Introductions to PartitionsP

Introduction to partitions

General

Interest in partitions appeared in the 17th century when G. W. Leibniz (1669) investigated the number of ways a given positive integer can be decomposed into a sum of smaller integers. Later, L. Euler (1740) also used partitions in his work. But extensive investigation of partitions began in the 20th century with the works of S. Ramanujan (1917) and G. H. Hardy. In particular G. H. Hardy (1920) introduced the notations $p(n)$ and $q(n)$ to represent the two most commonly used types of paritITIONS.

Definitions of partitions

The partition functions discussed here include two basic functions that describe the structure of integer numbers—the number of unrestricted partitions of an integer (partitions P) $p(n)$, and the number of partitions of an integer into distinct parts (partitions Q) $q(n)$.

Partitions P

For nonnegative integer n, the function $p(n)$ is the number of unrestricted partitions of the positive integer n into a sum of strictly positive numbers that add up to n independent of the order, when repetitions are allowed.

The function $p(n)$ can be described by the following formulas:

$$p(n) = \left([t^n] \prod_{k=1}^{\infty} \frac{1}{1-t^k}\right) /;\ n \in \mathbb{N}$$

$$p(n) = 0 /;\ n \in \mathbb{Z} \land n < 0,$$

where $([t^n]\, f(t))$ (with $f(t) = \prod_{k=1}^{\infty} \frac{1}{1-t^k}$) is the coefficient of the t^n term in the series expansion around $t = 0$ of the function $f(t)$: $f(t) = \sum_{n=0}^{\infty} ([t^n]\, f(t))\, t^n$.

Example: There are seven possible ways to express 5 as a sum of nonnegative integers: $5 = 0 + 5 = 1 + 4 = 2 + 3 = 1 + 1 + 3 = 1 + 2 + 2 = 1 + 1 + 1 + 2 = 1 + 1 + 1 + 1 + 1$. For this reason $p(5) = 7$.

Partitions Q

For nonnegative integer n, the function $q(n)$ is the number of restricted partitions of the positive integer n into a sum of distinct positive numbers that add up to n when order does not matter and repetitions are not allowed.

The function $q(n)$ can be described by the following formulas:

$$q(n) = \left([t^n] \prod_{k=1}^{\infty} (t^k + 1)\right) /;\ n \in \mathbb{N}$$

$q(n) == 0 \,/;\, n \in \mathbb{Z} \land n < 0,$

where $([t^n]\, f(t))$ (with $f(t) == \prod_{k=1}^{\infty} (t^k + 1)$) is the coefficient of the t^n term in the series expansion around $t = 0$ of the function $f(t)$: $f(t) == \sum_{n=0}^{\infty} ([t^n]\, f(t))\, t^n$.

Example: There are three possible ways to express 5 as a sum of nonnegative integers without repetitions: $5 == 0 + 5 == 1 + 4 == 2 + 3$. For this reason $q(5) == 3$.

Connections within the group of the partitions and with other function groups

Representations through related functions

The partition functions $p(n)$ and $q(n)$ are connected by the following formula:

$$p(n) == \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} q(n - 2k)\, p(k).$$

The best-known properties and formulas of partitions

Simple values at zero and infinity

The partition functions $p(n)$ and $q(n)$ are defined for zero and infinity values of argument n by the following rules:

$$p(0) == 1$$

$$q(0) == 1$$

$$p(\infty) == \infty$$

$$q(\infty) == \infty.$$

Specific values for specialized variables

The following table represents the values of the partitions $p(n)$ and $q(n)$ for $0 \leq n \leq 10$ and some powers of 10:

n	0	1	2	3	4	5	6	7	8	9	10	…	100	1000	…
$p(n)$	1	1	2	3	5	7	11	15	22	30	42	…	190 569 292	24 061 467 864 032 622 473 692 149 727 991	…
$q(n)$	1	1	1	2	2	3	4	5	6	8	10	…	444 793	8 635 565 795 744 155 161 506	…

Analyticity

The partition functions $p(n)$ and $q(n)$ are non-analytical functions that are defined only for integers.

Periodicity

The partition functions $p(n)$ and $q(n)$ do not have periodicity.

Parity and symmetry

The partition functions $p(n)$ and $q(n)$ do not have symmetry.

Series representations

The partition functions $p(n)$ and $q(n)$ have the following series representations:

$$p(n) = \frac{1}{\pi\sqrt{2}} \sum_{k=1}^{\infty} A(k, n)\sqrt{k}\, \frac{\partial\left(\sinh\left(\frac{1}{k}\pi\sqrt{\frac{2}{3}}\sqrt{n-\frac{1}{24}}\right)\left(n-\frac{1}{24}\right)^{-\frac{1}{2}}\right)}{\partial n}$$

$$p(n) = \frac{\pi^2}{9\sqrt{3}} \sum_{k=1}^{\infty} \frac{A(k, n)}{k^{5/2}}\, {}_0F_1\left(;\frac{5}{2};\frac{\left(n-\frac{1}{24}\right)\pi^2}{6k^2}\right)$$

$$q(n) = \frac{1}{\sqrt{2}} \sum_{k=1}^{\infty} A(2k-1, n)\, \frac{\partial J_0\left(\frac{\pi i}{2k-1}\sqrt{\frac{1}{3}}\sqrt{n+\frac{1}{24}}\right)}{\partial n}$$

$$q(n) = \frac{\pi^2\sqrt{2}}{24} \sum_{k=1}^{\infty} \frac{A(2k-1, n)}{(1-2k)^2}\, {}_0F_1\left(;2;\frac{\pi^2\left(n+\frac{1}{24}\right)}{12(1-2k)^2}\right),$$

where $A(k, n)$ is a special case of a generalized Kloosterman sum:

$$A(k, n) = \sum_{h=1}^{k} \delta_{\gcd(h,k),1} \exp\left(\pi i \sum_{j=1}^{k-1} \frac{1}{k} j\left(\frac{hj}{k} - \left\lfloor \frac{hj}{k} \right\rfloor - \frac{1}{2}\right) - \frac{2\pi i h n}{k}\right).$$

Asymptotic series expansions

The partition functions $p(n)$ and $q(n)$ have the following asymptotic series expansions:

$$p(n) \propto \frac{1}{4n\sqrt{3}} \exp\left(\sqrt{\frac{2}{3}}\sqrt{n}\,\pi\right)\left(1+O\left(\frac{1}{n}\right)\right) /; (n \to \infty)$$

$$q(n) \propto \frac{1}{4\sqrt[4]{3}\, n^{3/4}} \exp\left(\pi\sqrt{\frac{n}{3}}\right)\left(1+O\left(\frac{1}{n}\right)\right) /; (n \to \infty).$$

Generating functions

The partition functions $p(n)$ and $q(n)$ can be represented as the coefficients of their generating functions:

$$p(n) = \left([t^n]\prod_{k=1}^{\infty} \frac{1}{1-t^k}\right) /; n \in \mathbb{N}$$

$$p(n) = \left([t^n]\ 1 \Big/ \left(\sum_{k=-\infty}^{\infty} (-1)^k\, t^{\frac{1}{2}(3k^2+k)}\right)\right) /; n \in \mathbb{N}$$

$$p(n) = \left([t^n]\ \sqrt[3]{\frac{2\sqrt[8]{t}}{\vartheta_1'\left(0, \sqrt{t}\right)}}\right) /; n \in \mathbb{N}$$

$$q(n) = \left([t^n] \prod_{k=1}^{\infty} \left(1 + t^k\right) \right) /; n \in \mathbb{N}$$

$$q(n) = \left([t^n] \prod_{k=1}^{\infty} \frac{1}{1 - t^{2k-1}} \right) /; n \in \mathbb{N},$$

where $([t^n] f(t))$ is the coefficient of the t^n term in the series expansion around $t = 0$ of the function $f(t)$, $f(t) = \sum_{n=0}^{\infty} ([t^n] f(t)) t^n$.

Identities

The partition functions $p(n)$ and $q(n)$ satisfy numerous identities, for example:

$$p(n) = \frac{1}{n} \sum_{k=1}^{n} \sigma_1(k) \, p(n-k)$$

$$p(n) = \sum_{k=1}^{n} (-1)^{k-1} \left(p\left(n - \frac{1}{2}\left(3 k^2 - k\right)\right) + p\left(n - \frac{1}{2}\left(3 k^2 + k\right)\right) \right)$$

$$p(2 n + 1) = p(n) - \sum_{k=1}^{\infty} (-1)^k \left(p\left(-3 k^2 - k + 2 n + 1\right) + p\left(-3 k^2 + k + 2 n + 1\right) \right) + \sum_{k=1}^{\infty} \left(p\left(-4 k^2 + 3 k + n\right) + p\left(-4 k^2 - 3 k + n\right) \right)$$

$$q(n) = \frac{1}{n} \sum_{k=1}^{n} \sigma_1(k) \, q(n-k) - \frac{2}{n} \sum_{k=1}^{\left\lfloor \frac{n}{2} \right\rfloor} \sigma_1(k) \, q(n - 2 k) /; n \in \mathbb{N}^+.$$

Complex characteristics

As real valued functions, the partitions $p(n)$ and $q(n)$ have the following complex characteristics:

n	$p(n)$	$q(n)$
Abs	$\lvert p(n) \rvert = p(n)$	$\lvert q(n) \rvert = q(n)$
Arg	$\operatorname{Arg}(p(n)) = 0$	$\operatorname{Arg}(q(n)) = 0$
Re	$\operatorname{Re}(p(n)) = p(n)$	$\operatorname{Re}(q(n)) = q(n)$
Im	$\operatorname{Im}(p(n)) = 0$	$\operatorname{Im}(q(n)) = 0$
Conjugate	$\overline{p(n)} = p(n)$	$\overline{q(n)} = q(n)$
Sign	$\operatorname{sgn}(p(n)) = 1$	$\operatorname{sgn}(q(n)) = 1$

Summation

There exist just a few formulas including finite and infinite summation of partitions, for example:

$$\sum_{k=\left\lceil -\frac{1}{6}\left(\sqrt{24 n+1} + 1\right) \right\rceil}^{\left\lfloor \frac{1}{6}\left(\sqrt{24 n+1} - 1\right) \right\rfloor} (-1)^k \, p\left(n - \frac{1}{2} k (3 k + 1)\right) = 0$$

$$\sum_{k=0}^{\infty} p(k) \, t^k = \prod_{k=1}^{\infty} \frac{1}{1 - t^k}.$$

Inequalities

The partitions $p(n)$ and $q(n)$ satisfy various inequalities, for example:

$$p(n) \leq \frac{1}{2}(p(n-1) + p(n+1)) \,/;\, n \in \mathbb{N}^+$$

$$q(n) \leq \frac{1}{2}(q(n-1) + q(n+1)) \,/;\, n - 3 \in \mathbb{N}^+.$$

Congruence properties

The $p(n)$ partitions have the following congruence properties:

$$p(5\,n + 4) \bmod 5 == 0$$

$$p(7\,n + 5) \bmod 7 == 0$$

$$p(11\,n + 6) \bmod 11 == 0$$

$$p(n) \bmod \left(5^{k_1}\, 7^{k_2+1}\, 11^{k_3}\right) == 0 \,/;\, (24\,n) \bmod \left(5^{k_1}\, 7^{k_2}\, 11^{k_3}\right) == 1 \bigwedge k_1 \in \mathbb{N} \bigwedge k_2 \in \mathbb{N} \bigwedge k_3 \in \mathbb{N}.$$

Zeros

The $p(n)$ and $q(n)$ partitions have the following unique zeros:

$$p(n) == 0 \,/;\, n \in \mathbb{Z} \land n < 0$$

$$q(n) == 0 \,/;\, n \in \mathbb{Z} \land n < 0.$$

Applications of partitions

Partitions are used in number theory and other fields of mathematics.

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