqrt{3(5 + \sqrt{5})}},$$

$$\left. \frac{16}{\sqrt{2} - \sqrt{6} - \sqrt{10} + \sqrt{30} + 2\sqrt{5 + \sqrt{5}} + 2\sqrt{3(5 + \sqrt{5})}} \right\}$$

If the denominator contains squares of integers other than 2, the results always contain complex numbers (meaning that the imaginary number $i = \sqrt{-1}$ appears unavoidably).

$$\left\{ \text{Sinh}\left[\frac{\pi i}{9}\right], \text{Cosh}\left[\frac{\pi i}{9}\right], \text{Tanh}\left[\frac{\pi i}{9}\right], \text{Coth}\left[\frac{\pi i}{9}\right], \text{Csch}\left[\frac{\pi i}{9}\right], \text{Sech}\left[\frac{\pi i}{9}\right] \right\}$$

$$\left\{ i \text{Sin}\left[\frac{\pi}{9}\right], \text{Cos}\left[\frac{\pi}{9}\right], i \text{Tan}\left[\frac{\pi}{9}\right], -i \text{Cot}\left[\frac{\pi}{9}\right], -i \text{Csc}\left[\frac{\pi}{9}\right], \text{Sec}\left[\frac{\pi}{9}\right] \right\}$$

FunctionExpand[%] // Together

$$\left\{ \frac{1}{8} \left(2^{2/3} (-1 - i\sqrt{3})^{1/3} + i 2^{2/3} \sqrt{3} (-1 - i\sqrt{3})^{1/3} - 2^{2/3} (-1 + i\sqrt{3})^{1/3} + i 2^{2/3} \sqrt{3} (-1 + i\sqrt{3})^{1/3} \right), \right.$$

$$\frac{1}{8} \left(2^{2/3} (-1 - i\sqrt{3})^{1/3} + i 2^{2/3} \sqrt{3} (-1 - i\sqrt{3})^{1/3} + 2^{2/3} (-1 + i\sqrt{3})^{1/3} - i 2^{2/3} \sqrt{3} (-1 + i\sqrt{3})^{1/3} \right),$$

$$\frac{-i (-1 - i\sqrt{3})^{1/3} + \sqrt{3} (-1 - i\sqrt{3})^{1/3} + i (-1 + i\sqrt{3})^{1/3} + \sqrt{3} (-1 + i\sqrt{3})^{1/3}}{-i (-1 - i\sqrt{3})^{1/3} + \sqrt{3} (-1 - i\sqrt{3})^{1/3} - i (-1 + i\sqrt{3})^{1/3} - \sqrt{3} (-1 + i\sqrt{3})^{1/3}},$$

$$\frac{-i (-1 - i\sqrt{3})^{1/3} + \sqrt{3} (-1 - i\sqrt{3})^{1/3} - i (-1 + i\sqrt{3})^{1/3} - \sqrt{3} (-1 + i\sqrt{3})^{1/3}}{-i (-1 - i\sqrt{3})^{1/3} + \sqrt{3} (-1 - i\sqrt{3})^{1/3} + i (-1 + i\sqrt{3})^{1/3} + \sqrt{3} (-1 + i\sqrt{3})^{1/3}},$$

$$-(8i) / \left(-i 2^{2/3} (-1 - i\sqrt{3})^{1/3} + 2^{2/3} \sqrt{3} (-1 - i\sqrt{3})^{1/3} + i 2^{2/3} (-1 + i\sqrt{3})^{1/3} + 2^{2/3} \sqrt{3} (-1 + i\sqrt{3})^{1/3} \right),$$

$$-(8i) / \left(-i 2^{2/3} (-1 - i\sqrt{3})^{1/3} + 2^{2/3} \sqrt{3} (-1 - i\sqrt{3})^{1/3} - i 2^{2/3} (-1 + i\sqrt{3})^{1/3} - 2^{2/3} \sqrt{3} (-1 + i\sqrt{3})^{1/3} \right) \left. \right\}$$

Here the function `RootReduce` is used to express the previous algebraic numbers as numbered roots of polynomial equations.

RootReduce[Simplify[%]]

$$\left\{ \text{Root}\left[3 + 36 \#1^2 + 96 \#1^4 + 64 \#1^6 \ \&, 4\right], \text{Root}\left[-1 - 6 \#1 + 8 \#1^3 \ \&, 3\right], \right.$$

$$\text{Root}\left[3 + 27 \#1^2 + 33 \#1^4 + \#1^6 \ \&, 4\right], \text{Root}\left[1 + 33 \#1^2 + 27 \#1^4 + 3 \#1^6 \ \&, 3\right],$$

$$\text{Root}\left[64 + 96 \#1^2 + 36 \#1^4 + 3 \#1^6 \ \&, 5\right], \text{Root}\left[-8 + 6 \#1^2 + \#1^3 \ \&, 3\right] \left. \right\}$$

The function `FunctionExpand` also reduces hyperbolic expressions with compound arguments or compositions, including hyperbolic functions, to simpler forms. Here are some examples.

FunctionExpand[Coth[$\sqrt{-z^2}$]]

$$\frac{\sqrt{-z} \operatorname{Cot}[z]}{\sqrt{z}}$$

Tanh $\left[\sqrt{i z^2}\right]$ // **FunctionExpand**

$$\frac{(-1)^{3/4} \sqrt{-(-1)^{3/4} z} \sqrt{(-1)^{3/4} z} \operatorname{Tanh}\left[(-1)^{1/4} z\right]}{z}$$

{Sinh $\left[\sqrt{z^2}\right]$, **Cosh** $\left[\sqrt{z^2}\right]$, **Tanh** $\left[\sqrt{z^2}\right]$,
Coth $\left[\sqrt{z^2}\right]$, **Csch** $\left[\sqrt{z^2}\right]$, **Sech** $\left[\sqrt{z^2}\right]$ } // **FunctionExpand**

$$\left\{ \frac{\sqrt{-i z} \sqrt{i z} \operatorname{Sinh}[z]}{z}, \operatorname{Cosh}[z], \frac{\sqrt{-i z} \sqrt{i z} \operatorname{Tanh}[z]}{z}, \right. \\ \left. \frac{\sqrt{-i z} \sqrt{i z} \operatorname{Coth}[z]}{z}, \frac{\sqrt{-i z} \sqrt{i z} \operatorname{Csch}[z]}{z}, \operatorname{Sech}[z] \right\}$$

Applying **Simplify** to the last expression gives a more compact result.

Simplify[%]

$$\left\{ \frac{\sqrt{z^2} \operatorname{Sinh}[z]}{z}, \operatorname{Cosh}[z], \frac{\sqrt{z^2} \operatorname{Tanh}[z]}{z}, \frac{\sqrt{z^2} \operatorname{Coth}[z]}{z}, \frac{\sqrt{z^2} \operatorname{Csch}[z]}{z}, \operatorname{Sech}[z] \right\}$$

Here are some similar examples.

Sinh $[2 \operatorname{ArcTanh}[z]]$ // **FunctionExpand**

$$\frac{2 z}{1 - z^2}$$

Cosh $\left[\frac{\operatorname{ArcCoth}[z]}{2}\right]$ // **FunctionExpand**

$$\frac{\sqrt{1 + \frac{\sqrt{-i z} \sqrt{i z}}{\sqrt{(-1+z)(1+z)}}}}{\sqrt{2}}$$

{Sinh $[2 \operatorname{ArcSinh}[z]]$, **Cosh** $[2 \operatorname{ArcCosh}[z]]$, **Tanh** $[2 \operatorname{ArcTanh}[z]]$,
Coth $[2 \operatorname{ArcCoth}[z]]$, **Csch** $[2 \operatorname{ArcCsch}[z]]$, **Sech** $[2 \operatorname{ArcSech}[z]]$ } // **FunctionExpand**

$$\left\{ 2 z \sqrt{i(-i+z)} \sqrt{-i(i+z)}, z^2 + (-1+z)(1+z), -\frac{2(-1+z)z(1+z)}{(1-z^2)(1+z^2)}, \right. \\ \left. \frac{1}{2} \left(1 - \frac{1}{z^2}\right) z \left(\frac{1}{(-1+z)(1+z)} + \frac{z^2}{(-1+z)(1+z)}\right), \frac{\sqrt{-z} z^{3/2}}{2 \sqrt{-1-z^2}}, \frac{z^2}{2-z^2} \right\}$$

$$\left\{ \text{Sinh}\left[\frac{\text{ArcSinh}[z]}{2}\right], \text{Cosh}\left[\frac{\text{ArcCosh}[z]}{2}\right], \text{Tanh}\left[\frac{\text{ArcTanh}[z]}{2}\right], \right. \\ \left. \text{Coth}\left[\frac{\text{ArcCoth}[z]}{2}\right], \text{Csch}\left[\frac{\text{ArcCsch}[z]}{2}\right], \text{Sech}\left[\frac{\text{ArcSech}[z]}{2}\right] \right\} // \text{FunctionExpand}$$

$$\left\{ \frac{z \sqrt{-1 + \sqrt{i(-i+z)} \sqrt{-i(i+z)}}}{\sqrt{2} \sqrt{-i z} \sqrt{i z}}, \frac{\sqrt{1+z}}{\sqrt{2}}, \frac{z}{1 + \sqrt{1-z} \sqrt{1+z}}, \right. \\ \left. z \left(1 + \frac{\sqrt{(-1+z)(1+z)}}{\sqrt{-i z} \sqrt{i z}} \right), \frac{\sqrt{2} \sqrt{-\frac{i}{z}} \sqrt{\frac{i}{z}} z}{\sqrt{-1 + \frac{\sqrt{-1-z^2}}{\sqrt{-z} \sqrt{z}}}}, \frac{\sqrt{2} \sqrt{-z}}{\sqrt{-1-z}} \right\}$$

Simplify[%]

$$\left\{ \frac{z \sqrt{-1 + \sqrt{1+z^2}}}{\sqrt{2} \sqrt{z^2}}, \frac{\sqrt{1+z}}{\sqrt{2}}, \frac{z}{1 + \sqrt{1-z^2}}, z + \frac{\sqrt{z^2} \sqrt{-1+z^2}}{z}, \frac{\sqrt{2} \sqrt{\frac{1}{z^2}} z}{\sqrt{-1 + \sqrt{1 + \frac{1}{z^2}}}}, \frac{\sqrt{2}}{\sqrt{1 + \frac{1}{z}}}} \right\}$$

FullSimplify

The function `FullSimplify` tries a wider range of transformations than the function `Simplify` and returns the simplest form it finds. Here are some examples that contrast the results of applying these functions to the same expressions.

$$\text{Cosh}\left[\frac{1}{2} \text{Log}[1 - i z] - \frac{1}{2} \text{Log}[1 + i z]\right] // \text{Simplify}$$

$$\text{Cosh}\left[\frac{1}{2} (\text{Log}[1 - i z] - \text{Log}[1 + i z])\right]$$

% // FullSimplify

$$\frac{1}{\sqrt{1+z^2}}$$

$$\left\{ \text{Sinh}\left[-\text{Log}\left[i z + \sqrt{1-z^2}\right]\right], \text{Cosh}\left[-\text{Log}\left[i z + \sqrt{1-z^2}\right]\right], \right. \\ \text{Tanh}\left[-\text{Log}\left[i z + \sqrt{1-z^2}\right]\right], \text{Coth}\left[-\text{Log}\left[i z + \sqrt{1-z^2}\right]\right], \\ \left. \text{Csch}\left[-\text{Log}\left[i z + \sqrt{1-z^2}\right]\right], \text{Sech}\left[-\text{Log}\left[i z + \sqrt{1-z^2}\right]\right] \right\} // \text{Simplify}$$

$$\left\{ -i z, \frac{1 - z^2 + i z \sqrt{1 - z^2}}{i z + \sqrt{1 - z^2}}, -\frac{-1 + \left(i z + \sqrt{1 - z^2} \right)^2}{1 + \left(i z + \sqrt{1 - z^2} \right)^2}, \right.$$

$$\left. -\frac{1 + \left(i z + \sqrt{1 - z^2} \right)^2}{-1 + \left(i z + \sqrt{1 - z^2} \right)^2}, \frac{i}{z}, \frac{2 \left(i z + \sqrt{1 - z^2} \right)}{1 + \left(i z + \sqrt{1 - z^2} \right)^2} \right\}$$

$$\left\{ \text{Sinh} \left[-\text{Log} \left[i z + \sqrt{1 - z^2} \right] \right], \text{Cosh} \left[-\text{Log} \left[i z + \sqrt{1 - z^2} \right] \right], \right.$$

$$\text{Tanh} \left[-\text{Log} \left[i z + \sqrt{1 - z^2} \right] \right], \text{Coth} \left[-\text{Log} \left[i z + \sqrt{1 - z^2} \right] \right],$$

$$\left. \text{Csch} \left[-\text{Log} \left[i z + \sqrt{1 - z^2} \right] \right], \text{Sech} \left[-\text{Log} \left[i z + \sqrt{1 - z^2} \right] \right] \right\} // \text{FullSimplify}$$

$$\left\{ -i z, \sqrt{1 - z^2}, -\frac{i z}{\sqrt{1 - z^2}}, \frac{i \sqrt{1 - z^2}}{z}, \frac{i}{z}, \frac{1}{\sqrt{1 - z^2}} \right\}$$

Operations performed by specialized *Mathematica* functions

Series expansions

Calculating the series expansion of hyperbolic functions to hundreds of terms can be done in seconds. Here are some examples.

`Series[Sinh[z], {z, 0, 5}]`

$$z + \frac{z^3}{6} + \frac{z^5}{120} + O[z]^6$$

`Normal[%]`

$$z + \frac{z^3}{6} + \frac{z^5}{120}$$

`Series[{Sinh[z], Cosh[z], Tanh[z], Coth[z], Csch[z], Sech[z]}, {z, 0, 3}]`

$$\left\{ z + \frac{z^3}{6} + O[z]^4, 1 + \frac{z^2}{2} + O[z]^4, z - \frac{z^3}{3} + O[z]^4, \right.$$

$$\left. \frac{1}{z} + \frac{z}{3} - \frac{z^3}{45} + O[z]^4, \frac{1}{z} - \frac{z}{6} + \frac{7 z^3}{360} + O[z]^4, 1 - \frac{z^2}{2} + O[z]^4 \right\}$$

`Series[Coth[z], {z, 0, 100}] // Timing`

$$\left\{ 0.79 \text{ Second}, \frac{1}{z} + \frac{z}{3} - \frac{z^3}{45} + \frac{2 z^5}{945} - \frac{z^7}{4725} + \frac{2 z^9}{93555} - \frac{1382 z^{11}}{638512875} + \right.$$

$$\left. \frac{4 z^{13}}{18243225} - \frac{3617 z^{15}}{162820783125} + \frac{87734 z^{17}}{38979295480125} - \frac{349222 z^{19}}{1531329465290625} + \right.$$

$$\begin{aligned}
& \frac{310\,732\,z^{21}}{13\,447\,856\,940\,643\,125} - \frac{472\,728\,182\,z^{23}}{201\,919\,571\,963\,756\,521\,875} + \frac{2\,631\,724\,z^{25}}{11\,094\,481\,976\,030\,578\,125} - \\
& \frac{13\,571\,120\,588\,z^{27}}{564\,653\,660\,170\,076\,273\,671\,875} + \frac{13\,785\,346\,041\,608\,z^{29}}{5\,660\,878\,804\,669\,082\,674\,070\,015\,625} - \\
& \frac{7\,709\,321\,041\,217\,z^{31}}{31\,245\,110\,285\,511\,170\,603\,633\,203\,125} + \frac{303\,257\,395\,102\,z^{33}}{12\,130\,454\,581\,433\,748\,587\,292\,890\,625} - \\
& \frac{52\,630\,543\,106\,106\,954\,746\,z^{35}}{20\,777\,977\,561\,866\,588\,586\,487\,628\,662\,044\,921\,875} + \frac{616\,840\,823\,966\,644\,z^{37}}{2\,403\,467\,618\,492\,375\,776\,343\,276\,883\,984\,375} - \\
& \frac{522\,165\,436\,992\,898\,244\,102\,z^{39}}{20\,080\,431\,172\,289\,638\,826\,798\,401\,128\,390\,556\,640\,625} + \\
& \frac{6\,080\,390\,575\,672\,283\,210\,764\,z^{41}}{2\,307\,789\,189\,818\,960\,127\,712\,594\,427\,864\,667\,427\,734\,375} - \\
& \frac{10\,121\,188\,937\,927\,645\,176\,372\,z^{43}}{37\,913\,679\,547\,025\,773\,526\,706\,908\,457\,776\,679\,169\,921\,875} + \\
& \frac{207\,461\,256\,206\,578\,143\,748\,856\,z^{45}}{7\,670\,102\,214\,448\,301\,053\,033\,358\,480\,610\,212\,529\,462\,890\,625} - \\
& \frac{11\,218\,806\,737\,995\,635\,372\,498\,255\,094\,z^{47}}{4\,093\,648\,603\,384\,274\,996\,519\,698\,921\,478\,879\,580\,162\,286\,669\,921\,875} + \\
& \frac{79\,209\,152\,838\,572\,743\,713\,996\,404\,z^{49}}{285\,258\,771\,457\,546\,764\,463\,363\,635\,252\,374\,414\,183\,254\,365\,234\,375} - \\
& \frac{246\,512\,528\,657\,073\,833\,030\,130\,766\,724\,z^{51}}{8\,761\,982\,491\,474\,419\,367\,550\,817\,114\,626\,909\,562\,924\,278\,968\,505\,859\,375} + \\
& \frac{233\,199\,709\,079\,078\,899\,371\,344\,990\,501\,528\,z^{53}}{81\,807\,125\,729\,900\,063\,867\,074\,959\,072\,425\,603\,825\,198\,823\,017\,351\,806\,640\,625} - \\
& \frac{1\,416\,795\,959\,607\,558\,144\,963\,094\,708\,378\,988\,z^{55}}{4\,905\,352\,087\,939\,496\,310\,826\,487\,207\,538\,302\,184\,255\,342\,959\,123\,162\,841\,796\,875} + \\
& \frac{23\,305\,824\,372\,104\,839\,134\,357\,731\,308\,699\,592\,z^{57}}{796\,392\,368\,980\,577\,121\,745\,974\,726\,570\,063\,253\,238\,310\,542\,073\,919\,837\,646\,484\,375} - \\
& \frac{9\,721\,865\,123\,870\,044\,576\,322\,439\,952\,638\,561\,968\,331\,928\,z^{59}}{3\,278\,777\,586\,273\,629\,598\,615\,520\,165\,380\,455\,583\,231\,003\,564\,645\,636\,125\,000\,418\,914\,794\,921\,875} + \\
& \frac{6\,348\,689\,256\,302\,894\,731\,330\,601\,216\,724\,328\,336\,z^{61}}{21\,132\,271\,510\,899\,613\,925\,529\,439\,369\,536\,628\,424\,678\,570\,233\,931\,462\,891\,949\,462\,890\,625} - \\
& \frac{106\,783\,830\,147\,866\,529\,886\,385\,444\,979\,142\,647\,942\,017\,z^{63}}{3\,508\,062\,732\,166\,890\,409\,707\,514\,582\,539\,928\,001\,638\,766\,051\,683\,792\,497\,378\,070\,587\,158\,203\,125} + \\
& \frac{(267\,745\,458\,568\,424\,664\,373\,021\,714\,282\,169\,516\,771\,254\,382\,z^{65})}{86\,812\,790\,293\,146\,213\,360\,651\,966\,604\,262\,937\,105\,495\,141\,563\,588\,806\,888\,204\,273\,501\,373\,291\,015} : \\
& \frac{625 - (250\,471\,004\,320\,250\,327\,955\,196\,022\,920\,428\,000\,776\,938\,z^{67})}{801\,528\,196\,428\,242\,695\,121\,010\,267\,455\,843\,804\,062\,822\,357\,897\,831\,858\,125\,102\,407\,684\,326\,171\,875} : \\
& + \frac{(172\,043\,582\,552\,384\,800\,434\,637\,321\,986\,040\,823\,829\,878\,646\,884\,z^{69})}{5\,433\,748\,964\,547\,053\,581\,149\,916\,185\,708\,338\,218\,048\,392\,402\,830\,337\,634\,114\,958\,370\,880\,742\,156} : \\
& \frac{982\,421\,875 - (11\,655\,909\,923\,339\,888\,220\,876\,554\,489\,282\,134\,730\,564\,976\,603\,688\,520\,858\,z^{71})}{3\,633\,348\,205\,269\,879\,230\,856\,840\,004\,304\,821\,536\,968\,049\,780\,112\,803\,650\,817\,771\,432\,558\,560\,793} :
\end{aligned}$$

$$\begin{aligned}
& 458\ 452\ 606\ 201\ 171\ 875 + \\
& (3\ 692\ 153\ 220\ 456\ 342\ 488\ 035\ 683\ 646\ 645\ 690\ 290\ 452\ 790\ 030\ 604\ z^{73}) / \\
& 11\ 359\ 005\ 221\ 796\ 317\ 918\ 049\ 302\ 062\ 760\ 294\ 302\ 183\ 889\ 391\ 189\ 419\ 445\ 133\ 951\ 612\ 582\ 060\ 536 \ ; \\
& 346\ 435\ 546\ 875 - (5\ 190\ 545\ 015\ 986\ 394\ 254\ 249\ 936\ 008\ 544\ 252\ 611\ 445\ 319\ 542\ 919\ 116\ z^{75}) / \\
& 157\ 606\ 197\ 452\ 423\ 911\ 112\ 934\ 066\ 120\ 799\ 083\ 442\ 801\ 465\ 302\ 753\ 194\ 801\ 233\ 578\ 624\ 576\ 089 \ ; \\
& 941\ 806\ 793\ 212\ 890\ 625 + \\
& (255\ 290\ 071\ 123\ 323\ 586\ 643\ 187\ 098\ 799\ 718\ 199\ 072\ 122\ 692\ 536\ 861\ 835\ 992\ z^{77}) / \\
& 76\ 505\ 736\ 228\ 426\ 953\ 173\ 738\ 238\ 352\ 183\ 101\ 801\ 688\ 392\ 812\ 244\ 485\ 181\ 277\ 127\ 930\ 109\ 049\ 138 \ ; \\
& 257\ 655\ 704\ 498\ 291\ 015\ 625 - \\
& (9\ 207\ 568\ 598\ 958\ 915\ 293\ 871\ 149\ 938\ 038\ 093\ 699\ 588\ 515\ 745\ 502\ 577\ 839\ 313\ 734\ z^{79}) / \\
& 27\ 233\ 582\ 984\ 369\ 795\ 892\ 070\ 228\ 410\ 001\ 578\ 355\ 986\ 013\ 571\ 390\ 071\ 723\ 225\ 259\ 349\ 721\ 067\ 988 \ ; \\
& 068\ 852\ 863\ 296\ 604\ 156\ 494\ 140\ 625 + \\
& (163\ 611\ 136\ 505\ 867\ 886\ 519\ 332\ 147\ 296\ 221\ 453\ 678\ 803\ 514\ 884\ 902\ 772\ 183\ 572\ z^{81}) / \\
& 4\ 776\ 089\ 171\ 877\ 348\ 057\ 451\ 105\ 924\ 101\ 750\ 653\ 118\ 402\ 745\ 283\ 825\ 543\ 113\ 171\ 217\ 116\ 857\ 704 \ ; \\
& 024\ 700\ 607\ 798\ 175\ 811\ 767\ 578\ 125 - \\
& (8\ 098\ 304\ 783\ 741\ 161\ 440\ 924\ 524\ 640\ 446\ 924\ 039\ 959\ 669\ 564\ 792\ 363\ 509\ 124\ 335\ 729\ 908\ z^{83}) / \\
& 2\ 333\ 207\ 846\ 470\ 426\ 678\ 843\ 707\ 227\ 616\ 712\ 214\ 909\ 162\ 634\ 745\ 895\ 349\ 325\ 948\ 586\ 531\ 533\ 393 \ ; \\
& 530\ 725\ 143\ 500\ 144\ 033\ 328\ 342\ 437\ 744\ 140\ 625 + \\
& (122\ 923\ 650\ 124\ 219\ 284\ 385\ 832\ 157\ 660\ 699\ 813\ 260\ 991\ 755\ 656\ 444\ 452\ 420\ 836\ 648\ z^{85}) / \\
& 349\ 538\ 086\ 043\ 843\ 717\ 584\ 559\ 187\ 055\ 386\ 621\ 548\ 470\ 304\ 913\ 596\ 772\ 372\ 737\ 435\ 524\ 697\ 231 \ ; \\
& 069\ 047\ 713\ 981\ 709\ 496\ 784\ 210\ 205\ 078\ 125 - \\
& (476\ 882\ 359\ 517\ 824\ 548\ 362\ 004\ 154\ 188\ 840\ 670\ 307\ 545\ 554\ 753\ 464\ 961\ 562\ 516\ 323\ 845\ 108\ z^{87}) / \\
& 13\ 383\ 510\ 964\ 174\ 348\ 021\ 497\ 060\ 628\ 653\ 950\ 829\ 663\ 288\ 548\ 327\ 870\ 152\ 944\ 013\ 988\ 358\ 928\ 114 \ ; \\
& 528\ 962\ 242\ 087\ 062\ 453\ 152\ 690\ 410\ 614\ 013\ 671\ 875 + \\
& (1\ 886\ 491\ 646\ 433\ 732\ 479\ 814\ 597\ 361\ 998\ 744\ 134\ 040\ 407\ 919\ 471\ 435\ 385\ 970\ 472\ 345\ 164\ 676\ 056 \\
& \ z^{89}) / \\
& 522\ 532\ 651\ 330\ 971\ 490\ 226\ 753\ 590\ 247\ 329\ 744\ 050\ 384\ 290\ 675\ 644\ 135\ 735\ 656\ 667\ 608\ 610\ 471 \ ; \\
& 400\ 391\ 047\ 234\ 539\ 824\ 350\ 830\ 981\ 313\ 610\ 076\ 904\ 296\ 875 - \\
& (450\ 638\ 590\ 680\ 882\ 618\ 431\ 105\ 331\ 665\ 591\ 912\ 924\ 988\ 342\ 163\ 281\ 788\ 877\ 675\ 244\ 114\ 763\ 912 \\
& \ z^{91}) / \\
& 1\ 231\ 931\ 818\ 039\ 911\ 948\ 327\ 467\ 370\ 123\ 161\ 265\ 684\ 460\ 571\ 086\ 659\ 079\ 080\ 437\ 659\ 781\ 065\ 743 \ ; \\
& 269\ 173\ 212\ 919\ 832\ 661\ 978\ 537\ 311\ 246\ 395\ 111\ 083\ 984\ 375 + \\
& (415\ 596\ 189\ 473\ 955\ 564\ 121\ 634\ 614\ 268\ 323\ 814\ 113\ 534\ 779\ 643\ 471\ 190\ 276\ 158\ 333\ 713\ 923\ 216 \\
& \ z^{93}) / \\
& 11\ 213\ 200\ 675\ 690\ 943\ 223\ 287\ 032\ 785\ 929\ 540\ 201\ 272\ 600\ 687\ 465\ 377\ 745\ 332\ 153\ 847\ 964\ 679\ 254 \ ; \\
& 692\ 602\ 138\ 023\ 498\ 144\ 562\ 090\ 675\ 557\ 613\ 372\ 802\ 734\ 375 - \\
& (423\ 200\ 899\ 194\ 533\ 026\ 195\ 195\ 456\ 219\ 648\ 467\ 346\ 087\ 908\ 778\ 120\ 468\ 301\ 277\ 466\ 840\ 101\ 336 \ ; \\
& 699\ 974\ 518\ z^{95}) / \\
& 112\ 694\ 926\ 530\ 960\ 148\ 011\ 367\ 752\ 417\ 874\ 063\ 473\ 378\ 698\ 369\ 880\ 587\ 800\ 838\ 274\ 234\ 349\ 237 \ ; \\
& 591\ 647\ 453\ 413\ 782\ 021\ 538\ 312\ 594\ 164\ 677\ 406\ 144\ 702\ 434\ 539\ 794\ 921\ 875 + \\
& (5\ 543\ 531\ 483\ 502\ 489\ 438\ 698\ 050\ 411\ 951\ 314\ 743\ 456\ 505\ 773\ 755\ 468\ 368\ 087\ 670\ 306\ 121\ 873\ 229 \ ; \\
& 244\ z^{97}) / \\
& 14\ 569\ 479\ 835\ 935\ 377\ 894\ 165\ 191\ 004\ 250\ 040\ 526\ 616\ 509\ 162\ 234\ 077\ 285\ 176\ 247\ 476\ 968\ 227\ 225 \ ; \\
& 810\ 918\ 346\ 966\ 001\ 491\ 701\ 692\ 846\ 112\ 140\ 419\ 483\ 184\ 814\ 453\ 125 - \\
& (378\ 392\ 151\ 276\ 488\ 501\ 180\ 909\ 732\ 277\ 974\ 887\ 490\ 811\ 366\ 132\ 267\ 744\ 533\ 542\ 784\ 817\ 245\ 581 \ ; \\
& 660\ 788\ 990\ 844\ z^{99}) / \\
& 9\ 815\ 205\ 420\ 757\ 514\ 710\ 108\ 178\ 059\ 369\ 553\ 458\ 327\ 392\ 260\ 750\ 404\ 049\ 930\ 407\ 987\ 933\ 582\ 359 \ ; \\
& 080\ 767\ 225\ 644\ 716\ 670\ 692\ 512\ 152\ 512\ 547\ 892\ 166\ 922\ 988\ 160\ 818\ 188\ 452\ 125 \ ; \dots \lfloor -101 \rfloor
\end{aligned}$$

Mathematica comes with the add-on package `DiscreteMath`RSolve`` that allows finding the general terms of series for many functions. After loading this package, and using the package function `SeriesTerm`, the following n^{th} term for odd hyperbolic functions can be evaluated.

```
<< DiscreteMath`RSolve`
```

```
SeriesTerm[{Sinh[z], Cosh[z], Tanh[z], Coth[z], Csch[z], Sech[z]}, {z, 0, n}] z^n
```

$$\left\{ z^n \text{If} \left[\text{Odd}[n], \frac{1}{n!}, 0 \right], z^n \text{If} \left[\text{Even}[n], \frac{1}{n!}, 0 \right], \right. \\ z^n \text{If} \left[\text{Odd}[n], \frac{2^{1+n} (-1 + 2^{1+n}) \text{BernoulliB}[1+n]}{(1+n)!}, 0 \right], \\ \left. \frac{2^{1+n} z^n \text{BernoulliB}[1+n]}{(1+n)!}, \frac{2^{1+n} z^n \text{BernoulliB} \left[1+n, \frac{1}{2} \right]}{(1+n)!}, \frac{z^n \text{EulerE}[n]}{n!} \right\}$$

Here is a quick check of the last result.

This series should be evaluated to $\{\sinh(z), \cosh(z), \tanh(z), \coth(z), \operatorname{csch}(z), \operatorname{sech}(z)\}$, which can be concluded from the following relation.

```
Sum[#, {n, 0, 100}] & /@% -
```

```
Series[{Sinh[z], Cosh[z], Tanh[z], Coth[z], Csch[z], Sech[z]}, {z, 0, 100}]
```

$$\left\{ O[z]^{101}, O[z]^{101}, O[z]^{101}, -\frac{1}{z} - 1 + O[z]^{101}, -\frac{1}{z} + O[z]^{101}, O[z]^{101} \right\}$$

Differentiation

Mathematica can evaluate derivatives of hyperbolic functions of an arbitrary positive integer order.

```
D[Sinh[z], z]
```

```
Cosh[z]
```

```
Sinh[z] // D[#, z] &
```

```
Cosh[z]
```

```
∂z{Sinh[z], Cosh[z], Tanh[z], Coth[z], Csch[z], Sech[z]}
```

$$\{\cosh[z], \sinh[z], \operatorname{sech}[z]^2, -\operatorname{csch}[z]^2, -\coth[z] \operatorname{csch}[z], -\operatorname{sech}[z] \tanh[z]\}$$

```
∂{z,2}{Sinh[z], Cosh[z], Tanh[z], Coth[z], Csch[z], Sech[z]}
```

$$\{\sinh[z], \cosh[z], -2 \operatorname{sech}[z]^2 \tanh[z], 2 \coth[z] \operatorname{csch}[z]^2, \\ \coth[z]^2 \operatorname{csch}[z] + \operatorname{csch}[z]^3, -\operatorname{sech}[z]^3 + \operatorname{sech}[z] \tanh[z]^2\}$$

```
Table[D[{Sinh[z], Cosh[z], Tanh[z], Coth[z], Csch[z], Sech[z]}, {z, n}], {n, 4}]
```

$$\left\{ \left\{ \text{Cosh}[z], \text{Sinh}[z], \text{Sech}[z]^2, -\text{Csch}[z]^2, -\text{Coth}[z] \text{Csch}[z], -\text{Sech}[z] \text{Tanh}[z] \right\}, \right. \\ \left. \left\{ \text{Sinh}[z], \text{Cosh}[z], -2 \text{Sech}[z]^2 \text{Tanh}[z], 2 \text{Coth}[z] \text{Csch}[z]^2, \right. \right. \\ \left. \left. \text{Coth}[z]^2 \text{Csch}[z] + \text{Csch}[z]^3, -\text{Sech}[z]^3 + \text{Sech}[z] \text{Tanh}[z]^2 \right\}, \right. \\ \left\{ \text{Cosh}[z], \text{Sinh}[z], -2 \text{Sech}[z]^4 + 4 \text{Sech}[z]^2 \text{Tanh}[z]^2, -4 \text{Coth}[z]^2 \text{Csch}[z]^2 - 2 \text{Csch}[z]^4, \right. \\ \left. -\text{Coth}[z]^3 \text{Csch}[z] - 5 \text{Coth}[z] \text{Csch}[z]^3, 5 \text{Sech}[z]^3 \text{Tanh}[z] - \text{Sech}[z] \text{Tanh}[z]^3 \right\}, \\ \left\{ \text{Sinh}[z], \text{Cosh}[z], 16 \text{Sech}[z]^4 \text{Tanh}[z] - 8 \text{Sech}[z]^2 \text{Tanh}[z]^3, \right. \\ \left. 8 \text{Coth}[z]^3 \text{Csch}[z]^2 + 16 \text{Coth}[z] \text{Csch}[z]^4, \text{Coth}[z]^4 \text{Csch}[z] + 18 \text{Coth}[z]^2 \text{Csch}[z]^3 + \right. \\ \left. \left. 5 \text{Csch}[z]^5, 5 \text{Sech}[z]^5 - 18 \text{Sech}[z]^3 \text{Tanh}[z]^2 + \text{Sech}[z] \text{Tanh}[z]^4 \right\} \right\}$$

Finite summation

Mathematica can calculate finite sums that contain hyperbolic functions. Here are two examples.

$$\text{Sum}[\text{Sinh}[a k], \{k, 0, n\}]$$

$$\frac{-1 + e^{a+a n}}{2 (-1 + e^a)} - \frac{e^{-a n} (-1 + e^{a+a n})}{2 (-1 + e^a)}$$

$$\sum_{k=0}^n (-1)^k \text{sinh}[a k]$$

$$-\frac{e^a + (-e^{-a})^n}{2 (1 + e^a)} + \frac{1 + e^a (-e^a)^n}{2 (1 + e^a)}$$

Infinite summation

Mathematica can calculate infinite sums that contain hyperbolic functions. Here are some examples.

$$\sum_{k=1}^{\infty} z^k \text{sinh}[k x]$$

$$-\frac{z}{2 (e^x - z)} - \frac{e^x z}{2 (-1 + e^x z)}$$

$$\sum_{k=1}^{\infty} \frac{\text{sinh}[k x]}{k!}$$

$$\frac{1}{2} (1 - e^{e^{-x}}) + \frac{1}{2} (-1 + e^{e^x})$$

$$\sum_{k=1}^{\infty} \frac{\text{Cosh}[k x]}{k}$$

$$-\frac{1}{2} \text{Log}[1 - e^{-x}] - \frac{1}{2} \text{Log}[1 - e^x]$$

Finite products

Mathematica can calculate some finite symbolic products that contain the hyperbolic functions. Here are two examples.

$$\prod_{k=1}^{n-1} \text{Sinh}\left[\frac{\pi k i}{n}\right]$$

$$\left(\frac{i}{2}\right)^{-1+n} n$$

$$\prod_{k=1}^{n-1} \text{Cosh}\left[z + \frac{\pi k i}{n}\right]$$

$$-(-1)^n 2^{1-n} \text{Sech}[z] \text{Sin}\left[\frac{1}{2} n (\pi + 2 i z)\right]$$

Infinite products

Mathematica can calculate infinite products that contain hyperbolic functions. Here are some examples.

$$\prod_{k=1}^{\infty} \text{Exp}\left[z^k \text{Sinh}[k x]\right]$$

$$e^{-\frac{(-1+e^{2x})z}{2(e^x-z)(-1+e^x z)}}$$

$$\prod_{k=1}^{\infty} \text{Exp}\left[\frac{\text{Cosh}[k x]}{k!}\right]$$

$$e^{\frac{1}{2}(-2+e^{-x}+e^x)}$$

Indefinite integration

Mathematica can calculate a huge set of doable indefinite integrals that contain hyperbolic functions. Here are some examples.

$$\int \text{sinh}[7 z] dz$$

$$\frac{1}{7} \text{Cosh}[7 z]$$

$$\int \left\{ \{\text{Sinh}[z], \text{Sinh}[z]^a\}, \{\text{Cosh}[z], \text{Cosh}[z]^a\}, \{\text{Tanh}[z], \text{Tanh}[z]^a\}, \right. \\ \left. \{\text{Coth}[z], \text{Coth}[z]^a\}, \{\text{Csch}[z], \text{Csch}[z]^a\}, \{\text{Sech}[z], \text{Sech}[z]^a\} \right\} dz$$

$$\left\{ \left\{ \text{Cosh}[z], \right. \right. \\ \left. \left. -\text{Cosh}[z] \text{Hypergeometric2F1}\left[\frac{1}{2}, \frac{1-a}{2}, \frac{3}{2}, \text{Cosh}[z]^2\right] \text{Sinh}[z]^{1+a} \left(-\text{Sinh}[z]^2\right)^{\frac{1}{2}(-1-a)}\right\}, \right. \\ \left. \left\{ \text{Sinh}[z], -\frac{\text{Cosh}[z]^{1+a} \text{Hypergeometric2F1}\left[\frac{1+a}{2}, \frac{1}{2}, \frac{3+a}{2}, \text{Cosh}[z]^2\right] \text{Sinh}[z]}{(1+a) \sqrt{-\text{Sinh}[z]^2}} \right\}, \right. \\ \left. \left\{ \text{Log}[\text{Cosh}[z]], \frac{\text{Hypergeometric2F1}\left[\frac{1+a}{2}, 1, 1+\frac{1+a}{2}, \text{Tanh}[z]^2\right] \text{Tanh}[z]^{1+a}}{1+a} \right\}, \right. \\ \left. \left\{ \text{Log}[\text{Sinh}[z]], \frac{\text{Coth}[z]^{1+a} \text{Hypergeometric2F1}\left[\frac{1+a}{2}, 1, 1+\frac{1+a}{2}, \text{Coth}[z]^2\right]}{1+a} \right\}, \right. \\ \left. \left\{ -\text{Log}\left[\text{Cosh}\left[\frac{z}{2}\right]\right] + \text{Log}\left[\text{Sinh}\left[\frac{z}{2}\right]\right], \right. \right. \\ \left. \left. -\text{Cosh}[z] \text{Csch}[z]^{-1+a} \text{Hypergeometric2F1}\left[\frac{1}{2}, \frac{1+a}{2}, \frac{3}{2}, \text{Cosh}[z]^2\right] \left(-\text{Sinh}[z]^2\right)^{\frac{1}{2}(-1+a)}\right\}, \right. \\ \left. \left\{ 2 \text{ArcTan}\left[\text{Tanh}\left[\frac{z}{2}\right]\right], -\frac{\text{Hypergeometric2F1}\left[\frac{1-a}{2}, \frac{1}{2}, \frac{3-a}{2}, \text{Cosh}[z]^2\right] \text{Sech}[z]^{-1+a} \text{Sinh}[z]}{(1-a) \sqrt{-\text{Sinh}[z]^2}} \right\} \right\}$$

Definite integration

Mathematica can calculate wide classes of definite integrals that contain hyperbolic functions. Here are some examples.

$$\int_0^{\pi/2} \sqrt[3]{\text{Sinh}[z]} \, dz \\ - \frac{(-1)^{1/3} \sqrt{\pi} \text{Gamma}\left[\frac{2}{3}\right]}{2 \text{Gamma}\left[\frac{7}{6}\right]} + (-1)^{1/3} \text{Cosh}\left[\frac{\pi}{2}\right] \text{Hypergeometric2F1}\left[\frac{1}{2}, \frac{1}{3}, \frac{3}{2}, \text{Cosh}\left[\frac{\pi}{2}\right]^2\right]$$

$$\int_1^{\pi/2} \left\{ \sqrt{\text{Sinh}[z]}, \sqrt{\text{Cosh}[z]}, \sqrt{\text{Tanh}[z]}, \sqrt{\text{Coth}[z]}, \sqrt{\text{Csch}[z]}, \sqrt{\text{Sech}[z]} \right\} dz$$

$$\begin{aligned}
 & \left\{ 2 (-1)^{1/4} \text{EllipticE} \left[\left(\frac{1}{4} - \frac{i}{4} \right) \pi, 2 \right] - 2 (-1)^{1/4} \text{EllipticE} \left[\frac{1}{4} (-2i + \pi), 2 \right], \right. \\
 & 2i \text{EllipticE} \left[\frac{i}{2}, 2 \right] - 2i \text{EllipticE} \left[\frac{i\pi}{4}, 2 \right], \\
 & \frac{1}{2} \left(i \text{Log} \left[1 - i \sqrt{\frac{-1 + e^2}{1 + e^2}} \right] - i \text{Log} \left[1 + i \sqrt{\frac{-1 + e^2}{1 + e^2}} \right] - \right. \\
 & \quad i \text{Log} \left[1 - i \sqrt{\frac{-1 + e^\pi}{1 + e^\pi}} \right] + i \text{Log} \left[1 + i \sqrt{\frac{-1 + e^\pi}{1 + e^\pi}} \right] + \text{Log} \left[1 - \sqrt{\text{Tanh}[1]} \right] - \\
 & \quad \left. \text{Log} \left[1 + \sqrt{\text{Tanh}[1]} \right] - \text{Log} \left[1 - \sqrt{\text{Tanh} \left[\frac{\pi}{2} \right]} \right] + \text{Log} \left[1 + \sqrt{\text{Tanh} \left[\frac{\pi}{2} \right]} \right] \right), \\
 & \frac{1}{2} i \left(\text{Log} \left[1 - i \sqrt{\frac{1 + e^2}{-1 + e^2}} \right] - \text{Log} \left[1 + i \sqrt{\frac{1 + e^2}{-1 + e^2}} \right] - \text{Log} \left[1 - i \sqrt{\frac{1 + e^\pi}{-1 + e^\pi}} \right] + \right. \\
 & \quad \text{Log} \left[1 + i \sqrt{\frac{1 + e^\pi}{-1 + e^\pi}} \right] - i \text{Log} \left[-1 + \sqrt{\text{Coth}[1]} \right] + i \text{Log} \left[1 + \sqrt{\text{Coth}[1]} \right] + \\
 & \quad \left. i \text{Log} \left[-1 + \sqrt{\text{Coth} \left[\frac{\pi}{2} \right]} \right] - i \text{Log} \left[1 + \sqrt{\text{Coth} \left[\frac{\pi}{2} \right]} \right] \right), \\
 & 2 (-1)^{3/4} \text{EllipticF} \left[\left(\frac{1}{4} - \frac{i}{4} \right) \pi, 2 \right] - 2 (-1)^{3/4} \text{EllipticF} \left[\frac{1}{4} (-2i + \pi), 2 \right], \\
 & 2i \text{EllipticF} \left[\frac{i}{2}, 2 \right] - 2i \text{EllipticF} \left[\frac{i\pi}{4}, 2 \right] \left. \right\} \\
 & \int_1^{\frac{\pi}{4}} \{ \{ \text{Sinh}[z], \text{Sinh}[z]^a \}, \{ \text{Cosh}[z], \text{Cosh}[z]^a \}, \{ \text{Tanh}[z], \text{Tanh}[z]^a \}, \\
 & \quad \{ \text{Coth}[z], \text{Coth}[z]^a \}, \{ \text{Csch}[z], \text{Csch}[z]^a \}, \{ \text{Sech}[z], \text{Sech}[z]^a \} \} dz
 \end{aligned}$$

$$\begin{aligned}
 & \left\{ \left\{ -\text{Cosh}[1] + \text{Cosh}\left[\frac{\pi}{4}\right], \right. \right. \\
 & \quad (-1)^{-\frac{1}{2}-\frac{a}{2}} \text{Cosh}[1] \text{Hypergeometric2F1}\left[\frac{1}{2}, \frac{1-a}{2}, \frac{3}{2}, \text{Cosh}[1]^2\right] \text{Sinh}[1]^{1+2\left(-\frac{1}{2}-\frac{a}{2}\right)+a} - \\
 & \quad (-1)^{-\frac{1}{2}-\frac{a}{2}} \text{Cosh}\left[\frac{\pi}{4}\right] \text{Hypergeometric2F1}\left[\frac{1}{2}, \frac{1-a}{2}, \frac{3}{2}, \text{Cosh}\left[\frac{\pi}{4}\right]^2\right] \text{Sinh}\left[\frac{\pi}{4}\right]^{1+2\left(-\frac{1}{2}-\frac{a}{2}\right)+a} \right\}, \\
 & \left\{ -\text{Sinh}[1] + \text{Sinh}\left[\frac{\pi}{4}\right], -\frac{i \text{Cosh}[1]^{1+a} \text{Hypergeometric2F1}\left[\frac{1+a}{2}, \frac{1}{2}, \frac{3+a}{2}, \text{Cosh}[1]^2\right]}{1+a} + \right. \\
 & \quad \left. \frac{i \text{Cosh}\left[\frac{\pi}{4}\right]^{1+a} \text{Hypergeometric2F1}\left[\frac{1+a}{2}, \frac{1}{2}, \frac{3+a}{2}, \text{Cosh}\left[\frac{\pi}{4}\right]^2\right]}{1+a} \right\}, \\
 & \left\{ -\text{Log}[\text{Cosh}[1]] + \text{Log}\left[\text{Cosh}\left[\frac{\pi}{4}\right]\right], -\frac{\text{Hypergeometric2F1}\left[\frac{1+a}{2}, 1, 1 + \frac{1+a}{2}, \text{Tanh}[1]^2\right] \text{Tanh}[1]^{1+a}}{1+a} + \right. \\
 & \quad \left. \frac{\text{Hypergeometric2F1}\left[\frac{1+a}{2}, 1, 1 + \frac{1+a}{2}, \text{Tanh}\left[\frac{\pi}{4}\right]^2\right] \text{Tanh}\left[\frac{\pi}{4}\right]^{1+a}}{1+a} \right\}, \\
 & \left\{ -\text{Log}[\text{Sinh}[1]] + \text{Log}\left[\text{Sinh}\left[\frac{\pi}{4}\right]\right], -\frac{\text{Coth}[1]^{1+a} \text{Hypergeometric2F1}\left[\frac{1+a}{2}, 1, 1 + \frac{1+a}{2}, \text{Coth}[1]^2\right]}{1+a} + \right. \\
 & \quad \left. \frac{\text{Coth}\left[\frac{\pi}{4}\right]^{1+a} \text{Hypergeometric2F1}\left[\frac{1+a}{2}, 1, 1 + \frac{1+a}{2}, \text{Coth}\left[\frac{\pi}{4}\right]^2\right]}{1+a} \right\}, \\
 & \left\{ \text{Log}\left[\text{Cosh}\left[\frac{1}{2}\right]\right] - \text{Log}\left[\text{Cosh}\left[\frac{\pi}{8}\right]\right] - \text{Log}\left[\text{Sinh}\left[\frac{1}{2}\right]\right] + \text{Log}\left[\text{Sinh}\left[\frac{\pi}{8}\right]\right], \right. \\
 & \quad (-1)^{-\frac{1}{2}+\frac{a}{2}} \text{Cosh}[1] \text{Csch}[1]^{-1+a} \text{Hypergeometric2F1}\left[\frac{1}{2}, \frac{1+a}{2}, \frac{3}{2}, \text{Cosh}[1]^2\right] \text{Sinh}[1]^{2\left(-\frac{1}{2}+\frac{a}{2}\right)} - \\
 & \quad (-1)^{-\frac{1}{2}+\frac{a}{2}} \text{Cosh}\left[\frac{\pi}{4}\right] \text{Csch}\left[\frac{\pi}{4}\right]^{-1+a} \text{Hypergeometric2F1}\left[\frac{1}{2}, \frac{1+a}{2}, \frac{3}{2}, \text{Cosh}\left[\frac{\pi}{4}\right]^2\right] \text{Sinh}\left[\frac{\pi}{4}\right]^{2\left(-\frac{1}{2}+\frac{a}{2}\right)} \right\}, \\
 & \left\{ -2 \text{ArcTan}\left[\text{Tanh}\left[\frac{1}{2}\right]\right] + 2 \text{ArcTan}\left[\text{Tanh}\left[\frac{\pi}{8}\right]\right], \right. \\
 & \quad \frac{i \text{Hypergeometric2F1}\left[\frac{1-a}{2}, \frac{1}{2}, \frac{3-a}{2}, \text{Cosh}[1]^2\right] \text{Sech}[1]^{-1+a}}{-1+a} - \\
 & \quad \left. \frac{i \text{Hypergeometric2F1}\left[\frac{1-a}{2}, \frac{1}{2}, \frac{3-a}{2}, \text{Cosh}\left[\frac{\pi}{4}\right]^2\right] \text{Sech}\left[\frac{\pi}{4}\right]^{-1+a}}{-1+a} \right\} \\
 & \int_0^\infty \left\{ \frac{1}{a+b \text{Sinh}[z]}, \frac{1}{a+b \text{Cosh}[z]}, \right. \\
 & \quad \left. \frac{1}{a+b \text{Tanh}[z]}, \frac{1}{a+b \text{Coth}[z]}, \frac{1}{a+b \text{Csch}[z]}, \frac{1}{a+b \text{Sech}[z]} \right\} dz
 \end{aligned}$$

$$\left\{ -\frac{1}{\sqrt{-a^2 - b^2}} \left(i \left(\text{Log} \left[1 - \frac{ia}{\sqrt{-a^2 - b^2}} \right] - \text{Log} \left[1 + \frac{ia}{\sqrt{-a^2 - b^2}} \right] + \right. \right. \right. \\ \left. \left. \left. \text{Log} \left[\frac{ia - ib + \sqrt{-a^2 - b^2}}{\sqrt{-a^2 - b^2}} \right] - \text{Log} \left[\frac{-ia + ib + \sqrt{-a^2 - b^2}}{\sqrt{-a^2 - b^2}} \right] \right) \right) , \\ -\frac{1}{\sqrt{-a^2 + b^2}} \left(i \left(\text{Log} \left[1 - \frac{ia}{\sqrt{-a^2 + b^2}} \right] - \text{Log} \left[1 + \frac{ia}{\sqrt{-a^2 + b^2}} \right] - \right. \right. \\ \left. \left. \left. \text{Log} \left[\frac{-ia - ib + \sqrt{-a^2 + b^2}}{\sqrt{-a^2 + b^2}} \right] + \text{Log} \left[\frac{ia + ib + \sqrt{-a^2 + b^2}}{\sqrt{-a^2 + b^2}} \right] \right) \right) , \\ \frac{b (\text{Log} [2a] - \text{Log} [a + b])}{a^2 - b^2} , \frac{b (\text{Log} [-a - b] - \text{Log} [-2b])}{-a^2 + b^2} , \\ \frac{1}{a \sqrt{-a^2 - b^2}} \\ \left(i b \left(\text{Log} \left[\frac{-ib + \sqrt{-a^2 - b^2}}{\sqrt{-a^2 - b^2}} \right] - \text{Log} \left[\frac{ia - ib + \sqrt{-a^2 - b^2}}{\sqrt{-a^2 - b^2}} \right] - \right. \right. \\ \left. \left. \left. \text{Log} \left[\frac{ib + \sqrt{-a^2 - b^2}}{\sqrt{-a^2 - b^2}} \right] + \text{Log} \left[\frac{-ia + ib + \sqrt{-a^2 - b^2}}{\sqrt{-a^2 - b^2}} \right] \right) \right) , \\ \frac{1}{a \sqrt{a^2 - b^2}} \left(i b \left(\text{Log} \left[1 - \frac{ib}{\sqrt{a^2 - b^2}} \right] - \text{Log} \left[1 + \frac{ib}{\sqrt{a^2 - b^2}} \right] - \right. \right. \\ \left. \left. \left. \text{Log} \left[\frac{-ia - ib + \sqrt{a^2 - b^2}}{\sqrt{a^2 - b^2}} \right] + \text{Log} \left[\frac{ia + ib + \sqrt{a^2 - b^2}}{\sqrt{a^2 - b^2}} \right] \right) \right) \right\}$$

Limit operation

Mathematica can calculate limits that contain hyperbolic functions. Here are some examples.

$$\text{Limit} \left[\frac{\text{Sinh}[z]}{z} + \text{Cosh}[z]^3, z \rightarrow 0 \right]$$

2

$$\text{Limit} \left[\left(\frac{\text{Tanh}[x]}{x} \right)^{\frac{1}{x^2}}, x \rightarrow 0 \right]$$

$$\frac{1}{e^{1/3}}$$

$$\text{Limit}\left[\frac{\text{Sinh}\left[\sqrt{z^2}\right]}{z}, z \rightarrow 0, \text{Direction} \rightarrow 1\right]$$

-1

$$\text{Limit}\left[\frac{\text{Sinh}\left[\sqrt{z^2}\right]}{z}, z \rightarrow 0, \text{Direction} \rightarrow -1\right]$$

1

Solving equations

The next input solves equations that contain hyperbolic functions. The message indicates that the multivalued functions are used to express the result and that some solutions might be absent.

$$\text{Solve}\left[\text{Tanh}[z]^2 + 3 \text{Sinh}\left[z + \frac{\pi}{6}\right] = 4, z\right]$$

Solve::ifun: Inverse functions are being used by Solve, so some solutions may not be found.

$$\left\{ \left\{ z \rightarrow -\text{ArcSech}\left[3 \left(-2 \text{Root}\left[1 + 6 \#1^2 - 6 \text{Sinh}\left[\frac{\pi}{6}\right] \#1^3 + 9 \#1^4 + 9 \text{Cosh}\left[\frac{\pi}{6}\right]^2 \#1^4 - 18 \text{Sinh}\left[\frac{\pi}{6}\right] \#1^5 - 9 \text{Cosh}\left[\frac{\pi}{6}\right]^2 \#1^6 + 9 \text{Sinh}\left[\frac{\pi}{6}\right]^2 \#1^6 \&, 1\right] - 3 \text{Root}\left[1 + 6 \#1^2 - 6 \text{Sinh}\left[\frac{\pi}{6}\right] \#1^3 + 9 \#1^4 + 9 \text{Cosh}\left[\frac{\pi}{6}\right]^2 \#1^4 - 18 \text{Sinh}\left[\frac{\pi}{6}\right] \#1^5 - 9 \text{Cosh}\left[\frac{\pi}{6}\right]^2 \#1^6 + 9 \text{Sinh}\left[\frac{\pi}{6}\right]^2 \#1^6 \&, 1\right]^3 - 3 \text{Cosh}\left[\frac{\pi}{6}\right]^2 \text{Root}\left[1 + 6 \#1^2 - 6 \text{Sinh}\left[\frac{\pi}{6}\right] \#1^3 + 9 \#1^4 + 9 \text{Cosh}\left[\frac{\pi}{6}\right]^2 \#1^4 - 18 \text{Sinh}\left[\frac{\pi}{6}\right] \#1^5 - 9 \text{Cosh}\left[\frac{\pi}{6}\right]^2 \#1^6 + 9 \text{Sinh}\left[\frac{\pi}{6}\right]^2 \#1^6 \&, 1\right]^3 + 3 \text{Cosh}\left[\frac{\pi}{6}\right]^2 \text{Root}\left[1 + 6 \#1^2 - 6 \text{Sinh}\left[\frac{\pi}{6}\right] \#1^3 + 9 \#1^4 + 9 \text{Cosh}\left[\frac{\pi}{6}\right]^2 \#1^4 - 18 \text{Sinh}\left[\frac{\pi}{6}\right] \#1^5 - 9 \text{Cosh}\left[\frac{\pi}{6}\right]^2 \#1^6 + 9 \text{Sinh}\left[\frac{\pi}{6}\right]^2 \#1^6 \&, 1\right]^5 + 2 \text{Root}\left[1 + 6 \#1^2 - 6 \text{Sinh}\left[\frac{\pi}{6}\right] \#1^3 + 9 \#1^4 + 9 \text{Cosh}\left[\frac{\pi}{6}\right]^2 \#1^4 - 18 \text{Sinh}\left[\frac{\pi}{6}\right] \#1^5 - 9 \text{Cosh}\left[\frac{\pi}{6}\right]^2 \#1^6 + 9 \text{Sinh}\left[\frac{\pi}{6}\right]^2 \#1^6 \&, 1\right]^2 \text{Sinh}\left[\frac{\pi}{6}\right] + 6 \text{Root}\left[1 + 6 \#1^2 - 6 \text{Sinh}\left[\frac{\pi}{6}\right] \#1^3 + 9 \#1^4 + 9 \text{Cosh}\left[\frac{\pi}{6}\right]^2 \#1^4 - 18 \text{Sinh}\left[\frac{\pi}{6}\right] \#1^5 - 9 \text{Cosh}\left[\frac{\pi}{6}\right]^2 \#1^6 + 9 \text{Sinh}\left[\frac{\pi}{6}\right]^2 \#1^6 \&, 1\right]^4 \text{Sinh}\left[\frac{\pi}{6}\right] - 3 \text{Root}\left[1 + 6 \#1^2 - 6 \text{Sinh}\left[\frac{\pi}{6}\right] \#1^3 + 9 \#1^4 + 9 \text{Cosh}\left[\frac{\pi}{6}\right]^2 \#1^4 - 18 \text{Sinh}\left[\frac{\pi}{6}\right] \#1^5 - 9 \text{Cosh}\left[\frac{\pi}{6}\right]^2 \#1^6 + 9 \text{Sinh}\left[\frac{\pi}{6}\right]^2 \#1^6 \&, 1\right]^2 \right. \right. \right.$$

$$\begin{aligned}
 & 9 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^2 \#1^6 + 9 \operatorname{Sinh}\left[\frac{\pi}{6}\right]^2 \#1^6 \&, 6\right]^3 - \\
 & 3 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^2 \operatorname{Root}\left[1 + 6 \#1^2 - 6 \operatorname{Sinh}\left[\frac{\pi}{6}\right] \#1^3 + 9 \#1^4 + 9 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^2 \#1^4 - \right. \\
 & \quad \left. 18 \operatorname{Sinh}\left[\frac{\pi}{6}\right] \#1^5 - 9 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^2 \#1^6 + 9 \operatorname{Sinh}\left[\frac{\pi}{6}\right]^2 \#1^6 \&, 6\right]^3 + \\
 & 3 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^2 \operatorname{Root}\left[1 + 6 \#1^2 - 6 \operatorname{Sinh}\left[\frac{\pi}{6}\right] \#1^3 + 9 \#1^4 + 9 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^2 \#1^4 - \right. \\
 & \quad \left. 18 \operatorname{Sinh}\left[\frac{\pi}{6}\right] \#1^5 - 9 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^2 \#1^6 + 9 \operatorname{Sinh}\left[\frac{\pi}{6}\right]^2 \#1^6 \&, 6\right]^5 + \\
 & 2 \operatorname{Root}\left[1 + 6 \#1^2 - 6 \operatorname{Sinh}\left[\frac{\pi}{6}\right] \#1^3 + 9 \#1^4 + 9 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^2 \#1^4 - 18 \operatorname{Sinh}\left[\frac{\pi}{6}\right] \#1^5 - \right. \\
 & \quad \left. 9 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^2 \#1^6 + 9 \operatorname{Sinh}\left[\frac{\pi}{6}\right]^2 \#1^6 \&, 6\right]^2 \operatorname{Sinh}\left[\frac{\pi}{6}\right] + \\
 & 6 \operatorname{Root}\left[1 + 6 \#1^2 - 6 \operatorname{Sinh}\left[\frac{\pi}{6}\right] \#1^3 + 9 \#1^4 + 9 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^2 \#1^4 - 18 \operatorname{Sinh}\left[\frac{\pi}{6}\right] \#1^5 - \right. \\
 & \quad \left. 9 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^2 \#1^6 + 9 \operatorname{Sinh}\left[\frac{\pi}{6}\right]^2 \#1^6 \&, 6\right]^4 \operatorname{Sinh}\left[\frac{\pi}{6}\right] - \\
 & 3 \operatorname{Root}\left[1 + 6 \#1^2 - 6 \operatorname{Sinh}\left[\frac{\pi}{6}\right] \#1^3 + 9 \#1^4 + 9 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^2 \#1^4 - 18 \operatorname{Sinh}\left[\frac{\pi}{6}\right] \#1^5 - \right. \\
 & \quad \left. 9 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^2 \#1^6 + 9 \operatorname{Sinh}\left[\frac{\pi}{6}\right]^2 \#1^6 \&, 6\right]^5 \operatorname{Sinh}\left[\frac{\pi}{6}\right]^2 \left. \right\} \left. \right\} \left. \right\}
 \end{aligned}$$

Complete solutions can be obtained by using the function Reduce.

Reduce[Sinh[x] == a, x] // InputForm

```
// InputForm = C[1] ∈ Integers &&
(x == I * Pi - ArcSinh[a] + (2 * I) * Pi * C[1] || x == ArcSinh[a] + (2 * I) * Pi * C[1])
```

Reduce[Cosh[x] == a, x] // InputForm

```
// InputForm =
C[1] ∈ Integers && (x == -ArcCosh[a] + (2 * I) * Pi * C[1] || x == ArcCosh[a] + (2 * I) * Pi * C[1])
```

Reduce[Tanh[x] == a, x] // InputForm

```
// InputForm = C[1] ∈ Integers && -1 + a^2 ≠ 0 && x == ArcTanh[a] + I * Pi * C[1]
```

Reduce[Coth[x] == a, x] // InputForm

```
// InputForm = C[1] ∈ Integers && -1 + a^2 ≠ 0 && x == ArcCoth[a] + I * Pi * C[1]
```

Reduce[Csch[x] == a, x] // InputForm

```
// InputForm = C[1] ∈ Integers && a ≠ 0 &&
(x == I * Pi - ArcSinh[a^(-1)] + (2 * I) * Pi * C[1] || x == ArcSinh[a^(-1)] + (2 * I) * Pi * C[1])
```

Reduce[Sech[x] == a, x] // InputForm

```
// InputForm = C[1] ∈ Integers && a ≠ 0 &&
(x == -ArcCosh[a^(-1)] + (2 * I) * Pi * C[1] || x == ArcCosh[a^(-1)] + (2 * I) * Pi * C[1])
```

Solving differential equations

Here are differential equations whose linear-independent solutions are hyperbolic functions. The solutions of the simplest second-order linear ordinary differential equation with constant coefficients can be represented through $\sinh(z)$ and $\cosh(z)$.

```
{DSolve[w' [z] - w[z] == 0, w[z], z],
 DSolve[w' [z] + w[z]^2 - 1 == 0, w[z], z]} // (ExpToTrig // @ #) &
{{w[z] → C[1] Cosh[z] + C[2] Cosh[z] + C[1] Sinh[z] - C[2] Sinh[z]},
 {{w[z] →  $\frac{\text{Cosh}[2 z] + \text{Cosh}[2 C[1]] + \text{Sinh}[2 z] + \text{Sinh}[2 C[1]]}{\text{Cosh}[2 z] - \text{Cosh}[2 C[1]] + \text{Sinh}[2 z] - \text{Sinh}[2 C[1]]}$ }}}
```

All hyperbolic functions satisfy first-order nonlinear differential equations. In carrying out the algorithm to solve the nonlinear differential equation, *Mathematica* has to solve a transcendental equation. In doing so, the generically multivariate inverse of a function is encountered, and a message is issued that a solution branch is potentially missed.

```
DSolve[{w' [z] ==  $\sqrt{1 + w[z]^2}$ , w[0] == 0}, w[z], z]
```

Solve::ifun: Inverse functions are being used by Solve, so some solutions may not be found.

```
{{w[z] → Sinh[z]}}
```

```
DSolve[{w' [z] ==  $\sqrt{-1 + w[z]^2}$ , w[0] == 1}, w[z], z] // FullSimplify
```

Solve::ifun: Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information. More...

```
{{w[z] → Cosh[z]}}
```

```
DSolve[{w' [z] + w[z]^2 - 1 == 0, w[0] == 0}, w[z], z] // FullSimplify
```

Solve::ifun: Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information. More...

```
{{w[z] → Tanh[z]}}
```

```
DSolve[{w' [z] - w[z]^2 + 1 == 0, w[ $\frac{\pi i}{2}$ ] == 0}, w[z], z] // FullSimplify
```

Solve::ifun: Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information. More...

```
{{w[z] → -Coth[z]}}
```

Integral transforms

Mathematica supports the main integral transforms like direct and inverse Fourier, Laplace, and Z transforms that can give results containing classical or generalized functions. Here are some transforms of hyperbolic functions.

`LaplaceTransform[Sinh[t], t, s]`

$$\frac{1}{-1 + s^2}$$

`LaplaceTransform[Cosh[t], t, s]`

$$\frac{s}{-1 + s^2}$$

`FourierTransform[Csch[t], t, s]`

$$i \sqrt{\frac{\pi}{2}} \operatorname{Tanh}\left[\frac{\pi s}{2}\right]$$

`FourierTransform[Sech[t], t, s]`

$$\sqrt{\frac{\pi}{2}} \operatorname{Sech}\left[\frac{\pi s}{2}\right]$$

Plotting

Mathematica has built-in functions for 2D and 3D graphics. Here are some examples.

```
Plot[Sin[Sinh[Sum[z^k, {k, 0, 5}]]], {z, -3/2, 4/5}, PlotRange -> All, PlotPoints -> 120];
```

```
Plot3D[Re[Tanh[x + i y]], {x, -2, 2}, {y, -2, 2},
  PlotPoints -> 240, PlotRange -> {-5, 5},
  ClipFill -> None, Mesh -> False, AxesLabel -> {"x", "y", None}];
```

```
ContourPlot[Arg[Sech[1/(x + i y)]]], {x, -1/4, 1/4}, {y, -1/3, 1/3},
  PlotPoints -> 400, PlotRange -> {-pi, pi}, FrameLabel -> {"x", "y", None, None},
  ColorFunction -> Hue, ContourLines -> False, Contours -> 200];
```

Introduction to the Hyperbolic Sine Function in *Mathematica*

Overview

The following shows how the hyperbolic sine function is implemented in *Mathematica*. Examples of evaluating *Mathematica* functions applied to various numeric and exact expressions that involve the hyperbolic sine function or return it are shown. These involve numeric and symbolic calculations and plots.

Notations

Mathematica forms of notations

Following *Mathematica*'s general naming convention, function names in `StandardForm` are just the capitalized versions of their traditional mathematics names. This shows the hyperbolic sine function in `StandardForm`.

```
Sinh[z]
```

```
Sinh[z]
```

This shows the hyperbolic sine function in `TraditionalForm`.

```
% // TraditionalForm
```

```
sinh(z)
```

Additional forms of notations

Mathematica also recognizes the most popular forms of notations for the sine function that are used in other programming languages. Here are three examples: `CForm`, `TeXForm`, and `FortranForm`.

```
{CForm[Sinh[2 Pi z]], TeXForm[Sinh[2 Pi z]], FortranForm[Sinh[2 Pi z]]}
```

```
{Sinh (2 * Pi * z), \sinh (2 \, \pi \, z), Sinh (2 * Pi * z)}
```

Automatic evaluations and transformations

Evaluation for exact, machine-number, and high-precision arguments

For the exact argument $z = \pi i/4$, *Mathematica* returns an exact result.

```
Sinh[ $\frac{i \pi}{4}$ ]
```

$$\frac{i}{\sqrt{2}}$$

```
Sinh[z] /. z ->  $\frac{\pi i}{4}$ 
```

$$\frac{i}{\sqrt{2}}$$

For a machine-number argument (a numerical argument with a decimal point and not too many digits), a machine number is also returned.

```
Sinh[5.]
```

```
74.2032
```

```
Sinh[z] /. z -> 3.
```

```
10.0179
```

The next inputs calculate 100-digit approximations at $z = 1$ and $z = 2$.

```
N[Sinh[z] /. z -> 1, 100]
```

```
1.1752011936438014568823818505956008151557179813340958702295654130133075673043238956
07117452089623392
```

```
N[Sinh[2], 100]
```

```
3.6268604078470187676682139828012617048863420123211357213094844749342502109887850367
23607181294232373
```

```
Sinh[2] // N[#, 100] &
```

```
3.6268604078470187676682139828012617048863420123211357213094844749342502109887850367
23607181294232373
```

It is possible to calculate thousands of digits for the hyperbolic sine function within a second. The next input calculates 10000 digits for $\sinh(1)$ and analyzes the frequency of the digit k in the resulting decimal number.

```
Map[Function[w, {First[#], Length[#]} & /@ Split[Sort[First[RealDigits[w]]]],  
N[{Sinh[z]} /. z -> 1, 10 000]]
```

```
{{{0, 980}, {1, 994}, {2, 996}, {3, 1014},  
{4, 986}, {5, 1001}, {6, 1017}, {7, 1020}, {8, 981}, {9, 1011}}}
```

Here is a 50-digit approximation for the hyperbolic sine function at the complex argument $z = 3 + 2i$.

```
N[Sinh[3 + 2 i], 50]
```

```
-4.1689069599665643507548130588537548435735656047581 +  
9.1544991469114295734672995446098325591588605687652 i
```

```
{N[Sinh[z] /. z -> 3 + 2 i, 50], Sinh[3 + 2 i] // N[#, 50] &}
```

```
{-4.1689069599665643507548130588537548435735656047581 +  
9.1544991469114295734672995446098325591588605687652 i,  
-4.1689069599665643507548130588537548435735656047581 +  
9.1544991469114295734672995446098325591588605687652 i}
```

Mathematica automatically evaluates mathematical functions with machine precision, if the arguments of the function are machine-number elements. In this case, only six digits after the decimal point are shown in the results. The remaining digits are suppressed, but can be displayed using the function `InputForm`.

```
{Sinh[2.], N[Sinh[2]], N[Sinh[2], 16], N[Sinh[2], 5], N[Sinh[2], 20]}
```

```
{3.62686, 3.62686, 3.62686, 3.62686, 3.6268604078470187677}
```

```
% // InputForm
```

```
{3.6268604078470186, 3.6268604078470186, 3.6268604078470186, 3.6268604078470186,  
3.626860407847018767668213982801262`20}
```

Simplification of the argument

Mathematica knows the symmetry and periodicity of the hyperbolic sine function. Here are some examples.

```
Sinh[-3]
```

```
-Sinh[3]
```

`{Sinh[-z], Sinh[z + π i], Sinh[z + 2 π i], Sinh[z + 342 π i], Sinh[-z + 21 π i]}`
`{-Sinh[z], -Sinh[z], Sinh[z], Sinh[z], Sinh[z]}`

Mathematica automatically simplifies the composition of the direct and inverse hyperbolic sine functions into its argument.

`Sinh[ArcSinh[z]]`

`z`

Mathematica also automatically simplifies the composition of the direct and any of the inverse hyperbolic functions into algebraic functions of the argument.

`{Sinh[ArcSinh[z]], Sinh[ArcCosh[z]], Sinh[ArcTanh[z]],`
`Sinh[ArcCoth[z]], Sinh[ArcCsch[z]], Sinh[ArcSech[z]]}`

$$\left\{ z, \sqrt{\frac{-1+z}{1+z}} (1+z), \frac{z}{\sqrt{1-z^2}}, \frac{1}{\sqrt{1-\frac{1}{z^2}} z}, \frac{1}{z}, \frac{\sqrt{\frac{1-z}{1+z}} (1+z)}{z} \right\}$$

If the argument has the structure $\pi k i/2 + z$ or $\pi k i/2 - z$, and $\pi k i/2 + i z$ or $\pi k i/2 - i z$ with integer k , the hyperbolic sine function can be automatically transformed into hyperbolic or trigonometric sine or cosine functions.

`Sinh[$\frac{\pi i}{2} - 4$]`

`i Cosh[4]`

`{Sinh[$\frac{\pi i}{2} - z$], Sinh[$\frac{\pi i}{2} + z$], Sinh[$-\frac{\pi i}{2} - z$], Sinh[$-\frac{\pi i}{2} + z$], Sinh[$\pi i - z$], Sinh[$\pi i + z$]}`

`{i Cosh[z], i Cosh[z], -i Cosh[z], -i Cosh[z], Sinh[z], -Sinh[z]}`

`Sinh[i 5]`

`i Sin[5]`

`{Sinh[i z], Sinh[$\frac{\pi i}{2} - i z$], Sinh[$\frac{\pi i}{2} + i z$], Sinh[$\pi i - i z$], Sinh[$\pi i + i z$]}`

`{i Sin[z], i Cos[z], i Cos[z], i Sin[z], -i Sin[z]}`

Simplification of simple expressions containing the hyperbolic sine function

Sometimes simple arithmetic operations containing the hyperbolic sine function can automatically produce other hyperbolic functions.

`1/Sinh[4]`

`csch(4)`

$$\{1/\text{Sinh}[z], 1/\text{Sinh}[\pi i/2 - z], \text{Sinh}[\pi i/2 - z]/\text{Sinh}[z], \text{Sinh}[z]/\text{Sinh}[\pi i/2 - z], 1/\text{Sinh}[\pi i/2 - z], \text{Sinh}[\pi i/2 - z]/\text{Sinh}[z]^2\}$$

$$\{\text{Csch}[z], -i \text{Sech}[z], i \text{Coth}[z], -i \text{Tanh}[z], -i \text{Sech}[z], i \text{Coth}[z] \text{Csch}[z]\}$$

The hyperbolic sine function arising as special cases from more general functions

The hyperbolic sine function can be treated as a particular case of other more general special functions. For example, $\sinh(z)$ can appear automatically from Bessel, Mathieu, Jacobi, hypergeometric, and Meijer functions for appropriate values of their parameters.

$$\{\text{BesselI}\left[\frac{1}{2}, z\right], \text{MathieuS}[1, 0, i z], \text{JacobiSC}[z, 1], \text{HypergeometricPFQ}\left[\{\}, \left\{\frac{3}{2}\right\}, \frac{z^2}{4}\right], \text{MeijerG}\left[\{\{\}, \{\}\}, \left\{\left\{\frac{1}{2}\right\}, \{0\}\right\}, -\frac{z^2}{4}\right]\}$$

$$\left\{\frac{\sqrt{\frac{2}{\pi}} \text{Sinh}[z]}{\sqrt{z}}, i \text{Sinh}[z], \text{Sinh}[z], \frac{\text{Sinh}\left[\sqrt{z^2}\right]}{\sqrt{z^2}}, \frac{\sqrt{-z^2} \text{Sinh}[z]}{\sqrt{\pi} z}\right\}$$

Equivalence transformations carried out by specialized *Mathematica* functions

General remarks

Automatic evaluation and transformations can sometimes be inconvenient: They act in only one direction and the result can be overly complicated. For example, almost everybody prefers using $\sinh(z)/2$ instead of the more complicated $\cosh(\pi i/2 - z) \sinh(\pi i/6)$. *Mathematica* provides automatic transformation of the second expression into the first one. But compact expressions like $\sinh(2z) \sinh(\pi i/16)$ should not be automatically expanded into

more complicated $i \sinh(z) \cosh(z) \left(2 - (2 + 2^{1/2})^{1/2}\right)^{1/2} \sqrt{2 - \sqrt{2 + \sqrt{2}}}$. For these purposes *Mathematica* has special commands. Some are demonstrated in the next section.

TrigExpand

The function `TrigExpand` expands out trigonometric and hyperbolic functions. In more detail, it splits up sums and integer multiples that appear in the arguments of trigonometric and hyperbolic functions, and then expands out the products of trigonometric and hyperbolic functions into sums of powers, using trigonometric and hyperbolic identities where possible. Here are some examples.

```
TrigExpand[Sinh[x - y]]
Cosh[y] Sinh[x] - Cosh[x] Sinh[y]

Sinh[4 z] // TrigExpand
4 Cosh[z]^3 Sinh[z] + 4 Cosh[z] Sinh[z]^3

Sinh[2 z]^2 // TrigExpand
```


$$-\frac{1}{2} + \frac{\text{Cosh}[z]^4}{2} + 3 \text{Cosh}[z]^2 \text{Sinh}[z]^2 + \frac{\text{Sinh}[z]^4}{2}$$

TrigExpand[{Sinh[x + y + z], Sinh[3 z]}]

$$\{\text{Cosh}[y] \text{Cosh}[z] \text{Sinh}[x] + \text{Cosh}[x] \text{Cosh}[z] \text{Sinh}[y] + \text{Cosh}[x] \text{Cosh}[y] \text{Sinh}[z] + \text{Sinh}[x] \text{Sinh}[y] \text{Sinh}[z], 3 \text{Cosh}[z]^2 \text{Sinh}[z] + \text{Sinh}[z]^3\}$$

TrigFactor

The function `TrigFactor` factors trigonometric and hyperbolic functions. In more detail, it splits up sums and integer multiples that appear in the arguments of trigonometric and hyperbolic functions, and then factors the resulting polynomials into trigonometric and hyperbolic functions, using trigonometric and hyperbolic identities where possible. Here are some examples.

TrigFactor[Sinh[x] + Sinh[y]]

$$2 \text{Cosh}\left[\frac{x}{2} - \frac{y}{2}\right] \text{Sinh}\left[\frac{x}{2} + \frac{y}{2}\right]$$

i Sinh[x] - Cosh[y] // TrigFactor

$$-\left(\text{Cosh}\left[\frac{x}{2} - \frac{y}{2}\right] - i \text{Sinh}\left[\frac{x}{2} - \frac{y}{2}\right]\right) \left(\text{Cosh}\left[\frac{x}{2} + \frac{y}{2}\right] - i \text{Sinh}\left[\frac{x}{2} + \frac{y}{2}\right]\right)$$

TrigReduce

The function `TrigReduce` rewrites the products and powers of trigonometric and hyperbolic functions in terms of trigonometric and hyperbolic functions with combined arguments. In more detail, it typically yields a linear expression involving trigonometric and hyperbolic functions with more complicated arguments. `TrigReduce` is approximately inverse to `TrigExpand` and `TrigFactor`. Here are some examples.

TrigReduce[Sinh[x] Sinh[y]]

$$\frac{1}{2} (-\text{Cosh}[x - y] + \text{Cosh}[x + y])$$

Sinh[x] Cosh[y] // TrigReduce

$$\frac{1}{2} (\text{Sinh}[x - y] + \text{Sinh}[x + y])$$

Table[TrigReduce[Sinh[z]^n], {n, 2, 5}]

$$\left\{\frac{1}{2} (-1 + \text{Cosh}[2 z]), \frac{1}{4} (-3 \text{Sinh}[z] + \text{Sinh}[3 z]), \frac{1}{8} (3 - 4 \text{Cosh}[2 z] + \text{Cosh}[4 z]), \frac{1}{16} (10 \text{Sinh}[z] - 5 \text{Sinh}[3 z] + \text{Sinh}[5 z])\right\}$$

TrigReduce[TrigExpand[{Sinh[x + y + z], Sinh[3 z], Sinh[x] Sinh[y]}]]

$$\{\text{Sinh}[x + y + z], \text{Sinh}[3 z], \frac{1}{2} (-\text{Cosh}[x - y] + \text{Cosh}[x + y])\}$$

TrigFactor[Sinh[x] + Sinh[y]] // TrigReduce

Sinh[x] + Sinh[y]

TrigToExp

The function `TrigToExp` converts trigonometric and hyperbolic functions to exponentials. It tries, where possible, to give results that do not involve explicit complex numbers. Here are some examples.

TrigToExp[Sinh[z]]

$$-\frac{e^{-z}}{2} + \frac{e^z}{2}$$

Sinh[a z] + Sinh[b z] // TrigToExp

$$-\frac{1}{2} e^{-a z} + \frac{e^{a z}}{2} - \frac{e^{-b z}}{2} + \frac{e^{b z}}{2}$$

ExpToTrig

The function `ExpToTrig` converts exponentials to trigonometric and hyperbolic functions. It is approximately inverse to `TrigToExp`. Here are some examples.

ExpToTrig[TrigToExp[Sinh[z]]]

Sinh[z]

{ $\alpha e^{-x\beta} + \alpha e^{x\beta}$, $\alpha e^{-x\beta} + \gamma e^{x\beta}$ } // ExpToTrig

{ $2 \alpha \text{Cosh}[x \beta]$, $\alpha \text{Cosh}[x \beta] + \gamma \text{Cosh}[x \beta] - \alpha \text{Sinh}[x \beta] + \gamma \text{Sinh}[x \beta]$ }

ComplexExpand

The function `ComplexExpand` expands expressions assuming that all the variables are real. The value option `TargetFunctions` is a list of functions from the set {`Re`, `Im`, `Abs`, `Arg`, `Conjugate`, `Sign`}. `ComplexExpand` tries to give results in terms of the functions specified. Here are some examples.

ComplexExpand[Sinh[x + i y]]

i Cosh[x] Sin[y] + Cos[y] Sinh[x]

Sinh[x + i y] + Sinh[x - i y] // ComplexExpand

2 Cos[y] Sinh[x]

ComplexExpand[Re[Sinh[x + i y]], TargetFunctions -> {Re, Im}]

Cos[y] Sinh[x]

ComplexExpand[Im[Sinh[x + i y]], TargetFunctions -> {Re, Im}]

Cosh[x] Sin[y]

ComplexExpand[Abs[Sinh[x + i y]], TargetFunctions -> {Re, Im}]

$$\sqrt{\cosh[x]^2 \sin[y]^2 + \cos[y]^2 \sinh[x]^2}$$

```
ComplexExpand[Abs[Sinh[x + i y]], TargetFunctions -> {Re, Im}] //
Simplify[#, {x, y} ∈ Reals] &
```

$$\frac{\sqrt{-\cos[2 y] + \cosh[2 x]}}{\sqrt{2}}$$

```
ComplexExpand[Re[Sinh[x + i y]] + Im[Sinh[x + i y]], TargetFunctions -> {Re, Im}]
```

$$\cosh[x] \sin[y] + \cos[y] \sinh[x]$$

```
ComplexExpand[Arg[Sinh[x + i y]], TargetFunctions -> {Re, Im}]
```

$$\text{ArcTan}[\cos[y] \sinh[x], \cosh[x] \sin[y]]$$

```
ComplexExpand[Arg[Sinh[x + i y]], TargetFunctions -> {Re, Im}] //
Simplify[#, {x, y} ∈ Reals] &
```

$$\text{ArcTan}[\cos[y] \sinh[x], \cosh[x] \sin[y]]$$

```
ComplexExpand[Conjugate[Sinh[x + i y]], TargetFunctions -> {Re, Im}] // Simplify
```

$$-i \cosh[x] \sin[y] + \cos[y] \sinh[x]$$

Simplify

The function `Simplify` performs a sequence of algebraic transformations on its argument, and returns the simplest form it finds. Here are some examples.

$$\text{Simplify}\left[\frac{\sinh[2 \text{ArcSinh}[z]]}{\sqrt{1+z^2}}\right]$$

$$2 z$$

$$\{\text{Simplify}[\sinh[2 z] / \sinh[z]], \sinh[2 z] / \cosh[z] // \text{Simplify}\}$$

$$\{2 \cosh[z], 2 \sinh[z]\}$$

Here is a large collection of identities involving hyperbolic functions. All are written as one large logical conjunction.

$$\text{Simplify}[\#] \ \& \ /@ \left(\text{Cosh}[z]^2 - \text{Sinh}[z]^2 == 1 \wedge \right.$$

$$\text{Sinh}[z]^2 == \frac{\text{Cosh}[2z] - 1}{2} \wedge$$

$$\text{Sinh}[2z] == 2 \text{Sinh}[z] \text{Cosh}[z] \wedge$$

$$\text{Sinh}[a + b] == \text{Sinh}[a] \text{Cosh}[b] + \text{Cosh}[a] \text{Sinh}[b] \wedge$$

$$\text{Sinh}[a - b] == \text{Sinh}[a] \text{Cosh}[b] - \text{Cosh}[a] \text{Sinh}[b] \wedge$$

$$\text{Sinh}[a] + \text{Sinh}[b] == 2 \text{Sinh}\left[\frac{a+b}{2}\right] \text{Cosh}\left[\frac{a-b}{2}\right] \wedge$$

$$\text{Sinh}[a] - \text{Sinh}[b] == 2 \text{Cosh}\left[\frac{a+b}{2}\right] \text{Sinh}\left[\frac{a-b}{2}\right] \wedge$$

$$A \text{Sinh}[z] + B \text{Cosh}[z] == A \sqrt{1 - \frac{B^2}{A^2}} \text{Sinh}\left[z + \text{ArcTanh}\left[\frac{B}{A}\right]\right] \wedge$$

$$\text{Sinh}[a] \text{Sinh}[b] == \frac{\text{Cosh}[a+b] - \text{Cosh}[a-b]}{2} \wedge$$

$$\text{Sinh}[a] \text{Cosh}[b] == \frac{\text{Sinh}[a+b] + \text{Sinh}[a-b]}{2} \wedge$$

$$\text{Sinh}[z_1 + z_2]^4 - 2 \left(\text{Sinh}[z_1]^2 + \text{Sinh}[z_2]^2 + 2 \text{Sinh}[z_1]^2 \text{Sinh}[z_2]^2 \right) \text{Sinh}[z_1 + z_2]^2 +$$

$$\left. \left(\text{Sinh}[z_1]^2 - \text{Sinh}[z_2]^2 \right)^2 == 0 \right)$$

True

The function `Simplify` has the `Assumption` option. For example, *Mathematica* knows that $\sinh(x) \geq 0$ for all real $x \geq 0$, and knows about the periodicity of hyperbolic functions for the symbolic integer coefficient k of $k\pi i$.

`Simplify[Sinh[x] ≥ 0, x ∈ Reals ∧ x ≥ 0]`

True

`Sinh[x] ≥ 0 // Simplify[#, x ≥ 0] &`

True

`Simplify[{Sinh[z + 2 k π i], Sinh[z + k π i] / Sinh[z]}, k ∈ Integers]`

{Sinh[z], (-1)^k}

Mathematica also knows that the composition of the inverse and direct hyperbolic sine produces the value of the inner argument under the corresponding restriction.

`ArcSinh[Sinh[z]]`

ArcSinh[Sinh[z]]

Simplify[ArcSinh[Sinh[z]], -π/2 < Im[z] < π/2]

z

FunctionExpand (and Together)

While the hyperbolic sine function auto-evaluates for simple fractions of π i, for more complicated cases it stays as a hyperbolic sine function to avoid the build up of large expressions. Using the function `FunctionExpand`, the hyperbolic sine function can sometimes be transformed into explicit radicals. Here are some examples.

$$\left\{ \sinh\left[\frac{\pi i}{16}\right], \sinh\left[\frac{\pi i}{60}\right] \right\}$$

$$\left\{ i \sin\left[\frac{\pi}{16}\right], i \sin\left[\frac{\pi}{60}\right] \right\}$$

FunctionExpand[%]

$$\left\{ \frac{1}{2} i \sqrt{2 - \sqrt{2 + \sqrt{2}}}, \right. \\ \left. i \left(-\frac{\frac{1}{8} \sqrt{3} (-1 + \sqrt{5}) - \frac{1}{4} \sqrt{\frac{1}{2} (5 + \sqrt{5})}}{\sqrt{2}} + \frac{\frac{1}{8} (-1 + \sqrt{5}) - \frac{1}{4} \sqrt{\frac{3}{2} (5 + \sqrt{5})}}{\sqrt{2}} \right) \right\}$$

Together[%]

$$\left\{ \frac{1}{2} i \sqrt{2 - \sqrt{2 + \sqrt{2}}}, \frac{1}{16} i \left(-\sqrt{2} - \sqrt{6} + \sqrt{10} + \sqrt{30} + 2\sqrt{5 + \sqrt{5}} - 2\sqrt{3(5 + \sqrt{5})} \right) \right\}$$

If the denominator contains squares of integers other than 2, the results always contain complex numbers deeply inside of expression (meaning that the imaginary number $i = \sqrt{-1}$ appears unavoidably).

$$\sinh\left[\frac{\pi i}{9}\right]$$

$$i \sin\left[\frac{\pi}{9}\right]$$

FunctionExpand[%] // **Together**

$$\frac{1}{8} \left(2^{2/3} (-1 - i \sqrt{3})^{1/3} + i 2^{2/3} \sqrt{3} (-1 - i \sqrt{3})^{1/3} - 2^{2/3} (-1 + i \sqrt{3})^{1/3} + i 2^{2/3} \sqrt{3} (-1 + i \sqrt{3})^{1/3} \right)$$

Here the function `RootReduce` is used to express the previous algebraic numbers as the roots of polynomial equations.

RootReduce[**Simplify**[%]]

$$\text{Root}\left[3 + 36 \#1^2 + 96 \#1^4 + 64 \#1^6 \&, 4\right]$$

The function `FunctionExpand` also reduces hyperbolic expressions with compound arguments or compositions, including inverse hyperbolic functions, to simpler ones. Here are some examples.

$$\left\{ \text{Sinh}\left[\sqrt{z^2}\right], \text{Sinh}\left[\frac{\text{ArcSinh}[z]}{2}\right], \text{Sinh}\left[3 \text{ArcSinh}[z]\right] \right\} // \text{FunctionExpand}$$

$$\left\{ \frac{\sqrt{-i z} \sqrt{i z} \text{Sinh}[z]}{z}, \frac{z \sqrt{-1 + \sqrt{i}(-i+z)} \sqrt{-i(i+z)}}{\sqrt{2} \sqrt{-i z} \sqrt{i z}}, -i(i z^3 + 3 i z(1+z^2)) \right\}$$

Applying `Simplify` to the last expression gives a more compact result.

```
Simplify[%]
```

$$\left\{ \frac{\sqrt{z^2} \text{Sinh}[z]}{z}, \frac{z \sqrt{-1 + \sqrt{1+z^2}}}{\sqrt{2} \sqrt{z^2}}, z(3+4z^2) \right\}$$

FullSimplify

The function `FullSimplify` tries a wider range of transformations than `Simplify` and returns the simplest form it finds. Here are some examples that contrast the results of applying the functions `Simplify` and `FullSimplify` to the same expressions.

```
set1 = {Sinh[Log[z + Sqrt[1 + z^2]]],
  Sinh[Log[Sqrt[-1 + z/2] * Sqrt[1 + z/2] + z/2]], Sinh[-Log[1 - z]/2 + Log[1 + z]/2],
  Sinh[-Log[1 - z^(-1)]/2 + Log[1 + z^(-1)]/2], Sinh[Log[Sqrt[1 + z^(-2)] + z^(-1)]],
  Sinh[Log[Sqrt[-1 + z^(-1)] * Sqrt[1 + z^(-1)] + z^(-1)]]}
```

$$\left\{ \frac{-1 + \left(z + \sqrt{1+z^2}\right)^2}{2\left(z + \sqrt{1+z^2}\right)}, \frac{-1 + \left(\sqrt{-1 + \frac{z}{2}} \sqrt{1 + \frac{z}{2}} + \frac{z}{2}\right)^2}{2\left(\sqrt{-1 + \frac{z}{2}} \sqrt{1 + \frac{z}{2}} + \frac{z}{2}\right)}, -\text{Sinh}\left[\frac{1}{2} \text{Log}[1 - z] - \frac{1}{2} \text{Log}[1 + z]\right], \right.$$

$$\left. -\text{Sinh}\left[\frac{1}{2} \text{Log}\left[1 - \frac{1}{z}\right] - \frac{1}{2} \text{Log}\left[1 + \frac{1}{z}\right]\right], \frac{-1 + \left(\sqrt{1 + \frac{1}{z^2}} + \frac{1}{z}\right)^2}{2\left(\sqrt{1 + \frac{1}{z^2}} + \frac{1}{z}\right)}, \frac{-1 + \left(\sqrt{-1 + \frac{1}{z}} \sqrt{1 + \frac{1}{z}} + \frac{1}{z}\right)^2}{2\left(\sqrt{-1 + \frac{1}{z}} \sqrt{1 + \frac{1}{z}} + \frac{1}{z}\right)} \right\}$$

```
set1 // Simplify
```

$$\left\{ z, \frac{-4 + z^2 + \sqrt{-2 + z} z \sqrt{2 + z}}{2 \left(z + \sqrt{-2 + z} \sqrt{2 + z} \right)}, -\text{Sinh} \left[\frac{1}{2} \left(\text{Log}[1 - z] - \text{Log}[1 + z] \right) \right], \right.$$

$$\left. \text{Sinh} \left[\frac{1}{2} \left(\text{Log} \left[1 + \frac{1}{z} \right] - \text{Log} \left[\frac{-1 + z}{z} \right] \right) \right], \frac{1}{z}, \frac{1 + \sqrt{-1 + \frac{1}{z}} \sqrt{1 + \frac{1}{z}} z - z^2}{z \left(1 + \sqrt{-1 + \frac{1}{z}} \sqrt{1 + \frac{1}{z}} z \right)} \right\}$$

set1 // FullSimplify

$$\left\{ z, \frac{1}{2} \sqrt{-2 + z} \sqrt{2 + z}, \frac{z}{\sqrt{1 - z^2}}, \right.$$

$$\left. \text{Sinh} \left[\frac{1}{2} \left(\text{Log} \left[1 + \frac{1}{z} \right] - \text{Log}[-1 + z] + \text{Log}[z] \right) \right], \frac{1}{z}, \sqrt{-1 + \frac{1}{z}} \sqrt{1 + \frac{1}{z}} \right\}$$

Operations carried out by specialized *Mathematica* functions

Series expansions

Calculating the series expansion of a hyperbolic sine function to hundreds of terms can be done in seconds.

Series[Sinh[z], {z, 0, 3}]

$$z + \frac{z^3}{6} + O[z]^4$$

Normal[%]

$$z + \frac{z^3}{6}$$

Series[Sinh[z], {z, 0, 100}] // Timing

$$\left\{ 0. \text{ Second}, z + \frac{z^3}{6} + \frac{z^5}{120} + \frac{z^7}{5040} + \frac{z^9}{362880} + \frac{z^{11}}{39916800} + \frac{z^{13}}{6227020800} + \frac{z^{15}}{1307674368000} + \right.$$

$$\frac{z^{17}}{355687428096000} + \frac{z^{19}}{121645100408832000} + \frac{z^{21}}{51090942171709440000} +$$

$$\frac{z^{23}}{25852016738884976640000} + \frac{z^{25}}{15511210043330985984000000} +$$

$$\frac{z^{27}}{10888869450418352160768000000} + \frac{z^{29}}{8841761993739701954543616000000} +$$

$$\frac{z^{31}}{8222838654177922817725562880000000} + \frac{z^{33}}{8683317618811886495518194401280000000} +$$

$$\left. \frac{z^{35}}{10333147966386144929666651337523200000000} \right\}$$

$$\begin{aligned}
& \frac{z^{37}}{13\ 763\ 753\ 091\ 226\ 345\ 046\ 315\ 979\ 581\ 580\ 902\ 400\ 000\ 000} + \\
& \frac{z^{39}}{20\ 397\ 882\ 081\ 197\ 443\ 358\ 640\ 281\ 739\ 902\ 897\ 356\ 800\ 000\ 000} + \\
& \frac{z^{41}}{33\ 452\ 526\ 613\ 163\ 807\ 108\ 170\ 062\ 053\ 440\ 751\ 665\ 152\ 000\ 000\ 000} + \\
& \frac{z^{43}}{60\ 415\ 263\ 063\ 373\ 835\ 637\ 355\ 132\ 068\ 513\ 997\ 507\ 264\ 512\ 000\ 000\ 000} + \\
& \frac{z^{45}}{119\ 622\ 220\ 865\ 480\ 194\ 561\ 963\ 161\ 495\ 657\ 715\ 064\ 383\ 733\ 760\ 000\ 000\ 000} + \\
& \frac{z^{47}}{258\ 623\ 241\ 511\ 168\ 180\ 642\ 964\ 355\ 153\ 611\ 979\ 969\ 197\ 632\ 389\ 120\ 000\ 000\ 000} + \\
& \frac{z^{49}}{608\ 281\ 864\ 034\ 267\ 560\ 872\ 252\ 163\ 321\ 295\ 376\ 887\ 552\ 831\ 379\ 210\ 240\ 000\ 000\ 000} + \\
& \frac{z^{51}}{1\ 551\ 118\ 753\ 287\ 382\ 280\ 224\ 243\ 016\ 469\ 303\ 211\ 063\ 259\ 720\ 016\ 986\ 112\ 000\ 000\ 000\ 000} + \\
& \frac{z^{53}}{4\ 274\ 883\ 284\ 060\ 025\ 564\ 298\ 013\ 753\ 389\ 399\ 649\ 690\ 343\ 788\ 366\ 813\ 724\ 672\ 000\ 000\ 000\ 000} + z^{55} / \\
& \frac{z^{57}}{12\ 696\ 403\ 353\ 658\ 275\ 925\ 965\ 100\ 847\ 566\ 516\ 959\ 580\ 321\ 051\ 449\ 436\ 762\ 275\ 840\ 000\ 000\ 000\ 000} + \\
& \frac{z^{59}}{40\ 526\ 919\ 504\ 877\ 216\ 755\ 680\ 601\ 905\ 432\ 322\ 134\ 980\ 384\ 796\ 226\ 602\ 145\ 184\ 481\ 280\ 000\ 000\ 000} + \\
& \frac{z^{61}}{138\ 683\ 118\ 545\ 689\ 835\ 737\ 939\ 019\ 720\ 389\ 406\ 345\ 902\ 876\ 772\ 687\ 432\ 540\ 821\ 294\ 940\ 160\ 000} + \\
& \frac{z^{63}}{507\ 580\ 213\ 877\ 224\ 798\ 800\ 856\ 812\ 176\ 625\ 227\ 226\ 004\ 528\ 988\ 036\ 003\ 099\ 405\ 939\ 480\ 985\ 600} + \\
& \frac{z^{65}}{1\ 982\ 608\ 315\ 404\ 440\ 064\ 116\ 146\ 708\ 361\ 898\ 137\ 544\ 773\ 690\ 227\ 268\ 628\ 106\ 279\ 599\ 612\ 729\ 753} + \\
& \frac{z^{67}}{8\ 247\ 650\ 592\ 082\ 470\ 666\ 723\ 170\ 306\ 785\ 496\ 252\ 186\ 258\ 551\ 345\ 437\ 492\ 922\ 123\ 134\ 388\ 955\ 774} + \\
& \frac{z^{69}}{36\ 471\ 110\ 918\ 188\ 685\ 288\ 249\ 859\ 096\ 605\ 464\ 427\ 167\ 635\ 314\ 049\ 524\ 593\ 701\ 628\ 500\ 267\ 962\ 436} + \\
& \frac{z^{71}}{171\ 122\ 452\ 428\ 141\ 311\ 372\ 468\ 338\ 881\ 272\ 839\ 092\ 270\ 544\ 893\ 520\ 369\ 393\ 648\ 040\ 923\ 257\ 279} + \\
& \frac{z^{73}}{850\ 478\ 588\ 567\ 862\ 317\ 521\ 167\ 644\ 239\ 926\ 010\ 288\ 584\ 608\ 120\ 796\ 235\ 886\ 430\ 763\ 388\ 588\ 680} + \\
& \frac{z^{75}}{4\ 470\ 115\ 461\ 512\ 684\ 340\ 891\ 257\ 138\ 125\ 051\ 110\ 076\ 800\ 700\ 282\ 905\ 015\ 819\ 080\ 092\ 370\ 422\ 104} + \\
& \frac{z^{77}}{24\ 809\ 140\ 811\ 395\ 398\ 091\ 946\ 477\ 116\ 594\ 033\ 660\ 926\ 243\ 886\ 570\ 122\ 837\ 795\ 894\ 512\ 655\ 842\ 677} + \\
& \frac{z^{79}}{145\ 183\ 092\ 028\ 285\ 869\ 634\ 070\ 784\ 086\ 308\ 284\ 983\ 740\ 379\ 224\ 208\ 358\ 846\ 781\ 574\ 688\ 061\ 991} + \\
& \frac{z^{81}}{894\ 618\ 213\ 078\ 297\ 528\ 685\ 144\ 171\ 539\ 831\ 652\ 069\ 808\ 216\ 779\ 571\ 907\ 213\ 868\ 063\ 227\ 837\ 990} +
\end{aligned}$$


```

5 797 126 020 747 367 985 879 734 231 578 109 105 412 357 244 731 625 958 745 865 049 716 390 179 \
693 892 056 256 184 534 249 745 940 480 000 000 000 000 000 000 + z83 /
39 455 239 697 206 586 511 897 471 180 120 610 571 436 503 407 643 446 275 224 357 528 369 751 562 \
996 629 334 879 591 940 103 770 870 906 880 000 000 000 000 000 000 + z85 /
281 710 411 438 055 027 694 947 944 226 061 159 480 056 634 330 574 206 405 101 912 752 560 026 \
159 795 933 451 040 286 452 340 924 018 275 123 200 000 000 000 000 000 000 + z87 /
2 107 757 298 379 527 717 213 600 518 699 389 595 229 783 738 061 356 212 322 972 511 214 654 115 \
727 593 174 080 683 423 236 414 793 504 734 471 782 400 000 000 000 000 000 000 + z89 /
16 507 955 160 908 461 081 216 919 262 453 619 309 839 666 236 496 541 854 913 520 707 833 171 034 \
378 509 739 399 912 570 787 600 662 729 080 382 999 756 800 000 000 000 000 000 000 + z91 /
135 200 152 767 840 296 255 166 568 759 495 142 147 586 866 476 906 677 791 741 734 597 153 670 \
771 559 994 765 685 283 954 750 449 427 751 168 336 768 008 192 000 000 000 000 000 000 000 +
z93 /
1 156 772 507 081 641 574 759 205 162 306 240 436 214 753 229 576 413 535 186 142 281 213 246 807 \
121 467 315 215 203 289 516 844 845 303 838 996 289 387 078 090 752 000 000 000 000 000 000 000 +
z95 /
10 329 978 488 239 059 262 599 702 099 394 727 095 397 746 340 117 372 869 212 250 571 234 293 987 \
594 703 124 871 765 375 385 424 468 563 282 236 864 226 607 350 415 360 000 000 000 000 000 000 \
000 + z97 /
96 192 759 682 482 119 853 328 425 949 563 698 712 343 813 919 172 976 158 104 477 319 333 745 612 \
481 875 498 805 879 175 589 072 651 261 284 189 679 678 167 647 067 832 320 000 000 000 000 000 \
000 000 + z99 /
933 262 154 439 441 526 816 992 388 562 667 004 907 159 682 643 816 214 685 929 638 952 175 999 \
932 299 156 089 414 639 761 565 182 862 536 979 208 272 237 582 511 852 109 168 640 000 000 000 \
000 000 000 000 + O[z]101 }

```

Mathematica comes with the add-on package `DiscreteMath`RSolve`` that allows finding the general terms of series for many functions. After loading this package, and using the package function `SeriesTerm`, the following n^{th} term of $\sinh(z)$ can be evaluated.

```

<< DiscreteMath`RSolve`
SeriesTerm[Sinh[z], {z, 0, n}] z^n
z^n If[Odd[n], 1/n!, 0]

```

This result can be verified by the following process. The previous expression is not zero for $n = 2k + 1$ and the corresponding sum has the following value.

```

Sum[1/(2k+1)! z2k+1, {k, 0, ∞}]
z Sinh[√z2]
-----
√z2
FunctionExpand[%]

```

`Sinh[z]`

Differentiation

Mathematica can evaluate derivatives of the hyperbolic sine function of an arbitrary positive integer order.

`∂z Sinh[z]`

`Cosh[z]`

`∂{z,2} Sinh[z]`

`Sinh[z]`

`Table[D[Sinh[z], {z, n}], {n, 10}]`

`{Cosh[z], Sinh[z], Cosh[z], Sinh[z],
Cosh[z], Sinh[z], Cosh[z], Sinh[z], Cosh[z], Sinh[z]}`

Finite summation

Mathematica can calculate finite symbolic sums that contain the hyperbolic sine function. Here are some examples.

$$\sum_{k=1}^n \text{Sinh}[a k]$$

$$\frac{e^a (-1 + e^{a n})}{2 (-1 + e^a)} - \frac{e^{-a n} (-1 + e^{a n})}{2 (-1 + e^a)}$$

$$\sum_{k=1}^n (-1)^k \text{Sinh}[a k]$$

$$-\frac{1 + (-e^{-a})^n}{2 (1 + e^a)} + \frac{e^a (-1 + (-e^a)^n)}{2 (1 + e^a)}$$

Infinite summation

Mathematica can calculate infinite sums that contain the hyperbolic sine function. Here are some examples.

$$\sum_{k=1}^{\infty} z^k \text{Sinh}[k x]$$

$$-\frac{z}{2 (e^x - z)} - \frac{e^x z}{2 (-1 + e^x z)}$$

$$\sum_{k=1}^{\infty} \frac{\text{Sinh}[k x]}{k!}$$

$$\frac{1}{2} (1 - e^{e^{-x}}) + \frac{1}{2} (-1 + e^{e^x})$$

$$\sum_{k=1}^{\infty} \frac{\text{Sinh}[k x]}{k}$$

$$\frac{1}{2} \operatorname{Log}[1 - e^{-x}] - \frac{1}{2} \operatorname{Log}[1 - e^x]$$

Finite products

Mathematica can calculate some doable finite symbolic products that contain the hyperbolic sine function. Here are two examples.

$$\prod_{k=1}^{n-1} \sinh\left[\frac{k \pi i}{n}\right]$$

$$\left(\frac{i}{2}\right)^{-1+n} n$$

$$\prod_{k=1}^{n-1} \operatorname{Sinh}\left[z + \frac{\pi k i}{n}\right]$$

$$\left(\frac{i}{2}\right)^{-1+n} \operatorname{Csch}[z] \operatorname{Sinh}[n z]$$

Indefinite integration

Mathematica can calculate a huge set of doable indefinite integrals that contain the hyperbolic sine function. Here are some examples.

$$\int \sinh[z] \, dz$$

$$\operatorname{Cosh}[z]$$

$$\int \sinh[z]^a \, dz$$

$$-\operatorname{Cosh}[z] \operatorname{Hypergeometric2F1}\left[\frac{1}{2}, \frac{1-a}{2}, \frac{3}{2}, \operatorname{Cosh}[z]^2\right] \operatorname{Sinh}[z]^{1+a} (-\operatorname{Sinh}[z]^2)^{\frac{1}{2}(-1-a)}$$

Definite integration

Mathematica can calculate wide classes of definite integrals that contain the hyperbolic sine function. Here are some examples.

$$\int_0^b \frac{\sinh[z]}{z} \, dz$$

$$\operatorname{SinhIntegral}[b]$$

$$\int_0^\infty e^{a t} \sinh[b t] \, dt$$

$$\text{If}\left[0 > \operatorname{Re}[a + b] \ \&\& \operatorname{Re}[-a + b] > 0 \ \&\& \operatorname{Re}[a] < 0, \frac{b}{a^2 - b^2}, \int_0^\infty e^{a t} \sinh[b t] \, dt\right]$$

Limit operation

Mathematica can calculate limits that contain the hyperbolic sine function. Here are some examples.

$$\text{Limit}\left[\frac{\text{Sinh}[2 z]}{z}, z \rightarrow 0\right]$$

2

$$\text{Limit}\left[\frac{\text{Sinh}\left[\sqrt{z^2}\right]}{z}, z \rightarrow 0, \text{Direction} \rightarrow 1\right]$$

-1

$$\text{Limit}\left[\frac{\text{Sinh}\left[\sqrt{z^2}\right]}{z}, z \rightarrow 0, \text{Direction} \rightarrow -1\right]$$

1

Solving equations

The next inputs solve two equations that contain the hyperbolic sine function. Because of the multivalued nature of the inverse hyperbolic sine function, the message indicates that only some of the possible solutions are returned.

$$\text{Solve}\left[\text{Sinh}[z]^2 + 3 \text{Sinh}[z + \text{Pi}/6] = 4, z\right]$$

Solve::ifun: Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information. More...

$$\left\{\left\{z \rightarrow \text{ArcCosh}\left[-\frac{3}{2} \text{Sinh}\left[\frac{\pi}{6}\right] - \frac{1}{2} \sqrt{\left(10 + 9 \text{Cosh}\left[\frac{\pi}{6}\right]^2 + \frac{1}{3} \left(-10 - 9 \text{Cosh}\left[\frac{\pi}{6}\right]^2 + 9 \text{Sinh}\left[\frac{\pi}{6}\right]^2\right) + \left(12 \left(25 + 9 \text{Cosh}\left[\frac{\pi}{6}\right]^2\right) + 540 \text{Sinh}\left[\frac{\pi}{6}\right]^2 + \left(-10 - 9 \text{Cosh}\left[\frac{\pi}{6}\right]^2 + 9 \text{Sinh}\left[\frac{\pi}{6}\right]^2\right)^2\right) / \left(3 \left(\frac{1}{2} \left(16000 + 17280 \text{Cosh}\left[\frac{\pi}{6}\right]^2 + 972 \text{Cosh}\left[\frac{\pi}{6}\right]^4 - 1458 \text{Cosh}\left[\frac{\pi}{6}\right]^6 + 21600 \text{Sinh}\left[\frac{\pi}{6}\right]^2 - 1944 \text{Cosh}\left[\frac{\pi}{6}\right]^2 \text{Sinh}\left[\frac{\pi}{6}\right]^2 + 4374 \text{Cosh}\left[\frac{\pi}{6}\right]^4 \text{Sinh}\left[\frac{\pi}{6}\right]^2 + 9720 \text{Sinh}\left[\frac{\pi}{6}\right]^4 - 4374 \text{Cosh}\left[\frac{\pi}{6}\right]^2 \text{Sinh}\left[\frac{\pi}{6}\right]^4 + 1458 \text{Sinh}\left[\frac{\pi}{6}\right]^6 + \sqrt{\left(-223948800 \text{Cosh}\left[\frac{\pi}{6}\right]^4 - 332563968 \text{Cosh}\left[\frac{\pi}{6}\right]^6 - 161558064 \text{Cosh}\left[\frac{\pi}{6}\right]^8 - 25509168 \text{Cosh}\left[\frac{\pi}{6}\right]^{10} - 75582720 \text{Cosh}\left[\frac{\pi}{6}\right]^4 \text{Sinh}\left[\frac{\pi}{6}\right]^2 + 170061120 \text{Cosh}\left[\frac{\pi}{6}\right]^6 \text{Sinh}\left[\frac{\pi}{6}\right]^2 + 76527504 \text{Cosh}\left[\frac{\pi}{6}\right]^8 \text{Sinh}\left[\frac{\pi}{6}\right]^2 + 68024448 \text{Cosh}\left[\frac{\pi}{6}\right]^4 \text{Sinh}\left[\frac{\pi}{6}\right]^4 - 76527504 \text{Cosh}\left[\frac{\pi}{6}\right]^6 \text{Sinh}\left[\frac{\pi}{6}\right]^4 + 25509168 \text{Cosh}\left[\frac{\pi}{6}\right]^4 \text{Sinh}\left[\frac{\pi}{6}\right]^6\right)}\right)}\right]^{1/3}\right)\right\}\right\} +$$

$$\begin{aligned}
 & \frac{1}{3} \left(\frac{1}{2} \left(16\,000 + 17\,280 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^2 + 972 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^4 - 1458 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^6 + 21\,600 \operatorname{Sinh}\left[\frac{\pi}{6}\right]^2 - \right. \right. \\
 & \quad 1944 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^2 \operatorname{Sinh}\left[\frac{\pi}{6}\right]^2 + 4374 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^4 \operatorname{Sinh}\left[\frac{\pi}{6}\right]^2 + 9720 \operatorname{Sinh}\left[\frac{\pi}{6}\right]^4 - \\
 & \quad 4374 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^2 \operatorname{Sinh}\left[\frac{\pi}{6}\right]^4 + 1458 \operatorname{Sinh}\left[\frac{\pi}{6}\right]^6 + \sqrt{\left(-223\,948\,800 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^4 - \right.} \\
 & \quad 332\,563\,968 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^6 - 161\,558\,064 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^8 - 25\,509\,168 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^{10} - \\
 & \quad 75\,582\,720 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^4 \operatorname{Sinh}\left[\frac{\pi}{6}\right]^2 + 170\,061\,120 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^6 \operatorname{Sinh}\left[\frac{\pi}{6}\right]^2 + \\
 & \quad 76\,527\,504 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^8 \operatorname{Sinh}\left[\frac{\pi}{6}\right]^2 + 68\,024\,448 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^4 \operatorname{Sinh}\left[\frac{\pi}{6}\right]^4 - \\
 & \quad \left. \left. \left. 76\,527\,504 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^6 \operatorname{Sinh}\left[\frac{\pi}{6}\right]^4 + 25\,509\,168 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^4 \operatorname{Sinh}\left[\frac{\pi}{6}\right]^6 \right) \right) \right)^{1/3} \right) - \\
 & \frac{1}{2} \sqrt{\left(10 + 9 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^2 + 9 \operatorname{Sinh}\left[\frac{\pi}{6}\right]^2 - \frac{1}{3} \left(-10 - 9 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^2 + 9 \operatorname{Sinh}\left[\frac{\pi}{6}\right]^2 \right) - \right.} \\
 & \quad \left. \left(12 \left(25 + 9 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^2 \right) + 540 \operatorname{Sinh}\left[\frac{\pi}{6}\right]^2 + \left(-10 - 9 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^2 + 9 \operatorname{Sinh}\left[\frac{\pi}{6}\right]^2 \right)^2 \right) \right) / \\
 & \quad \left(3 \left(\frac{1}{2} \left(16\,000 + 17\,280 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^2 + 972 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^4 - 1458 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^6 + 21\,600 \operatorname{Sinh}\left[\frac{\pi}{6}\right]^2 - \right. \right. \right. \\
 & \quad 1944 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^2 \operatorname{Sinh}\left[\frac{\pi}{6}\right]^2 + 4374 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^4 \operatorname{Sinh}\left[\frac{\pi}{6}\right]^2 + 9720 \operatorname{Sinh}\left[\frac{\pi}{6}\right]^4 - \\
 & \quad 4374 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^2 \operatorname{Sinh}\left[\frac{\pi}{6}\right]^4 + 1458 \operatorname{Sinh}\left[\frac{\pi}{6}\right]^6 + \sqrt{\left(-223\,948\,800 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^4 - \right.} \\
 & \quad 332\,563\,968 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^6 - 161\,558\,064 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^8 - 25\,509\,168 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^{10} - \\
 & \quad 75\,582\,720 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^4 \operatorname{Sinh}\left[\frac{\pi}{6}\right]^2 + 170\,061\,120 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^6 \operatorname{Sinh}\left[\frac{\pi}{6}\right]^2 + \\
 & \quad 76\,527\,504 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^8 \operatorname{Sinh}\left[\frac{\pi}{6}\right]^2 + 68\,024\,448 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^4 \operatorname{Sinh}\left[\frac{\pi}{6}\right]^4 - \\
 & \quad \left. \left. \left. 76\,527\,504 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^6 \operatorname{Sinh}\left[\frac{\pi}{6}\right]^4 + 25\,509\,168 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^4 \operatorname{Sinh}\left[\frac{\pi}{6}\right]^6 \right) \right) \right)^{1/3} \right) - \\
 & \frac{1}{3} \left(\frac{1}{2} \left(16\,000 + 17\,280 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^2 + 972 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^4 - 1458 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^6 + 21\,600 \operatorname{Sinh}\left[\frac{\pi}{6}\right]^2 - \right. \right. \\
 & \quad 1944 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^2 \operatorname{Sinh}\left[\frac{\pi}{6}\right]^2 + 4374 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^4 \operatorname{Sinh}\left[\frac{\pi}{6}\right]^2 + 9720 \operatorname{Sinh}\left[\frac{\pi}{6}\right]^4 -
 \end{aligned}$$

$$\begin{aligned}
 & \left(76\,527\,504 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^8 \operatorname{Sinh}\left[\frac{\pi}{6}\right]^2 + 68\,024\,448 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^4 \operatorname{Sinh}\left[\frac{\pi}{6}\right]^4 - \right. \\
 & \left. 76\,527\,504 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^6 \operatorname{Sinh}\left[\frac{\pi}{6}\right]^4 + 25\,509\,168 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^4 \operatorname{Sinh}\left[\frac{\pi}{6}\right]^6 \right) \Bigg)^{1/3} + \\
 & \frac{1}{2} \sqrt{\left(10 + 9 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^2 + 9 \operatorname{Sinh}\left[\frac{\pi}{6}\right]^2 - \frac{1}{3} \left(-10 - 9 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^2 + 9 \operatorname{Sinh}\left[\frac{\pi}{6}\right]^2 \right) - \right. \\
 & \left. \left(12 \left(25 + 9 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^2 \right) + 540 \operatorname{Sinh}\left[\frac{\pi}{6}\right]^2 + \left(-10 - 9 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^2 + 9 \operatorname{Sinh}\left[\frac{\pi}{6}\right]^2 \right)^2 \right) \right) / \\
 & \left(3 \left(\frac{1}{2} \left(16\,000 + 17\,280 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^2 + 972 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^4 - 1458 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^6 + 21\,600 \operatorname{Sinh}\left[\frac{\pi}{6}\right]^2 - \right. \right. \right. \\
 & \left. 1944 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^2 \operatorname{Sinh}\left[\frac{\pi}{6}\right]^2 + 4374 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^4 \operatorname{Sinh}\left[\frac{\pi}{6}\right]^2 + 9720 \operatorname{Sinh}\left[\frac{\pi}{6}\right]^4 - \right. \\
 & \left. 4374 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^2 \operatorname{Sinh}\left[\frac{\pi}{6}\right]^4 + 1458 \operatorname{Sinh}\left[\frac{\pi}{6}\right]^6 + \sqrt{\left(-223\,948\,800 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^4 - \right. \right. \right. \\
 & \left. 332\,563\,968 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^6 - 161\,558\,064 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^8 - 25\,509\,168 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^{10} - \right. \\
 & \left. 75\,582\,720 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^4 \operatorname{Sinh}\left[\frac{\pi}{6}\right]^2 + 170\,061\,120 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^6 \operatorname{Sinh}\left[\frac{\pi}{6}\right]^2 + \right. \\
 & \left. 76\,527\,504 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^8 \operatorname{Sinh}\left[\frac{\pi}{6}\right]^2 + 68\,024\,448 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^4 \operatorname{Sinh}\left[\frac{\pi}{6}\right]^4 - \right. \\
 & \left. \left. \left. 76\,527\,504 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^6 \operatorname{Sinh}\left[\frac{\pi}{6}\right]^4 + 25\,509\,168 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^4 \operatorname{Sinh}\left[\frac{\pi}{6}\right]^6 \right) \right) \right) \Bigg)^{1/3} - \\
 & \frac{1}{3} \left(\frac{1}{2} \left(16\,000 + 17\,280 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^2 + 972 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^4 - 1458 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^6 + 21\,600 \operatorname{Sinh}\left[\frac{\pi}{6}\right]^2 - \right. \right. \\
 & \left. 1944 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^2 \operatorname{Sinh}\left[\frac{\pi}{6}\right]^2 + 4374 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^4 \operatorname{Sinh}\left[\frac{\pi}{6}\right]^2 + 9720 \operatorname{Sinh}\left[\frac{\pi}{6}\right]^4 - \right. \\
 & \left. 4374 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^2 \operatorname{Sinh}\left[\frac{\pi}{6}\right]^4 + 1458 \operatorname{Sinh}\left[\frac{\pi}{6}\right]^6 + \sqrt{\left(-223\,948\,800 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^4 - \right. \right. \right. \\
 & \left. 332\,563\,968 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^6 - 161\,558\,064 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^8 - 25\,509\,168 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^{10} - \right. \\
 & \left. 75\,582\,720 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^4 \operatorname{Sinh}\left[\frac{\pi}{6}\right]^2 + 170\,061\,120 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^6 \operatorname{Sinh}\left[\frac{\pi}{6}\right]^2 + \right. \\
 & \left. 76\,527\,504 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^8 \operatorname{Sinh}\left[\frac{\pi}{6}\right]^2 + 68\,024\,448 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^4 \operatorname{Sinh}\left[\frac{\pi}{6}\right]^4 - \right. \\
 & \left. \left. \left. 76\,527\,504 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^6 \operatorname{Sinh}\left[\frac{\pi}{6}\right]^4 + 25\,509\,168 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^4 \operatorname{Sinh}\left[\frac{\pi}{6}\right]^6 \right) \right) \right) \Bigg)^{1/3} -
 \end{aligned}$$

$$\begin{aligned}
 & \left(12 \left(25 + 9 \operatorname{Cosh} \left[\frac{\pi}{6} \right]^2 \right) + 540 \operatorname{Sinh} \left[\frac{\pi}{6} \right]^2 + \right. \\
 & \quad \left. \left(-10 - 9 \operatorname{Cosh} \left[\frac{\pi}{6} \right]^2 + 9 \operatorname{Sinh} \left[\frac{\pi}{6} \right]^2 \right)^2 \right) / \\
 & \left(3 \left(\frac{1}{2} \left(16\,000 + 17\,280 \operatorname{Cosh} \left[\frac{\pi}{6} \right]^2 + 972 \operatorname{Cosh} \left[\frac{\pi}{6} \right]^4 - 1458 \operatorname{Cosh} \left[\frac{\pi}{6} \right]^6 + 21\,600 \operatorname{Sinh} \left[\frac{\pi}{6} \right]^2 - \right. \right. \right. \\
 & \quad 1944 \operatorname{Cosh} \left[\frac{\pi}{6} \right]^2 \operatorname{Sinh} \left[\frac{\pi}{6} \right]^2 + 4374 \operatorname{Cosh} \left[\frac{\pi}{6} \right]^4 \operatorname{Sinh} \left[\frac{\pi}{6} \right]^2 + 9720 \operatorname{Sinh} \left[\frac{\pi}{6} \right]^4 - \\
 & \quad 4374 \operatorname{Cosh} \left[\frac{\pi}{6} \right]^2 \operatorname{Sinh} \left[\frac{\pi}{6} \right]^4 + 1458 \operatorname{Sinh} \left[\frac{\pi}{6} \right]^6 + \sqrt{\left(-223\,948\,800 \operatorname{Cosh} \left[\frac{\pi}{6} \right]^4 - \right.} \\
 & \quad 332\,563\,968 \operatorname{Cosh} \left[\frac{\pi}{6} \right]^6 - 161\,558\,064 \operatorname{Cosh} \left[\frac{\pi}{6} \right]^8 - 25\,509\,168 \operatorname{Cosh} \left[\frac{\pi}{6} \right]^{10} - \\
 & \quad 75\,582\,720 \operatorname{Cosh} \left[\frac{\pi}{6} \right]^4 \operatorname{Sinh} \left[\frac{\pi}{6} \right]^2 + 170\,061\,120 \operatorname{Cosh} \left[\frac{\pi}{6} \right]^6 \operatorname{Sinh} \left[\frac{\pi}{6} \right]^2 + \\
 & \quad 76\,527\,504 \operatorname{Cosh} \left[\frac{\pi}{6} \right]^8 \operatorname{Sinh} \left[\frac{\pi}{6} \right]^2 + 68\,024\,448 \operatorname{Cosh} \left[\frac{\pi}{6} \right]^4 \operatorname{Sinh} \left[\frac{\pi}{6} \right]^4 - \\
 & \quad \left. \left. \left. \left. 76\,527\,504 \operatorname{Cosh} \left[\frac{\pi}{6} \right]^6 \operatorname{Sinh} \left[\frac{\pi}{6} \right]^4 + 25\,509\,168 \operatorname{Cosh} \left[\frac{\pi}{6} \right]^4 \operatorname{Sinh} \left[\frac{\pi}{6} \right]^6 \right) \right) \right) \right)^{1/3} \right) - \\
 & \frac{1}{3} \left(\frac{1}{2} \left(16\,000 + 17\,280 \operatorname{Cosh} \left[\frac{\pi}{6} \right]^2 + 972 \operatorname{Cosh} \left[\frac{\pi}{6} \right]^4 - 1458 \operatorname{Cosh} \left[\frac{\pi}{6} \right]^6 + 21\,600 \operatorname{Sinh} \left[\frac{\pi}{6} \right]^2 - \right. \right. \right. \\
 & \quad 1944 \operatorname{Cosh} \left[\frac{\pi}{6} \right]^2 \operatorname{Sinh} \left[\frac{\pi}{6} \right]^2 + 4374 \operatorname{Cosh} \left[\frac{\pi}{6} \right]^4 \operatorname{Sinh} \left[\frac{\pi}{6} \right]^2 + 9720 \operatorname{Sinh} \left[\frac{\pi}{6} \right]^4 - \\
 & \quad 4374 \operatorname{Cosh} \left[\frac{\pi}{6} \right]^2 \operatorname{Sinh} \left[\frac{\pi}{6} \right]^4 + 1458 \operatorname{Sinh} \left[\frac{\pi}{6} \right]^6 + \sqrt{\left(-223\,948\,800 \operatorname{Cosh} \left[\frac{\pi}{6} \right]^4 - \right.} \\
 & \quad 332\,563\,968 \operatorname{Cosh} \left[\frac{\pi}{6} \right]^6 - 161\,558\,064 \operatorname{Cosh} \left[\frac{\pi}{6} \right]^8 - 25\,509\,168 \operatorname{Cosh} \left[\frac{\pi}{6} \right]^{10} - \\
 & \quad 75\,582\,720 \operatorname{Cosh} \left[\frac{\pi}{6} \right]^4 \operatorname{Sinh} \left[\frac{\pi}{6} \right]^2 + 170\,061\,120 \operatorname{Cosh} \left[\frac{\pi}{6} \right]^6 \operatorname{Sinh} \left[\frac{\pi}{6} \right]^2 + \\
 & \quad 76\,527\,504 \operatorname{Cosh} \left[\frac{\pi}{6} \right]^8 \operatorname{Sinh} \left[\frac{\pi}{6} \right]^2 + 68\,024\,448 \operatorname{Cosh} \left[\frac{\pi}{6} \right]^4 \operatorname{Sinh} \left[\frac{\pi}{6} \right]^4 - \\
 & \quad \left. \left. \left. \left. 76\,527\,504 \operatorname{Cosh} \left[\frac{\pi}{6} \right]^6 \operatorname{Sinh} \left[\frac{\pi}{6} \right]^4 + 25\,509\,168 \operatorname{Cosh} \left[\frac{\pi}{6} \right]^4 \operatorname{Sinh} \left[\frac{\pi}{6} \right]^6 \right) \right) \right) \right)^{1/3} + \\
 & \left(240 \operatorname{Sinh} \left[\frac{\pi}{6} \right] - 216 \operatorname{Sinh} \left[\frac{\pi}{6} \right]^3 + 24 \operatorname{Sinh} \left[\frac{\pi}{6} \right] \left(-10 - 9 \operatorname{Cosh} \left[\frac{\pi}{6} \right]^2 + 9 \operatorname{Sinh} \left[\frac{\pi}{6} \right]^2 \right) \right) / \\
 & \left(4 \sqrt{\left(10 + 9 \operatorname{Cosh} \left[\frac{\pi}{6} \right]^2 + \frac{1}{3} \left(-10 - 9 \operatorname{Cosh} \left[\frac{\pi}{6} \right]^2 + 9 \operatorname{Sinh} \left[\frac{\pi}{6} \right]^2 \right) \right)} \right)
 \end{aligned}$$

$$\left(12 \left(25 + 9 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^2\right) + 540 \operatorname{Sinh}\left[\frac{\pi}{6}\right]^2 + \left(-10 - 9 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^2 + 9 \operatorname{Sinh}\left[\frac{\pi}{6}\right]^2\right)^2\right) /$$
$$\left[3 \left(\frac{1}{2} \left(16\,000 + 17\,280 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^2 + 972 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^4 - 1458 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^6 + 21\,600\right.\right.\right.$$
$$\operatorname{Sinh}\left[\frac{\pi}{6}\right]^2 - 1944 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^2 \operatorname{Sinh}\left[\frac{\pi}{6}\right]^2 + 4374 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^4 \operatorname{Sinh}\left[\frac{\pi}{6}\right]^2 +$$
$$9720 \operatorname{Sinh}\left[\frac{\pi}{6}\right]^4 - 4374 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^2 \operatorname{Sinh}\left[\frac{\pi}{6}\right]^4 + 1458 \operatorname{Sinh}\left[\frac{\pi}{6}\right]^6 +$$
$$\sqrt{\left(-223\,948\,800 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^4 - 332\,563\,968 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^6 - 161\,558\,064\right.$$
$$\operatorname{Cosh}\left[\frac{\pi}{6}\right]^8 - 25\,509\,168 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^{10} - 75\,582\,720 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^4 \operatorname{Sinh}\left[\frac{\pi}{6}\right]^2 +$$
$$170\,061\,120 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^6 \operatorname{Sinh}\left[\frac{\pi}{6}\right]^2 + 76\,527\,504 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^8 \operatorname{Sinh}\left[\frac{\pi}{6}\right]^2 +$$
$$68\,024\,448 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^4 \operatorname{Sinh}\left[\frac{\pi}{6}\right]^4 - 76\,527\,504 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^6$$
$$\left.\left.\left.\left.\left.\left.\left.\left.\left.\left.\operatorname{Sinh}\left[\frac{\pi}{6}\right]^4 + 25\,509\,168 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^4 \operatorname{Sinh}\left[\frac{\pi}{6}\right]^6\right)\right)\right)\right)\right)\right)^{1/3}\right) + \frac{1}{3}$$
$$\left(\frac{1}{2} \left(16\,000 + 17\,280 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^2 + 972 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^4 - 1458 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^6 +$$
$$21\,600 \operatorname{Sinh}\left[\frac{\pi}{6}\right]^2 - 1944 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^2 \operatorname{Sinh}\left[\frac{\pi}{6}\right]^2 + 4374 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^4 \operatorname{Sinh}\left[\frac{\pi}{6}\right]^2 +$$
$$9720 \operatorname{Sinh}\left[\frac{\pi}{6}\right]^4 - 4374 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^2 \operatorname{Sinh}\left[\frac{\pi}{6}\right]^4 + 1458 \operatorname{Sinh}\left[\frac{\pi}{6}\right]^6 +$$
$$\sqrt{\left(-223\,948\,800 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^4 - 332\,563\,968 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^6 - 161\,558\,064\right.$$
$$\operatorname{Cosh}\left[\frac{\pi}{6}\right]^8 - 25\,509\,168 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^{10} - 75\,582\,720 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^4 \operatorname{Sinh}\left[\frac{\pi}{6}\right]^2 +$$
$$170\,061\,120 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^6 \operatorname{Sinh}\left[\frac{\pi}{6}\right]^2 + 76\,527\,504 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^8 \operatorname{Sinh}\left[\frac{\pi}{6}\right]^2 +$$
$$68\,024\,448 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^4 \operatorname{Sinh}\left[\frac{\pi}{6}\right]^4 - 76\,527\,504 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^6 \operatorname{Sinh}\left[\frac{\pi}{6}\right]^4 +$$
$$\left.\left.\left.\left.\left.\left.\left.\left.\left.\left.\operatorname{Cosh}\left[\frac{\pi}{6}\right]^4 \operatorname{Sinh}\left[\frac{\pi}{6}\right]^6\right)\right)\right)\right)\right)\right)^{1/3}\right)\right)\right)\right],$$
$$\{z \rightarrow -\operatorname{ArcCosh}\left[-\frac{3}{2} \operatorname{Sinh}\left[\frac{\pi}{6}\right] + \frac{1}{2} \sqrt{\left(10 + 9 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^2 + \frac{1}{3} \left(-10 - 9 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^2 + 9 \operatorname{Sinh}\left[\frac{\pi}{6}\right]^2\right) +$$

$$\begin{aligned}
 & \left(12 \left(25 + 9 \operatorname{Cosh} \left[\frac{\pi}{6} \right]^2 \right) + 540 \operatorname{Sinh} \left[\frac{\pi}{6} \right]^2 + \left(-10 - 9 \operatorname{Cosh} \left[\frac{\pi}{6} \right]^2 + 9 \operatorname{Sinh} \left[\frac{\pi}{6} \right]^2 \right)^2 \right) / \\
 & \left(3 \left(\frac{1}{2} \left(16\,000 + 17\,280 \operatorname{Cosh} \left[\frac{\pi}{6} \right]^2 + 972 \operatorname{Cosh} \left[\frac{\pi}{6} \right]^4 - 1458 \operatorname{Cosh} \left[\frac{\pi}{6} \right]^6 + 21\,600 \operatorname{Sinh} \left[\frac{\pi}{6} \right]^2 - \right. \right. \right. \\
 & \quad 1944 \operatorname{Cosh} \left[\frac{\pi}{6} \right]^2 \operatorname{Sinh} \left[\frac{\pi}{6} \right]^2 + 4374 \operatorname{Cosh} \left[\frac{\pi}{6} \right]^4 \operatorname{Sinh} \left[\frac{\pi}{6} \right]^2 + 9720 \operatorname{Sinh} \left[\frac{\pi}{6} \right]^4 - \\
 & \quad 4374 \operatorname{Cosh} \left[\frac{\pi}{6} \right]^2 \operatorname{Sinh} \left[\frac{\pi}{6} \right]^4 + 1458 \operatorname{Sinh} \left[\frac{\pi}{6} \right]^6 + \sqrt{\left(-223\,948\,800 \operatorname{Cosh} \left[\frac{\pi}{6} \right]^4 - \right.} \\
 & \quad 332\,563\,968 \operatorname{Cosh} \left[\frac{\pi}{6} \right]^6 - 161\,558\,064 \operatorname{Cosh} \left[\frac{\pi}{6} \right]^8 - 25\,509\,168 \operatorname{Cosh} \left[\frac{\pi}{6} \right]^{10} - \\
 & \quad 75\,582\,720 \operatorname{Cosh} \left[\frac{\pi}{6} \right]^4 \operatorname{Sinh} \left[\frac{\pi}{6} \right]^2 + 170\,061\,120 \operatorname{Cosh} \left[\frac{\pi}{6} \right]^6 \operatorname{Sinh} \left[\frac{\pi}{6} \right]^2 + \\
 & \quad 76\,527\,504 \operatorname{Cosh} \left[\frac{\pi}{6} \right]^8 \operatorname{Sinh} \left[\frac{\pi}{6} \right]^2 + 68\,024\,448 \operatorname{Cosh} \left[\frac{\pi}{6} \right]^4 \operatorname{Sinh} \left[\frac{\pi}{6} \right]^4 - \\
 & \quad \left. \left. \left. 76\,527\,504 \operatorname{Cosh} \left[\frac{\pi}{6} \right]^6 \operatorname{Sinh} \left[\frac{\pi}{6} \right]^4 + 25\,509\,168 \operatorname{Cosh} \left[\frac{\pi}{6} \right]^4 \operatorname{Sinh} \left[\frac{\pi}{6} \right]^6 \right) \right) \right)^{1/3} \right) + \\
 & \frac{1}{3} \left(\frac{1}{2} \left(16\,000 + 17\,280 \operatorname{Cosh} \left[\frac{\pi}{6} \right]^2 + 972 \operatorname{Cosh} \left[\frac{\pi}{6} \right]^4 - 1458 \operatorname{Cosh} \left[\frac{\pi}{6} \right]^6 + 21\,600 \operatorname{Sinh} \left[\frac{\pi}{6} \right]^2 - \right. \right. \right. \\
 & \quad 1944 \operatorname{Cosh} \left[\frac{\pi}{6} \right]^2 \operatorname{Sinh} \left[\frac{\pi}{6} \right]^2 + 4374 \operatorname{Cosh} \left[\frac{\pi}{6} \right]^4 \operatorname{Sinh} \left[\frac{\pi}{6} \right]^2 + 9720 \operatorname{Sinh} \left[\frac{\pi}{6} \right]^4 - \\
 & \quad 4374 \operatorname{Cosh} \left[\frac{\pi}{6} \right]^2 \operatorname{Sinh} \left[\frac{\pi}{6} \right]^4 + 1458 \operatorname{Sinh} \left[\frac{\pi}{6} \right]^6 + \sqrt{\left(-223\,948\,800 \operatorname{Cosh} \left[\frac{\pi}{6} \right]^4 - \right.} \\
 & \quad 332\,563\,968 \operatorname{Cosh} \left[\frac{\pi}{6} \right]^6 - 161\,558\,064 \operatorname{Cosh} \left[\frac{\pi}{6} \right]^8 - 25\,509\,168 \operatorname{Cosh} \left[\frac{\pi}{6} \right]^{10} - \\
 & \quad 75\,582\,720 \operatorname{Cosh} \left[\frac{\pi}{6} \right]^4 \operatorname{Sinh} \left[\frac{\pi}{6} \right]^2 + 170\,061\,120 \operatorname{Cosh} \left[\frac{\pi}{6} \right]^6 \operatorname{Sinh} \left[\frac{\pi}{6} \right]^2 + \\
 & \quad 76\,527\,504 \operatorname{Cosh} \left[\frac{\pi}{6} \right]^8 \operatorname{Sinh} \left[\frac{\pi}{6} \right]^2 + 68\,024\,448 \operatorname{Cosh} \left[\frac{\pi}{6} \right]^4 \operatorname{Sinh} \left[\frac{\pi}{6} \right]^4 - \\
 & \quad \left. \left. \left. 76\,527\,504 \operatorname{Cosh} \left[\frac{\pi}{6} \right]^6 \operatorname{Sinh} \left[\frac{\pi}{6} \right]^4 + 25\,509\,168 \operatorname{Cosh} \left[\frac{\pi}{6} \right]^4 \operatorname{Sinh} \left[\frac{\pi}{6} \right]^6 \right) \right) \right)^{1/3} \right) + \\
 & \frac{1}{2} \sqrt{\left(10 + 9 \operatorname{Cosh} \left[\frac{\pi}{6} \right]^2 + 9 \operatorname{Sinh} \left[\frac{\pi}{6} \right]^2 - \frac{1}{3} \left(-10 - 9 \operatorname{Cosh} \left[\frac{\pi}{6} \right]^2 + 9 \operatorname{Sinh} \left[\frac{\pi}{6} \right]^2 \right) - \right.} \\
 & \quad \left(12 \left(25 + 9 \operatorname{Cosh} \left[\frac{\pi}{6} \right]^2 \right) + 540 \operatorname{Sinh} \left[\frac{\pi}{6} \right]^2 + \left(-10 - 9 \operatorname{Cosh} \left[\frac{\pi}{6} \right]^2 + 9 \operatorname{Sinh} \left[\frac{\pi}{6} \right]^2 \right)^2 \right) / \\
 & \quad \left(3 \left(\frac{1}{2} \left(16\,000 + 17\,280 \operatorname{Cosh} \left[\frac{\pi}{6} \right]^2 + 972 \operatorname{Cosh} \left[\frac{\pi}{6} \right]^4 - 1458 \operatorname{Cosh} \left[\frac{\pi}{6} \right]^6 + 21\,600 \operatorname{Sinh} \left[\frac{\pi}{6} \right]^2 - \right. \right. \right.
 \end{aligned}$$

$$\begin{aligned}
 & 1944 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^2 \operatorname{Sinh}\left[\frac{\pi}{6}\right]^2 + 4374 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^4 \operatorname{Sinh}\left[\frac{\pi}{6}\right]^2 + 9720 \operatorname{Sinh}\left[\frac{\pi}{6}\right]^4 - \\
 & 4374 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^2 \operatorname{Sinh}\left[\frac{\pi}{6}\right]^4 + 1458 \operatorname{Sinh}\left[\frac{\pi}{6}\right]^6 + \sqrt{\left(-223\,948\,800 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^4 - \right.} \\
 & \quad \left. 332\,563\,968 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^6 - 161\,558\,064 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^8 - 25\,509\,168 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^{10} - \right.} \\
 & \quad \left. 75\,582\,720 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^4 \operatorname{Sinh}\left[\frac{\pi}{6}\right]^2 + 170\,061\,120 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^6 \operatorname{Sinh}\left[\frac{\pi}{6}\right]^2 + \right.} \\
 & \quad \left. 76\,527\,504 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^8 \operatorname{Sinh}\left[\frac{\pi}{6}\right]^2 + 68\,024\,448 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^4 \operatorname{Sinh}\left[\frac{\pi}{6}\right]^4 - \right.} \\
 & \quad \left. 76\,527\,504 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^6 \operatorname{Sinh}\left[\frac{\pi}{6}\right]^4 + 25\,509\,168 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^4 \operatorname{Sinh}\left[\frac{\pi}{6}\right]^6\right)\right)^{1/3} \Bigg) - \\
 & \frac{1}{3} \left(\frac{1}{2} \left(16\,000 + 17\,280 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^2 + 972 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^4 - 1458 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^6 + 21\,600 \operatorname{Sinh}\left[\frac{\pi}{6}\right]^2 - \right. \right. \\
 & \quad 1944 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^2 \operatorname{Sinh}\left[\frac{\pi}{6}\right]^2 + 4374 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^4 \operatorname{Sinh}\left[\frac{\pi}{6}\right]^2 + 9720 \operatorname{Sinh}\left[\frac{\pi}{6}\right]^4 - \\
 & \quad 4374 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^2 \operatorname{Sinh}\left[\frac{\pi}{6}\right]^4 + 1458 \operatorname{Sinh}\left[\frac{\pi}{6}\right]^6 + \sqrt{\left(-223\,948\,800 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^4 - \right.} \\
 & \quad \left. 332\,563\,968 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^6 - 161\,558\,064 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^8 - 25\,509\,168 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^{10} - \right.} \\
 & \quad \left. 75\,582\,720 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^4 \operatorname{Sinh}\left[\frac{\pi}{6}\right]^2 + 170\,061\,120 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^6 \operatorname{Sinh}\left[\frac{\pi}{6}\right]^2 + \right.} \\
 & \quad \left. 76\,527\,504 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^8 \operatorname{Sinh}\left[\frac{\pi}{6}\right]^2 + 68\,024\,448 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^4 \operatorname{Sinh}\left[\frac{\pi}{6}\right]^4 - \right.} \\
 & \quad \left. 76\,527\,504 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^6 \operatorname{Sinh}\left[\frac{\pi}{6}\right]^4 + 25\,509\,168 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^4 \operatorname{Sinh}\left[\frac{\pi}{6}\right]^6\right)\right)^{1/3} + \\
 & \left(240 \operatorname{Sinh}\left[\frac{\pi}{6}\right] - 216 \operatorname{Sinh}\left[\frac{\pi}{6}\right]^3 + 24 \operatorname{Sinh}\left[\frac{\pi}{6}\right] \left(-10 - 9 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^2 + 9 \operatorname{Sinh}\left[\frac{\pi}{6}\right]^2 \right) \right) / \\
 & \left(4 \sqrt{\left(10 + 9 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^2 + \frac{1}{3} \left(-10 - 9 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^2 + 9 \operatorname{Sinh}\left[\frac{\pi}{6}\right]^2 \right) + \right. \right.} \\
 & \quad \left. \left. \left(12 \left(25 + 9 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^2 \right) + 540 \operatorname{Sinh}\left[\frac{\pi}{6}\right]^2 + \left(-10 - 9 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^2 + 9 \operatorname{Sinh}\left[\frac{\pi}{6}\right]^2 \right)^2 \right) \right) / \right.} \\
 & \quad \left. \left. \left(3 \left(\frac{1}{2} \left(16\,000 + 17\,280 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^2 + 972 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^4 - 1458 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^6 + 21\,600 \right. \right. \right. \right. \right. \right. \right. \right. \\
 & \quad \quad \left. \left. \operatorname{Sinh}\left[\frac{\pi}{6}\right]^2 - 1944 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^2 \operatorname{Sinh}\left[\frac{\pi}{6}\right]^2 + 4374 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^4 \operatorname{Sinh}\left[\frac{\pi}{6}\right]^2 + \right. \right. \\
 & \quad \quad \left. \left. 9720 \operatorname{Sinh}\left[\frac{\pi}{6}\right]^4 - 4374 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^2 \operatorname{Sinh}\left[\frac{\pi}{6}\right]^4 + 1458 \operatorname{Sinh}\left[\frac{\pi}{6}\right]^6 + \right. \right.
 \end{aligned}$$

$$\sqrt{\left(-223\,948\,800 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^4 - 332\,563\,968 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^6 - 161\,558\,064 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^8 - 25\,509\,168 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^{10} - 75\,582\,720 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^4 \operatorname{Sinh}\left[\frac{\pi}{6}\right]^2 + 170\,061\,120 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^6 \operatorname{Sinh}\left[\frac{\pi}{6}\right]^2 + 76\,527\,504 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^8 \operatorname{Sinh}\left[\frac{\pi}{6}\right]^2 + 68\,024\,448 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^4 \operatorname{Sinh}\left[\frac{\pi}{6}\right]^4 - 76\,527\,504 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^6 \operatorname{Sinh}\left[\frac{\pi}{6}\right]^4 + 25\,509\,168 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^4 \operatorname{Sinh}\left[\frac{\pi}{6}\right]^6\right)\right)^{1/3}} + \frac{1}{3} \left(\frac{1}{2} \left(16\,000 + 17\,280 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^2 + 972 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^4 - 1458 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^6 + 21\,600 \operatorname{Sinh}\left[\frac{\pi}{6}\right]^2 - 1944 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^2 \operatorname{Sinh}\left[\frac{\pi}{6}\right]^2 + 4374 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^4 \operatorname{Sinh}\left[\frac{\pi}{6}\right]^2 + 9720 \operatorname{Sinh}\left[\frac{\pi}{6}\right]^4 - 4374 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^2 \operatorname{Sinh}\left[\frac{\pi}{6}\right]^4 + 1458 \operatorname{Sinh}\left[\frac{\pi}{6}\right]^6 + \sqrt{\left(-223\,948\,800 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^4 - 332\,563\,968 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^6 - 161\,558\,064 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^8 - 25\,509\,168 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^{10} - 75\,582\,720 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^4 \operatorname{Sinh}\left[\frac{\pi}{6}\right]^2 + 170\,061\,120 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^6 \operatorname{Sinh}\left[\frac{\pi}{6}\right]^2 + 76\,527\,504 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^8 \operatorname{Sinh}\left[\frac{\pi}{6}\right]^2 + 68\,024\,448 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^4 \operatorname{Sinh}\left[\frac{\pi}{6}\right]^4 - 76\,527\,504 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^6 \operatorname{Sinh}\left[\frac{\pi}{6}\right]^4 + 25\,509\,168 \operatorname{Cosh}\left[\frac{\pi}{6}\right]^4 \operatorname{Sinh}\left[\frac{\pi}{6}\right]^6\right)\right)^{1/3} \right) \right) \right)$$

Solve[Sin[x] == a, x]

Solve::ifun: Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information. More...

```
{{x -> ArcSin[a]}}
```

A complete solution of the previous equation can be obtained using the function Reduce.

Reduce[Sinh[x] == a, x] // InputForm

```
C[1] ∈ Integers && (x == I*Pi - ArcSinh[a] + (2*I)*Pi*C[1] || x == ArcSinh[a] + (2*I)*Pi*C[1])
```

Solving differential equations

Here are differential equations whose linear independent solutions include the hyperbolic sine function. The solutions of the simplest second-order linear ordinary differential equation with constant coefficients can be represented using $\sinh(z)$ and $\cosh(z)$.

```
DSolve[w''[z] - w[z] == 0, w[z], z] //
  FullSimplify[#, ComplexityFunction -> (100 Count[#, E^_, ∞] &)] & // Simplify
{{w[z] -> (C[1] + C[2]) Cosh[z] + (C[1] - C[2]) Sinh[z]}}
```

```
dsol1 = DSolve[4 w[z] - 5 w''[z] + w^(4)[z] == 0, w, z] /. Exp[x_] -> Cosh[x] + Sinh[x]
{{w -> Function[{z}, (Cosh[-2 z] + Sinh[-2 z]) C[1] +
  (Cosh[-z] + Sinh[-z]) C[2] + (Cosh[z] + Sinh[z]) C[3] + (Cosh[2 z] + Sinh[2 z]) C[4]]}}
```

In the last input, the differential equation was solved for $w(z)$. If the argument is suppressed, the result is returned as a pure function (in the sense of the λ -calculus).

```
dsol2 =
  DSolve[4 w[z] - 5 w''[z] + w^(4)[z] == 0, w[z], z] /. Exp[x_] -> Cosh[x] + Sinh[x] // Simplify
{{w[z] -> (C[1] + C[4]) Cosh[2 z] +
  (-C[2] + C[3]) Sinh[z] + Cosh[z] (C[2] + C[3] - 2 (C[1] - C[4]) Sinh[z])}}
```

The advantage of such a pure function is that it can be used for different arguments, derivatives, and more.

```
w'[ξ] /. dsol1
{C[2] (-Cosh[ξ] + Sinh[ξ]) + C[3] (Cosh[ξ] + Sinh[ξ]) +
  C[1] (-2 Cosh[2 ξ] + 2 Sinh[2 ξ]) + C[4] (2 Cosh[2 ξ] + 2 Sinh[2 ξ])}
```

```
w'[ξ] /. dsol2
{w'[ξ]}
```

In carrying out the algorithm to solve the following nonlinear differential equation, *Mathematica* has to solve a transcendental equation. In doing so, the generically multivariate inverse of a function is encountered, and a message is issued warning that a solution branch is potentially missed.

```
DSolve[{w'[z] == Sqrt[1 + w[z]^2], w[0] == 0}, w[z], z]
Solve::ifun: Inverse functions are being used by Solve, so some solutions may not be found.
{{w[z] -> Sinh[z]}}
```

Integral transforms

Mathematica supports the main integral transforms, such as the direct and inverse Laplace and Fourier transforms that can give results that contain classical or generalized functions.

```
LaplaceTransform[Sinh[t], t, s]
```

$$\frac{1}{-1 + s^2}$$

FourierTransform[1/Sinh[t], t, s]

$$i \sqrt{\frac{\pi}{2}} \operatorname{Tanh}\left[\frac{\pi s}{2}\right]$$

FourierSinTransform[1/Sinh[t], t, s]

$$\sqrt{\frac{\pi}{2}} \operatorname{Tanh}\left[\frac{\pi s}{2}\right]$$

FourierCosTransform[1/Sinh[t], t, s]

$$\frac{1}{2\sqrt{2\pi}} \left(-4 \operatorname{EulerGamma} - \operatorname{PolyGamma}\left[0, \frac{1}{2} - \frac{is}{2}\right] - \operatorname{PolyGamma}\left[0, \frac{1}{2} + \frac{is}{2}\right] + \right. \\ \left. \operatorname{PolyGamma}\left[0, -\frac{is}{2}\right] + \operatorname{PolyGamma}\left[0, \frac{is}{2}\right] - 2 \operatorname{PolyGamma}[0, -is] - 2 \operatorname{PolyGamma}[0, is] \right)$$

Plotting

Mathematica has built-in functions for 2D and 3D graphics. Here are some examples.

```
Plot[Sin[Sinh[z]], {z, -4, 4}, PlotPoints -> 200];
```

```
Plot3D[Re[Sinh[x + i y]], {x, -2, 2}, {y, -π, π},
  PlotPoints -> 240, PlotRange -> {-5, 5},
  ClipFill -> None, Mesh -> False, AxesLabel -> {"x", "y", None}];
```

```
ContourPlot[Arg[Sinh[1/(x + i y)]], {x, -1/2, 1/2}, {y, -1/2, 1/2},
  PlotPoints -> 400, PlotRange -> {-π, π}, FrameLabel -> {"x", "y", None, None},
  ColorFunction -> Hue, ContourLines -> False, Contours -> 200];
```

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