Introductions to LucasL

Introduction to the Fibonacci and Lucas numbers

The sequence now known as Fibonacci numbers (sequence 0, 1, 1, 2, 3, 5, 8, 13...) first appeared in the work of an ancient Indian mathematician, Pingala (450 or 200 BC). Pingala's work with the mountain of cadence (now known as Pascal's triangle) made him the first known person to have looked into Fibonacci numbers. Next, another Indian mathematician, Virahanka (6th century AD), took note of the Fibonacci sequence through analysis of a completely different problem. Virahanka considered the following problem: assuming that lines of n units are composed of syllables that can be long or short—a long syllable takes twice as long as a short syllable to articulate—and each line of n units takes the same time to articulate no matter how it is composed, how many different combinations of syllables are there for each line of length n? Research of this question was continued by the Indian scholar Hemachandra and the Indian mathematician Gopala in the 12th century. Almost a half century later, the sequence was studied by the man whose name is most heavily linked to Fibonacci numbers, Leonardo of Pisa, a.k.a. Fibonacci (1202). Fibonacci considered the famous growth of an idealized rabbit population problem.

Later, European mathematicians began to study various aspects of Fibonacci numbers. Researchers included J. Kepler (1608), A. Girard (1634), R. Simpson (1753), É. Léger (1837), É. Lucas (1870, 1876–1880), G. H. Hardy, and E. M. Wright (1938). From this group, it was Francois Edouard Anatole Lucas (1870, 1876–1880) who gave Fibonacci numbers their name. He also investigated a similar sequence (sequence 2, 1, 3, 4, 7, 11, 18, 29, …), which was later coined Lucas numbers. In many works these sequences are notated F_n and L_n ($n = 0, 1, 2, \ldots$) to represent the first letters of the last names Fibonacci and Lucas. Eventually, it was established that both sequences can be analytically extended on complex z-planes and that they satisfy the same three-term recurrence relation, reflecting that the Fibonacci and Lucas numbers are the sums of two neighboring terms:

$$w(z) = w(z-2) + w(z-1) \;/;\; w(z) = c_1 F_z + c_2 L_z.$$

For integer arguments, Fibonacci and Lucas numbers can be elegantly represented through the symmetric relations (including the golden ratio $\phi = \left(1 + \sqrt{5}\right) / 2$):

$$F_n = \frac{\phi^n - (1-\phi)^n}{\phi - (1-\phi)} \;/;\; n \in \mathbb{Z}$$

$$L_n = \frac{\phi^n + (1-\phi)^n}{\phi + (1-\phi)} \;/;\; n \in \mathbb{Z}.$$

Definitions of the Fibonacci and Lucas numbers:

For any complex ν, the Fibonacci numbers F_ν and Lucas numbers L_ν are defined by the formulas:

$$F_\nu == \frac{\phi^\nu - \cos(\nu\pi)\,\phi^{-\nu}}{\sqrt{5}}$$

$$L_\nu = \phi^\nu + \phi^{-\nu} \cos(\pi\nu),$$

where ϕ is the golden ratio $\phi = \left(1 + \sqrt{5}\right) \big/ 2$.

Connections within the group of the Fibonacci and Lucas numbers and with other function groups

Representations through more general functions

The Fibonacci and Lucas numbers F_ν and L_ν have the following representations through more general functions including some hypergeometric functions and Meijer G functions:

$$F_\nu == F_\nu(1)$$

$$F_n == i^{n-1}\, U_{n-1}\left(-\frac{i}{2}\right) /;\, n \in \mathbb{N}.$$

F_ν	L_ν
$F_\nu == \frac{\nu}{2}\cos^2\left(\frac{\pi\nu}{2}\right) {}_2F_1\left(1-\frac{\nu}{2}, \frac{\nu}{2}+1; \frac{3}{2}; -\frac{1}{4}\right) + \sin^2\left(\frac{\pi\nu}{2}\right) {}_2F_1\left(\frac{1-\nu}{2}, \frac{\nu+1}{2}; \frac{1}{2}; -\frac{1}{4}\right)$	$L_\nu = 2\cos^2\left(\frac{\pi\nu}{2}\right) {}_2F_1\left(-\frac{\nu}{2}, \frac{\nu}{2}; \frac{1}{2}; -\frac{1}{4}\right) + \nu\sin^2\left(\frac{\pi\nu}{2}\right) {}_2F_1\left(\frac{1-\nu}{2}, \frac{\nu+1}{2}; \frac{3}{2}; -\frac{1}{4}\right)$
$F_\nu == {}_2F_1\left(\frac{1-\nu}{2}, 1-\frac{\nu}{2}; 1-\nu; -4\right) - \cos(\pi\nu)\, {}_2F_1\left(\frac{\nu+1}{2}, \frac{\nu}{2}+1; \nu+1; -4\right) /;\, \nu \notin \mathbb{Z}$	$L_\nu = {}_2F_1\left(-\frac{\nu}{2}, \frac{1}{2}-\frac{\nu}{2}; 1-\nu; -4\right) + \cos(\pi\nu)\, {}_2F_1\left(\frac{\nu}{2}, \frac{\nu+1}{2}; \nu+1; -4\right) /;\, \nu \notin \mathbb{Z}$
$F_\nu == \frac{1}{2} e^{\frac{i\pi\nu}{2}} \left(\nu\left(\frac{i\sin(\pi\nu)}{2} - \cos(\pi\nu)\right) {}_2F_1\left(1-\frac{\nu}{2}, \frac{\nu}{2}+1; \frac{3}{2}; \frac{5}{4}\right) + \frac{\sin(\pi\nu)}{\sqrt{5}}\, {}_2F_1\left(\frac{\nu+1}{2}, \frac{1-\nu}{2}; \frac{1}{2}; \frac{5}{4}\right)\right)$	$L_\nu = 2\left(\cos^3\left(\frac{\pi\nu}{2}\right) - i\sin^3\left(\frac{\pi\nu}{2}\right)\right) {}_2F_1\left(-\frac{\nu}{2}, \frac{\nu}{2}; \frac{1}{2}; \frac{5}{4}\right) + \frac{\sqrt{5}}{2} e^{\frac{i\pi\nu}{2}} \nu\sin(\pi\nu)\, {}_2F_1\left(\frac{1-\nu}{2}, \frac{\nu+1}{2}; \frac{3}{2}; \frac{5}{4}\right)$
$F_n == \frac{n}{2^{n-1}}\, {}_2F_1\left(\frac{1-n}{2}, 1-\frac{n}{2}; \frac{3}{2}; 5\right) /;\, n \in \mathbb{Z}$	$L_n = \frac{1}{2^{n-1}}\, {}_2F_1\left(\frac{1-n}{2}, -\frac{n}{2}; \frac{1}{2}; 5\right) /;\, n \in \mathbb{Z}$
$F_n == {}_2F_1\left(\frac{1-n}{2}, 1-\frac{n}{2}; 1-n; -4\right) /;\, n-1 \in \mathbb{N}^+$	$L_n = {}_2F_1\left(\frac{1-n}{2}, -\frac{n}{2}; 1-n; -4\right) /;\, n \in \mathbb{N}^+$
$F_\nu == \frac{\sin(\pi\nu)}{2\sqrt{\pi}} G_{3,3}^{2,2}\left(\frac{1}{4} \,\middle\vert\, \begin{matrix} \frac{\nu+1}{2}, \frac{1-\nu}{2}, \frac{\nu}{2} \\ 0, \frac{1}{2}, \frac{\nu}{2} \end{matrix}\right) /;\, \nu \notin \mathbb{Z}$	$L_\nu = -\frac{\nu\sin(\pi\nu)}{2\sqrt{\pi}} G_{3,3}^{2,2}\left(\frac{1}{4} \,\middle\vert\, \begin{matrix} \frac{\nu}{2}+1, 1-\frac{\nu}{2}, \frac{\nu+1}{2} \\ 0, \frac{1}{2}, \frac{\nu+1}{2} \end{matrix}\right) /;\, \nu \notin \mathbb{Z}$
$F_\nu == \frac{1}{2^\nu\sqrt{\pi}} G_{2,2}^{1,2}\left(4 \,\middle\vert\, \begin{matrix} \frac{\nu+1}{2}, \frac{\nu}{2} \\ 0, \nu \end{matrix}\right) - \frac{\cos(\nu\pi)}{2^{-\nu}\sqrt{\pi}} G_{2,2}^{1,2}\left(4 \,\middle\vert\, \begin{matrix} \frac{1-\nu}{2}, -\frac{\nu}{2} \\ 0, -\nu \end{matrix}\right) /;\, \nu \notin \mathbb{Z}$	$L_\nu = -\frac{\nu\sin(\pi\nu)}{2\sqrt{\pi}} G_{3,3}^{2,2}\left(4 \,\middle\vert\, \begin{matrix} \frac{1}{2}, 1, \frac{1-\nu}{2} \\ \frac{\nu}{2}, -\frac{\nu}{2}, \frac{1-\nu}{2} \end{matrix}\right) /;\, \nu \notin \mathbb{Z}$

Representations of Fibonacci and Lucas numbers through each other and through elementary functions

The Fibonacci and Lucas numbers F_ν and L_ν can be represented through each other by the following formulas:

F_ν	L_ν
$F_\nu = \frac{1}{5}\left(L_{\nu-1} + L_{\nu+1}\right)$	$L_\nu = F_{\nu-1} + F_{\nu+1}$
$F_\nu = \frac{2L_{2\nu+1} - L_{2\nu}}{5L_\nu} - \frac{\phi^{-2\nu}\sin^2(\pi\nu)}{\sqrt{5}\, L_\nu}$	$L_\nu = \frac{F_{2\nu}}{F_\nu} - \frac{\phi^{-2\nu}\sin^2(\pi\nu)}{\sqrt{5}\, F_\nu}$
$F_n = \frac{(-1)^m\left(2L_{n-m+1} - L_{n-m}\right) + 2L_{m+n+1} - L_{m+n}}{5L_m} /;\, m \in \mathbb{Z} \land n \in \mathbb{Z}$	$L_n = \frac{(-1)^n F_{m-n} + F_{m+n}}{F_m} /;\, m \in \mathbb{Z} \land m \neq 0 \land n \in \mathbb{Z}$

The Fibonacci and Lucas numbers F_ν and L_ν have the following representations through elementary functions:

F_ν	L_ν
$F_\nu = \frac{2\, e^{\nu \log(w)} - e^{(\pi i - \log(w))\, \nu} - e^{-(\pi i + \log(w))\, \nu}}{2\sqrt{5}} /;\ w = \frac{1+\sqrt{5}}{2}$	$L_\nu = (\cos(\pi\nu)+1)\cosh\left(\nu\, \text{csch}^{-1}(2)\right) - (\cos(\pi\nu)-1)\sinh($
$F_\nu = \frac{2\,\phi^\nu - e^{\nu(-i\pi - \log(\phi))} - e^{\nu(i\pi - \log(\phi))}}{2\sqrt{5}}$	$L_\nu = 2^{-\nu}\left(\cos(\pi\nu)\left(-1+\sqrt{5}\right)^\nu + \left(1+\sqrt{5}\right)^\nu\right)$
$F_\nu = \frac{2\,\phi^\nu - e^{\nu(-i\pi - \log(\phi))} - e^{\nu(i\pi - \log(\phi))}}{2\sqrt{5}}$	$L_\nu = 2\cos\left(\nu \csc^{-1}\left(\frac{2}{\sqrt{5}}\right)\right)\left(\cos^3\left(\frac{\pi\nu}{2}\right) - i\sin^3\left(\frac{\pi\nu}{2}\right)\right) + e^{\frac{i\pi\nu}{2}}\sin(\pi\nu)\sin\left(\nu\csc^{-1}\left(\frac{2}{\sqrt{5}}\right)\right)$
$F_\nu = \frac{1}{\sqrt{5}}\exp\left(-\nu\,\text{csch}^{-1}(2)\right)\left(\exp\left(2\nu\,\text{csch}^{-1}(2)\right) - \cos(\pi\nu)\right)$	$L_\nu = \phi^\nu + (1-\phi)^\nu + (\cos(\pi\nu) - (-1)^\nu)\,\phi^{-\nu}$
$F_\nu = -\frac{i}{\sqrt{5}}\, e^{\frac{i\pi\nu}{2}}\left(\sin(\pi\nu)\cos\left(\nu\sin^{-1}\left(\frac{\sqrt{5}}{2}\right)\right) - (2\cos(\pi\nu) - i\sin(\pi\nu))\sin\left(\nu\sin^{-1}\left(\frac{\sqrt{5}}{2}\right)\right)\right)$	
$F_\nu = \frac{1}{\sqrt{5}}\left((1-\cos(\pi\nu))\cosh(\nu\log(\phi)) + (\cos(\pi\nu)+1)\sinh(\nu\log(\phi))\right)$	
$F_\nu = \frac{1}{\sqrt{5}}\left(2\sin\left(\frac{\pi\nu}{2}\right)\sin\left(\nu\csc^{-1}\left(\frac{2}{\sqrt{5}}\right)\right) + \left(1+e^{i\pi\nu}\right)\sinh\left(\nu\,\text{csch}^{-1}(2)\right)\right)$	

$$F_\nu = \frac{e^{\frac{i\pi\nu}{2}}}{\sqrt{5}}\left((2\, i\cos(\pi\nu) + \sin(\pi\nu))\sin\left(2\nu\sin^{-1}\left(\frac{\sqrt{2-i}}{2}\right)\right) - i\sin(\pi\nu)\cos\left(2\nu\sin^{-1}\left(\frac{\sqrt{2-i}}{2}\right)\right)\right)$$

$$F_\nu = \frac{1}{\sqrt{5}}\, e^{-\frac{i\pi\nu}{2}}\left(i\sin(\pi\nu)\cos\left(2\nu\sin^{-1}\left(\frac{\sqrt{2+i}}{2}\right)\right) + (\sin(\pi\nu) - 2\, i\cos(\pi\nu))\sin\left(2\nu\sin^{-1}\left(\frac{\sqrt{2+i}}{2}\right)\right)\right)$$

$$F_n = \frac{i^{n-1}\sin(n\, z)}{\sin(z)} /;\ z = i\log\left(\frac{\sqrt{5}+1}{2}\right) + \frac{\pi}{2} \bigwedge n \in \mathbb{Z}.$$

The best-known properties and formulas of the Fibonacci and Lucas numbers

Simple values at zero and infinity

The Fibonacci and Lucas numbers F_ν and L_ν have the following values at zero and infinity:

F_ν	L_ν
$F_0 = 0$	$L_0 = 2$
$F_\infty = \infty$	$L_\infty = \infty$

Specific values for specialized variables

The Fibonacci and Lucas numbers F_n and L_n with integer argument n can be represented by the following formulas:

F_ν	L_ν
$F_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right) /; n \in \mathbb{Z}$	$L_n = \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n /; n \in \mathbb{Z}$
$F_{-n} = \frac{(-1)^{n-1}}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right) /; n \in \mathbb{Z}$	$L_{-n} = (-1)^n \left(\frac{1+\sqrt{5}}{2}\right)^n + (-1)^n \left(\frac{1-\sqrt{5}}{2}\right)^n /; n \in \mathbb{Z}$
$F_n = \frac{1}{5}\left(\lfloor \phi^{n-1} \rceil + \lfloor \phi^{n+1} \rceil\right) /; n \in \mathbb{Z} \land n > 2$	$L_n = \lfloor \phi^n \rceil /; n \in \mathbb{Z} \land n > 1$
$F_n = \frac{\phi^n - (1-\phi)^n}{\sqrt{5}} /; n \in \mathbb{Z}$	$L_n = \phi^n + (1-\phi)^n /; n \in \mathbb{Z}$
$F_{n+1} = \left\lfloor \frac{1}{2}\left(\left(1+\sqrt{5}\right) F_n + 1\right) \right\rfloor /; n \in \mathbb{Z} \land n > 1$	$L_{n+1} = \left\lfloor \frac{1}{2}\left(\left(1+\sqrt{5}\right) L_n + 1\right) \right\rfloor /; n \in \mathbb{Z} \land n > 3$
$F_n = \frac{\phi^n - (1-\phi)^n}{\phi - (1-\phi)} /; n \in \mathbb{Z}$	$L_n = \frac{\phi^n + (1-\phi)^n}{\phi + (1-\phi)} /; n \in \mathbb{Z}$

For the cases of integer arguments $n /; 0 \leq n \leq 25$, the values of the Fibonacci and Lucas numbers F_n and L_n can be described by the following table:

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	…
$\boldsymbol{F_n}$	0	1	1	2	3	5	8	13	21	34	55	89	144	233	377	610	987	1597	2584	4181	6765	…
L_n	2	1	3	4	7	11	18	29	47	76	123	199	322	521	843	1364	2207	3571	5778	9349	15 127	…

Analyticity

The Fibonacci and Lucas numbers F_ν and L_ν are entire analytical functions of ν that are defined over the whole complex ν-plane:

Periodicity

The Fibonacci and Lucas numbers F_ν and L_ν do not have periodicity.

Parity and symmetry

The Fibonacci and Lucas numbers F_ν and L_ν generically do not have parity, but they have mirror symmetry:

F_ν	L_ν
$F_{-n} = (-1)^{n-1} F_n /; n \in \mathbb{Z}$	$L_{-n} = (-1)^n L_n /; n \in \mathbb{Z}$
$F_{\overline{\nu}} = \overline{F_\nu}$	$L_{\overline{\nu}} = \overline{L_\nu}$

Poles and essential singularities

The Fibonacci and Lucas numbers F_ν and L_ν have only the singular point $\nu = \tilde{\infty}$. It is an essential singular point.

Branch points and branch cuts

The Fibonacci and Lucas numbers F_ν and L_ν do not have branch points and branch cuts over the complex ν-plane.

Series representations

The Fibonacci and Lucas numbers F_ν and L_ν have the following series expansions (which converge in the whole complex ν-plane):

F_ν	L_ν
$F_\nu \propto F_{\nu_0} + \frac{1}{\sqrt{5}} \left(2\left(1+\sqrt{5}\right)\right)^{-\nu_0} \left(\text{csch}^{-1}(2) \left(1+\sqrt{5}\right)^{2\nu_0} + 4^{\nu_0} \left(\text{csch}^{-1}(2) \cos(\pi\nu_0) + \pi \sin(\pi\nu_0)\right)\right) (\nu-\nu_0) + \ldots /; (\nu \to \nu_0)$	$L_\nu \propto L_{\nu_0} + \left(\phi^{\nu_0} \text{csch}^{-1}(2) + \frac{1}{2} \phi^{-\nu_0} \left(e^{i\pi\nu_0} \left(i\pi - \text{csch}^{-1}(2)\right) - e^{-i\pi\nu_0} \left(i\pi + \text{csch}^{-1}(2)\right)\right)\right) (\nu-\nu_0) + .$
$F_\nu = \sum_{k=0}^{\infty} \frac{1}{k!} \left(F_{\nu_0} \text{csch}^{-1}(2)^k + \frac{1}{\sqrt{5}} \left(2^{\nu_0-1} \left(1+\sqrt{5}\right)^{-\nu_0} e^{-i\pi\nu_0} \left(\text{csch}^{-1}(2)^k - (-1)^k \left(i\pi + \text{csch}^{-1}(2)\right)^k + e^{2i\pi\nu_0} \left(\text{csch}^{-1}(2)^k - \left(i\pi - \text{csch}^{-1}(2)\right)^k\right)\right)\right)\right) (\nu-\nu_0)^k$	$L_\nu = \sum_{k=0}^{\infty} \frac{1}{k!} \left(\phi^{\nu_0} \text{csch}^{-1}(2)^k + \frac{1}{2} \phi^{-\nu_0} \left(e^{\pi i \nu_0} \left(\pi i - \text{csc} \right.\right.\right.$ $\left.\left.\left. (-1)^k e^{-\pi i \nu_0} \left(\pi i + \text{csch}^{-1}(2)\right)^k\right)\right) (\nu - \nu$
$F_\nu \propto F_{\nu_0} (1 + O(\nu-\nu_0))$	$L_\nu \propto L_{\nu_0} (1 + O(\nu-\nu_0))$
$F_\nu \propto \frac{2\log(\phi)\nu}{\sqrt{5}} + \frac{\pi^2\nu^2}{2\sqrt{5}} + \frac{1}{\sqrt{5}} \left(\frac{\log^3(\phi)}{3} - \frac{\pi^2}{2}\log(\phi)\right) \nu^3 + \ldots /; (\nu \to 0)$	$L_\nu \propto 2 + \left(\log^2(\phi) - \frac{\pi^2}{2}\right)\nu^2 + \frac{\pi^2 \log(\phi)}{2} \nu^3 + \ldots /; (\nu \to 0)$
$F_\nu = \frac{1}{2\sqrt{5}} \sum_{k=1}^{\infty} \frac{\left(2\,\text{csch}^{-1}(2)^k - \left(-i\pi - \text{csch}^{-1}(2)\right)^k - \left(i\pi - \text{csch}^{-1}(2)\right)^k\right) \nu^k}{k!}$	$L_\nu = \sum_{k=0}^{\infty} \frac{\text{csch}^{-1}(2)^k + \frac{1}{2}\left(\left(\pi i - \text{csch}^{-1}(2)\right)^k + (-1)^k \left(\pi i + \text{csch}^{-1}(2)\right)^k\right)}{k!} \nu$
$F_\nu \propto \frac{2\log(\phi)}{\sqrt{5}} \nu + O\left(\nu^2\right) /; (\nu \to 0)$	$L_\nu \propto 2 + O\left(\nu^2\right) /; (\nu \to 0)$

Asymptotic series expansions

The asymptotic behavior of the Fibonacci and Lucas numbers F_ν and L_ν is described by the following formulas:

F_ν	L_ν
$F_\nu \propto \frac{\phi^\nu}{\sqrt{5}} /; (\nu \to \infty)$	$L_\nu \propto \phi^\nu /; (\nu \to \infty)$
$F_\nu \propto \begin{cases} \frac{\phi^\nu}{\sqrt{5}} & \text{Im}(\nu) < 0 \land \text{Re}(\nu) - \pi \lvert\text{Im}(\nu)\rvert > 0 \\ -\frac{e^{i\nu\pi - \nu\,\text{csch}^{-1}(2)}}{2\sqrt{5}} & \text{Im}(\nu) < 0 \land \pi\,\text{Im}(\nu) + \text{Re}(\nu) < 0 \\ -\frac{e^{-\text{csch}^{-1}(2)\nu - i\pi\nu}}{2\sqrt{5}} & \text{Im}(\nu) > 0 \land \text{Re}(\nu) - \pi\,\text{Im}(\nu) < 0 \\ \frac{\phi^\nu - \cos(\nu\pi)\phi^{-\nu}}{\sqrt{5}} & \text{True} \end{cases} /; (\lvert\nu\rvert \to \infty)$	$L_\nu \propto \begin{cases} \phi^\nu & \text{Im}(\nu) < 0 \land \text{Re}(\nu) - \pi \lvert\text{Im}(\nu)\rvert > \\ \frac{1}{2} e^{i\nu\pi - \nu\,\text{csch}^{-1}(2)} & \text{Im}(\nu) < 0 \land \text{Re}(\nu) + \pi\,\text{Im}(\nu) < 0 \\ \frac{1}{2} e^{-i\pi\nu - \text{csch}^{-1}(2)\nu} & \text{Im}(\nu) > 0 \land \text{Re}(\nu) - \pi\,\text{Im}(\nu) < 0 \\ \phi^{-\nu}\cos(\nu\pi) + \phi^\nu & \text{True} \end{cases}$
$F_\nu \propto \frac{\phi^\nu - \cos(\nu\pi)\phi^{-\nu}}{\sqrt{5}} /; (\lvert\nu\rvert \to \infty)$	$L_\nu \propto \phi^\nu + \cos(\nu\pi)\phi^{-\nu} /; (\lvert\nu\rvert \to \infty)$

Other series representations

The Fibonacci and Lucas numbers F_n and L_n for integer nonnegative n can be represented through the following sums involving binomials:

$$F_n = \sum_{k=0}^{n-1} \binom{n-k-1}{k} /; n \in \mathbb{N}$$

$$F_n = \sum_{k=0}^{n-1} \binom{k}{n-k-1} /; n \in \mathbb{N}$$

$$F_n = \sum_{k=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \binom{n-k-1}{k} /;\, n \in \mathbb{N}$$

$$F_{2n+1} = \sum_{k=0}^{n} \binom{k+n}{2k} /;\, n \in \mathbb{N}$$

$$F_{2n} = \sum_{k=0}^{n-1} \binom{k+n}{2k+1} /;\, n \in \mathbb{N}$$

$$F_n = \frac{2^{1-n}\sqrt{\pi}}{\Gamma\left(\frac{n}{2}\right)} \sum_{k=0}^{n-1} \frac{(n-k-1)!\left(1-\frac{n}{2}\right)_k (-4)^k}{k!\,\Gamma\left(\frac{n+1}{2}-k\right)} /;\, n \in \mathbb{N}$$

$$F_n = \sum_{k=1}^{n} 5^{\frac{n-k}{2}}\, i^{k-1} \binom{2n-k}{k-1} \exp\left(i\,(n-k)\tan^{-1}(-2)\right) /;\, n \in \mathbb{N}$$

$$F_n = \frac{1}{2^{n-1}} \sum_{k=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \binom{n}{2k+1} 5^k /;\, n \in \mathbb{N}$$

$$F_n = (-i)^{n-1} \sum_{k=0}^{n-1} \binom{k+n}{2k+1} (i-2)^k /;\, n \in \mathbb{N}$$

$$F_n^2 = \frac{n}{5^{n-2\left\lfloor \frac{n+1}{2} \right\rfloor+1}} \sum_{k=0}^{n-1} \frac{1}{k+1} \binom{k+n}{2k+1} (-5)^{k\left(n-2\left\lfloor \frac{n}{2} \right\rfloor\right)} /;\, n \in \mathbb{N}$$

$$L_n = 2^{1-n} \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \binom{n}{2k} 5^k /;\, n \in \mathbb{N}.$$

Integral representations

The Fibonacci and Lucas numbers F_n and L_n have the following integral representations on the real axis:

$$F_{2n} = \frac{n}{2}\left(\frac{3}{2}\right)^{n-1} \int_0^{\pi} \left(1+\frac{\sqrt{5}}{3}\cos(t)\right)^{n-1} \sin(t)\,dt /;\, n \in \mathbb{Z}$$

$$L_{2n+1} = \frac{1}{4}\left(\frac{3}{2}\right)^{n-1} \int_0^{\pi} \left(\frac{1}{3}\sqrt{5}\cos(t)+1\right)^{n-1} \left(5n+\sqrt{5}\,(n+1)\cos(t)+3\right)\sin(t)\,dt /;\, n \in \mathbb{Z}.$$

Generating functions

The Fibonacci and Lucas numbers F_n and L_n can be represented as the coefficients of the series of the corresponding generating functions:

$$F_n = \left([t^n]\,\frac{t}{1-t-t^2}\right) /;\, n \in \mathbb{N}$$

$$L_n = \left([t^n] \, \frac{2-t}{1-t-t^2} \right) /; n \in \mathbb{N}.$$

Transformations: Addition formulas

The Fibonacci and Lucas numbers F_n and L_n satisfy numerous addition formulas:

$$F_{m+2n} = \sum_{k=0}^{\infty} \binom{n}{k} F_{k+m} /; m \in \mathbb{N} \land n \in \mathbb{N}$$

$$F_{m+2n} = \sum_{k=0}^{\infty} 2^{n-k} \binom{n}{k} F_{m-k} /; m \in \mathbb{N} \land n \in \mathbb{N}$$

$$F_{m+3n} = \sum_{k=0}^{\infty} 2^{k} \binom{n}{k} F_{k+m} /; m \in \mathbb{N} \land n \in \mathbb{N}$$

$$F_{2n+1} = F_{n+1}^2 + F_n^2 /; n \in \mathbb{N}$$

F_ν	L_ν
$F_{m+n} = F_{n+1} F_m + F_{m-1} F_n /; m \in \mathbb{Z} \land n \in \mathbb{Z}$; $F_{m+n} = \frac{1}{2} (F_n L_m + F_m L_n) /; n \in \mathbb{Z} \land m \in \mathbb{Z}$; $F_{m+n} = \sum_{k=0}^{\infty} \binom{n}{k} F_{m-k} /; m \in \mathbb{N} \land n \in \mathbb{N}$	$L_{m+n} = \frac{1}{2} (5 F_m F_n + L_m L_n) /; n \in \mathbb{Z} \land m \in \mathbb{Z}$
$F_{m-n} = (-1)^n (F_m F_{n+1} - F_n F_{m+1}) /; m \in \mathbb{Z} \land n \in \mathbb{Z}$; $F_{m-n} = \frac{1}{2} (-1)^n (F_m L_n - F_n L_m) /; n \in \mathbb{Z} \land m \in \mathbb{Z}$	$L_{m-n} = \frac{1}{2} (-1)^{n-1} (5 F_n F_m - L_n L_m) /; n \in \mathbb{Z} \land m \in \mathbb{Z}$
$F_{\nu+1} = \frac{1}{2} (F_\nu + L_\nu)$	$L_{\nu+1} = \frac{1}{2} (5 F_\nu + L_\nu)$

Transformations: Multiple arguments

The Fibonacci and Lucas numbers F_n and L_n satisfy numerous identities, for example the following multiple argument formulas:

$$F_{2\nu} = L_\nu F_\nu + \frac{\sin^2(\pi \nu) \phi^{-2\nu}}{\sqrt{5}}$$

F_ν	L_ν
$F_{2\nu} == F_{\nu-1} F_\nu + F_{\nu+1} F_\nu + \frac{\phi^{-2\nu} \sin^2(\pi \nu)}{\sqrt{5}}$	$L_{2\nu} = \frac{1}{5} \left(3 L_\nu^2 - 2 L_{\nu+1} L_\nu + 2 L_{\nu+1}^2 - 5 \phi^{-2\nu} \text{si}\right.$
$F_{2\nu+1} == F_{\nu-1} F_{\nu+1} + F_{\nu+2} F_\nu - \frac{\phi^{-2\nu-1} \sin^2(\pi \nu)}{\sqrt{5}}$	$L_{2\nu+1} = \frac{1}{5} \left(-L_\nu^2 + 4 L_{\nu+1} L_\nu + L_{\nu+1}^2 + 5 \phi^{-2\nu-1}\right.$
$F_{2n} = L_n F_n /; n \in \mathbb{Z};$ $F_{2n} == F_{n-p} F_{n+p-1} + F_{n-p+1} F_{n+p} /; n \in \mathbb{N} \bigwedge p \in \mathbb{N}$	$L_{2n} = \frac{1}{2} \left(5 F_n^2 + L_n^2\right) /; n \in \mathbb{Z}$
$F_{2n} == \sum_{k=0}^{n} \binom{n}{k} F_k /; n \in \mathbb{N}$	$L_{2n} == \sum_{k=0}^{n} \binom{n}{k} L_k /; n \in \mathbb{N}$
$F_{3n} == \sum_{k=0}^{n} \binom{n}{k} F_k 2^k /; n \in \mathbb{N}$	$L_{3n} == \sum_{k=0}^{n} \binom{n}{k} 2^k L_k /; n \in \mathbb{N}$
$F_{2\nu} = 3 F_{2(\nu-1)} - F_{2(\nu-2)}$	$L_{2\nu} = 3 L_{2(\nu-1)} - L_{2(\nu-2)}$
$F_{m\nu} = L_m F_{m(\nu-1)} - (-1)^m F_{m(\nu-2)} /; m \in \mathbb{Z}$	$L_{m\nu} = L_m L_{m(\nu-1)} - (-1)^m L_{m(\nu-2)} /; m \in \mathbb{Z}$
$F_{mn} == F_n \sum_{k=0}^{\left\lfloor \frac{m-1}{2} \right\rfloor} \binom{m-k-1}{k} (-1)^{k(n-1)} L_n^{m-2k-1} /; m \in \mathbb{N}^+ \bigwedge n \in \mathbb{Z}$	$L_{mn} = \sum_{k=0}^{\left\lfloor \frac{m}{2} \right\rfloor} \frac{m}{m-k} \binom{m-k}{k} (-1)^{k(n+1)} L_n^{m-2k} /;$
$F_{mn} = \frac{1}{2^{m-1}} \sum_{k=0}^{\left\lfloor \frac{m-1}{2} \right\rfloor} \binom{m}{2k+1} F_n^{2k+1} L_n^{-2k+m-1} 5^k /; m \in \mathbb{N}^+ \bigwedge n \in \mathbb{Z}$	$L_{mn} = L_n^{m-2\left\lfloor \frac{m}{2} \right\rfloor} \sum_{k=0}^{\left\lfloor \frac{m}{2} \right\rfloor} \left(\frac{m}{m-k}\right)^{2\left\lfloor \frac{m}{2} \right\rfloor - m+1} \left(\begin{matrix} 2\left\lfloor \frac{m}{2} \right\rfloor - \\ k \end{matrix}\right.$ $(-1)^{kn} 5^{\left\lfloor \frac{m}{2} \right\rfloor - k} F_n^{2\left\lfloor \frac{m}{2} \right\rfloor - 2k} /; n \in \mathbb{Z} \bigwedge n \neq 0 \bigwedge m$
$F_{mn} == \sum_{k=0}^{m} \binom{m}{k} F_n^k F_{n-1}^{m-k} F_k /; m-1 \in \mathbb{N}^+ \bigwedge n \in \mathbb{N}^+ \bigwedge n \geq m$	$L_{mn} = 2^{1-m} \sum_{k=0}^{\left\lfloor \frac{m}{2} \right\rfloor} \binom{m}{2k} F_n^{2k} L_n^{m-2k} 5^k /; n \in \mathbb{N}$
$F_{2mn} == L_n \sum_{k=0}^{m-1} \binom{2m-k-1}{k} (-1)^{kn} F_n^{-2k+2m-1} 5^{-k+m-1} /; m \in \mathbb{N}^+ \bigwedge n \in \mathbb{Z}$	$L_{mn} = \sum_{k=0}^{m} \binom{m}{k} L_k F_n^k F_{n-1}^{m-k} /; n \in \mathbb{Z} \bigwedge n \neq$
$F_{(2m-1)n} == \sum_{k=0}^{m-1} \frac{2m-1}{2m-k-1} \binom{2m-k-1}{k} (-1)^{kn} F_n^{2m-2k-1} 5^{m-k-1} /; m \in \mathbb{N}^+ \bigwedge n \in \mathbb{Z}$	

Transformations: Products and powers of the direct function

The Fibonacci and Lucas numbers F_n and L_n satisfy numerous identities for products and powers:

F_ν	L_ν
$F_{\nu+1} F_{\nu-1} == F_\nu^2 + \cos(\nu \pi)$	$L_{\nu+1} L_{\nu-1} = L_\nu^2 - 5 \cos(\pi \nu)$
$F_n F_m = \frac{1}{5} (L_{m+n} - (-1)^n L_{m-n}) /; m \in \mathbb{Z} \land n \in \mathbb{Z}$	$L_m L_n = (-1)^n L_{m-n} + L_{m+n} /; m \in \mathbb{Z} \land n \in \mathbb{Z}$
$F_\nu^2 == F_{\nu+1} F_{\nu-1} - \cos(\nu \pi)$	$L_\nu^2 = L_{\nu-1} L_{\nu+1} + 5 \cos(\pi \nu)$
$F_n^2 = \frac{1}{5} (L_{2n} - 2(-1)^n) /; n \in \mathbb{Z}$	$L_n^2 = L_{2n} + 2(-1)^n /; n \in \mathbb{Z}$
$F_n^3 = \frac{1}{5} \left(3(-1)^{n+1} F_n + F_{3n}\right) /; n \in \mathbb{Z}$	$L_n^3 = L_{3n} + 3(-1)^n L_n /; n \in \mathbb{Z}$
$F_n^4 == F_{n-2} F_{n-1} F_{n+1} F_{n+2} + 1 /; n \in \mathbb{Z}$	$L_n^4 = L_{4n} + 4(-1)^n L_{2n} + 6 /; n \in \mathbb{Z}$
$F_n^m == \frac{1}{2} 5^{-\left\lfloor \frac{m}{2} \right\rfloor} \sum_{k=0}^{m} \binom{m}{k} (-1)^{k(n+1)} ((1 + (-1)^m) F_{-2kn+mn+1} -$ $(-1)^m F_{(m-2k)n}) /; n \in \mathbb{Z} \bigwedge m \in \mathbb{N}^+$	$L_n^m = \frac{1}{2} \sum_{j=0}^{m} (-1)^{jn} \binom{m}{j} L_{n(m-2j)} /; n \in \mathbb{Z} \bigwedge m \in \mathbb{N}^+$

Identities

The Fibonacci and Lucas numbers F_ν and L_ν are solutions of the following simple difference equation with constant coefficients:

$w(z) = w(z-2) + w(z-1)$ /; $w(z) = c_1 F_z + c_2 L_z$.

The Fibonacci and Lucas numbers F_ν and L_ν satisfy numerous recurrence identities:

F_ν	L_ν
$F_\nu = F_{\nu+2} - F_{\nu+1}$	$L_\nu = L_{\nu+2} - L_{\nu+1}$
$F_\nu = F_{\nu-1} + F_{\nu-2}$	$L_\nu = L_{\nu-2} + L_{\nu-1}$
$F_\nu = \frac{F_{\nu+1} - \phi^\nu}{1-\phi}$	$L_\nu = \frac{L_{\nu+1} - \sqrt{5}\,\phi^\nu}{1-\phi}$
$F_\nu = (1-\phi) F_{\nu-1} + \phi^{\nu-1}$	$L_\nu = (1-\phi) L_{\nu-1} + \sqrt{5}\,\phi^{\nu-1}$
$F_\nu = i^{m+1} U_{\frac{m-1}{2}}\left(-\frac{3}{2}\right) F_{m+\nu} + i^m U_{\frac{m}{2}-1}\left(-\frac{3}{2}\right) F_{m+\nu+1}$ /; $m \in \mathbb{N}^+$	$L_\nu = i^{m+1} U_{\frac{m-1}{2}}\left(-\frac{3}{2}\right) L_{m+\nu} + i^m U_{\frac{m}{2}-1}\left(-\frac{3}{2}\right) L_{m+\nu+1}$ /; $m \in \mathbb{N}^+$
$F_\nu = i^{1-m} U_{\frac{m-1}{2}}\left(-\frac{3}{2}\right) F_{\nu-m} - (-i)^m U_{\frac{m}{2}-1}\left(-\frac{3}{2}\right) F_{\nu-m-1}$ /; $m \in \mathbb{N}^+$	$L_\nu = i^{1-m} U_{\frac{m-1}{2}}\left(-\frac{3}{2}\right) L_{\nu-m} - (-i)^m U_{\frac{m}{2}-1}\left(-\frac{3}{2}\right) L_{\nu-m-1}$ /; $m \in \mathbb{N}^+$
$F_\nu^2 - F_{\nu+1} F_{\nu-1} = -\cos(\nu\pi)$	$L_\nu^2 - L_{\nu+1} L_{\nu-1} = 5\cos(\pi\nu)$

Other identities for Fibonacci and Lucas numbers F_ν and L_ν are just functional identities:

$$F_{k+n} F_{l+n} - F_{k+l+n} F_n = (-1)^n F_k F_l \text{ /; } k \in \mathbb{N} \bigwedge l \in \mathbb{N} \bigwedge n \in \mathbb{N}$$

$$F_{k+n-1} F_{n-k} + F_{n-k+1} F_{k+n} = F_{2n} \text{ /; } k \in \mathbb{N} \bigwedge n \in \mathbb{N}$$

$$F_n^2 - F_{m+n} F_{n-m} = (-1)^{n-m} F_m^2 \text{ /; } n \in \mathbb{Z} \land m \in \mathbb{Z}$$

$$4 F_{2n+1} F_{2n+2} F_{2n+3} F_{2n+4} - (2 F_{2n+2} F_{2n+3} + 1)^2 + 1 = 0 \text{ /; } n \in \mathbb{N}$$

$$\frac{1}{F_{a+n} F_{b+n} F_{c+n}} = \frac{(-1)^{b+n}}{F_{c-b} F_{a-b} F_{b+n}} + \frac{(-1)^{c+n}}{F_{a-c} F_{b-c} F_{c+n}} + \frac{(-1)^{a+n}}{F_{b-a} F_{c-b} F_{a+n}} \text{ /; } n \in \mathbb{N} \bigwedge a \in \mathbb{N}^+ \bigwedge b \in \mathbb{N}^+ \bigwedge c \in \mathbb{N}^+ \bigwedge a \neq b \bigwedge a \neq c \bigwedge b \neq c$$

$$F_{n+1} = \left\lfloor \phi F_n + \frac{1}{2} \right\rfloor \text{ /; } n - 1 \in \mathbb{N}^+$$

$$F_{\gcd(m,n)} = \gcd(F_m, F_n) \text{ /; } m \in \mathbb{Z} \land n \in \mathbb{Z}$$

$$F_m \sum_{k=1}^{n} \frac{(-1)^k}{F_k F_{k+m}} = F_n \sum_{k=1}^{m} \frac{(-1)^k}{F_k F_{k+n}} \text{ /; } m \in \mathbb{N} \land n \in \mathbb{N}$$

$$\tan^{-1}\left(\frac{1}{F_{2n+1}}\right) + \tan^{-1}\left(\frac{1}{F_{2n+2}}\right) = \tan^{-1}\left(\frac{1}{F_{2n}}\right) \text{ /; } n \in \mathbb{N}^+$$

$$\sum_{k=1}^{n} \frac{(-1)^k}{L_k L_{k+m}} = \frac{F_n}{F_m} \sum_{k=1}^{m} \frac{(-1)^k}{L_k L_{k+n}} \text{ /; } n \in \mathbb{Z} \land m \in \mathbb{Z} \land n\,m > 0.$$

Complex characteristics

The Fibonacci and Lucas numbers F_ν and L_ν have the following complex characteristics for complex arguments:

$x + i\, y$ /; $x \in \mathbb{R} \wedge y \in \mathbb{R}$	$F_{x+i\, y}$
Abs	$\lvert F_{x+i\,y} \rvert = \frac{1}{\sqrt{10}} \sqrt{\left(\phi^{-2x} \left(\cosh^2(\pi y) - 4\, \phi^{2x} \cos(\pi x) \cos(2\, y \log(\phi)) \cosh(\pi y) + 2\, \phi^{4x} + \sinh^2(\pi y) + \cos(2\, \pi x) + 4\, \phi^{2x} \sin(\pi x) \sin(2\, y \log(\phi)) \sinh(\pi y)\right)\right)}$
Arg	$\arg(F_{x+i\,y}) = \tan^{-1}\left(\phi^{-x} \left(\phi^{2x} \cos(y \log(\phi)) - \cos(\pi x) \cosh(\pi y) \cos(y \log(\phi)) + \sin(\pi x) \sin(y \log(\phi)) \sinh(\pi y)\right), \phi^{-x} \left(\phi^{2x} \sin(y \log(\phi)) + \cos(\pi x) \cosh(\pi y) \sin(y \log(\phi)) + \cos(y \log(\phi)) \sin(\pi x) \sinh(\pi y)\right)\right)$
Re	$\operatorname{Re}(F_{x+i\,y}) = \frac{\phi^{-x}}{\sqrt{5}} \left(\phi^{2x} \cos(y \log(\phi)) - \cos(\pi x) \cosh(\pi y) \cos(y \log(\phi)) + \sin(\pi x) \sin(y \log(\phi)) \sinh(\pi y)\right)$
Im	$\operatorname{Im}(F_{x+i\,y}) = \frac{\phi^{-x}}{\sqrt{5}} \left(\phi^{2x} \sin(y \log(\phi)) + \cos(\pi x) \cosh(\pi y) \sin(y \log(\phi)) + \cos(y \log(\phi)) \sin(\pi x) \sinh(\pi y)\right)$
Conjugate	$\overline{F_{x+i\,y}} = \frac{\phi^{-x}}{\sqrt{5}} \left(\phi^{2x} (\cos(y \log(\phi)) - i \sin(y \log(\phi))) + \sin(\pi x) \sinh(\pi y) (\sin(y \log(\phi)) - i \cos(y \log(\phi))) - \cos(\pi x) \cosh(\pi y) (\cos(y \log(\phi)) + i \sin(y \log(\phi)))\right)$
Sign	$\operatorname{sgn}(F_{x+i\,y}) = \left(\sqrt{2}\ \phi^{-x} \left(\cos(\pi x) \cosh(\pi y) (i \sin(y \log(\phi)) - \cos(y \log(\phi))) + \phi^{2x} (\cos(y \log(\phi)) + i \sin(y \log(\phi))\right.\right.$ $\left.\left. \sin(\pi x) (i \cos(y \log(\phi)) + \sin(y \log(\phi))) \sinh(\pi y)\right)\right) \Big/ \left(\sqrt{\left(\phi^{-2x} \left(\cosh^2(\pi y) - 4\, \phi^{2x} \cos(\pi x) \cos(2\, y \log(\phi)) \cosh(\pi y) + 2\, \phi^{4x} + \sinh^2(\pi y) + \cos(2\, \pi x) + 4\, \phi^{2x} \sin(\pi x) \sin(2\, y \log(\phi)) \sinh(\pi y)\right)\right)}\right)$

$x + i\, y$ /; $x \in \mathbb{R} \wedge y \in \mathbb{R}$	$L_{x+i\, y}$
Abs	$\lvert L_{x+i\,y} \rvert = \frac{1}{\sqrt{2}} \sqrt{\left(\left(1+\sqrt{5}\right)^{-2x} \left(4^x \cos(2\, \pi x) + 2 \left(\left(3+\sqrt{5}\right)^{2x} + 2^{-2\, i\, y} \left(1+\sqrt{5}\right)^{2\,(x - i\, y)} \left(\cos(\pi\,(x - i\, y)) \left(1+\sqrt{5}\right)^{4\, i\, y} + 2^{4\, i\, y} \cos(\pi\,(x + i\, y))\right)\right) + 4^x \cosh(2\, \pi y)\right)\right)}$
Arg	$\arg(L_{x+i\,y}) = \tan^{-1}(\phi^x \cos(y \log(\phi)) + \phi^{-x} \cos(\pi x) \cosh(\pi y) \cos(y \log(\phi)) - \phi^{-x} \sin(\pi x) \sin(y \log(\phi)) \sinh(\pi y), \phi^x \sin(y \log(\phi)) - \phi^{-x} \cos(\pi x) \cosh(\pi y) \sin(y \log(\phi)) - \phi^{-x} \cos(y \log(\phi)) \sin(\pi x) \sinh(\pi y))$
Re	$\operatorname{Re}(L_{x+i\,y}) = \phi^x \cos(y \log(\phi)) + \phi^{-x} \cos(\pi x) \cosh(\pi y) \cos(y \log(\phi)) - \phi^{-x} \sin(\pi x) \sin(y \log(\phi)) \sinh(\pi y)$
Im	$\operatorname{Im}(L_{x+i\,y}) = \phi^x \sin(y \log(\phi)) - \phi^{-x} \cos(\pi x) \cosh(\pi y) \sin(y \log(\phi)) - \phi^{-x} \cos(y \log(\phi)) \sin(\pi x) \sinh(\pi y)$
Conjugate	$\overline{L_{x+i\,y}} = \phi^x \cos(y \log(\phi)) + \phi^{-x} \cos(\pi x) \cosh(\pi y) \cos(y \log(\phi)) - \phi^{-x} \sin(\pi x) \sin(y \log(\phi)) \sinh(\pi y) - i\, (\phi^x \sin(y \log(\phi)) - \phi^{-x} \cos(\pi x) \cosh(\pi y) \sin(y \log(\phi)) - \phi^{-x} \cos(y \log(\phi)) \sin(\pi x) \sinh(\pi y))$
Sign	$\operatorname{sgn}(L_{x+i\,y}) = \phi^x \cos(y \log(\phi)) + \left(2^{\frac{1}{2}-x} \left(1+\sqrt{5}\right)^{-x} \left(i \sin\left(y \operatorname{csch}^{-1}(2)\right) \left(1+\sqrt{5}\right)^{2x} + 4^x \cos(\pi\,(x + i\, y)) \left(\cos\left(y \operatorname{csch}^{-1}(2)\right) - i \sin\left(y \operatorname{csch}^{-1}(2)\right)\right)\right)\right) \Big/ \left(\sqrt{\left(\left(1+\sqrt{5}\right)^{-2x} \left(4^x \cos(2\, \pi x) + 2 \left(\left(3+\sqrt{5}\right)^{2x} + 2^{-2\, i\, y} \left(1+\sqrt{5}\right)^{2\,(x - i\, y)} \left(\cos(\pi\,(x - i\, y)) \left(1+\sqrt{5}\right)^{4\, i\, y} + 2^{4\, i\, y} \cos(\pi\,(x + i\, y))\right)\right) + 4^x \cosh(\right.\right.}\right.$

Differentiation

The Fibonacci and Lucas numbers F_ν and L_ν have the following representations for derivatives of the first and n^{th} orders or the arbitrary fractional order α:

F_ν	L_ν
$\frac{\partial F_\nu}{\partial \nu} == \frac{\phi^{-\nu} (\phi^{2\nu} \log(\phi) + \cos(\pi \nu) \log(\phi) + \pi \sin(\pi \nu))}{\sqrt{5}}$	$\frac{\partial L_\nu}{\partial \nu} = \phi^\nu \log(\phi) - \phi^{-\nu} \cos(\pi \nu) \log(\phi) - \phi^{-\nu} \pi \sin(\pi \nu)$
$\frac{\partial^n F_\nu}{\partial \nu^n} == \frac{1}{\sqrt{5}} \left(\phi^\nu \log^n(\phi) - \frac{1}{2} (-1)^n \phi^{-\nu} \left(e^{-i \pi \nu} (\log(\phi) + i \pi)^n + e^{i \pi \nu} (\log(\phi) - i \pi)^n\right)\right) /; n \in \mathbb{N}$	$\frac{\partial^n L_\nu}{\partial \nu^n} = \phi^\nu \log^n(\phi) + \frac{1}{2} \phi^{-\nu} \left(e^{\pi i \nu} (\pi i - \log(\phi))^n + (-1)^n e^{-\pi i \nu} (\pi i + \log(\phi))^n\right) /; n \in \mathbb{N}$
$\frac{\partial^\alpha F_\nu}{\partial \nu^\alpha} == \frac{\nu^{-\alpha}}{2\sqrt{5}} \left(\left(\nu \left(\pi i - \text{csch}^{-1}(2)\right)\right)^\alpha \exp\left(\left(i \pi - \text{csch}^{-1}(2)\right) \nu\right) \left(Q\left(-\alpha, \left(i \pi - \text{csch}^{-1}(2)\right) \nu\right) - 1\right) + \exp\left(-\left(i \pi + \text{csch}^{-1}(2)\right) \nu\right) \left(\nu \left(-i \pi - \text{csch}^{-1}(2)\right)\right)^\alpha \left(Q\left(-\alpha, -\left(i \pi + \text{csch}^{-1}(2)\right) \nu\right) - 1\right) - 2 \nu^\alpha \text{csch}^{-1}(2)^\alpha \exp\left(\nu \, \text{csch}^{-1}(2)\right) \left(Q\left(-\alpha, \nu \, \text{csch}^{-1}(2)\right) - 1\right)\right)$	$\frac{\partial^\alpha L_\nu}{\partial \nu^\alpha} = -\frac{\nu^{-\alpha}}{2} ((Q(-\alpha, (i \pi - \text{csch}^{-1}(2)) \nu) - 1) e^{(i \pi - \text{csch}^{-1}(2)}$ $(\nu (\pi i - \text{csch}^{-1}(2)))^\alpha + (Q(-\alpha, -(i \pi + \text{csch}^{-}$ $e^{-(i \pi + \text{csch}^{-1}(2)) \nu} (\nu (-i \pi - \text{csch}^{-1}(2)))^\alpha + 2 (Q(-\alpha$ $e^{\nu \, \text{csch}^{-1}(2)} \nu^\alpha \text{csch}^{-1}(2)^\alpha)$

Differential equations

The Fibonacci and Lucas numbers F_ν and L_ν satisfy the following third-order linear differential equation:

$$w^{(3)}(\nu) + \log(\phi) w''(\nu) + \left(\pi^2 - \log^2(\phi)\right) w'(\nu) - \log(\phi) \left(\log^2(\phi) + \pi^2\right) w(\nu) == 0 /; w(\nu) = c_1 L_\nu + c_2 F_\nu + c_3 \phi^{-\nu} \sin(\pi \nu),$$

where c_1, c_2, and c_3 are arbitrary constants.

Indefinite integration

Some indefinite integrals for Fibonacci and Lucas numbers F_ν and L_ν can be evaluated as follows:

F_ν	L_ν
$\int F_\nu \, d\nu == \frac{1}{\sqrt{5}} \left(\frac{\phi^{-\nu} (\log(\phi) \cos(\pi \nu) - \pi \sin(\pi \nu))}{\log^2(\phi) + \pi^2} + \frac{\phi^\nu}{\log(\phi)}\right)$	$\int L_\nu \, d\nu = \frac{\phi^{-\nu} (\pi \sin(\pi \nu) - \cos(\pi \nu) \log(\phi))}{\log^2(\phi) + \pi^2} + \frac{\phi^\nu}{\log(\phi)}$
$\int \nu^{\alpha-1} F_\nu \, d\nu == \frac{\nu^\alpha}{2\sqrt{5}} \left(-2 (-\nu)^{-\alpha} \Gamma\left(\alpha, -\nu \, \text{csch}^{-1}(2)\right) \text{csch}^{-1}(2)^{-\alpha} + \left(\nu \left(-i \pi + \text{csch}^{-1}(2)\right)\right)^{-\alpha} \Gamma\left(\alpha, \nu \left(-i \pi + \text{csch}^{-1}(2)\right)\right) + \left(\nu \left(i \pi + \text{csch}^{-1}(2)\right)\right)^{-\alpha} \Gamma\left(\alpha, \nu \left(i \pi + \text{csch}^{-1}(2)\right)\right)\right)$	$\int \nu^{\alpha-1} L_\nu \, d\nu = -\frac{1}{2} \nu^\alpha (2 E_{1-\alpha}(-\nu \, \text{csch}^{-1}(2)) + E_{1-\alpha}(\nu (-i \pi +$ $E_{1-\alpha}(\nu (i \pi + \text{csch}^{-1}(2))))$

Laplace transforms

Laplace transforms $\mathcal{L}_\nu[f(\nu)](z)$ of the Fibonacci and Lucas numbers F_ν and L_ν can be represented by the following formulas:

F_ν	L_ν
$\mathcal{L}_\nu[F_\nu](z) == \frac{1}{\sqrt{5} \left(z - \text{csch}^{-1}(2)\right)} - \frac{z + \text{csch}^{-1}(2)}{\sqrt{5} \left(\left(z + \text{csch}^{-1}(2)\right)^2 + \pi^2\right)} /; \text{Re}(z) > \log(\phi)$	$\mathcal{L}_\nu[L_\nu](z) = \frac{1}{z - \text{csch}^{-1}(2)} + \frac{z + \text{csch}^{-1}(2)}{\left(z + \text{csch}^{-1}(2)\right)^2 + \pi^2} /; \text{Re}(z) > \log(\phi)$

Summation

There exist many formulas for finite summation of Fibonacci and Lucas numbers, for example:

F_n /; $n \in \mathbb{N}$	L_n /; $n \in \mathbb{N}$
$\sum_{k=0}^{n} F_k = F_{n+2} - 1$	$\sum_{k=0}^{n} L_k = L_{n+2} - 1$
$\sum_{k=0}^{n} \binom{n}{k} F_k = F_{2n}$	$\sum_{k=0}^{n} \binom{n}{k} L_k = L_{2n}$
$\sum_{k=0}^{n} \binom{n}{k} F_k 2^k = F_{3n}$	$\sum_{k=0}^{n} \binom{n}{k} 2^k L_k = L_{3n}$
$\sum_{k=0}^{n} F_k z^k = \frac{z(z^n(z F_n + F_{n+1}) - 1)}{z^2 + z - 1}$	$\sum_{k=0}^{n} L_k z^k = \frac{(z L_n + L_{n+1}) z^{n+1} + z - 2}{z^2 + z - 1}$
$\sum_{k=0}^{n} F_{k p+q} z^k = \frac{F_q - F_{(n+1) p+q} z^{n+1} + (-1)^p F_{n p+q} z^{n+2} - (-1)^p F_{q-p} z}{(-1)^p z^2 - L_p z + 1}$ /; $p \in \mathbb{Z} \land q \in \mathbb{Z}$	$\sum_{k=0}^{n} L_{k p+q} z^k = \frac{L_q - z^{n+1} L_{(n+1) p+q} + (-1)^p z^{n+2} L_{n p+q} - (-1)^p z L_{q-p}}{(-1)^p z^2 - z L_p + 1}$ /; $p \in \mathbb{Z} \land q \in \mathbb{Z}$
$\sum_{k=0}^{n} F_k F_{n-k} = \frac{1}{5} (n L_n - F_n)$	$\sum_{k=0}^{n} L_k L_{n-k} = F_n + (n+2) L_n$
$\sum_{k=0}^{n} F_k^2 = F_n F_{n+1}$	$\sum_{k=1}^{n} L_k^2 = L_n L_{n+1} - 2$

Here are some corresponding infinite sums:

$$\sum_{k=1}^{\infty} F_k z^k = -\frac{z}{z^2 + z - 1}$$

$$\sum_{k=1}^{\infty} \frac{1}{F_{2k-1}} = \frac{1}{4} \sqrt{5}\, \vartheta_2\left(0, \frac{2}{3 + \sqrt{5}}\right)^2$$

$$\sum_{k=1}^{\infty} \frac{1}{F_k F_{k+2}} = 1$$

$$\sum_{k=1}^{\infty} \sin\left(\frac{n \pi F_{k-1}}{2 F_{k+1} F_k}\right) \cos\left(\frac{n \pi F_{k+2}}{2 F_{k+1} F_k}\right) = 0 \text{ /; } n \in \mathbb{Z}$$

$$\sum_{k=0}^{\infty} F_k F_{k+1} F_{k+2} z^k = \frac{2z}{(-z^2 + z + 1)(-z^2 - 4z + 1)}$$

$$\sum_{k=1}^{\infty} L_k z^k = -\frac{z(2z + 1)}{z^2 + z - 1}.$$

And here are some multiple sums:

$$\sum_{m_1=0}^{n} \sum_{m_2=0}^{n} \cdots \sum_{m_k=0}^{n} \delta_{n, \sum_{j=1}^{k} m_j} \prod_{j=1}^{k} F_{m_j+1} = \sum_{j=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \binom{k - j + n - 1}{k - 1} \binom{n - j}{j} \text{ /; } n \in \mathbb{N} \land k \in \mathbb{N}^+$$

$$\sum_{k_1=1}^{n} \sum_{k_2=1}^{n} \cdots \sum_{k_p=1}^{n} \delta_{n - \sum_{j=1}^{p} k_j} \prod_{j=1}^{p} F_{k_j} = F_{n,p} \text{ /; } n \in \mathbb{N} \land p \in \mathbb{N}^+ \land F_{n,p} = \frac{2}{5}\left(\frac{n-1}{p-1} + 1\right) F_{n-1,p-1} + \frac{1}{5}\left(\frac{n}{p-1} - 1\right) F_{n,p-1} \bigwedge F_{n,1} = F_n.$$

Limit operation

Some formulas including limit operations with Fibonacci and Lucas numbers F_ν and L_ν take on symmetrical forms:

$$\lim_{\nu \to \infty} \frac{F_\nu}{L_\nu} = \frac{1}{\sqrt{5}}$$

F_ν	L_ν
$\lim_{\nu \to \infty} \frac{F_{\alpha+\nu}}{F_\nu} = \phi^\alpha$	$\lim_{\nu \to \infty} \frac{L_{\alpha+\nu}}{L_\nu} = \phi^\alpha$
$\lim_{\vee \to \infty} \frac{\sum_{k=0}^{m-1} F_{\nu+k}}{F_{m+\nu} - F_\nu} = \phi \;/;\; m \in \mathbb{N}^+$	$\lim_{\nu \to \infty} \frac{\sum_{k=0}^{m-1} L_{k+\nu}}{L_{m+\nu} - L_\nu} = \phi \;/;\; m \in \mathbb{N}^+$

Other identities

The Fibonacci numbers F_n can be obtained from the evaluation of some determinates, for example:

$$F_n = \left| \begin{pmatrix} \begin{cases} 1 & \text{if } k == l \\ i & \text{if } |k - l| == 1 \\ 0 & \text{else} \end{cases} \end{pmatrix}_{\substack{1 \le k \le n \\ 1 \le l \le n}} \right|.$$

Applications of the Fibonacci and Lucas numbers

Fibonacci and Lucas numbers have numerous applications throughout algebraic coding theory, linear sequential circuits, quasicrystals, phyllotaxies, biomathematics, and computer science.

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