

# Introductions to Inverse JacobiCS

## Introduction to the inverse Jacobi elliptic functions

### General

The inverses of the Jacobi elliptic functions  $\text{cd}^{-1}(z|m)$ ,  $\text{cn}^{-1}(z|m)$ ,  $\text{cs}^{-1}(z|m)$ ,  $\text{dc}^{-1}(z|m)$ ,  $\text{dn}^{-1}(z|m)$ ,  $\text{ds}^{-1}(z|m)$ ,  $\text{nc}^{-1}(z|m)$ ,  $\text{nd}^{-1}(z|m)$ ,  $\text{ns}^{-1}(z|m)$ ,  $\text{sc}^{-1}(z|m)$ ,  $\text{sd}^{-1}(z|m)$ , and  $\text{sn}^{-1}(z|m)$  can be represented through elliptic integrals. They first appeared in a paper by N. H. Abel (1826) who studied the so-called hyperelliptic and Abelian integrals. Later A. G. Greenhill (1892) paid some attention to these functions. This interest was continued by L. M. Milne-Thompson (1948).

### Definitions of the inverse Jacobi functions

The inverses of the twelve Jacobi elliptic functions  $\text{cd}^{-1}(z|m)$ ,  $\text{cn}^{-1}(z|m)$ ,  $\text{cs}^{-1}(z|m)$ ,  $\text{dc}^{-1}(z|m)$ ,  $\text{dn}^{-1}(z|m)$ ,  $\text{ds}^{-1}(z|m)$ ,  $\text{nc}^{-1}(z|m)$ ,  $\text{nd}^{-1}(z|m)$ ,  $\text{ns}^{-1}(z|m)$ ,  $\text{sc}^{-1}(z|m)$ ,  $\text{sd}^{-1}(z|m)$ , and  $\text{sn}^{-1}(z|m)$  are defined by the following formulas:

$$z = \text{cd}(w|m); w = \text{cd}^{-1}(z|m) \quad \text{cd}^{-1}(z|m) = \int_z^1 \frac{1}{\sqrt{1-t^2} \sqrt{1-mt^2}} dt; -1 < z < 1 \wedge m < 1$$

$$z = \text{cn}(w|m); w = \text{cn}^{-1}(z|m) \quad \text{cn}^{-1}(z|m) = \int_z^1 \frac{1}{\sqrt{1-t^2} \sqrt{m^2-m+1}} dt; -1 < z < 1 \wedge m(z^2-1) > -1$$

$$z = \text{cs}(w|m); w = \text{cs}^{-1}(z|m) \quad \text{cs}^{-1}(z|m) = \int_z^\infty \frac{1}{\sqrt{t^2+1} \sqrt{t^2-m+1}} dt; z \in \mathbb{R} \wedge z^2-m > -1$$

$$z = \text{dc}(w|m); w = \text{dc}^{-1}(z|m) \quad \text{dc}^{-1}(z|m) = \int_1^z \frac{1}{\sqrt{t^2-1} \sqrt{t^2-m}} dt; z \in \mathbb{R} \wedge z^2 > 1 \wedge z^2-m > 0 \wedge m < 1$$

$$z = \text{dn}(w|m); w = \text{dn}^{-1}(z|m) \quad \text{dn}^{-1}(z|m) = \int_z^1 \frac{1}{\sqrt{1-t^2} \sqrt{t^2+m-1}} dt; -1 < z < 1 \wedge z^2+m > 1$$

$$z = \text{ds}(w|m); w = \text{ds}^{-1}(z|m) \quad \text{ds}^{-1}(z|m) = \int_z^\infty \frac{1}{\sqrt{t^2+m} \sqrt{t^2+m-1}} dt; z \in \mathbb{R} \wedge z^2+m > 1$$

$$z = \text{nc}(w|m); w = \text{nc}^{-1}(z|m) \quad \text{nc}^{-1}(z|m) = \int_1^z \frac{1}{\sqrt{t^2-1} \sqrt{(1-m)t^2+m}} dt; z \in \mathbb{R} \wedge z^2 > 1 \wedge (1-m)z^2+m > 0$$

$$z = \text{nd}(w|m); w = \text{nd}^{-1}(z|m) \quad \text{nd}^{-1}(z|m) = \int_1^z \frac{1}{\sqrt{t^2-1} \sqrt{1-(1-m)t^2}} dt; z \in \mathbb{R} \wedge z^2 > 1 \wedge (1-m)z^2 < 1 \wedge m > 0$$

$$z = \text{ns}(w|m); w = \text{ns}^{-1}(z|m) \quad \text{ns}^{-1}(z|m) = \int_z^\infty \frac{1}{\sqrt{t^2-1} \sqrt{t^2-m}} dt; z \in \mathbb{R} \wedge z^2 > 1 \wedge z^2 > m$$

$$z = \text{sc}(w|m); w = \text{sc}^{-1}(z|m) \quad \text{sc}^{-1}(z|m) = \int_0^z \frac{1}{\sqrt{t^2+1} \sqrt{(1-m)t^2+1}} dt; z \in \mathbb{R} \wedge (1-m)z^2 > -1$$

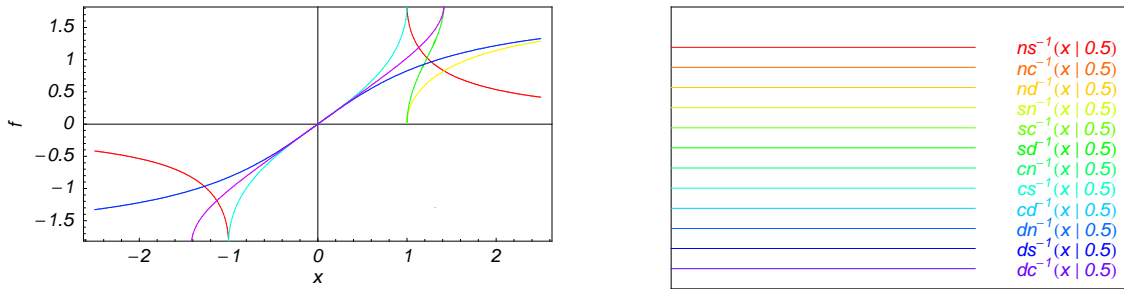
$$z = \text{sd}(w|m); w = \text{sd}^{-1}(z|m) \quad \text{sd}^{-1}(z|m) = \int_0^z \frac{1}{\sqrt{mt^2+1} \sqrt{1-(1-m)t^2}} dt; z \in \mathbb{R} \wedge m z^2 > -1 \wedge (1-m)z^2 < 1$$

$$z = \operatorname{sn}(w | m) /; w = \operatorname{sn}^{-1}(z | m) \quad \operatorname{sn}^{-1}(z | m) = \int_0^z \frac{1}{\sqrt{1-t^2} \sqrt{1-mt^2}} dt /; -1 < z < 1 \wedge m z^2 < 1$$

It is obvious that the inverses of the twelve Jacobi elliptic functions are actually the definite elliptic integrals and can be expressed through the Legendre elliptic integrals.

### A quick look at the inverse Jacobi functions

Here is a quick look at the graphics for the inverse Jacobi functions along the real axis.



### Connections within the group of inverse Jacobi functions and with other related function groups

#### Representations through more general functions

The inverses of the Jacobi elliptic functions  $\operatorname{cd}^{-1}(z | m)$ ,  $\operatorname{cn}^{-1}(z | m)$ ,  $\operatorname{cs}^{-1}(z | m)$ ,  $\operatorname{dc}^{-1}(z | m)$ ,  $\operatorname{dn}^{-1}(z | m)$ ,  $\operatorname{ds}^{-1}(z | m)$ ,  $\operatorname{nc}^{-1}(z | m)$ ,  $\operatorname{nd}^{-1}(z | m)$ ,  $\operatorname{ns}^{-1}(z | m)$ ,  $\operatorname{sc}^{-1}(z | m)$ ,  $\operatorname{sd}^{-1}(z | m)$ , and  $\operatorname{sn}^{-1}(z | m)$  can be represented through hypergeometric functions of two variables (including the Appell  $F_1$  function) by the following formulas:

$$\begin{aligned} \operatorname{cd}^{-1}(z | m) &= K(m) - z F_{1 \times 0 \times 0}^{1 \times 1 \times 1} \left( \begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ \frac{3}{2} \end{matrix}; m z^2, z^2 \right) & \operatorname{cn}^{-1}(z | m) &= K(m) - \frac{1}{\sqrt{1-m}} z F_{1 \times 1}^{1 \times 1} \\ \operatorname{cs}^{-1}(z | m) &= \frac{1}{z} F_{1 \times 0 \times 1}^{2 \times 0 \times 1} \left( \begin{matrix} \frac{1}{2}, 1; \frac{1}{2} \\ \frac{3}{2}; 1 \end{matrix}; -\frac{m}{z^2}, \frac{m}{z^2} \right) & \operatorname{dc}^{-1}(z | m) &= K(m) - \frac{1}{z} F_{1 \times 0 \times 0}^{1 \times 1 \times 1} \left( \begin{matrix} \frac{1}{2} \\ \frac{3}{2} \end{matrix}; z^2, m \right) \\ \operatorname{dn}^{-1}(z | m) &= \frac{1}{\sqrt{m-1}} K\left(\frac{1}{1-m}\right) - \frac{z\sqrt{1-z^2}}{\sqrt{z^2-1}} F_{0 \times 1 \times 0}^{1 \times 1 \times 1} \left( \begin{matrix} 1; \frac{1}{2}, \frac{1}{2} \\ \frac{3}{2}; \end{matrix}; z^2, m \right) & \operatorname{ds}^{-1}(z | m) &= \frac{1}{z} \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_{2k}^2 z^{-2k}}{\left(\frac{3}{2}\right)_k} F_{1 \times 1}^{1 \times 1} \\ \operatorname{nc}^{-1}(z | m) &= K(m) - \frac{1}{z} \left( F_{2 \times 0 \times 0}^{1 \times 2 \times 2} \left( \begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 1 \\ \frac{3}{2}, 1; \end{matrix}; -\frac{m}{z^2}, \frac{1}{z^2} \right) + \frac{m}{2} F_{1 \times 1 \times 1}^{2 \times 1 \times 1} \left( \begin{matrix} \frac{3}{2}, \frac{3}{2}, \frac{1}{2}, 1 \\ 2; \frac{3}{2}, \frac{3}{2}; \end{matrix}; -\frac{m}{z^2}, m \right) \right) & \operatorname{nd}^{-1}(z | m) &= i K(1-m) + \frac{\sqrt{1-z^2}}{\sqrt{z^2-1}} z \\ \operatorname{ns}^{-1}(z | m) &= \frac{1}{z} F_{1 \times 0 \times 0}^{1 \times 1 \times 1} \left( \begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ \frac{3}{2}; \end{matrix}; \frac{m}{z^2}, \frac{1}{z^2} \right) & \operatorname{sc}^{-1}(z | m) &= z F_{1 \times 0 \times 0}^{1 \times 1 \times 1} \left( \begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ \frac{3}{2}; \end{matrix}; \frac{m}{z^2}, \frac{1}{z^2} \right) \\ \operatorname{sd}^{-1}(z | m) &= z F_{1 \times 0 \times 0}^{1 \times 1 \times 1} \left( \begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ \frac{3}{2}; \end{matrix}; (1-m)z^2, -mz^2 \right) & \operatorname{sn}^{-1}(z | m) &= z F_{1 \times 0 \times 0}^{1 \times 1 \times 1} \left( \begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ \frac{3}{2}; \end{matrix}; (1-m)z^2, -mz^2 \right) \\ \operatorname{cd}^{-1}(z | m) &= K(m) - z F_1 \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{3}{2}; z^2, m z^2 \right) /; -1 < z < 1 \wedge m < 1 \end{aligned}$$

$$\operatorname{cn}^{-1}(z|m) = \sqrt{1-z^2} F_1\left(\frac{1}{2}; \frac{1}{2}, \frac{1}{2}, \frac{3}{2}; 1-z^2, m(1-z^2)\right); -1 < z < 1 \wedge -1 < m < 1$$

$$\operatorname{cs}^{-1}(z|m) = \frac{1}{z} F_1\left(\frac{1}{2}; \frac{1}{2}, \frac{1}{2}, \frac{3}{2}; -\frac{1}{z^2}, \frac{m-1}{z^2}\right); z \in \mathbb{R} \wedge m-z^2 < 1$$

$$\operatorname{dc}^{-1}(z|m) = \frac{1}{\sqrt{m}} \left( K\left(\frac{1}{m}\right) - z F_1\left(\frac{1}{2}; \frac{1}{2}, \frac{1}{2}, \frac{3}{2}; z^2, \frac{z^2}{m}\right) \right); z > 1 \wedge m < 1$$

$$\operatorname{dn}^{-1}(z|m) = \frac{1}{\sqrt{m-1}} \left( K\left(\frac{1}{1-m}\right) - z F_1\left(\frac{1}{2}; \frac{1}{2}, \frac{1}{2}, \frac{3}{2}; z^2, \frac{z^2}{1-m}\right) \right); -1 < z < 1 \wedge z^2 + m > 1 \wedge m > 0$$

$$\operatorname{ds}^{-1}(z|m) = \frac{1}{z} F_1\left(\frac{1}{2}; \frac{1}{2}, \frac{1}{2}, \frac{3}{2}; \frac{1-m}{z^2}, -\frac{m}{z^2}\right); z^2 + m > 1$$

$$\operatorname{nc}^{-1}(z|m) = \frac{i}{\sqrt{m}} K\left(1-\frac{1}{m}\right) - \frac{iz}{\sqrt{m}} F_1\left(\frac{1}{2}; \frac{1}{2}, \frac{1}{2}, \frac{3}{2}; z^2, \left(1-\frac{1}{m}\right)z^2\right); -1 < z < 1 \wedge m \in \mathbb{R}$$

$$\operatorname{nd}^{-1}(z|m) = i \left( K(1-m) - z F_1\left(\frac{1}{2}; \frac{1}{2}, \frac{1}{2}, \frac{3}{2}; z^2, (1-m)z^2\right) \right); -1 < z < 1 \wedge -1 < m < 1$$

$$\operatorname{ns}^{-1}(z|m) = \frac{1}{z} F_1\left(\frac{1}{2}; \frac{1}{2}, \frac{1}{2}, \frac{3}{2}; \frac{1}{z^2}, \frac{m}{z^2}\right); (z > 1 \vee z < -1) \wedge z^2 > m$$

$$\operatorname{sc}^{-1}(z|m) = z F_1\left(\frac{1}{2}; \frac{1}{2}, \frac{1}{2}, \frac{3}{2}; -z^2, (m-1)z^2\right); z \in \mathbb{R} \wedge (m-1)z^2 < 1$$

$$\operatorname{sd}^{-1}(z|m) = z F_1\left(\frac{1}{2}; \frac{1}{2}, \frac{1}{2}, \frac{3}{2}; -mz^2, (1-m)z^2\right); z \in \mathbb{R} \wedge (1-m)z^2 < 1$$

$$\operatorname{sn}^{-1}(z|m) = z F_1\left(\frac{1}{2}; \frac{1}{2}, \frac{1}{2}, \frac{3}{2}; z^2, mz^2\right); -1 < z < 1 \wedge mz^2 < 1.$$

### Representations through related equivalent functions

The twelve inverses of the Jacobi elliptic functions  $\operatorname{cd}^{-1}(z|m)$ ,  $\operatorname{cn}^{-1}(z|m)$ ,  $\operatorname{cs}^{-1}(z|m)$ ,  $\operatorname{dc}^{-1}(z|m)$ ,  $\operatorname{dn}^{-1}(z|m)$ ,  $\operatorname{ds}^{-1}(z|m)$ ,  $\operatorname{nc}^{-1}(z|m)$ ,  $\operatorname{nd}^{-1}(z|m)$ ,  $\operatorname{ns}^{-1}(z|m)$ ,  $\operatorname{sc}^{-1}(z|m)$ ,  $\operatorname{sd}^{-1}(z|m)$ , and  $\operatorname{sn}^{-1}(z|m)$  can be represented through incomplete and complete elliptic integrals  $F(\sin^{-1}(z)|m)$  and  $K(m)$  by the following formulas:

$$\operatorname{cd}^{-1}(z|m) = K(m) - F(\sin^{-1}(z)|m); m \notin (1, \infty)$$

$$\operatorname{cn}^{-1}(z|m) = F(\cos^{-1}(z)|m); -1 < z < 1 \wedge m \in \mathbb{R}$$

$$\operatorname{cs}^{-1}(z|m) = -i F\left(i \sinh^{-1}\left(\frac{1}{z}\right) \middle| 1-m\right); z > 0 \wedge m \in \mathbb{R}$$

$$\operatorname{dc}^{-1}(z|m) = \frac{1}{\sqrt{m}} \left( K\left(\frac{1}{m}\right) - F\left(\sin^{-1}(z) \middle| \frac{1}{m}\right) \right); -1 < z < 1 \wedge m > 1 \vee z > 1 \wedge m < 1$$

$$\operatorname{dn}^{-1}(z|m) = \frac{1}{\sqrt{m-1}} \left( K\left(\frac{1}{1-m}\right) - F\left(\sin^{-1}(z) \middle| \frac{1}{1-m}\right) \right); z < 1 \wedge m > 1$$

$$\operatorname{ds}^{-1}(z|m) = \frac{1}{\sqrt{1-m}} F\left(\sin^{-1}\left(\frac{\sqrt{1-m}}{z}\right) \middle| \frac{m}{m-1}\right); z > 1 \wedge m > 1$$

$$\operatorname{nc}^{-1}(z|m) = \frac{i}{\sqrt{m}} \left( F\left(\sin^{-1}(z) \middle| \frac{m-1}{m}\right) - K\left(\frac{m-1}{m}\right) \right); z > 1 \wedge m > 0$$

$$\operatorname{nd}^{-1}(z|m) = i(F(\sin^{-1}(z)|1-m) - K(1-m)); z > 1 \wedge m > 1$$

$$\operatorname{ns}^{-1}(z|m) = F\left(\sin^{-1}\left(\frac{1}{z}\right) \middle| m\right); (z < -1 \vee z > 1) \wedge m < 1$$

$$\operatorname{sc}^{-1}(z|m) = -i F(i \sinh^{-1}(z) | 1-m); |z| < 1$$

$$\operatorname{sd}^{-1}(z|m) = -\frac{i}{\sqrt{m}} F\left(i \sinh^{-1}(\sqrt{m} z) \middle| \frac{m-1}{m}\right); |z| < 1 \wedge |m| < 1$$

$$\operatorname{sn}^{-1}(z|m) = F(\sin^{-1}(z) | m); |z| < 1 \wedge |m| < 1.$$

The twelve inverse Jacobi functions can also be expressed through the elliptic logarithm  $\operatorname{elog}(z_1, z_2; a, b)$  and the complete elliptic integral  $K(m)$  by the following formulas:

$$\operatorname{cd}^{-1}(z|m) = K(m) + \frac{\sqrt{z_2^2}}{z_2} \operatorname{elog}(z_1, z_2; a, b);$$

$$\{a, b, z_1\} = \left\{-m-1, m, \frac{1}{z^2}\right\} \wedge z_1^3 + a z_1^2 + b z_1 - z_2^2 = 0 \wedge (0 < z < 1 \wedge m \in \mathbb{R}) \vee z < -1 \wedge m > 1$$

$$\operatorname{cn}^{-1}(z|m) = -\frac{i \sqrt{z_2^2}}{z_2 \sqrt{m}} \left( K\left(1 - \frac{1}{m}\right) + \operatorname{elog}(z_1, z_2; a, b) \right);$$

$$\{a, b, z_1\} = \left\{\frac{1}{m} - 2, 1 - \frac{1}{m}, z^2\right\} \wedge z_1^3 + a z_1^2 + b z_1 - z_2^2 = 0 \wedge 0 < z < 1 \wedge 0 < m < 1$$

$$\operatorname{cs}^{-1}(z|m) = -\frac{\sqrt{z_2^2}}{z_2} \operatorname{elog}(z_1, z_2; a, b); \{a, b, z_1\} = \{2-m, 1-m, z^2\} \wedge z_1^3 + a z_1^2 + b z_1 - z_2^2 = 0 \wedge z > 0 \wedge m < 1$$

$$\operatorname{dc}^{-1}(z|m) = K(m) + \frac{\sqrt{z_2^2}}{z_2} \operatorname{elog}(z_1, z_2; a, b); \{a, b, z_1\} = \{-m-1, m, z^2\} \wedge z_1^3 + a z_1^2 + b z_1 - z_2^2 = 0 \wedge z > 1 \wedge m < 1$$

$$\operatorname{dn}^{-1}(z|m) = -\frac{i \sqrt{z_2^2}}{z_2} (K(1-m) + \operatorname{elog}(z_1, z_2; a, b));$$

$$\{a, b, z_1\} = \{m-2, 1-m, z^2\} \wedge z_1^3 + a z_1^2 + b z_1 - z_2^2 = 0 \wedge 0 < z < 1 \wedge m > 1$$

$$\operatorname{ds}^{-1}(z|m) = -\frac{\sqrt{z_2^2}}{z_2} \operatorname{elog}(z_1, z_2; a, b); \{a, b, z_1\} = \{2m-1, m(m-1), z^2\} \wedge z_1^3 + a z_1^2 + b z_1 - z_2^2 = 0 \wedge z > 0 \wedge m > 1$$

$$\text{nc}^{-1}(z | m) = \frac{1}{\sqrt{1-m}} \left( \frac{\sqrt{z_2^2}}{z_2} \text{elog}(z_1, z_2; a, b) - K\left(\frac{m}{m-1}\right) \right) /;$$

$$\{a, b, z_1\} = \left\{ \frac{2m-1}{1-m}, \frac{m}{m-1}, z^2 \right\} \bigwedge z_1^3 + a z_1^2 + b z_1 - z_2^2 = 0 \bigwedge 0 < z < 1 \bigwedge m > 1$$

$$\text{nd}^{-1}(z | m) = -\frac{i \sqrt{z_2^2}}{z_2} (K(1-m) + \text{elog}(z_1, z_2; a, b)) /;$$

$$\{a, b, z_1\} = \left\{ m-2, 1-m, \frac{1}{z^2} \right\} \bigwedge z_1^3 + a z_1^2 + b z_1 - z_2^2 = 0 \bigwedge z > 1 \bigwedge m > 1$$

$$\text{ns}^{-1}(z | m) = -\frac{\sqrt{z_2^2}}{z_2} \text{elog}(z_1, z_2; a, b) /; \{a, b, z_1\} = \{-m-1, m, z^2\} \bigwedge z_1^3 + a z_1^2 + b z_1 - z_2^2 = 0 \bigwedge z > 0 \bigwedge m < 1$$

$$\text{sc}^{-1}(z | m) = -\frac{\sqrt{z_2^2}}{z_2} \text{elog}(z_1, z_2; a, b) /; \{a, b, z_1\} = \left\{ 2-m, 1-m, \frac{1}{z^2} \right\} \bigwedge z_1^3 + a z_1^2 + b z_1 - z_2^2 = 0 \bigwedge z > 0 \bigwedge m < 1$$

$$\text{sd}^{-1}(z | m) = -\frac{\sqrt{z_2^2}}{z_2} \text{elog}(z_1, z_2; a, b) /; \{a, b, z_1\} = \left\{ 2m-1, m(m-1), \frac{1}{z^2} \right\} \bigwedge z_1^3 + a z_1^2 + b z_1 - z_2^2 = 0 \bigwedge z > 0 \bigwedge m > 1$$

$$\text{sn}^{-1}(z | m) = -z \text{elog}\left(1, \sqrt{a+b+1}; a, b\right) /; \{a, b\} = \{-z^2(m+1), m z^4\} \bigwedge |z| < 1.$$

### Relations to inverse functions

The twelve inverses of the Jacobi elliptic functions  $\text{cd}^{-1}(z | m)$ ,  $\text{cn}^{-1}(z | m)$ ,  $\text{cs}^{-1}(z | m)$ ,  $\text{dc}^{-1}(z | m)$ ,  $\text{dn}^{-1}(z | m)$ ,  $\text{ds}^{-1}(z | m)$ ,  $\text{nc}^{-1}(z | m)$ ,  $\text{nd}^{-1}(z | m)$ ,  $\text{ns}^{-1}(z | m)$ ,  $\text{sc}^{-1}(z | m)$ ,  $\text{sd}^{-1}(z | m)$ , and  $\text{sn}^{-1}(z | m)$  are connected with the corresponding direct Jacobi functions by the following formulas:

$$\begin{aligned} \text{cd}(\text{cd}^{-1}(z | m) | m) &= z & \text{cn}(\text{cn}^{-1}(z | m) | m) &= z & \text{cs}(\text{cs}^{-1}(z | m) | m) &= z \\ \text{dc}(\text{dc}^{-1}(z | m) | m) &= z & \text{dn}(\text{dn}^{-1}(z | m) | m) &= z & \text{ds}(\text{ds}^{-1}(z | m) | m) &= z \\ \text{nc}(\text{nc}^{-1}(z | m) | m) &= z & \text{nd}(\text{nd}^{-1}(z | m) | m) &= z & \text{ns}(\text{ns}^{-1}(z | m) | m) &= z \\ \text{sc}(\text{sc}^{-1}(z | m) | m) &= z & \text{sd}(\text{sd}^{-1}(z | m) | m) &= z & \text{sn}(\text{sn}^{-1}(z | m) | m) &= z. \end{aligned}$$

### Representations through other inverse Jacobi functions

The twelve inverses of the Jacobi elliptic functions  $\text{cd}^{-1}(z | m)$ ,  $\text{cn}^{-1}(z | m)$ ,  $\text{cs}^{-1}(z | m)$ ,  $\text{dc}^{-1}(z | m)$ ,  $\text{dn}^{-1}(z | m)$ ,  $\text{ds}^{-1}(z | m)$ ,  $\text{nc}^{-1}(z | m)$ ,  $\text{nd}^{-1}(z | m)$ ,  $\text{ns}^{-1}(z | m)$ ,  $\text{sc}^{-1}(z | m)$ ,  $\text{sd}^{-1}(z | m)$ , and  $\text{sn}^{-1}(z | m)$  are interconnected by formulas that include the complete elliptic integral  $K(m)$ , rational functions, simple powers, and arithmetical operations of the arguments of other inverse Jacobi functions. These formulas can be divided into the following eleven groups:

Representations of  $\text{cd}^{-1}(z | m)$  through other inverse Jacobi functions are:

$$\text{cd}^{-1}(z | m) = K(m) - \text{cn}^{-1}\left(\sqrt{1-z^2} \mid m\right) /; 0 < z < 1 \wedge m < 1$$

$$\operatorname{cd}^{-1}(z|m) = \frac{1}{\sqrt{m}} \left( K\left(\frac{1}{m}\right) + i \operatorname{cs}^{-1}\left(-iz \left| 1 - \frac{1}{m}\right.\right) \right); 0 < z < 1 \wedge 0 < m < 1$$

$$\operatorname{cd}^{-1}(z|m) = \operatorname{dc}^{-1}\left(\frac{1}{z} \middle| m\right); z < 0 \wedge m < 0 \vee z < 1 \wedge m < 1$$

$$\operatorname{cd}^{-1}(z|m) = -\frac{i}{\sqrt{m}} \operatorname{dn}^{-1}\left(z \middle| \frac{m-1}{m}\right); -1 < z < 1 \wedge m > 1$$

$$\operatorname{cd}^{-1}(z|m) = K(m) - \frac{1}{\sqrt{1-m}} \operatorname{ds}^{-1}\left(\frac{1}{\sqrt{1-m}z} \middle| \frac{m}{m-1}\right); z > 0 \wedge m \in \mathbb{R}$$

$$\operatorname{cd}^{-1}(z|m) = -i \operatorname{nd}^{-1}(z|1-m); z \in \mathbb{R} \wedge m < 0$$

$$\operatorname{cd}^{-1}(z|m) = \frac{1}{\sqrt{m}} \left( \operatorname{ns}^{-1}\left(z \middle| \frac{1}{m}\right) - K\left(\frac{1}{m}\right) \right); -1 < z < 1 \wedge m < 0$$

$$\operatorname{cd}^{-1}(z|m) = K(m) - i \operatorname{sc}^{-1}(-iz|1-m); z \in \mathbb{R} \wedge m \in \mathbb{R}$$

$$\operatorname{cd}^{-1}(z|m) = K(m) - \frac{1}{\sqrt{1-m}} \operatorname{sd}^{-1}\left(z\sqrt{1-m} \middle| \frac{m}{m-1}\right); -1 < z < 1 \wedge m < 1$$

$$\operatorname{cd}^{-1}(z|m) = K(m) - \operatorname{sn}^{-1}(z|m); z \in \mathbb{R} \wedge m \in \mathbb{R}.$$

Representations of  $\operatorname{cn}^{-1}(z|m)$  through other inverse Jacobi functions are:

$$\operatorname{cn}^{-1}(z|m) = K(m) - \operatorname{cd}^{-1}\left(\sqrt{1-z^2} \middle| m\right); 0 < z < 1 \wedge m < 1$$

$$\operatorname{cn}^{-1}(z|m) = i \operatorname{cs}^{-1}\left(\frac{1}{\sqrt{z^2-1}} \middle| 1-m\right); z > 1 \wedge m > 1$$

$$\operatorname{cn}^{-1}(z|m) = K(m) - \frac{1}{\sqrt{m}} \operatorname{dc}^{-1}\left(\sqrt{1-z^2} \middle| \frac{1}{m}\right); 0 < z < 1 \wedge 0 < m < 1$$

$$\operatorname{cn}^{-1}(z|m) = \frac{1}{\sqrt{m}} \operatorname{dn}^{-1}\left(z \middle| \frac{1}{m}\right); -1 < z < 1 \wedge m < 1$$

$$\operatorname{cn}^{-1}(z|m) = \frac{1}{\sqrt{1-m}} \operatorname{ds}^{-1}\left(\frac{1}{\sqrt{(z^2-1)(m-1)}} \middle| \frac{m}{m-1}\right); z > 1 \wedge m > 1$$

$$\operatorname{cn}^{-1}(z|m) = -i \operatorname{nc}^{-1}(z|1-m); -1 < z < 1 \wedge m \in \mathbb{R}$$

$$\operatorname{cn}^{-1}(z|m) = K(m) + i \operatorname{nd}^{-1}\left(\sqrt{1-z^2} \middle| 1-m\right); z > 0 \wedge m \in \mathbb{R}$$

$$\operatorname{cn}^{-1}(z|m) = \operatorname{ns}^{-1}\left(\frac{1}{\sqrt{1-z^2}} \middle| m\right); 0 < z < 1 \wedge m \in \mathbb{R}$$

$$\operatorname{cn}^{-1}(z|m) = -i \operatorname{sc}^{-1}\left(i\sqrt{1-z^2} \mid 1-m\right); 0 < z < 1 \wedge m \in \mathbb{R}$$

$$\operatorname{cn}^{-1}(z|m) = -\frac{i}{\sqrt{1-m}} \operatorname{sd}^{-1}\left(\sqrt{(m-1)(1-z^2)} \mid \frac{1}{1-m}\right); 0 < z < 1 \wedge 0 < m < 1$$

$$\operatorname{cn}^{-1}(z|m) = \frac{1}{\sqrt{1-m}} \left( K\left(\frac{m}{m-1}\right) - \operatorname{sn}^{-1}\left(z \mid \frac{m}{m-1}\right) \right); -1 < z < 1 \wedge m < 1.$$

Representations of  $\operatorname{cs}^{-1}(z|m)$  through other inverse Jacobi functions are:

$$\operatorname{cs}^{-1}(z|m) = iK(1-m) - \frac{i}{\sqrt{1-m}} \operatorname{cd}^{-1}\left(iz \mid \frac{1}{1-m}\right)$$

$$\operatorname{cs}^{-1}(z|m) = i \operatorname{cn}^{-1}\left(\frac{\sqrt{z^2+1}}{z} \mid 1-m\right); z > 0 \wedge m > 0$$

$$\operatorname{cs}^{-1}(z|m) = i \left( \frac{1}{\sqrt{1-m}} \operatorname{dc}^{-1}\left(\frac{i}{z} \mid \frac{1}{1-m}\right) - K(1-m) \right); 0 < z < 1 \wedge 0 < m < 1$$

$$\operatorname{cs}^{-1}(z|m) = i \left( \frac{1}{\sqrt{m-1}} \operatorname{dn}^{-1}\left(\frac{i}{z} \mid \frac{m}{m-1}\right) - K(1-m) \right); z > 0 \wedge m < 0$$

$$\operatorname{cs}^{-1}(z|m) = \frac{i}{\sqrt{m}} \operatorname{ds}^{-1}\left(\frac{iz}{\sqrt{m}} \mid \frac{m-1}{m}\right); z \in \mathbb{R} \wedge m < 0$$

$$\operatorname{cs}^{-1}(z|m) = iK(1-m) - \frac{i}{\sqrt{m}} \operatorname{nc}^{-1}\left(iz \mid 1 - \frac{1}{m}\right); z \in \mathbb{R} \wedge m < 1$$

$$\operatorname{cs}^{-1}(z|m) = \operatorname{nd}^{-1}\left(\frac{i}{z} \mid m\right) - iK(1-m); z > 0 \wedge m \in \mathbb{R}$$

$$\operatorname{cs}^{-1}(z|m) = -i \operatorname{ns}^{-1}(-iz \mid 1-m)$$

$$\operatorname{cs}^{-1}(z|m) = \operatorname{sc}^{-1}\left(\frac{1}{z} \mid m\right)$$

$$\operatorname{cs}^{-1}(z|m) = \frac{1}{\sqrt{m}} \operatorname{sd}^{-1}\left(\frac{\sqrt{m}}{z} \mid \frac{1}{m}\right); z > 0 \wedge m \in \mathbb{R}$$

$$\operatorname{cs}^{-1}(z|m) = -i \operatorname{sn}^{-1}\left(\frac{i}{z} \mid 1-m\right); z > 0 \wedge m \in \mathbb{R}.$$

Representations of  $\operatorname{dc}^{-1}(z|m)$  through other inverse Jacobi functions are:

$$\operatorname{dc}^{-1}(z|m) = \operatorname{cd}^{-1}\left(\frac{1}{z} \mid m\right)$$

$$\operatorname{dc}^{-1}(z|m) = \frac{1}{\sqrt{m}} \left( K\left(\frac{1}{m}\right) - \operatorname{cn}^{-1}\left(\sqrt{1-z^2} \mid \frac{1}{m}\right) \right); 0 < z < 1 \wedge m > 1$$

$$\operatorname{dc}^{-1}(z|m) = \frac{1}{\sqrt{m}} \left( K\left(\frac{1}{m}\right) - i \operatorname{cs}^{-1}\left(\frac{i}{z} \middle| 1 - \frac{1}{m}\right) \right); z < 1 \wedge m > 1$$

$$\operatorname{dc}^{-1}(z|m) = 2iK(1-m) - i \operatorname{dn}^{-1}(z|1-m)$$

$$\operatorname{dc}^{-1}(z|m) = K(m) + \frac{i}{\sqrt{1-m}} \operatorname{ds}^{-1}\left(-\frac{iz}{\sqrt{1-m}} \middle| \frac{1}{1-m}\right); 0 < m < 1$$

$$\operatorname{dc}^{-1}(z|m) = \frac{1}{\sqrt{1-m}} \operatorname{nc}^{-1}\left(z \middle| \frac{m}{m-1}\right); z > 0 \wedge m > 1$$

$$\operatorname{dc}^{-1}(z|m) = -\frac{i}{\sqrt{m}} \operatorname{nd}^{-1}\left(z \middle| 1 - \frac{1}{m}\right); z > 0 \wedge m > 0$$

$$\operatorname{dc}^{-1}(z|m) = \frac{1}{\sqrt{m}} \left( K\left(\frac{1}{m}\right) - \operatorname{ns}^{-1}\left(\frac{1}{z} \middle| \frac{1}{m}\right) \right); z < 1 \wedge m > 1$$

$$\operatorname{dc}^{-1}(z|m) = \frac{1}{\sqrt{m}} \left( K\left(\frac{1}{m}\right) + i \operatorname{sc}^{-1}\left(iz \middle| \frac{m-1}{m}\right) \right); m > 1$$

$$\operatorname{dc}^{-1}(z|m) = \frac{1}{\sqrt{m}} K\left(\frac{1}{m}\right) - \frac{1}{\sqrt{1-m}} \operatorname{sd}^{-1}\left(\frac{z\sqrt{1-m}}{\sqrt{m}} \middle| \frac{m}{m-1}\right); -1 < z < 1 \wedge m > 1$$

$$\operatorname{dc}^{-1}(z|m) = \frac{1}{\sqrt{m}} \left( K\left(\frac{1}{m}\right) - \operatorname{sn}^{-1}\left(z \middle| \frac{1}{m}\right) \right); -1 < z < 1 \wedge m > 1 \vee z > 1 \wedge m < 1.$$

Representations of  $\operatorname{dn}^{-1}(z|m)$  through other inverse Jacobi functions are:

$$\operatorname{dn}^{-1}(z|m) = \frac{1}{\sqrt{m-1}} \operatorname{cd}^{-1}\left(z \middle| \frac{1}{1-m}\right); -1 < z < 1 \wedge m > 1$$

$$\operatorname{dn}^{-1}(z|m) = \frac{1}{\sqrt{m}} \operatorname{cn}^{-1}\left(z \middle| \frac{1}{m}\right); -1 < z < 1 \wedge m > 1$$

$$\operatorname{dn}^{-1}(z|m) = \frac{1}{\sqrt{1-m}} \left( iK\left(\frac{1}{1-m}\right) + \operatorname{cs}^{-1}\left(\frac{i}{z} \middle| \frac{m}{m-1}\right) \right); -1 < z < 1 \wedge m > 1$$

$$\operatorname{dn}^{-1}(z|m) = 2K(m) + i \operatorname{dc}^{-1}(z|1-m); -1 < z < 1 \wedge m < 0 \vee z \in \mathbb{R} \wedge m > 1$$

$$\operatorname{dn}^{-1}(z|m) = \frac{1}{\sqrt{m-1}} K\left(\frac{1}{1-m}\right) - \frac{1}{\sqrt{m}} \operatorname{ds}^{-1}\left(\frac{\sqrt{m-1}}{\sqrt{m}z} \middle| \frac{1}{m}\right); 0 < z < 1 \wedge m > 1$$

$$\operatorname{dn}^{-1}(z|m) = -\frac{i}{\sqrt{m}} \operatorname{nc}^{-1}\left(z \middle| 1 - \frac{1}{m}\right); z < 1 \wedge m > 1$$

$$\operatorname{dn}^{-1}(z|m) = \operatorname{nd}^{-1}\left(\frac{1}{z} \middle| m\right); -1 < z < 0 \wedge m < 0 \vee z < 1 \wedge m > 1$$



$$\operatorname{dn}^{-1}(z|m) = \frac{1}{\sqrt{m-1}} \left( K\left(\frac{1}{1-m}\right) - \operatorname{ns}^{-1}\left(\frac{1}{z} \middle| \frac{1}{1-m}\right) \right); -1 < z < 1 \wedge m > 1$$

$$\operatorname{dn}^{-1}(z|m) = \frac{1}{\sqrt{m-1}} \left( K\left(\frac{1}{1-m}\right) - i \operatorname{sc}^{-1}\left(-iz \middle| \frac{m}{m-1}\right) \right); -1 < z < 1 \wedge m > 1$$

$$\operatorname{dn}^{-1}(z|m) = \frac{1}{\sqrt{m-1}} K\left(\frac{1}{1-m}\right) + \frac{i}{\sqrt{m}} \operatorname{sd}^{-1}\left(\frac{i\sqrt{m}z}{\sqrt{m-1}} \middle| 1 - \frac{1}{m}\right); -1 < z < 1 \wedge m > 1$$

$$\operatorname{dn}^{-1}(z|m) = \frac{1}{\sqrt{m-1}} \left( \operatorname{sn}^{-1}\left(z \middle| \frac{1}{1-m}\right) - K\left(\frac{1}{1-m}\right) \right); z \in \mathbb{R} \wedge z^2 + m < 1 \wedge m > 0.$$

Representations of  $\operatorname{ds}^{-1}(z|m)$  through other inverse Jacobi functions are:

$$\operatorname{ds}^{-1}(z|m) = \frac{1}{\sqrt{1-m}} \left( K\left(\frac{m}{m-1}\right) - \operatorname{cd}^{-1}\left(\frac{\sqrt{1-m}}{z} \middle| \frac{m}{m-1}\right) \right); z > 0 \wedge m > 0$$

$$\operatorname{ds}^{-1}(z|m) = \frac{1}{\sqrt{1-m}} \operatorname{cn}^{-1}\left(\frac{\sqrt{z^2+m-1}}{z} \middle| \frac{m}{m-1}\right); z > 1 \wedge m > 1$$

$$\operatorname{ds}^{-1}(z|m) = \frac{i}{\sqrt{1-m}} \operatorname{cs}^{-1}\left(\frac{iz}{\sqrt{1-m}} \middle| \frac{1}{1-m}\right); z \in \mathbb{R} \wedge m > 1$$

$$\operatorname{ds}^{-1}(z|m) = \frac{i}{\sqrt{m}} \left( K\left(\frac{m-1}{m}\right) - \operatorname{dc}^{-1}\left(\frac{iz}{\sqrt{m}} \middle| \frac{m-1}{m}\right) \right); m > 1$$

$$\operatorname{ds}^{-1}(z|m) = \frac{1}{\sqrt{1-m}} K\left(\frac{m}{m-1}\right) + \frac{1}{\sqrt{m}} \operatorname{dn}^{-1}\left(\frac{\sqrt{1-m}}{z} \middle| \frac{1}{m}\right); z > 0 \wedge m > 1$$

$$\operatorname{ds}^{-1}(z|m) = \frac{1}{\sqrt{1-m}} K\left(\frac{m}{m-1}\right) + i \operatorname{nc}^{-1}\left(\frac{\sqrt{1-m}}{z} \middle| 1-m\right); z > 0 \wedge m < 1$$

$$\operatorname{ds}^{-1}(z|m) = \frac{1}{\sqrt{1-m}} \left( K\left(\frac{m}{m-1}\right) + i \operatorname{nd}^{-1}\left(\frac{\sqrt{1-m}}{z} \middle| \frac{1}{1-m}\right) \right); z > 0 \wedge m > 0$$

$$\operatorname{ds}^{-1}(z|m) = \frac{1}{\sqrt{1-m}} \operatorname{ns}^{-1}\left(\frac{z}{\sqrt{1-m}} \middle| \frac{m}{m-1}\right); z > 0 \wedge m > 0$$

$$\operatorname{ds}^{-1}(z|m) = \frac{i}{\sqrt{1-m}} \operatorname{sc}^{-1}\left(-\frac{i\sqrt{1-m}}{z} \middle| \frac{1}{1-m}\right); z > 0 \wedge m > 0$$

$$\operatorname{ds}^{-1}(z|m) = \operatorname{sd}^{-1}\left(\frac{1}{z} \middle| m\right); z > 0 \wedge m > 1$$

$$\operatorname{ds}^{-1}(z|m) = \frac{1}{\sqrt{1-m}} \operatorname{sn}^{-1}\left(\frac{\sqrt{1-m}}{z} \middle| \frac{m}{m-1}\right); z > 0 \wedge m > 0$$

Representations of  $\text{nc}^{-1}(z | m)$  through other inverse Jacobi functions are:

$$\text{nc}^{-1}(z | m) = \frac{i}{\sqrt{m}} \text{cd}^{-1}\left(z \left| \frac{m-1}{m} \right.\right); -1 < z < 1 \wedge m > 0$$

$$\text{nc}^{-1}(z | m) = i \text{cn}^{-1}(z | 1-m); -1 < z < 1 \wedge m \in \mathbb{R}$$

$$\text{nc}^{-1}(z | m) = \frac{1}{\sqrt{1-m}} \left( K\left(\frac{m}{m-1}\right) - i \text{cs}^{-1}\left(i z \left| \frac{1}{1-m} \right.\right) \right); z > 1 \wedge m < 1$$

$$\text{nc}^{-1}(z | m) = -\frac{1}{\sqrt{1-m}} \text{dc}^{-1}\left(z \left| \frac{m}{m-1} \right.\right); 0 < z < 1 \wedge m > 1$$

$$\text{nc}^{-1}(z | m) = -\frac{1}{\sqrt{m-1}} \text{dn}^{-1}\left(z \left| \frac{1}{1-m} \right.\right); -1 < z < 1 \wedge 0 < m < 1$$

$$\text{nc}^{-1}(z | m) = \frac{i}{\sqrt{m}} K\left(\frac{m-1}{m}\right) - i \text{ds}^{-1}\left(\frac{\sqrt{m}}{z} \left| 1-m \right.\right); z > 0 \wedge m > 0$$

$$\text{nc}^{-1}(z | m) = \frac{1}{\sqrt{m}} \text{nd}^{-1}\left(z \left| \frac{1}{m} \right.\right); -1 < z < 1 \wedge m > 0$$

$$\text{nc}^{-1}(z | m) = \frac{1}{\sqrt{1-m}} \left( K\left(\frac{m}{m-1}\right) - \text{ns}^{-1}\left(z \left| \frac{m}{m-1} \right.\right) \right); z > 1 \wedge m < 1$$

$$\text{nc}^{-1}(z | m) = \frac{1}{\sqrt{m}} \left( i K\left(\frac{m-1}{m}\right) - \text{sc}^{-1}\left(-i z \left| \frac{1}{m} \right.\right) \right); -1 < z < 1 \wedge m > 0$$

$$\text{nc}^{-1}(z | m) = \frac{i}{\sqrt{m}} K\left(\frac{m-1}{m}\right) - \text{sd}^{-1}\left(\frac{i z}{\sqrt{m}} \left| m \right.\right); -1 < z < 1 \wedge m > 0$$

$$\text{nc}^{-1}(z | m) = \frac{i}{\sqrt{m}} \left( K\left(\frac{m-1}{m}\right) - \text{sn}^{-1}\left(z \left| \frac{m-1}{m} \right.\right) \right); 0 < z < 1 \wedge m > 0.$$

Representations of  $\text{nd}^{-1}(z | m)$  through other inverse Jacobi functions are:

$$\text{nd}^{-1}(z | m) = i \text{cd}^{-1}(z | 1-m)$$

$$\text{nd}^{-1}(z | m) = -i \left( \text{cn}^{-1}\left(\sqrt{1-z^2} \left| 1-m \right.\right) - K(1-m) \right); 0 < z < 1 \wedge m > 0$$

$$\text{nd}^{-1}(z | m) = i K(1-m) + \text{cs}^{-1}\left(\frac{i}{z} \left| m \right.\right); -1 < z < 1 \wedge m > 0$$

$$\text{nd}^{-1}(z | m) = \frac{i}{\sqrt{1-m}} \text{dc}^{-1}\left(z \left| \frac{1}{1-m} \right.\right); -1 < z < 1 \wedge 0 < m < 1$$

$$\text{nd}^{-1}(z | m) = \text{dn}^{-1}\left(\frac{1}{z} \left| m \right.\right); z < -1 \wedge m < 0 \vee z > 1 \wedge m > 1$$

$$\operatorname{nd}^{-1}(z|m) = i \left( K(1-m) - \frac{1}{\sqrt{m}} \operatorname{ds}^{-1} \left( \frac{1}{\sqrt{m}z} \middle| \frac{m-1}{m} \right) \right); 0 < z < 1 \wedge m \in \mathbb{R}$$

$$\operatorname{nd}^{-1}(z|m) = \frac{1}{\sqrt{m}} \operatorname{nc}^{-1} \left( z \middle| \frac{1}{m} \right); -1 < z < 1 \wedge m > 1$$

$$\operatorname{nd}^{-1}(z|m) = \frac{i}{\sqrt{1-m}} \left( K \left( \frac{1}{1-m} \right) - \operatorname{ns}^{-1} \left( z \middle| \frac{1}{1-m} \right) \right); -1 < z < 1$$

$$\operatorname{nd}^{-1}(z|m) = i K(1-m) + \operatorname{sc}^{-1}(-iz|m); -1 < z < 1 \wedge m \in \mathbb{R}$$

$$\operatorname{nd}^{-1}(z|m) = i K(1-m) - \frac{1}{\sqrt{m}} \operatorname{sd}^{-1} \left( iz\sqrt{m} \middle| \frac{1}{m} \right)$$

$$\operatorname{nd}^{-1}(z|m) = -i (\operatorname{sn}^{-1}(z|1-m) - K(1-m)); z > 1 \wedge m > 1.$$

Representations of  $\operatorname{ns}^{-1}(z|m)$  through other inverse Jacobi functions are:

$$\operatorname{ns}^{-1}(z|m) = K(m) - \frac{1}{\sqrt{m}} \operatorname{cd}^{-1} \left( z \middle| \frac{1}{m} \right); -1 < z < 1 \wedge m < 0$$

$$\operatorname{ns}^{-1}(z|m) = \operatorname{cn}^{-1} \left( \frac{\sqrt{z^2-1}}{z} \middle| m \right); z > 1 \wedge m < 1$$

$$\operatorname{ns}^{-1}(iz|m) = i \operatorname{cs}^{-1}(-z|1-m)$$

$$\operatorname{ns}^{-1}(z|m) = K(m) - \frac{1}{\sqrt{m}} \operatorname{dc}^{-1} \left( \frac{1}{z} \middle| \frac{1}{m} \right); -1 < z < 1 \wedge m < 0$$

$$\operatorname{ns}^{-1}(z|m) = K(m) + \frac{i}{\sqrt{m}} \operatorname{dn}^{-1} \left( \frac{1}{z} \middle| 1 - \frac{1}{m} \right); z > -1 \wedge m > 1$$

$$\operatorname{ns}^{-1}(z|m) = \frac{1}{\sqrt{1-m}} \operatorname{ds}^{-1} \left( \frac{z}{\sqrt{1-m}} \middle| \frac{m}{m-1} \right); 0 < z < 1 \wedge m < 1$$

$$\operatorname{ns}^{-1}(z|m) = K(m) + \frac{i}{\sqrt{1-m}} \operatorname{nc}^{-1} \left( \frac{1}{z} \middle| \frac{1}{1-m} \right); z > 1 \wedge m \in \mathbb{R}$$

$$\operatorname{ns}^{-1}(z|m) = K(m) + \frac{i}{\sqrt{m}} \operatorname{nd}^{-1} \left( z \middle| 1 - \frac{1}{m} \right); -1 < z < 1 \wedge m < 0$$

$$\operatorname{ns}^{-1}(z|m) = -i \operatorname{sc}^{-1} \left( \frac{i}{z} \middle| 1-m \right); z > 1 \wedge m \in \mathbb{R}$$

$$\operatorname{ns}^{-1}(z|m) = \frac{i}{\sqrt{1-m}} \operatorname{sd}^{-1} \left( \frac{\sqrt{m-1}}{z} \middle| \frac{1}{1-m} \right); z > 0 \wedge m > 1$$

$$\operatorname{ns}^{-1}(z|m) = \operatorname{sn}^{-1} \left( \frac{1}{z} \middle| m \right); z \in \mathbb{R} \wedge m < 1.$$

Representations of  $\text{sc}^{-1}(z | m)$  through other inverse Jacobi functions are:

$$\text{sc}^{-1}(z | m) = i \left( \text{cd}^{-1}(iz | 1 - m) - K(1 - m) \right); z \in \mathbb{R} \wedge m \in \mathbb{R}$$

$$\text{sc}^{-1}(z | m) = i \text{cn}^{-1}\left(\sqrt{z^2 + 1} \mid 1 - m\right); 0 < z < 1 \wedge m > 1$$

$$\text{sc}^{-1}(z | m) = \text{cs}^{-1}\left(\frac{1}{z} \mid m\right); z > 0 \wedge m < 1$$

$$\text{sc}^{-1}(z | m) = i \left( K(1 - m) - \frac{1}{\sqrt{1 - m}} \text{dc}^{-1}\left(-iz \mid \frac{1}{1 - m}\right) \right); z \in \mathbb{R} \wedge 0 < m < 1$$

$$\text{sc}^{-1}(z | m) = \frac{1}{\sqrt{1 - m}} \text{dn}^{-1}\left(iz \mid 1 - \frac{1}{1 - m}\right) - i K(1 - m); z < 1 \wedge m < 0$$

$$\text{sc}^{-1}(z | m) = -\frac{i}{\sqrt{m}} \text{ds}^{-1}\left(-\frac{i}{z\sqrt{m}} \mid \frac{m - 1}{m}\right); z > 0 \wedge m > 1$$

$$\text{sc}^{-1}(z | m) = \frac{1}{\sqrt{m}} \text{nc}^{-1}\left(iz \mid \frac{1}{m}\right) - i K(1 - m); -1 < z < 1 \wedge m > 0$$

$$\text{sc}^{-1}(z | m) = \text{nd}^{-1}(iz | m) - i K(1 - m); z \in \mathbb{R} \wedge m \in \mathbb{R}$$

$$\text{sc}^{-1}(z | m) = -i \text{ns}^{-1}\left(-\frac{i}{z} \mid 1 - m\right); 0 < z < 1 \wedge m > 0$$

$$\text{sc}^{-1}(z | m) = \frac{1}{\sqrt{m}} \text{sd}^{-1}\left(z\sqrt{m} \mid \frac{1}{m}\right); z < 1 \wedge m < 0$$

$$\text{sc}^{-1}(z | m) = -i \text{sn}^{-1}(iz | 1 - m).$$

Representations of  $\text{sd}^{-1}(z | m)$  through other inverse Jacobi functions are:

$$\text{sd}^{-1}(z | m) = \frac{1}{\sqrt{1 - m}} \left( K\left(\frac{m}{m - 1}\right) - \text{cd}^{-1}\left(\sqrt{1 - m} z \mid \frac{m}{m - 1}\right) \right); z \in \mathbb{R} \wedge m > 1$$

$$\text{sd}^{-1}(z | m) = -i \text{cn}^{-1}\left(\sqrt{z^2 + 1} \mid 1 - m\right); 0 < z < 1 \wedge 0 < m < 1$$

$$\text{sd}^{-1}(z | m) = \text{cs}^{-1}\left(\frac{1}{z} \mid m\right); z > 0 \wedge m \in \mathbb{R}$$

$$\text{sd}^{-1}(z | m) = i \left( \frac{1}{\sqrt{1 - m}} \text{dc}^{-1}\left(iz \mid \frac{1}{1 - m}\right) - K(1 - m) \right); z \in \mathbb{R} \wedge 0 < m < 1$$

$$\text{sd}^{-1}(z | m) = i \left( \frac{1}{\sqrt{m - 1}} \text{dn}^{-1}\left(iz \mid \frac{m}{m - 1}\right) - K(1 - m) \right); -1 < z < 1 \wedge m > 1$$

$$\text{sd}^{-1}(z | m) = \text{ds}^{-1}\left(\frac{1}{z} \mid m\right); z > 0 \wedge m > 1$$

$$\operatorname{sc}^{-1}(z | m) = i K(1 - m) - \frac{1}{\sqrt{m}} \operatorname{nc}^{-1}\left(z \left| \frac{1}{m} \right.\right); -1 < z < 1 \wedge m > 1$$

$$\operatorname{sd}^{-1}(z | m) = \frac{1}{\sqrt{m}} \left( \operatorname{nd}^{-1}\left(i z \sqrt{m} \left| \frac{1}{m} \right.\right) - i K\left(1 - \frac{1}{m}\right) \right); z \in \mathbb{R} \wedge m > 1$$

$$\operatorname{sc}^{-1}(z | m) = -i \operatorname{ns}^{-1}\left(-\frac{i}{z} \left| 1 - m \right.\right); z > 0 \wedge m \in \mathbb{R}$$

$$\operatorname{sc}^{-1}(z | m) = -i \operatorname{sn}^{-1}(i z | 1 - m)$$

$$\operatorname{sd}^{-1}(z | m) = -\frac{i}{\sqrt{m}} \operatorname{sn}^{-1}\left(\sqrt{-m} z \left| \frac{m-1}{m} \right.\right); -1 < z < 1 \wedge m > 0.$$

Representations of  $\operatorname{sn}^{-1}(z | m)$  through other inverse Jacobi functions are:

$$\operatorname{sn}^{-1}(z | m) = K(m) - \operatorname{cd}^{-1}(z | m); z \in \mathbb{R} \wedge m \in \mathbb{R}$$

$$\operatorname{sn}^{-1}(z | m) = K(m) - \frac{1}{\sqrt{1-m}} \operatorname{cn}^{-1}\left(z \left| \frac{m}{m-1} \right.\right); -1 < z < 1 \wedge m < 1$$

$$\operatorname{sn}^{-1}(z | m) = K(m) + \frac{i}{\sqrt{m}} \operatorname{dn}^{-1}\left(z \left| \frac{m-1}{m} \right.\right); z < 0 \wedge m > 1$$

$$\operatorname{sn}^{-1}(z | m) = K(m) - \frac{1}{\sqrt{m}} \operatorname{dc}^{-1}\left(z \left| \frac{1}{m} \right.\right); z > 1 \wedge m > 1$$

$$\operatorname{sn}^{-1}(z | m) = K(m) + \frac{i}{\sqrt{m}} \operatorname{dn}^{-1}\left(z \left| \frac{m-1}{m} \right.\right); z < 0 \wedge m > 1$$

$$\operatorname{sn}^{-1}(z | m) = \operatorname{dn}^{-1}\left(\sqrt{1-mz^2} \left| m \right.\right); z > 1 \wedge m < 0$$

$$\operatorname{sn}^{-1}(z | m) = \frac{1}{\sqrt{1-m}} \operatorname{ds}^{-1}\left(\frac{1}{\sqrt{1-m} z} \left| \frac{m}{m-1} \right.\right); z > 0 \wedge m > 1$$

$$\operatorname{sn}^{-1}(z | m) = K(m) + \frac{i}{\sqrt{1-m}} \operatorname{nc}^{-1}\left(z \left| \frac{1}{1-m} \right.\right); -1 < z < 1 \wedge m < 1$$

$$\operatorname{sn}^{-1}(z | m) = K(m) + i \operatorname{nd}^{-1}(z | 1 - m); z \in \mathbb{R} \wedge m \in \mathbb{R}$$

$$\operatorname{sn}^{-1}(z | m) = \operatorname{ns}^{-1}\left(\frac{1}{z} \left| m \right.\right); -1 < z < 0 \wedge m < 0 \vee z > 0 \wedge m < 0$$

$$\operatorname{sn}^{-1}(z | m) = -i \operatorname{sc}^{-1}(i z | 1 - m)$$

$$\operatorname{sn}^{-1}(z | m) = -\frac{i}{\sqrt{1-m}} \operatorname{sd}^{-1}\left(\sqrt{m-1} z \left| \frac{1}{1-m} \right.\right); -1 < z < 1 \wedge m < 1.$$

## The best-known properties and formulas for inverse Jacobi functions

### Simple values at zero

The inverse Jacobi functions  $\text{cd}^{-1}(z|m)$ ,  $\text{cn}^{-1}(z|m)$ ,  $\text{cs}^{-1}(z|m)$ ,  $\text{dc}^{-1}(z|m)$ ,  $\text{dn}^{-1}(z|m)$ ,  $\text{ds}^{-1}(z|m)$ ,  $\text{nc}^{-1}(z|m)$ ,  $\text{nd}^{-1}(z|m)$ ,  $\text{ns}^{-1}(z|m)$ ,  $\text{sc}^{-1}(z|m)$ ,  $\text{sd}^{-1}(z|m)$ , and  $\text{sn}^{-1}(z|m)$  have the following simple values at the origin:

$$\begin{aligned} \text{cd}^{-1}(0|0) &= \frac{\pi}{2} & \text{cn}^{-1}(0|0) &= \frac{\pi}{2} & \text{cs}^{-1}(0|0) &= \frac{\pi}{2} \\ \text{dc}^{-1}(0|0) &= \infty & \text{dn}^{-1}(0|0) & & \text{ds}^{-1}(0|0) &= \infty \\ \text{nc}^{-1}(0|0) &= \infty & \text{nd}^{-1}(0|0) & & \text{ns}^{-1}(0|0) &= \infty \\ \text{sc}^{-1}(0|0) &= 0 & \text{sd}^{-1}(0|0) &= 0 & \text{sn}^{-1}(0|0) &= 0. \end{aligned}$$

**Specific values for specialized parameter values**

The inverse Jacobi functions  $\text{cd}^{-1}(z|m)$ ,  $\text{cn}^{-1}(z|m)$ ,  $\text{cs}^{-1}(z|m)$ ,  $\text{dc}^{-1}(z|m)$ ,  $\text{dn}^{-1}(z|m)$ ,  $\text{ds}^{-1}(z|m)$ ,  $\text{nc}^{-1}(z|m)$ ,  $\text{nd}^{-1}(z|m)$ ,  $\text{ns}^{-1}(z|m)$ ,  $\text{sc}^{-1}(z|m)$ ,  $\text{sd}^{-1}(z|m)$ , and  $\text{sn}^{-1}(z|m)$  can be represented through elementary functions when  $m = 0$  or  $m = 1$ . In these cases they degenerate into inverse trigonometric and inverse hyperbolic functions. If  $m = \frac{1}{2}$ , they can be represented through the elliptic integrals  $F(w(z)|n)$  and  $K(n)$ :

$$\begin{aligned} \text{cd}^{-1}(z|0) &= \cos^{-1}(z) & \text{cd}^{-1}\left(z\left|\frac{1}{2}\right.\right) &= K\left(\frac{1}{2}\right) - F\left(\sin^{-1}(z)\left|\frac{1}{2}\right.\right) & \text{cd}^{-1}(z|1) &= \infty \\ \text{cn}^{-1}(z|0) &= \cos^{-1}(z) & \text{cn}^{-1}\left(z\left|\frac{1}{2}\right.\right) &= F\left(\cos^{-1}(z)\left|\frac{1}{2}\right.\right) & \text{cn}^{-1}(z|1) &= \text{sech}^{-1}(z) \\ \text{cs}^{-1}(z|0) &= \cot^{-1}(z) & \text{cs}^{-1}\left(z\left|\frac{1}{2}\right.\right) &= -i F\left(i \sinh^{-1}\left(\frac{1}{z}\right)\left|\frac{1}{2}\right.\right) & \text{cs}^{-1}(z|1) &= \text{csch}^{-1}(z) \\ \text{dc}^{-1}(z|0) &= \sec^{-1}(z) & \text{dc}^{-1}\left(z\left|\frac{1}{2}\right.\right) &= \sqrt{2} \left(K(2) - F(\sin^{-1}(z)|2)\right); z > 1 & \text{dc}^{-1}(z|1) &= \infty \\ \text{dn}^{-1}(z|0) &= \infty & \text{dn}^{-1}\left(z\left|\frac{1}{2}\right.\right) &= \frac{8(1+i)\pi^{3/2}}{\Gamma(-\frac{1}{4})^2} - i\sqrt{2} F(\sin^{-1}(z)|2) & \text{dn}^{-1}(z|1) &= \text{sech}^{-1}(z) \\ \text{ds}^{-1}(z|0) &= \csc^{-1}(z) & \text{ds}^{-1}\left(z\left|\frac{1}{2}\right.\right) &= \sqrt{2} F\left(\sin^{-1}\left(\frac{1}{\sqrt{2}z}\right)\left|-1\right.\right); z > 1 & \text{ds}^{-1}(z|1) &= \text{csch}^{-1}(z) \\ \text{nc}^{-1}(z|0) &= \sec^{-1}(z) & \text{nc}^{-1}\left(z\left|\frac{1}{2}\right.\right) &= i\sqrt{2} \left(F(\sin^{-1}(z)|-1) - \frac{1}{4\sqrt{2}\pi} \Gamma\left(\frac{1}{4}\right)^2\right); z > 1 & \text{nc}^{-1}(z|1) &= \cosh^{-1}(z) \\ \text{nd}^{-1}(z|0) &= \infty & \text{nd}^{-1}\left(z\left|\frac{1}{2}\right.\right) &= i\sqrt{2} F\left(\frac{\pi}{4}\left|2\right.\right) - \sqrt{2} i F\left(\sin^{-1}\left(\frac{z}{\sqrt{2}}\right)\left|2\right.\right); -1 < z < 1 & \text{nd}^{-1}(z|1) &= \cosh^{-1}(z) \\ \text{ns}^{-1}(z|0) &= \csc^{-1}(z) & \text{ns}^{-1}\left(z\left|\frac{1}{2}\right.\right) &= \sqrt{2} F(\sin^{-1}(z)|2) + \frac{i\pi^{3/2}}{2\Gamma(\frac{3}{4})^2} & \text{ns}^{-1}(z|1) &= \coth^{-1}(z) \\ \text{sc}^{-1}(z|0) &= \tan^{-1}(z) & \text{sc}^{-1}\left(z\left|\frac{1}{2}\right.\right) &= -i\sqrt{2} F\left(i \sinh^{-1}\left(\frac{z}{\sqrt{2}}\right)\left|2\right.\right) & \text{sc}^{-1}(z|1) &= \sinh^{-1}(z) \\ \text{sd}^{-1}(z|0) &= \sin^{-1}(z) & \text{sd}^{-1}\left(z\left|\frac{1}{2}\right.\right) &= -i\sqrt{2} F\left(i \sinh^{-1}\left(\frac{z}{\sqrt{2}}\right)\left|-1\right.\right); z > -1 & \text{sd}^{-1}(z|1) &= \sinh^{-1}(z) \\ \text{sn}^{-1}(z|0) &= \sin^{-1}(z) & \text{sn}^{-1}\left(z\left|\frac{1}{2}\right.\right) &= F(\sin^{-1}(z)|\frac{1}{2}) & \text{sn}^{-1}(z|1) &= \tanh^{-1}(z). \end{aligned}$$

At the points  $z = -1, -1/2, 0, 1/2, 1$ , and  $i$ , the inverse Jacobi functions  $\text{cd}^{-1}(z|m)$ ,  $\text{cn}^{-1}(z|m)$ ,  $\text{cs}^{-1}(z|m)$ ,  $\text{dc}^{-1}(z|m)$ ,  $\text{dn}^{-1}(z|m)$ ,  $\text{ds}^{-1}(z|m)$ ,  $\text{nc}^{-1}(z|m)$ ,  $\text{nd}^{-1}(z|m)$ ,  $\text{ns}^{-1}(z|m)$ ,  $\text{sc}^{-1}(z|m)$ ,  $\text{sd}^{-1}(z|m)$ , and  $\text{sn}^{-1}(z|m)$  have the following representations through the elliptic integrals  $F(w(z, m)|n)$  and  $K(u(m))$ :

$$\begin{aligned} \text{cd}^{-1}(-1|m) &= 2K(m) & \text{cd}^{-1}\left(-\frac{1}{2}\left|m\right.\right) &= F\left(\frac{\pi}{6}\left|m\right.\right) + K(m) & \text{cd}^{-1}(0|m) &= K(m) \\ \text{cd}^{-1}\left(\frac{1}{2}\left|m\right.\right) &= K(m) - F\left(\frac{\pi}{6}\left|m\right.\right) & \text{cd}^{-1}(1|m) &= 0 & \text{cd}^{-1}(i|m) &= K(m) - F(\sin^{-1}(i)|m) \end{aligned}$$

$$\begin{aligned}
 \operatorname{cn}^{-1}(-1 | m) &= 2 K(m) & \operatorname{cn}^{-1}\left(-\frac{1}{2} | m\right) &= F\left(\frac{2\pi}{3} | m\right) & \operatorname{cn}^{-1}(0 | m) &= K(m) /; m \in \mathbb{R} \wedge m < 1 \\
 \operatorname{cn}^{-1}\left(\frac{1}{2} | m\right) &= F\left(\frac{\pi}{3} | m\right) & \operatorname{cn}^{-1}(1 | m) &= 0 & \operatorname{cn}^{-1}(i | m) &= \frac{1}{\sqrt{m-1}} \left( i \left( K\left(\frac{m}{m-1}\right) - F\left(\sin^{-1}(i) \left| \frac{m}{m-1} \right. \right) \right) \right) \\
 \operatorname{cs}^{-1}(-1 | m) &= \frac{1}{\sqrt{1-m}} \left( K\left(\frac{m}{m-1}\right) - i F\left(i \sinh^{-1}(1) \left| \frac{1}{1-m} \right. \right) \right) & \operatorname{cs}^{-1}\left(-\frac{1}{2} | m\right) &= -i F\left(i \sinh^{-1}(2) | 1-m\right) - \frac{2i}{\sqrt{1-m}} F\left(i \sinh^{-1}\left(\frac{1}{2}\right) \left| \frac{1}{1-m} \right. \right) & \operatorname{cs}^{-1}(0 | m) &= 0 \\
 \operatorname{cs}^{-1}\left(\frac{1}{2} | m\right) &= -i F\left(i \sinh^{-1}(2) | 1-m\right) & \operatorname{cs}^{-1}(1 | m) &= \frac{i}{\sqrt{1-m}} \left( F\left(i \sinh^{-1}(1) \left| \frac{1}{1-m} \right. \right) + K\left(\frac{m}{m-1}\right) \right) & \operatorname{cs}^{-1}(i | m) &= \frac{i}{\sqrt{1-m}} \left( K\left(\frac{m}{m-1}\right) - F\left(\sin^{-1}(i) \left| \frac{m}{m-1} \right. \right) \right) \\
 \operatorname{dc}^{-1}(-1 | m) &= \frac{2}{\sqrt{m}} K\left(\frac{1}{m}\right) /; m > 1 & \operatorname{dc}^{-1}\left(-\frac{1}{2} | m\right) &= \frac{1}{\sqrt{m}} \left( K\left(\frac{1}{m}\right) + F\left(\frac{\pi}{6} \left| \frac{1}{m} \right. \right) \right) /; m > 1 & \operatorname{dc}^{-1}(0 | m) &= \frac{1}{\sqrt{m}} K\left(\frac{1}{m}\right) /; m > 1 \\
 \operatorname{dc}^{-1}\left(\frac{1}{2} | m\right) &= \frac{1}{\sqrt{m}} \left( K\left(\frac{1}{m}\right) - F\left(\frac{\pi}{6} \left| \frac{1}{m} \right. \right) \right) /; m > 1 & \operatorname{dc}^{-1}(1 | m) &= 0 & \operatorname{dc}^{-1}(i | m) &= \frac{1}{\sqrt{m}} \left( K\left(\frac{1}{m}\right) - F\left(\sin^{-1}(i) \left| \frac{1}{m} \right. \right) \right) \\
 \operatorname{dn}^{-1}(-1 | m) &= \frac{2}{\sqrt{m-1}} K\left(\frac{1}{1-m}\right) /; m > 1 & \operatorname{dn}^{-1}\left(-\frac{1}{2} | m\right) &= -\frac{1}{\sqrt{m-1}} \left( F\left(\frac{\pi}{6} \left| \frac{1}{1-m} \right. \right) + K\left(\frac{1}{1-m}\right) \right) /; 0 < m < 1 & \operatorname{dn}^{-1}(0 | m) &= 0 \\
 \operatorname{dn}^{-1}\left(\frac{1}{2} | m\right) &= \frac{1}{\sqrt{m-1}} \left( K\left(\frac{1}{1-m}\right) - F\left(\frac{\pi}{6} \left| \frac{1}{1-m} \right. \right) \right) /; m > 1 & \operatorname{dn}^{-1}(1 | m) &= 0 & \operatorname{dn}^{-1}(i | m) &= \frac{1}{\sqrt{m-1}} \left( K\left(\frac{1}{1-m}\right) - F\left(\sin^{-1}(i) \left| \frac{1}{1-m} \right. \right) \right) \\
 \operatorname{ds}^{-1}(-1 | m) &= \frac{1}{\sqrt{m}} \left( K\left(\frac{1}{m}\right) + i F\left(\sin^{-1}\left(\frac{1}{\sqrt{1-m}}\right) \left| \frac{m-1}{m} \right. \right) \right) /; m > 1 & \operatorname{ds}^{-1}\left(-\frac{1}{2} | m\right) &= \frac{2i}{\sqrt{m}} F\left(\sin^{-1}\left(\frac{1}{2\sqrt{1-m}}\right) \left| \frac{m-1}{m} \right. \right) + \frac{1}{\sqrt{1-m}} F\left(\sin^{-1}(2) \left| \frac{m-1}{m} \right. \right) & \operatorname{ds}^{-1}(0 | m) &= 0 \\
 \operatorname{ds}^{-1}\left(\frac{1}{2} | m\right) &= \frac{1}{\sqrt{1-m}} F\left(\sin^{-1}(2\sqrt{1-m}) \left| \frac{m}{m-1} \right. \right) /; m > 1 & \operatorname{ds}^{-1}(1 | m) &= \frac{1}{\sqrt{m}} \left( K\left(\frac{1}{m}\right) - i F\left(\sin^{-1}\left(\frac{1}{\sqrt{1-m}}\right) \left| \frac{m-1}{m} \right. \right) \right) /; m > 1 & \operatorname{ds}^{-1}(i | m) &= \frac{i}{\sqrt{m}} \left( K\left(\frac{1}{m}\right) - F\left(\sin^{-1}(i) \left| \frac{1}{m} \right. \right) \right) \\
 \operatorname{nc}^{-1}(-1 | m) &= \frac{2i}{\sqrt{m}} K\left(\frac{m-1}{m}\right) & \operatorname{nc}^{-1}\left(-\frac{1}{2} | m\right) &= \frac{i}{\sqrt{m}} \left( K\left(\frac{m-1}{m}\right) + F\left(\frac{\pi}{6} \left| \frac{m-1}{m} \right. \right) \right) & \operatorname{nc}^{-1}(0 | m) &= \frac{i}{\sqrt{m}} K\left(\frac{m-1}{m}\right) \\
 \operatorname{nc}^{-1}\left(\frac{1}{2} | m\right) &= \frac{i}{\sqrt{m}} \left( K\left(\frac{m-1}{m}\right) - F\left(\frac{\pi}{6} \left| \frac{m-1}{m} \right. \right) \right) & \operatorname{nc}^{-1}(1 | m) &= 0 & \operatorname{nc}^{-1}(i | m) &= \frac{i}{\sqrt{m}} \left( K\left(\frac{m-1}{m}\right) - F\left(\sin^{-1}(i) \left| \frac{m-1}{m} \right. \right) \right) \\
 \operatorname{nd}^{-1}(-1 | m) &= 2i K(1-m) & \operatorname{nd}^{-1}\left(-\frac{1}{2} | m\right) &= i \left( F\left(\frac{\pi}{6} | 1-m\right) + K(1-m) \right) & \operatorname{nd}^{-1}(0 | m) &= i K(1-m) \\
 \operatorname{nd}^{-1}\left(\frac{1}{2} | m\right) &= i \left( K(1-m) - F\left(\frac{\pi}{6} | 1-m\right) \right) & \operatorname{nd}^{-1}(1 | m) &= 0 & \operatorname{nd}^{-1}(i | m) &= i K(1-m) - i F\left(i \sinh^{-1}(1) | 1-m\right) \\
 \operatorname{ns}^{-1}(-1 | m) &= -K(m) & \operatorname{ns}^{-1}\left(-\frac{1}{2} | m\right) &= K(m) - \frac{1}{\sqrt{m}} \left( F\left(\frac{\pi}{6} \left| \frac{1}{m} \right. \right) + K\left(\frac{1}{m}\right) \right) & \operatorname{ns}^{-1}(0 | m) &= K(m) - \frac{1}{\sqrt{m}} K\left(\frac{1}{m}\right) \\
 \operatorname{ns}^{-1}\left(\frac{1}{2} | m\right) &= \frac{1}{\sqrt{m}} \left( F\left(\frac{\pi}{6} \left| \frac{1}{m} \right. \right) - K\left(\frac{1}{m}\right) \right) + K(m) & \operatorname{ns}^{-1}(1 | m) &= K(m) & \operatorname{ns}^{-1}(i | m) &= K(m) - \frac{1}{\sqrt{m}} \left( K\left(\frac{1}{m}\right) - F\left(\sin^{-1}(i) \left| \frac{1}{m} \right. \right) \right) \\
 \operatorname{sc}^{-1}(-1 | m) &= i F\left(i \sinh^{-1}(1) | 1-m\right) & \operatorname{sc}^{-1}\left(-\frac{1}{2} | m\right) &= i F\left(i \sinh^{-1}\left(\frac{1}{2}\right) | 1-m\right) & \operatorname{sc}^{-1}(0 | m) &= 0 \\
 \operatorname{sc}^{-1}\left(\frac{1}{2} | m\right) &= -i F\left(i \sinh^{-1}\left(\frac{1}{2}\right) | 1-m\right) & \operatorname{sc}^{-1}(1 | m) &= -i F\left(i \sinh^{-1}(1) | 1-m\right) & \operatorname{sc}^{-1}(i | m) &= i K(1-m) \\
 \operatorname{sd}^{-1}(-1 | m) &= \frac{i}{\sqrt{m}} F\left(i \sinh^{-1}(\sqrt{m}) \left| \frac{m-1}{m} \right. \right) /; m > 0 & \operatorname{sd}^{-1}\left(-\frac{1}{2} | m\right) &= \frac{i}{\sqrt{m}} F\left(i \sinh^{-1}\left(\frac{\sqrt{m}}{2}\right) \left| \frac{m-1}{m} \right. \right) /; m > 0 & \operatorname{sd}^{-1}(0 | m) &= 0 \\
 \operatorname{sd}^{-1}\left(\frac{1}{2} | m\right) &= -\frac{i}{\sqrt{m}} F\left(i \sinh^{-1}\left(\frac{\sqrt{m}}{2}\right) \left| \frac{m-1}{m} \right. \right) /; m > 0 & \operatorname{sd}^{-1}(1 | m) &= -\frac{i}{\sqrt{m}} F\left(i \sinh^{-1}(\sqrt{m}) \left| \frac{m-1}{m} \right. \right) /; m > 0 & \operatorname{sd}^{-1}(i | m) &= \frac{i}{\sqrt{m}} F\left(i \sinh^{-1}(i) \left| \frac{m-1}{m} \right. \right) \\
 \operatorname{sn}^{-1}(-1 | m) &= -K(m) & \operatorname{sn}^{-1}\left(-\frac{1}{2} | m\right) &= -F\left(\frac{\pi}{6} | m\right) & \operatorname{sn}^{-1}(0 | m) &= 0 \\
 \operatorname{sn}^{-1}\left(\frac{1}{2} | m\right) &= F\left(\frac{\pi}{6} | m\right) & \operatorname{sn}^{-1}(1 | m) &= K(m) & \operatorname{sn}^{-1}(i | m) &= F\left(i \sinh^{-1}(1) | m\right).
 \end{aligned}$$

At the points  $m = \pm \infty$  or  $z = \pm \infty$ , the inverse Jacobi functions  $\operatorname{cd}^{-1}(z | m)$ ,  $\operatorname{cn}^{-1}(z | m)$ ,  $\operatorname{cs}^{-1}(z | m)$ ,  $\operatorname{dc}^{-1}(z | m)$ ,  $\operatorname{dn}^{-1}(z | m)$ ,  $\operatorname{ds}^{-1}(z | m)$ ,  $\operatorname{nc}^{-1}(z | m)$ ,  $\operatorname{nd}^{-1}(z | m)$ ,  $\operatorname{ns}^{-1}(z | m)$ ,  $\operatorname{sc}^{-1}(z | m)$ ,  $\operatorname{sd}^{-1}(z | m)$ , and  $\operatorname{sn}^{-1}(z | m)$  have the following values:

$$\begin{aligned}
 \operatorname{cd}^{-1}(z|\infty) &= 0 & \operatorname{cd}^{-1}(z|-\infty) &= 0 \\
 \operatorname{cd}^{-1}(\infty|m) &= \frac{1}{\sqrt{m}} K\left(\frac{1}{m}\right) & \operatorname{cd}^{-1}(-\infty|m) &= 2K(m) - \frac{1}{\sqrt{m}} K\left(\frac{1}{m}\right) \\
 \\
 \operatorname{cn}^{-1}(z|\infty) &= 0 & \operatorname{cn}^{-1}(z|-\infty) &= 0 \\
 \operatorname{cn}^{-1}(\infty|m) &= -\frac{i}{\sqrt{m}} K\left(1 - \frac{1}{m}\right) & \operatorname{cn}^{-1}(-\infty|m) &= \frac{2}{\sqrt{1-m}} K\left(\frac{m}{m-1}\right) + \frac{i}{\sqrt{m}} K\left(\frac{m-1}{m}\right) \\
 \\
 \operatorname{cs}^{-1}(z|\infty) &= 0 & \operatorname{cs}^{-1}(z|-\infty) &= 0 \\
 \operatorname{cs}^{-1}(\infty|m) &= 0 & \operatorname{cs}^{-1}(-\infty|m) &= \frac{2}{\sqrt{1-m}} K\left(\frac{m}{m-1}\right) \\
 \\
 \operatorname{dc}^{-1}(z|\infty) &= 0 & \operatorname{dc}^{-1}(z|-\infty) &= 0 \\
 \operatorname{dc}^{-1}(\infty|m) &= K(m) & \operatorname{dc}^{-1}(-\infty|m) &= -2iK(1-m) + \frac{2}{\sqrt{m}} K\left(\frac{1}{m}\right) - K(m) \ ; \ m > 1 \\
 \\
 \operatorname{dn}^{-1}(z|\infty) &= 0 & \operatorname{dn}^{-1}(z|-\infty) &= 0 \\
 \operatorname{dn}^{-1}(\infty|m) &= iK(1-m) & \operatorname{dn}^{-1}(-\infty|m) &= \frac{2}{\sqrt{m-1}} K\left(\frac{1}{1-m}\right) - iK(1-m) \ ; \ m > 1 \\
 \\
 \operatorname{ds}^{-1}(z|\infty) &= 0 & \operatorname{ds}^{-1}(z|-\infty) &= 0 \\
 \operatorname{ds}^{-1}(\infty|m) &= 0 & \operatorname{ds}^{-1}(-\infty|m) &= \frac{2}{\sqrt{m}} K\left(\frac{1}{m}\right) \ ; \ m > 1 \\
 \\
 \operatorname{nc}^{-1}(z|\infty) &= 0 & \operatorname{nc}^{-1}(z|-\infty) &= 0 \\
 \operatorname{nc}^{-1}(\infty|m) &= -\frac{1}{\sqrt{1-m}} K\left(\frac{m}{m-1}\right) & \operatorname{nc}^{-1}(-\infty|m) &= \frac{1}{\sqrt{1-m}} K\left(\frac{m}{m-1}\right) + \frac{2i}{\sqrt{m}} K\left(\frac{m-1}{m}\right) \\
 \\
 \operatorname{nd}^{-1}(z|\infty) &= 0 & \operatorname{nd}^{-1}(z|-\infty) &= 0 \\
 \operatorname{nd}^{-1}(\infty|m) &= -\frac{1}{\sqrt{m-1}} K\left(\frac{1}{1-m}\right) & \operatorname{nd}^{-1}(-\infty|m) &= \frac{1}{\sqrt{m-1}} K\left(\frac{1}{1-m}\right) + 2iK(1-m) \\
 \\
 \operatorname{ns}^{-1}(z|\infty) &= 0 & \operatorname{ns}^{-1}(z|-\infty) &= 0 \\
 \operatorname{ns}^{-1}(\infty|m) &= 0 & \operatorname{ns}^{-1}(-\infty|m) &= 2K(m) - \frac{2}{\sqrt{m}} K\left(\frac{1}{m}\right) \\
 \\
 \operatorname{sc}^{-1}(z|\infty) &= 0 & \operatorname{sc}^{-1}(z|-\infty) &= 0 \\
 \operatorname{sc}^{-1}(\infty|m) &= K(m) & \operatorname{sc}^{-1}(-\infty|m) &= -K(m) \\
 \\
 \operatorname{sd}^{-1}(z|\infty) &= 0 & \operatorname{sd}^{-1}(z|-\infty) &= 0 \\
 \operatorname{sd}^{-1}(\infty|m) &= \frac{1}{\sqrt{m-1}} K\left(\frac{1}{1-m}\right) \ ; \ m > 1 & \operatorname{sd}^{-1}(-\infty|m) &= -\frac{1}{\sqrt{m-1}} K\left(\frac{1}{1-m}\right) \ ; \ m > 1 \\
 \\
 \operatorname{sn}^{-1}(z|\infty) &= 0 & \operatorname{sn}^{-1}(z|-\infty) &= 0 \\
 \operatorname{sn}^{-1}(\infty|m) &= K(m) - \frac{1}{\sqrt{m}} K\left(\frac{1}{m}\right) \ ; \ m > 1 & \operatorname{sn}^{-1}(-\infty|m) &= \frac{1}{\sqrt{m}} K\left(\frac{1}{m}\right) - K(m) \ ; \ m > 1.
 \end{aligned}$$

### Analyticity

The inverse Jacobi functions  $\operatorname{cd}^{-1}(z|m)$ ,  $\operatorname{cn}^{-1}(z|m)$ ,  $\operatorname{cs}^{-1}(z|m)$ ,  $\operatorname{dc}^{-1}(z|m)$ ,  $\operatorname{dn}^{-1}(z|m)$ ,  $\operatorname{ds}^{-1}(z|m)$ ,  $\operatorname{nc}^{-1}(z|m)$ ,  $\operatorname{nd}^{-1}(z|m)$ ,  $\operatorname{ns}^{-1}(z|m)$ ,  $\operatorname{sc}^{-1}(z|m)$ ,  $\operatorname{sd}^{-1}(z|m)$ , and  $\operatorname{sn}^{-1}(z|m)$  are analytical functions of  $z$  and  $m$  that are defined over  $\mathbb{C}^2$ .



### Poles and essential singularities

The inverse Jacobi functions  $\text{cd}^{-1}(z|m)$ ,  $\text{cn}^{-1}(z|m)$ ,  $\text{cs}^{-1}(z|m)$ ,  $\text{dc}^{-1}(z|m)$ ,  $\text{dn}^{-1}(z|m)$ ,  $\text{ds}^{-1}(z|m)$ ,  $\text{nc}^{-1}(z|m)$ ,  $\text{nd}^{-1}(z|m)$ ,  $\text{ns}^{-1}(z|m)$ ,  $\text{sc}^{-1}(z|m)$ ,  $\text{sd}^{-1}(z|m)$ , and  $\text{sn}^{-1}(z|m)$  do not have poles and essential singularities with respect to  $z$  and  $m$ .

### Branch points and branch cuts

For fixed  $z$ , the point  $m = \infty$  is the branch point for all twelve inverse Jacobi functions. Other branch points are the following:  $m = \frac{1}{z^2}$  for  $\text{cd}^{-1}(z|m)$ ,  $m = \frac{1}{1-z^2}$  for  $\text{cn}^{-1}(z|m)$ ,  $m = 1 + z^2$  for  $\text{cs}^{-1}(z|m)$ ,  $m = z^2$  for  $\text{dc}^{-1}(z|m)$ ,  $m = 1 - z^2$  for  $\text{dn}^{-1}(z|m)$ ,  $m = -z^2$  and  $m = 1 - z^2$  for  $\text{ds}^{-1}(z|m)$ ,  $m = \frac{z^2}{z^2-1}$  for  $\text{nc}^{-1}(z|m)$ ,  $m = \frac{z^2-1}{z^2}$  for  $\text{nd}^{-1}(z|m)$ ,  $m = z^2$  for  $\text{ns}^{-1}(z|m)$ ,  $m = \frac{z^2+1}{z^2}$  for  $\text{sc}^{-1}(z|m)$ ,  $m = -\frac{1}{z^2}$  and  $m = \frac{z^2-1}{z^2}$  for  $\text{sd}^{-1}(z|m)$ , and  $m = \frac{1}{z^2}$  for  $\text{sn}^{-1}(z|m)$ .

For fixed  $m$ , the point  $m = \infty$  is the branch point for all twelve inverse Jacobi functions. There are four or five other branch points that include the following:  $z = \pm 1$ ,  $z = \pm \frac{1}{\sqrt{m}}$  for  $\text{cd}^{-1}(z|m)$ ,  $z = \pm i$ ,  $z = \pm \sqrt{m-1}$  for  $\text{cn}^{-1}(z|m)$ ,  $z = \pm i$ ,  $z = \pm \sqrt{m-1}$  for  $\text{cs}^{-1}(z|m)$ ,  $z = 0$ ,  $z = \pm 1$ ,  $z = \pm \sqrt{m}$  for  $\text{dc}^{-1}(z|m)$ ,  $z = 0$ ,  $z = \pm 1$ ,  $z = \pm \sqrt{1-m}$  for  $\text{dn}^{-1}(z|m)$ ,  $z = 0$ ,  $z = \pm \sqrt{-m}$ ,  $z = \pm \sqrt{1-m}$  for  $\text{ds}^{-1}(z|m)$ ,  $z = \pm 1$ ,  $z = \pm \sqrt{\frac{m}{m-1}}$  for  $\text{nc}^{-1}(z|m)$ ,  $z = \pm 1$ ,  $z = \pm \frac{1}{\sqrt{1-m}}$  for  $\text{nd}^{-1}(z|m)$ ,  $z = \pm 1$ ,  $z = \pm \sqrt{m}$  for  $\text{ns}^{-1}(z|m)$ ,  $z = \pm i$ ,  $z = \pm \frac{1}{\sqrt{m-1}}$  for  $\text{sc}^{-1}(z|m)$ ,  $z = \pm \frac{1}{\sqrt{-m}}$ ,  $z = \pm \frac{1}{\sqrt{1-m}}$  for  $\text{sd}^{-1}(z|m)$ , and  $z = \pm 1$ ,  $z = \pm \frac{1}{\sqrt{m}}$  for  $\text{sn}^{-1}(z|m)$ .

### Parity and symmetry

The inverse Jacobi functions  $\text{cd}^{-1}(z|m)$ ,  $\text{cn}^{-1}(z|m)$ ,  $\text{cs}^{-1}(z|m)$ ,  $\text{dc}^{-1}(z|m)$ ,  $\text{dn}^{-1}(z|m)$ ,  $\text{ds}^{-1}(z|m)$ ,  $\text{nc}^{-1}(z|m)$ ,  $\text{nd}^{-1}(z|m)$ ,  $\text{ns}^{-1}(z|m)$ ,  $\text{sc}^{-1}(z|m)$ ,  $\text{sd}^{-1}(z|m)$ , and  $\text{sn}^{-1}(z|m)$  have mirror symmetry:

$$\begin{aligned} \text{cd}^{-1}(\bar{z}|\bar{m}) &= \overline{\text{cd}^{-1}(z|m)} & \text{cn}^{-1}(\bar{z}|\bar{m}) &= \overline{\text{cn}^{-1}(z|m)} & \text{cs}^{-1}(\bar{z}|\bar{m}) &= \overline{\text{cs}^{-1}(z|m)} \\ \text{dc}^{-1}(\bar{z}|\bar{m}) &= \overline{\text{dc}^{-1}(z|m)} & \text{dn}^{-1}(\bar{z}|\bar{m}) &= \overline{\text{dn}^{-1}(z|m)} & \text{ds}^{-1}(\bar{z}|\bar{m}) &= \overline{\text{ds}^{-1}(z|m)} \\ \text{nc}^{-1}(\bar{z}|\bar{m}) &= \overline{\text{nc}^{-1}(z|m)} & \text{nd}^{-1}(\bar{z}|\bar{m}) &= \overline{\text{nd}^{-1}(z|m)} & \text{ns}^{-1}(\bar{z}|\bar{m}) &= \overline{\text{ns}^{-1}(z|m)} \\ \text{sc}^{-1}(\bar{z}|\bar{m}) &= \overline{\text{sc}^{-1}(z|m)} & \text{sd}^{-1}(\bar{z}|\bar{m}) &= \overline{\text{sd}^{-1}(z|m)} & \text{sn}^{-1}(\bar{z}|\bar{m}) &= \overline{\text{sn}^{-1}(z|m)}. \end{aligned}$$

Nine inverse Jacobi functions  $\text{cd}^{-1}(z|m)$ ,  $\text{cn}^{-1}(z|m)$ ,  $\text{cs}^{-1}(z|m)$ ,  $\text{dc}^{-1}(z|m)$ ,  $\text{dn}^{-1}(z|m)$ ,  $\text{ds}^{-1}(z|m)$ ,  $\text{nc}^{-1}(z|m)$ ,  $\text{nd}^{-1}(z|m)$ ,  $\text{ns}^{-1}(z|m)$  have the following quasi-reflection symmetry with respect to  $z$ :

$$\begin{aligned} \text{cd}^{-1}(-z|m) &= 2K(m) - \text{cd}^{-1}(z|m) & \text{cn}^{-1}(-z|m) &= \frac{2}{\sqrt{1-m}} F\left(\sin^{-1}(z) \middle| \frac{m}{m-1}\right) + \text{cn}^{-1}(z|m) \\ \text{cs}^{-1}(-z|m) &= \text{cs}^{-1}(z|m) - \frac{2i}{\sqrt{1-m}} F\left(i \sinh^{-1}(z) \middle| \frac{1}{1-m}\right) & \text{dc}^{-1}(-z|m) &= \frac{2}{\sqrt{m}} F\left(\sin^{-1}(z) \middle| \frac{1}{m}\right) + \text{dc}^{-1}(z|m) \\ \text{dn}^{-1}(-z|m) &= \text{dn}^{-1}(z|m) - \frac{2}{\sqrt{m-1}} F\left(\sin^{-1}(z) \middle| \frac{1}{1-m}\right) & \text{ds}^{-1}(-z|m) &= \frac{2i}{\sqrt{m}} F\left(\sin^{-1}\left(\frac{z}{\sqrt{1-m}}\right) \middle| \frac{m-1}{m}\right) + \text{ds}^{-1}(z|m) \\ \text{nc}^{-1}(-z|m) &= \frac{2i}{\sqrt{m}} F\left(\sin^{-1}(z) \middle| \frac{m-1}{m}\right) + \text{nc}^{-1}(z|m); m < 1 & \text{nd}^{-1}(-z|m) &= 2i F\left(\sin^{-1}(z) \middle| 1-m\right) + \text{nd}^{-1}(z|m) \end{aligned}$$

$$\operatorname{ns}^{-1}(-z|m) = \operatorname{ns}^{-1}(z|m) - \frac{2}{\sqrt{m}} F\left(\sin^{-1}(z) \middle| \frac{1}{m}\right).$$

The other three inverse Jacobi functions  $\operatorname{sc}^{-1}(z|m)$ ,  $\operatorname{sd}^{-1}(z|m)$ , and  $\operatorname{sn}^{-1}(z|m)$  are odd functions with respect to  $z$ :

$$\operatorname{sc}^{-1}(-z|m) = -\operatorname{sc}^{-1}(z|m) \quad \operatorname{sd}^{-1}(-z|m) = -\operatorname{sd}^{-1}(z|m) \quad \operatorname{sn}^{-1}(-z|m) = -\operatorname{sn}^{-1}(z|m).$$

### Series representations

The inverse Jacobi functions  $\operatorname{cd}^{-1}(z|m)$ ,  $\operatorname{cn}^{-1}(z|m)$ ,  $\operatorname{cs}^{-1}(z|m)$ ,  $\operatorname{dc}^{-1}(z|m)$ ,  $\operatorname{dn}^{-1}(z|m)$ ,  $\operatorname{ds}^{-1}(z|m)$ ,  $\operatorname{nc}^{-1}(z|m)$ ,  $\operatorname{nd}^{-1}(z|m)$ ,  $\operatorname{ns}^{-1}(z|m)$ ,  $\operatorname{sc}^{-1}(z|m)$ ,  $\operatorname{sd}^{-1}(z|m)$ , and  $\operatorname{sn}^{-1}(z|m)$  have the following series expansions at the point  $z = 0$ :

$$\operatorname{cd}^{-1}(z|m) = K(m) - z - \frac{m+1}{6} z^3 - \frac{3+2m+3m^2}{40} z^5 - \dots; (z \rightarrow 0)$$

$$\operatorname{cn}^{-1}(z|m) = K(m) - \frac{z}{\sqrt{1-m}} \left( 1 + \frac{2m-1}{6(m-1)} z^2 + \frac{3-8m+8m^2}{40(m-1)^2} z^4 + \dots \right); (z \rightarrow 0)$$

$$\operatorname{cs}^{-1}(z|m) = i \left( \frac{1}{\sqrt{1-m}} K\left(\frac{1}{1-m}\right) - K(1-m) - \frac{z}{\sqrt{m-1}} \left( 1 - \frac{m-2}{6(m-1)} z^2 + \frac{8-8m+3m^2}{40(m-1)^2} z^4 - \dots \right) \right); (z \rightarrow 0)$$

$$\operatorname{dc}^{-1}(z|m) = \frac{1}{\sqrt{m}} \left( K\left(\frac{1}{m}\right) - z - \frac{(1+m)z^3}{6m} - \frac{(3+2m+3m^2)z^5}{40m^2} - \dots \right); (z \rightarrow 0)$$

$$\operatorname{dn}^{-1}(z|m) = \frac{1}{\sqrt{m-1}} \left( K\left(\frac{1}{1-m}\right) - z - \frac{(2-m)z^3}{6(1-m)} - \frac{(8-8m+3m^2)z^5}{40(-1+m)^2} - \dots \right); (z \rightarrow 0)$$

$$\operatorname{ds}^{-1}(z|m) = \sqrt{\frac{1}{m}} K\left(\frac{1}{m}\right) - \frac{z}{\sqrt{m-1}\sqrt{m}} + \frac{(2m-1)z^3}{6(m-1)^{3/2}m^{3/2}} - \dots; (z \rightarrow 0)$$

$$\operatorname{nc}^{-1}(z|m) = i K(1-m) + \frac{\sqrt{1-z^2}}{\sqrt{m}\sqrt{z^2-1}} \left( z + \frac{2m-1}{6m} z^3 + \frac{8m^2-8m+3}{40m^2} z^5 + \dots \right); (z \rightarrow 0)$$

$$\operatorname{nd}^{-1}(z|m) = i K(1-m) + \frac{\sqrt{1-z^2}}{\sqrt{z^2-1}} \left( z + \frac{2-m}{6} z^3 + \frac{8-8m+3m^2}{40} z^5 + \dots \right); (z \rightarrow 0)$$

$$\operatorname{ns}^{-1}(z|m) = \frac{1}{\sqrt{-m}} \left( \left(-\frac{1}{m}\right)^{-1/2} K(m) + i K\left(\frac{1}{m}\right) \right) + \frac{1}{\sqrt{m}} \left( z + \frac{1+m}{6m} z^3 + \frac{3+2m+3m^2}{40m^2} z^5 + \dots \right); (z \rightarrow 0)$$

$$\operatorname{sc}^{-1}(z|m) = z + \frac{m-2}{6} z^3 + \frac{3m^2-8m+8}{40} z^5 + \dots; (z \rightarrow 0)$$

$$\operatorname{sd}^{-1}(z|m) = z + \frac{1-2m}{6} z^3 + \frac{3-8m+8m^2}{40} z^5 - \dots; (z \rightarrow 0)$$

$$\operatorname{sn}^{-1}(z|m) = z + \frac{1+m}{6} z^3 + \frac{3+2m+3m^2}{40} z^5 + \dots; (z \rightarrow 0).$$

The previous expansions are the particular cases of the following series representations of the twelve inverse Jacobi functions near the point  $z = 0$ :

$$\operatorname{cd}^{-1}(z|m) = K(m) - \sum_{k=0}^{\infty} \frac{m^k \left(\frac{1}{2}\right)_k}{(2k+1)k!} {}_2F_1\left(\frac{1}{2}, -k; \frac{1}{2} - k; \frac{1}{m}\right) z^{2k+1}$$

$$\operatorname{cn}^{-1}(z|m) = K(m) - \frac{1}{\sqrt{1-m}} \sum_{k=0}^{\infty} \frac{\left(\frac{m}{m-1}\right)^k \left(\frac{1}{2}\right)_k}{(2k+1)k!} {}_2F_1\left(\frac{1}{2}, -k; \frac{1}{2} - k; \frac{m-1}{m}\right) z^{2k+1}$$

$$\operatorname{cs}^{-1}(z|m) = i \left( \frac{1}{\sqrt{1-m}} K\left(\frac{1}{1-m}\right) - K(1-m) - \sum_{k=0}^{\infty} \frac{(m-1)^{-k-\frac{1}{2}} \left(\frac{1}{2}\right)_k}{(2k+1)k!} {}_2F_1\left(\frac{1}{2}, -k; \frac{1}{2} - k; 1-m\right) z^{2k+1} \right)$$

$$\operatorname{dc}^{-1}(z|m) = \frac{1}{\sqrt{m}} K\left(\frac{1}{m}\right) - \frac{1}{\sqrt{m}} \sum_{k=0}^{\infty} \frac{m^{-k} \left(\frac{1}{2}\right)_k}{(2k+1)k!} {}_2F_1\left(\frac{1}{2}, -k; \frac{1}{2} - k; m\right) z^{2k+1}$$

$$\operatorname{dn}^{-1}(z|m) = \frac{1}{\sqrt{m-1}} \left( K\left(\frac{1}{1-m}\right) - \sum_{k=0}^{\infty} \frac{(1-m)^{-k} \left(\frac{1}{2}\right)_k}{(2k+1)k!} {}_2F_1\left(\frac{1}{2}, -k; \frac{1}{2} - k; 1-m\right) z^{2k+1} \right)$$

$$\operatorname{ds}^{-1}(z|m) = \sqrt{\frac{1}{m}} K\left(\frac{1}{m}\right) - \frac{1}{\sqrt{m-1} \sqrt{m}} \sum_{k=0}^{\infty} \frac{(-1)^k m^{-k} \left(\frac{1}{2}\right)_k}{(2k+1)k!} {}_2F_1\left(\frac{1}{2}, -k; \frac{1}{2} - k; \frac{m}{m-1}\right) z^{2k+1}$$

$$\operatorname{nc}^{-1}(z|m) = i K(1-m) + \frac{\sqrt{1-z^2}}{\sqrt{z^2-1} \sqrt{m}} \sum_{k=0}^{\infty} \frac{\left(\frac{m-1}{m}\right)^k \left(\frac{1}{2}\right)_k}{(2k+1)k!} {}_2F_1\left(\frac{1}{2}, -k; \frac{1}{2} - k; \frac{m}{m-1}\right) z^{2k+1}$$

$$\operatorname{nd}^{-1}(z|m) = i K(1-m) + \frac{\sqrt{1-z^2}}{\sqrt{z^2-1}} \sum_{k=0}^{\infty} \frac{(1-m)^k \left(\frac{1}{2}\right)_k}{(2k+1)k!} {}_2F_1\left(\frac{1}{2}, -k; \frac{1}{2} - k; \frac{1}{1-m}\right) z^{2k+1}$$

$$\operatorname{ns}^{-1}(z|m) = \frac{1}{\sqrt{-m}} \left( \left(-\frac{1}{m}\right)^{-1/2} K(m) + i K\left(\frac{1}{m}\right) + \sum_{k=0}^{\infty} \frac{m^{-k-\frac{1}{2}} \left(\frac{1}{2}\right)_k}{(2k+1)k!} {}_2F_1\left(\frac{1}{2}, -k; \frac{1}{2} - k; m\right) z^{2k+1} \right)$$

$$\operatorname{sc}^{-1}(z|m) = \sum_{k=0}^{\infty} \frac{(m-1)^k \left(\frac{1}{2}\right)_k}{(2k+1)k!} {}_2F_1\left(\frac{1}{2}, -k; \frac{1}{2} - k; \frac{1}{1-m}\right) z^{2k+1}$$

$$\operatorname{sd}^{-1}(z|m) = \sum_{k=0}^{\infty} \frac{(1-m)^k \left(\frac{1}{2}\right)_k}{(2k+1)k!} {}_2F_1\left(\frac{1}{2}, -k; \frac{1}{2} - k; \frac{m}{m-1}\right) z^{2k+1}$$

$$\operatorname{sn}^{-1}(z|m) = \sum_{k=0}^{\infty} \frac{m^k \left(\frac{1}{2}\right)_k}{(2k+1)k!} {}_2F_1\left(\frac{1}{2}, -k; \frac{1}{2} - k; \frac{1}{m}\right) z^{2k+1}.$$

The inverse Jacobi functions  $\operatorname{cd}^{-1}(z|m)$ ,  $\operatorname{cn}^{-1}(z|m)$ ,  $\operatorname{cs}^{-1}(z|m)$ ,  $\operatorname{dc}^{-1}(z|m)$ ,  $\operatorname{dn}^{-1}(z|m)$ ,  $\operatorname{ds}^{-1}(z|m)$ ,  $\operatorname{nc}^{-1}(z|m)$ ,  $\operatorname{nd}^{-1}(z|m)$ ,  $\operatorname{ns}^{-1}(z|m)$ ,  $\operatorname{sc}^{-1}(z|m)$ ,  $\operatorname{sd}^{-1}(z|m)$ , and  $\operatorname{sn}^{-1}(z|m)$  have the following series expansions at the point  $m = 0$ :

$$\operatorname{cd}^{-1}(z|m) = \cos^{-1}(z) + \frac{1}{4} \left( \sqrt{1-z^2} z + \cos^{-1}(z) \right) m + \frac{3}{64} \left( z \sqrt{1-z^2} (2z^2+3) + \cos^{-1}(z) \right) m^2 + \dots /; (m \rightarrow 0)$$

$$\operatorname{cn}^{-1}(z|m) = \cos^{-1}(z) + \frac{1}{4} \left( \cos^{-1}(z) - z \sqrt{1-z^2} \right) m + \frac{3}{64} \left( (2z^2-5) \sqrt{1-z^2} z + 3 \cos^{-1}(z) \right) m^2 + \dots /; (m \rightarrow 0)$$

$$\operatorname{cs}^{-1}(z|m) = \cot^{-1}(z) + \frac{(z^2+1) \cot^{-1}(z) - z}{4(z^2+1)} m + \frac{3(-3z^3-5z+3(z^2+1)^2 \cot^{-1}(z))}{64(z^2+1)^2} m^2 + \dots /; (m \rightarrow 0)$$

$$\operatorname{dc}^{-1}(z|m) = \sec^{-1}(z) + \frac{1}{4z} \left( z \sec^{-1}(z) + \sqrt{1-\frac{1}{z^2}} \right) m + \frac{3}{64z^3} \left( 3 \sec^{-1}(z) z^3 + (3z^2+2) \sqrt{1-\frac{1}{z^2}} \right) m^2 + \dots /; (m \rightarrow 0)$$

$$\operatorname{dn}^{-1}(z|m) = \frac{\sqrt{1-m}}{\sqrt{m-1}} \left( -\frac{1}{2} \left( 1 + \frac{m}{4} + \frac{9m^2}{64} + \dots \right) \log(-m) + \log(4) + \frac{\log(4)-1}{4} m + \frac{3(6\log(4)-7)}{128} m^2 + \dots \right) - \frac{z \sqrt{1-z^2}}{\sqrt{z^2-1}} \left( \frac{\tanh^{-1}(z)}{z} + \frac{1}{4} \left( \frac{\tanh^{-1}(z)}{z} + \frac{1}{1-z^2} \right) m + \frac{3(-3z^3+5z+3(z^2-1)^2 \tanh^{-1}(z))}{64z(z^2-1)^2} m^2 + \dots \right) /; (m \rightarrow 0)$$

$$\operatorname{ds}^{-1}(z|m) =$$

$$\operatorname{csc}^{-1}(z) + \frac{1}{4} \left( \operatorname{csc}^{-1}(z) + \frac{z^2+1}{z(1-z^2)} \sqrt{1-\frac{1}{z^2}} \right) m + \frac{1}{64} \left( 9 \operatorname{csc}^{-1}(z) - \frac{(9z^6-12z^4-11z^2+6)}{z^3(z^2-1)^2} \sqrt{1-\frac{1}{z^2}} \right) m^2 + \dots /; (m \rightarrow 0)$$

$$\operatorname{nc}^{-1}(z|m) = \sec^{-1}(z) + \frac{1}{4z} \left( z \sec^{-1}(z) - \sqrt{1-\frac{1}{z^2}} \right) m + \frac{3}{64z^3} \left( 3 \sec^{-1}(z) z^3 + (2-5z^2) \sqrt{1-\frac{1}{z^2}} \right) m^2 + \dots /; (m \rightarrow 0)$$

$$\operatorname{nd}^{-1}(z|m) = \left( -\frac{i}{2} \left( 1 + \frac{m}{4} + \frac{9m^2}{64} + \dots \right) \log(m) + i \log(4) + \frac{i}{4} (\log(4)-1) m + \frac{3i}{128} (6\log(4)-7) m^2 + \dots \right) + \frac{\sqrt{1-z^2}}{\sqrt{z^2-1}} \left( \tanh^{-1}(z) + \frac{1}{4} \left( \frac{z}{z^2-1} + \tanh^{-1}(z) \right) m + \frac{3}{64} \left( \frac{(5z^2-3)z}{(z^2-1)^2} + 3 \tanh^{-1}(z) \right) m^2 + \dots \right) /; (m \rightarrow 0)$$

$$\operatorname{ns}^{-1}(z|m) = \operatorname{csc}^{-1}(z) - \frac{1}{4z} \left( \sqrt{1-\frac{1}{z^2}} - z \operatorname{csc}^{-1}(z) \right) m - \frac{3}{64z^3} \left( \sqrt{1-\frac{1}{z^2}} (3z^2+2) - 3z^3 \operatorname{csc}^{-1}(z) \right) m^2 - \dots /; (m \rightarrow 0)$$

$$\operatorname{sc}^{-1}(z|m) = \tan^{-1}(z) + \frac{(z^2 + 1) \tan^{-1}(z) - z}{4(z^2 + 1)} m + \frac{3(-5z^3 - 3z + 3(z^2 + 1)^2 \tan^{-1}(z))}{64(z^2 + 1)^2} m^2 + \dots ; (m \rightarrow 0)$$

$$\operatorname{sd}^{-1}(z|m) = \sin^{-1}(z) + \frac{z \sqrt{1 - z^2} (z^2 + 1) + (z^2 - 1) \sin^{-1}(z)}{4(z^2 - 1)} m -$$

$$\frac{z \sqrt{1 - z^2} (6z^6 - 11z^4 - 12z^2 + 9) - 9(z^2 - 1)^2 \sin^{-1}(z)}{64(z^2 - 1)^2} m^2 + \dots ; (m \rightarrow 0)$$

$$\operatorname{sn}^{-1}(z|m) = \sin^{-1}(z) - \frac{1}{4} \left( z \sqrt{1 - z^2} - \sin^{-1}(z) \right) m - \frac{3}{64} \left( z(2z^2 + 3) \sqrt{1 - z^2} - 3 \sin^{-1}(z) \right) m^2 + \dots ; (m \rightarrow 0).$$

The previous expansions are the particular cases of the following series representations of the twelve inverse Jacobi functions near the point  $m = 0$ :

$$\operatorname{cd}^{-1}(z|m) = \sum_{k=0}^{\infty} \left( \frac{\pi \left(\frac{1}{2}\right)_k \left(\frac{1}{2}\right)_k}{2k!^2} - \frac{\left(\frac{1}{2}\right)_k z^{2k+1}}{(2k+1)k!} {}_2F_1\left(\frac{1}{2}, k + \frac{1}{2}; k + \frac{3}{2}; z^2\right) \right) m^k$$

$$\operatorname{cn}^{-1}(z|m) = \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k}{k!} \left( \frac{\sqrt{\pi}}{2k!} \Gamma\left(k + \frac{1}{2}\right) - z {}_2F_1\left(\frac{1}{2}, \frac{1}{2} - k; \frac{3}{2}; z^2\right) \right) m^k$$

$$\operatorname{cs}^{-1}(z|m) = \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k z^{-2k-1}}{(2k+1)k!} {}_2F_1\left(k + \frac{1}{2}, k + 1; k + \frac{3}{2}; -\frac{1}{z^2}\right) m^k$$

$$\operatorname{dc}^{-1}(z|m) = \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k}{k!} \left( \frac{\sqrt{\pi} \Gamma\left(k + \frac{3}{2}\right)}{(2k+1)k!} - \frac{z^{-2k-1}}{2k+1} {}_2F_1\left(\frac{1}{2}, k + \frac{1}{2}; k + \frac{3}{2}; \frac{1}{z^2}\right) \right) m^k$$

$$\operatorname{dn}^{-1}(z|m) = \frac{1}{\sqrt{m-1}} K\left(\frac{1}{1-m}\right) - \frac{z \sqrt{1-z^2}}{\sqrt{z^2-1}} \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k}{k!} {}_2F_1\left(\frac{1}{2}, k + 1; \frac{3}{2}; z^2\right) m^k$$

$$\operatorname{ds}^{-1}(z|m) = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{z^{-2j-2k-1} (-1)^{j+k} \left(\frac{1}{2}\right)_j \left(\frac{1}{2}\right)_k}{(2j+2k+1)j!k!} {}_2F_1\left(j + \frac{1}{2}, j + k + \frac{1}{2}; j + k + \frac{3}{2}; \frac{1}{z^2}\right) m^{j+k}$$

$$\operatorname{nc}^{-1}(z|m) = \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k}{k!} \left( \frac{\sqrt{\pi}}{2k!} \Gamma\left(k + \frac{1}{2}\right) - \frac{1}{z} {}_2F_1\left(\frac{1}{2}, \frac{1}{2} - k; \frac{3}{2}; \frac{1}{z^2}\right) \right) m^k$$

$$\operatorname{nd}^{-1}(z|m) = i K(1-m) + \frac{\sqrt{1-z^2}}{\sqrt{z^2-1}} \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{1}{2}\right)_k z^{2k+1}}{(2k+1)k!} {}_2F_1\left(k + \frac{1}{2}, k + 1; k + \frac{3}{2}; z^2\right) m^k$$

$$\begin{aligned} \operatorname{ns}^{-1}(z|m) &= \sum_{k=0}^{\infty} \frac{z^{-2k-1} \left(\frac{1}{2}\right)_k}{k! (2k+1)} {}_2F_1\left(\frac{1}{2}, k + \frac{1}{2}; k + \frac{3}{2}; \frac{1}{z^2}\right) m^k \\ \operatorname{sc}^{-1}(z|m) &= \sum_{k=0}^{\infty} \frac{z^{2k+1} \left(\frac{1}{2}\right)_k}{k! (2k+1)} {}_2F_1\left(k + \frac{1}{2}, k + 1; k + \frac{3}{2}; -z^2\right) m^k \\ \operatorname{sd}^{-1}(z|m) &= \sum_{j=0}^{\infty} \sum_{k=0}^j \frac{(-1)^j \left(\frac{1}{2}\right)_{j-k} \left(\frac{1}{2}\right)_k}{(2j+1)(j-k)! k!} z^{2j+1} {}_2F_1\left(j + \frac{1}{2}, k + \frac{1}{2}; j + \frac{3}{2}; z^2\right) m^j \\ \operatorname{sn}^{-1}(z|m) &= \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k z^{2k+1}}{(2k+1) k!} {}_2F_1\left(\frac{1}{2}, k + \frac{1}{2}; k + \frac{3}{2}; z^2\right) m^k. \end{aligned}$$

### Integral representations

The inverse Jacobi functions  $\operatorname{cd}^{-1}(z|m)$ ,  $\operatorname{cn}^{-1}(z|m)$ ,  $\operatorname{cs}^{-1}(z|m)$ ,  $\operatorname{dc}^{-1}(z|m)$ ,  $\operatorname{dn}^{-1}(z|m)$ ,  $\operatorname{ds}^{-1}(z|m)$ ,  $\operatorname{nc}^{-1}(z|m)$ ,  $\operatorname{nd}^{-1}(z|m)$ ,  $\operatorname{ns}^{-1}(z|m)$ ,  $\operatorname{sc}^{-1}(z|m)$ ,  $\operatorname{sd}^{-1}(z|m)$ , and  $\operatorname{sn}^{-1}(z|m)$  have the following integral representations, which can be used for their definitions:

$$\begin{aligned} \operatorname{cd}^{-1}(z|m) &= \int_z^1 \frac{1}{\sqrt{1-t^2} \sqrt{1-mt^2}} dt /; -1 < z < 1 \wedge m < 1 \\ \operatorname{cn}^{-1}(z|m) &= \int_z^1 \frac{1}{\sqrt{1-t^2} \sqrt{mt^2-m+1}} dt /; -1 < z < 1 \wedge m(z^2-1) > -1 \\ \operatorname{cs}^{-1}(z|m) &= \int_z^{\infty} \frac{1}{\sqrt{t^2+1} \sqrt{t^2-m+1}} dt /; z \in \mathbb{R} \wedge z^2-m > -1 \\ \operatorname{dc}^{-1}(z|m) &= \int_1^z \frac{1}{\sqrt{t^2-1} \sqrt{t^2-m}} dt /; z \in \mathbb{R} \wedge z^2 > 1 \wedge z^2-m > 0 \wedge m < 1 \\ \operatorname{dn}^{-1}(z|m) &= \int_z^1 \frac{1}{\sqrt{1-t^2} \sqrt{t^2+m-1}} dt /; -1 < z < 1 \wedge z^2+m > 1 \\ \operatorname{ds}^{-1}(z|m) &= \int_z^{\infty} \frac{1}{\sqrt{t^2+m} \sqrt{t^2+m-1}} dt /; z \in \mathbb{R} \wedge z^2+m > 1 \\ \operatorname{nc}^{-1}(z|m) &= \int_1^z \frac{1}{\sqrt{t^2-1} \sqrt{(1-m)t^2+m}} dt /; z \in \mathbb{R} \wedge z^2 > 1 \wedge (1-m)z^2+m > 0 \\ \operatorname{nd}^{-1}(z|m) &= \int_1^z \frac{1}{\sqrt{t^2-1} \sqrt{1-(1-m)t^2}} dt /; z \in \mathbb{R} \wedge z^2 > 1 \wedge (1-m)z^2 < 1 \wedge m > 0 \end{aligned}$$

$$\operatorname{ns}^{-1}(z | m) = \int_z^\infty \frac{1}{\sqrt{t^2 - 1} \sqrt{t^2 - m}} dt /; z \in \mathbb{R} \wedge z^2 > 1 \wedge z^2 > m$$

$$\operatorname{sc}^{-1}(z | m) = \int_0^z \frac{1}{\sqrt{t^2 + 1} \sqrt{(1 - m)t^2 + 1}} dt /; z \in \mathbb{R} \wedge (1 - m)z^2 > -1$$

$$\operatorname{sd}^{-1}(z | m) = \int_0^z \frac{1}{\sqrt{mt^2 + 1} \sqrt{1 - (1 - m)t^2}} dt /; z \in \mathbb{R} \wedge mz^2 > -1 \wedge (1 - m)z^2 < 1$$

$$\operatorname{sn}^{-1}(z | m) = \int_0^z \frac{1}{\sqrt{1 - t^2} \sqrt{1 - mt^2}} dt /; -1 < z < 1 \wedge mz^2 < 1.$$

### Transformations

Some inverse Jacobi functions satisfy additional formulas, for example:

$$\operatorname{cn}^{-1}(z_1 | m) + \operatorname{cn}^{-1}(z_2 | m) = \operatorname{cn}^{-1} \left( \frac{z_1 z_2 - \sqrt{(1 - z_1^2)(m z_1^2 + (1 - m))(1 - z_2^2)(m z_2^2 + (1 - m))}}{1 - m(1 - z_1^2)(1 - z_2^2)} \middle| m \right)$$

$$\operatorname{dn}^{-1}(z_1 | m) + \operatorname{dn}^{-1}(z_2 | m) = \operatorname{dn}^{-1} \left( \frac{m z_1 z_2 + \sqrt{(1 - z_1^2)(z_1^2 + m - 1)(1 - z_2^2)(z_2^2 + m - 1)}}{m - (1 - z_1^2)(1 - z_2^2)} \middle| m \right)$$

$$\operatorname{sn}^{-1}(z_1 | m) + \operatorname{sn}^{-1}(z_2 | m) = \operatorname{sn}^{-1} \left( \frac{\sqrt{(1 - z_2^2)(1 - m z_2^2)} z_1 + \sqrt{(1 - z_1^2)(1 - m z_1^2)} z_2}{1 - m z_1^2 z_2^2} \middle| m \right) /;$$

$$z_1 \in \mathbb{R} \wedge z_2 \in \mathbb{R} \wedge m \in \mathbb{R} \wedge -\frac{1}{m} < z_1 < \frac{1}{m} \wedge -\frac{1}{m} < z_2 < \frac{1}{m}.$$

### Identities

The inverse Jacobi functions  $\operatorname{cd}^{-1}(z | m)$ ,  $\operatorname{cn}^{-1}(z | m)$ ,  $\operatorname{cs}^{-1}(z | m)$ ,  $\operatorname{dc}^{-1}(z | m)$ ,  $\operatorname{dn}^{-1}(z | m)$ ,  $\operatorname{ds}^{-1}(z | m)$ ,  $\operatorname{nc}^{-1}(z | m)$ ,  $\operatorname{nd}^{-1}(z | m)$ ,  $\operatorname{ns}^{-1}(z | m)$ ,  $\operatorname{sc}^{-1}(z | m)$ ,  $\operatorname{sd}^{-1}(z | m)$ , and  $\operatorname{sn}^{-1}(z | m)$  satisfy nonlinear functional equations:

$$((z_2^2 - 1)m z_1^2 - m z_2^2 + 1) \operatorname{cd}(w(z_1) + w(z_2) | m)^2 + 2(m - 1) z_1 z_2 \operatorname{cd}(w(z_1) + w(z_2) | m) + z_2^2 + z_1^2 (1 - m z_2^2) = 1 /; w(z) = \operatorname{cd}^{-1}(z | m)$$

$$((z_2^2 - 1)m z_1^2 - m z_2^2 + m - 1) \operatorname{cn}(w(z_1) + w(z_2) | m)^2 + 2 z_1 z_2 \operatorname{cn}(w(z_1) + w(z_2) | m) + (-m z_2^2 + m - 1) z_1^2 + (m - 1)(z_2^2 - 1) = 0 /; w(z) = \operatorname{cn}^{-1}(z | m)$$

$$(z_1^2 - z_2^2)^2 \operatorname{cs}(w(z_1) + w(z_2) | m)^4 + 2(-z_2^2 z_1^4 + (-z_2^4 + 2(m - 2) z_2^2 + m - 1) z_1^2 + (m - 1) z_2^2) \operatorname{cs}(w(z_1) + w(z_2) | m)^2 + (z_1^2 z_2^2 + m - 1)^2 = 0 /; w(z) = \operatorname{cs}^{-1}(z | m)$$

$$((z_2^2 - 1) z_1^2 - z_2^2 + m) \operatorname{dc}(w(z_1) + w(z_2) | m)^2 - 2(m - 1) z_1 z_2 \operatorname{dc}(w(z_1) + w(z_2) | m) + (z_2^2 - 1)m + z_1^2 (m - z_2^2) = 0 /; w(z) = \operatorname{dc}^{-1}(z | m)$$

$$(z_2^2 + m + (z_2^2 - 1)(-z_1^2) - 1) \operatorname{dn}(w(z_1) + w(z_2) | m)^2 - 2m z_1 z_2 \operatorname{dn}(w(z_1) + w(z_2) | m) + (z_2^2 + m - 1) z_1^2 + (m - 1)(z_2^2 - 1) = 0 /; w(z) = \operatorname{dn}^{-1}(z | m)$$

$$(z_1^2 - z_2^2)^2 \operatorname{ds}(w(z_1) + w(z_2) | m)^4 - 2(z_2^2 z_1^4 + (z_1^4 + (4m - 2)z_2^2 + (m - 1)m)z_1^2 + (m - 1)mz_2^2) \operatorname{ds}(w(z_1) + w(z_2) | m)^2 + ((m - 1)m - z_1^2 z_2^2)^2 = 0 ; w(z) = \operatorname{ds}^{-1}(z | m)$$

$$((m - 1)(z_2^2 - 1)z_1^2 - (m - 1)z_2^2 + m) \operatorname{nc}(w(z_1) + w(z_2) | m)^2 - 2z_1 z_2 \operatorname{nc}(w(z_1) + w(z_2) | m) + (z_2^2 - 1)m + z_1^2(m - (m - 1)z_2^2) = 0 ; w(z) = \operatorname{nc}^{-1}(z | m)$$

$$((m - 1)(z_2^2 - 1)z_1^2 - (m - 1)z_2^2 - 1) \operatorname{nd}(w(z_1) + w(z_2) | m)^2 + 2m z_1 z_2 \operatorname{nd}(w(z_1) + w(z_2) | m) - z_2^2 - z_1^2((m - 1)z_2^2 + 1) + 1 = 0 ; w(z) = \operatorname{nd}^{-1}(z | m)$$

$$(z_1^2 - z_2^2)^2 \operatorname{ns}(w(z_1) + w(z_2) | m)^4 + (-2z_2^2 z_1^4 + (-2z_2^4 + 4(m + 1)z_2^2 - 2m)z_1^2 - 2mz_2^2) \operatorname{ns}(w(z_1) + w(z_2) | m)^2 + (m - z_1^2 z_2^2)^2 = 0 ; w(z) = \operatorname{ns}^{-1}(z | m)$$

$$((m - 1)z_1^2 z_2^2 + 1)^2 \operatorname{sc}(w(z_1) + w(z_2) | m)^4 + 2(z_1^2((m - 1)z_1^4 + ((m - 1)z_1^2 + 2(m - 2)z_2^2 - 1) - z_2^2) \operatorname{sc}(w(z_1) + w(z_2) | m)^2 + (z_1^2 - z_2^2)^2) = 0 ; w(z) = \operatorname{sc}^{-1}(z | m)$$

$$((m - 1)m z_1^2 z_2^2 - 1)^2 \operatorname{sd}(w(z_1) + w(z_2) | m)^4 - 2(((m - 1)(z_1^2 + z_2^2)m + 4m - 2)z_2^2 + 1)z_1^2 + z_2^2) \operatorname{sd}(w(z_1) + w(z_2) | m)^2 + (z_1^2 - z_2^2)^2 = 0 ; w(z) = \operatorname{sd}^{-1}(z | m)$$

$$(m z_1^2 z_2^2 - 1)^2 \operatorname{sn}(w(z_1) + w(z_2) | m)^4 - 2(((m(z_1^2 + z_2^2) - 2(m + 1)z_2^2 + 1)z_1^2 + z_2^2) \operatorname{sn}(w(z_1) + w(z_2) | m)^2 + (z_1^2 - z_2^2)^2) = 0 ; w(z) = \operatorname{sn}^{-1}(z | m).$$

### Representations of derivatives

The derivatives of the inverse Jacobi functions  $\operatorname{cd}^{-1}(z | m)$ ,  $\operatorname{cn}^{-1}(z | m)$ ,  $\operatorname{cs}^{-1}(z | m)$ ,  $\operatorname{dc}^{-1}(z | m)$ ,  $\operatorname{dn}^{-1}(z | m)$ ,  $\operatorname{ds}^{-1}(z | m)$ ,  $\operatorname{nc}^{-1}(z | m)$ ,  $\operatorname{nd}^{-1}(z | m)$ ,  $\operatorname{ns}^{-1}(z | m)$ ,  $\operatorname{sc}^{-1}(z | m)$ ,  $\operatorname{sd}^{-1}(z | m)$ , and  $\operatorname{sn}^{-1}(z | m)$  with respect to variable  $z$  can be expressed through direct and inverse Jacobi functions:

$$\frac{\partial \operatorname{cd}^{-1}(z | m)}{\partial z} = \frac{1}{(m - 1) \operatorname{nd}(\operatorname{cd}^{-1}(z | m) | m) \operatorname{sd}(\operatorname{cd}^{-1}(z | m) | m)}$$

$$\frac{\partial \operatorname{cn}^{-1}(z | m)}{\partial z} = -\frac{1}{\operatorname{dn}(\operatorname{cn}^{-1}(z | m) | m) \operatorname{sn}(\operatorname{cn}^{-1}(z | m) | m)}$$

$$\frac{\partial \operatorname{cs}^{-1}(z | m)}{\partial z} = -\frac{1}{\operatorname{ds}(\operatorname{cs}^{-1}(z | m) | m) \operatorname{ns}(\operatorname{cs}^{-1}(z | m) | m)}$$

$$\frac{\partial \operatorname{dc}^{-1}(z | m)}{\partial z} = \frac{1}{(1 - m) \operatorname{nc}(\operatorname{dc}^{-1}(z | m) | m) \operatorname{sc}(\operatorname{dc}^{-1}(z | m) | m)}$$

$$\frac{\partial \operatorname{dn}^{-1}(z | m)}{\partial z} = -\frac{1}{m \operatorname{cn}(\operatorname{dn}^{-1}(z | m) | m) \operatorname{sn}(\operatorname{dn}^{-1}(z | m) | m)}$$

$$\frac{\partial \operatorname{ds}^{-1}(z | m)}{\partial z} = -\frac{1}{\operatorname{cs}(\operatorname{ds}^{-1}(z | m) | m) \operatorname{ns}(\operatorname{ds}^{-1}(z | m) | m)}$$



$$\frac{\partial \operatorname{nc}^{-1}(z|m)}{\partial z} = \frac{1}{\operatorname{dc}(\operatorname{nc}^{-1}(z|m)|m) \operatorname{sc}(\operatorname{nc}^{-1}(z|m)|m)}$$

$$\frac{\partial \operatorname{nd}^{-1}(z|m)}{\partial z} = \frac{1}{m \operatorname{cd}(\operatorname{nd}^{-1}(z|m)|m) \operatorname{sd}(\operatorname{nd}^{-1}(z|m)|m)}$$

$$\frac{\partial \operatorname{ns}^{-1}(z|m)}{\partial z} = -\frac{1}{\operatorname{cs}(\operatorname{ns}^{-1}(z|m)|m) \operatorname{ds}(\operatorname{ns}^{-1}(z|m)|m)}$$

$$\frac{\partial \operatorname{sc}^{-1}(z|m)}{\partial z} = \frac{1}{\operatorname{dc}(\operatorname{sc}^{-1}(z|m)|m) \operatorname{nc}(\operatorname{sc}^{-1}(z|m)|m)}$$

$$\frac{\partial \operatorname{sd}^{-1}(z|m)}{\partial z} = \frac{1}{\operatorname{cd}(\operatorname{sd}^{-1}(z|m)|m) \operatorname{nd}(\operatorname{sd}^{-1}(z|m)|m)}$$

$$\frac{\partial \operatorname{sn}^{-1}(z|m)}{\partial z} = \frac{1}{\operatorname{cn}(\operatorname{sn}^{-1}(z|m)|m) \operatorname{dn}(\operatorname{sn}^{-1}(z|m)|m)}$$

The previous formulas can be generalized to the following symbolic derivatives of the  $n^{\text{th}}$  order with respect to variable  $z$ :

$$\frac{\partial^n \operatorname{cd}^{-1}(z|m)}{\partial z^n} = \delta_n \operatorname{cd}^{-1}(z|m) - \sum_{j=0}^{n-1} \frac{(1-n)_{2(n-j)-2}}{(n-j-1)! (2z)^{n-2j-1}} \sum_{k=0}^j \binom{j}{k} \left(\frac{1}{2}\right)_k \left(\frac{1}{2}\right)_{j-k} m^{j-k} (1-z^2)^{-k-\frac{1}{2}} (1-mz^2)^{k-j-\frac{1}{2}} ; n \in \mathbb{N}$$

$$\frac{\partial^n \operatorname{cn}^{-1}(z|m)}{\partial z^n} = \delta_n \operatorname{cn}^{-1}(z|m) - \sum_{j=0}^{n-1} \frac{(1-n)_{2(n-j)-2}}{(n-j-1)! (2z)^{n-2j-1}} \sum_{k=0}^j (-1)^{j+k} \binom{j}{k} \left(\frac{1}{2}\right)_k \left(\frac{1}{2}\right)_{j-k} m^{j-k} (1-z^2)^{-k-\frac{1}{2}} (mz^2 - m + 1)^{k-j-\frac{1}{2}} ; n \in \mathbb{N}$$

$$\frac{\partial^n \operatorname{cs}^{-1}(z|m)}{\partial z^n} = \delta_n \operatorname{cs}^{-1}(z|m) - \sum_{j=0}^{n-1} \frac{(-1)^j (1-n)_{2(n-j)-2}}{(n-j-1)! (2z)^{-2j+n-1}} \sum_{k=0}^j \binom{j}{k} \left(\frac{1}{2}\right)_k \left(\frac{1}{2}\right)_{j-k} (z^2 + 1)^{-k-\frac{1}{2}} (z^2 - m + 1)^{k-j-\frac{1}{2}} ; n \in \mathbb{N}$$

$$\frac{\partial^n \operatorname{dc}^{-1}(z|m)}{\partial z^n} = \delta_n \operatorname{dc}^{-1}(z|m) - \sum_{j=0}^{n-1} \frac{(-1)^{j-1} (1-n)_{2(n-j)-2}}{(n-j-1)! (2z)^{n-2j-1}} \sum_{k=0}^j \binom{j}{k} \left(\frac{1}{2}\right)_k \left(\frac{1}{2}\right)_{j-k} (z^2 - 1)^{-k-\frac{1}{2}} (z^2 - m)^{k-j-\frac{1}{2}} ; n \in \mathbb{N}$$

$$\frac{\partial^n \operatorname{dn}^{-1}(z|m)}{\partial z^n} = \delta_n \operatorname{dn}^{-1}(z|m) - \sum_{j=0}^{n-1} \frac{(1-n)_{2(n-j)-2}}{(n-j-1)! (2z)^{n-2j-1}} \sum_{k=0}^j (-1)^{j+k} \binom{j}{k} \left(\frac{1}{2}\right)_k \left(\frac{1}{2}\right)_{j-k} (1-z^2)^{-k-\frac{1}{2}} (z^2 + m - 1)^{k-j-\frac{1}{2}} ; n \in \mathbb{N}$$

$$\frac{\partial^n \operatorname{ds}^{-1}(z|m)}{\partial z^n} = \delta_n \operatorname{ds}^{-1}(z|m) + \sum_{j=0}^{n-1} \frac{(1-n)_{2(n-j)-2}}{(n-j-1)! (2z)^{n-2j-1}} \sum_{k=0}^j (-1)^j \binom{j}{k} \left(\frac{1}{2}\right)_k \left(\frac{1}{2}\right)_{j-k} (z^2 + m - 1)^{-k-\frac{1}{2}} (z^2 + m)^{k-j-\frac{1}{2}} ; n \in \mathbb{N}$$

$$\frac{\partial^n \text{nc}^{-1}(z | m)}{\partial z^n} = \delta_n \text{nc}^{-1}(z | m) - \sum_{j=0}^{n-1} \frac{(1-n)_{2(n-j)-2}}{(n-j-1)! (2z)^{n-2j-1}} \sum_{k=0}^j (-1)^k \binom{j}{k} \left(\frac{1}{2}\right)_k \left(\frac{1}{2}\right)_{j-k} (m-1)^{j-k} (z^2-1)^{-k-\frac{1}{2}} ((1-m)z^2+m)^{k-j-\frac{1}{2}} ; n \in \mathbb{N}$$

$$\frac{\partial^n \text{nd}^{-1}(z | m)}{\partial z^n} = \delta_n \text{nd}^{-1}(z | m) - \sum_{j=0}^{n-1} \frac{(1-n)_{2(n-j)-2}}{(n-j-1)! (2z)^{n-2j-1}} \sum_{k=0}^j (-1)^k \binom{j}{k} \left(\frac{1}{2}\right)_k \left(\frac{1}{2}\right)_{j-k} (1-m)^{j-k} (z^2-1)^{-k-\frac{1}{2}} (1-(1-m)z^2)^{k-j-\frac{1}{2}} ; n \in \mathbb{N}$$

$$\frac{\partial^n \text{ns}^{-1}(z | m)}{\partial z^n} = \text{ns}^{-1}(z | m) \delta_n + \sum_{j=0}^{n-1} \frac{(-1)^{j-1} (1-n)_{2(n-j)-2}}{(n-j-1)! (2z)^{-2j+n-1}} \sum_{k=0}^j \binom{j}{k} \left(\frac{1}{2}\right)_k \left(\frac{1}{2}\right)_{j-k} (z^2-1)^{-k-\frac{1}{2}} (z^2-m)^{k-j-\frac{1}{2}} ; n \in \mathbb{N}$$

$$\frac{\partial^n \text{sc}^{-1}(z | m)}{\partial z^n} = \delta_n \text{sc}^{-1}(z | m) - \sum_{j=0}^{n-1} \frac{(-1)^{j-1} (1-n)_{2(n-j)-2}}{(n-j-1)! (2z)^{n-2j-1}} \sum_{k=0}^j \binom{j}{k} \left(\frac{1}{2}\right)_k \left(\frac{1}{2}\right)_{j-k} (1-m)^{j-k} (z^2+1)^{-k-\frac{1}{2}} ((1-m)z^2+1)^{k-j-\frac{1}{2}} ; n \in \mathbb{N}$$

$$\frac{\partial^n \text{sd}^{-1}(z | m)}{\partial z^n} = \delta_n \text{sd}^{-1}(z | m) + \sum_{j=0}^{n-1} \frac{(-1)^j m^j (1-n)_{2(n-j)-2}}{(n-j-1)! (2z)^{n-2j-1}} \sum_{k=0}^j \binom{j}{k} \left(\frac{1}{2}\right)_k \left(\frac{1}{2}\right)_{j-k} \left(\frac{m-1}{m}\right)^{j-k} (mz^2+1)^{-k-\frac{1}{2}} (1-(1-m)z^2)^{k-j-\frac{1}{2}} ; n \in \mathbb{N}$$

$$\frac{\partial^n \text{sn}^{-1}(z | m)}{\partial z^n} = \text{sn}^{-1}(z | m) \delta_n + \sum_{j=0}^{n-1} \frac{(1-n)_{2(n-j)-2}}{(n-j-1)! (2z)^{-2j+n-1}} \sum_{k=0}^j \binom{j}{k} \left(\frac{1}{2}\right)_k \left(\frac{1}{2}\right)_{j-k} m^{j-k} (1-z^2)^{-k-\frac{1}{2}} (1-mz^2)^{k-j-\frac{1}{2}} ; n \in \mathbb{N}$$

The derivatives of the inverse Jacobi functions  $\text{cd}^{-1}(z | m)$ ,  $\text{cn}^{-1}(z | m)$ ,  $\text{cs}^{-1}(z | m)$ ,  $\text{dc}^{-1}(z | m)$ ,  $\text{dn}^{-1}(z | m)$ ,  $\text{ds}^{-1}(z | m)$ ,  $\text{nc}^{-1}(z | m)$ ,  $\text{nd}^{-1}(z | m)$ ,  $\text{ns}^{-1}(z | m)$ ,  $\text{sc}^{-1}(z | m)$ ,  $\text{sd}^{-1}(z | m)$ , and  $\text{sn}^{-1}(z | m)$  with respect to variable  $m$  have more complicated representations that include direct and inverse Jacobi functions and the elliptic integral  $E(\text{am}(z | m) | m)$ :

$$\frac{\partial \text{cd}^{-1}(z | m)}{\partial m} = \frac{E(\text{am}(\text{cd}^{-1}(z | m) | m) | m) + (m-1) \text{cd}^{-1}(z | m)}{2(1-m)m}$$

$$\frac{\partial \text{cn}^{-1}(z | m)}{\partial m} = \frac{E(\text{am}(\text{cn}^{-1}(z | m) | m) | m) + (m-1) \text{cn}^{-1}(z | m) - m \text{cd}(\text{cn}^{-1}(z | m) | m) \text{sn}(\text{cn}^{-1}(z | m) | m)}{2(1-m)m}$$

$$\frac{\partial \text{cs}^{-1}(z | m)}{\partial m} = \frac{-E(\text{am}(\text{cs}^{-1}(z | m) | m) | m) + (1-m) \text{cs}^{-1}(z | m) + m \text{cd}(\text{cs}^{-1}(z | m) | m) \text{sn}(\text{cs}^{-1}(z | m) | m)}{2(m-1)m}$$

$$\frac{\partial \text{dc}^{-1}(z | m)}{\partial m} = \frac{E(\text{am}(\text{dc}^{-1}(z | m) | m) | m) - (1-m) \text{dc}^{-1}(z | m)}{2(1-m)m}$$

$$\frac{\partial \operatorname{dn}^{-1}(z|m)}{\partial m} = \frac{E(\operatorname{am}(\operatorname{dn}^{-1}(z|m)|m)|m) + (m-1)\operatorname{dn}^{-1}(z|m) - z \operatorname{sc}(\operatorname{dn}^{-1}(z|m)|m)}{2(1-m)m}$$

$$\frac{\partial \operatorname{ds}^{-1}(z|m)}{\partial m} = \frac{1}{2(m-1)m} (-E(\operatorname{am}(\operatorname{ds}^{-1}(z|m)|m)|m) + (1-m)\operatorname{ds}^{-1}(z|m) + m \operatorname{dn}(\operatorname{ds}^{-1}(z|m)|m) \operatorname{sc}(\operatorname{ds}^{-1}(z|m)|m))$$

$$\frac{\partial \operatorname{nc}^{-1}(z|m)}{\partial m} = \frac{1}{2(m-1)m} (-E(\operatorname{am}(\operatorname{nc}^{-1}(z|m)|m)|m) + (1-m)\operatorname{nc}^{-1}(z|m) + m \operatorname{cd}(\operatorname{nc}^{-1}(z|m)|m) \operatorname{sn}(\operatorname{nc}^{-1}(z|m)|m))$$

$$\frac{\partial \operatorname{nd}^{-1}(z|m)}{\partial m} = \frac{1}{2(m-1)m} \left( \frac{\operatorname{sc}(\operatorname{nd}^{-1}(z|m)|m)}{z} - E(\operatorname{am}(\operatorname{nd}^{-1}(z|m)|m)|m) + (1-m)\operatorname{nd}^{-1}(z|m) \right)$$

$$\frac{\partial \operatorname{ns}^{-1}(z|m)}{\partial m} = \frac{1}{2(m-1)m} \left( \frac{m \operatorname{cd}(\operatorname{ns}^{-1}(z|m)|m)}{z} - E(\operatorname{am}(\operatorname{ns}^{-1}(z|m)|m)|m) + (1-m)\operatorname{ns}^{-1}(z|m) \right)$$

$$\frac{\partial \operatorname{sc}^{-1}(z|m)}{\partial m} = \frac{1}{2(m-1)m} (-E(\operatorname{am}(\operatorname{sc}^{-1}(z|m)|m)|m) + (1-m)\operatorname{sc}^{-1}(z|m) + m \operatorname{cd}(\operatorname{sc}^{-1}(z|m)|m) \operatorname{sn}(\operatorname{sc}^{-1}(z|m)|m))$$

$$\frac{\partial \operatorname{sd}^{-1}(z|m)}{\partial m} = \frac{1}{2(m-1)m} (-E(\operatorname{am}(\operatorname{sd}^{-1}(z|m)|m)|m) + (1-m)\operatorname{sd}^{-1}(z|m) + m \operatorname{dn}(\operatorname{sd}^{-1}(z|m)|m) \operatorname{sc}(\operatorname{sd}^{-1}(z|m)|m))$$

$$\frac{\partial \operatorname{sn}^{-1}(z|m)}{\partial m} = \frac{E(\sin^{-1}(z|m)) - (1-m)F(\sin^{-1}(z|m)) - m z \operatorname{cd}(F(\sin^{-1}(z|m)|m))}{2(1-m)m}.$$

The previous formulas can be generalized to the following symbolic derivatives of the  $n^{\text{th}}$  order with respect to variable  $z$ :

$$\frac{\partial^n \operatorname{cd}^{-1}(z|m)}{\partial m^n} = \frac{\pi m^{-n}}{2} {}_2\tilde{F}_1\left(\frac{1}{2}, \frac{1}{2}; 1-n; m\right) - \frac{(-1)^n \sqrt{\pi} z^{2n+1}}{(2n+1)\Gamma(\frac{1}{2}-n)} F_1\left(n+\frac{1}{2}; \frac{1}{2}, n+\frac{1}{2}; n+\frac{3}{2}; z^2, m z^2\right); n \in \mathbb{N}$$

$$\frac{\partial^n \operatorname{cn}^{-1}(z|m)}{\partial m^n} = \frac{(-1)^n \sqrt{\pi} (1-z^2)^{n+\frac{1}{2}}}{(2n+1)\Gamma(\frac{1}{2}-n)} F_1\left(n+\frac{1}{2}; \frac{1}{2}, n+\frac{1}{2}; n+\frac{3}{2}; 1-z^2, m(1-z^2)\right); n \in \mathbb{N}$$

$$\frac{\partial^n \operatorname{cs}^{-1}(z|m)}{\partial m^n} = \frac{(-1)^n \sqrt{\pi} z^{-2n-1}}{(2n+1)\Gamma(\frac{1}{2}-n)} F_1\left(n+\frac{1}{2}; \frac{1}{2}, n+\frac{1}{2}; n+\frac{3}{2}; -\frac{1}{z^2}, -\frac{1-m}{z^2}\right); n \in \mathbb{N}$$

$$\frac{\partial^n \operatorname{dc}^{-1}(z|m)}{\partial m^n} = \frac{(-1)^{n-1} \sqrt{\pi} z^{-2n-1}}{(2n+1)\Gamma(\frac{1}{2}-n)} F_1\left(n+\frac{1}{2}; \frac{1}{2}, n+\frac{1}{2}; n+\frac{3}{2}; \frac{1}{z^2}, \frac{m}{z^2}\right) + \frac{\pi m^{-n}}{2} {}_2\tilde{F}_1\left(\frac{1}{2}, \frac{1}{2}; 1-n; m\right); n \in \mathbb{N}$$

$$\frac{\partial^n \operatorname{dn}^{-1}(z|m)}{\partial m^n} = \frac{\sqrt{\pi} (m-1)^{-n-\frac{1}{2}}}{2\Gamma(\frac{1}{2}-n)} \left( \pi {}_2F_1\left(\frac{1}{2}, n+\frac{1}{2}; 1; \frac{1}{1-m}\right) - 2z F_1\left(\frac{1}{2}; \frac{1}{2}, n+\frac{1}{2}; \frac{3}{2}; z^2, \frac{z^2}{1-m}\right) \right); n \in \mathbb{N}$$

$$\frac{\partial^n \text{ds}^{-1}(z | m)}{\partial m^n} = \frac{z^{-2n-1}}{2n+1} \sum_{k=0}^n \binom{n}{k} \left(\frac{1}{2} - k\right)_k \left(k - n + \frac{1}{2}\right)_{n-k} F_1\left(n + \frac{1}{2}; -k + n + \frac{1}{2}, k + \frac{1}{2}; n + \frac{3}{2}; \frac{1-m}{z^2}, -\frac{m}{z^2}\right);$$

$$|z| > 1 \wedge |m| > 1 \wedge n \in \mathbb{N}$$

$$\frac{\partial^n \text{nc}^{-1}(z | m)}{\partial m^n} = \frac{(-1)^n \sqrt{\pi}}{(2n+1)\Gamma\left(\frac{1}{2} - n\right)} \left(1 - \frac{1}{z^2}\right)^{n+\frac{1}{2}} F_1\left(n + \frac{1}{2}; \frac{1}{2}, n + \frac{1}{2}; n + \frac{3}{2}; 1 - \frac{1}{z^2}, m\left(1 - \frac{1}{z^2}\right)\right); n \in \mathbb{N}$$

$$\frac{\partial^n \text{nd}^{-1}(z | m)}{\partial m^n} = \frac{\pi i}{2} (m-1)^{-n} {}_2\tilde{F}_1\left(\frac{1}{2}, \frac{1}{2}; 1-n; 1-m\right) + \frac{i\sqrt{\pi} z^{2n+1}}{(2n+1)\Gamma\left(\frac{1}{2} - n\right)} F_1\left(n + \frac{1}{2}; \frac{1}{2}, n + \frac{1}{2}; n + \frac{3}{2}; z^2, (1-m)z^2\right); n \in \mathbb{N}$$

$$\frac{\partial^n \text{ns}^{-1}(z | m)}{\partial m^n} = \frac{(-1)^n \sqrt{\pi} z^{-2n-1}}{(2n+1)\Gamma\left(\frac{1}{2} - n\right)} F_1\left(n + \frac{1}{2}; \frac{1}{2}, n + \frac{1}{2}; n + \frac{3}{2}; \frac{1}{z^2}, \frac{m}{z^2}\right); |z| > 1 \wedge n \in \mathbb{N}$$

$$\frac{\partial^n \text{sc}^{-1}(z | m)}{\partial m^n} = \frac{(-1)^n \sqrt{\pi} z^{2n+1}}{(2n+1)\Gamma\left(\frac{1}{2} - n\right)} F_1\left(n + \frac{1}{2}; \frac{1}{2}, n + \frac{1}{2}; n + \frac{3}{2}; -z^2, (m-1)z^2\right); n \in \mathbb{N}$$

$$\frac{\partial^n \text{sd}^{-1}(z | m)}{\partial m^n} = \frac{z^{2n+1}}{2n+1} \sum_{k=0}^n \binom{n}{k} \left(\frac{1}{2} - k\right)_k \left(k - n + \frac{1}{2}\right)_{n-k} F_1\left(n + \frac{1}{2}; \frac{1}{2} - k + n, k + \frac{1}{2}; n + \frac{3}{2}; (1-m)z^2, -mz^2\right); n \in \mathbb{N}$$

$$\frac{\partial^n \text{sn}^{-1}(z | m)}{\partial m^n} = \frac{(-1)^n \sqrt{\pi} z^{2n+1}}{(2n+1)\Gamma\left(\frac{1}{2} - n\right)} F_1\left(n + \frac{1}{2}; \frac{1}{2}, n + \frac{1}{2}; n + \frac{3}{2}; z^2, mz^2\right); n \in \mathbb{N}$$

### Integration

The indefinite integrals of the twelve inverse Jacobi functions  $\text{cd}^{-1}(z | m)$ ,  $\text{cn}^{-1}(z | m)$ ,  $\text{cs}^{-1}(z | m)$ ,  $\text{dc}^{-1}(z | m)$ ,  $\text{dn}^{-1}(z | m)$ ,  $\text{ds}^{-1}(z | m)$ ,  $\text{nc}^{-1}(z | m)$ ,  $\text{nd}^{-1}(z | m)$ ,  $\text{ns}^{-1}(z | m)$ ,  $\text{sc}^{-1}(z | m)$ ,  $\text{sd}^{-1}(z | m)$ , and  $\text{sn}^{-1}(z | m)$  with respect to variable  $z$  can be expressed through direct and inverse Jacobi and elementary functions by the following formulas:

$$\int \text{cd}^{-1}(z | m) dz = z \text{cd}^{-1}(z | m) + \frac{\log(\sqrt{m} \text{sd}(\text{cd}^{-1}(z | m) | m) - \text{nd}(\text{cd}^{-1}(z | m) | m))}{\sqrt{m}}$$

$$\int \text{cn}^{-1}(z | m) dz = z \text{cn}^{-1}(z | m) + \frac{i}{\sqrt{m}} \log\left(\frac{i \text{dn}(\text{cn}^{-1}(z | m) | m)}{\sqrt{m}} - \text{sn}(\text{cn}^{-1}(z | m) | m)\right)$$

$$\int \text{cs}^{-1}(z | m) dz = z \text{cs}^{-1}(z | m) + \log(\text{ds}(\text{cs}^{-1}(z | m) | m) + \text{ns}(\text{cs}^{-1}(z | m) | m))$$

$$\int \text{dc}^{-1}(z | m) dz = \text{dc}^{-1}(z | m) z - \log(\text{nc}(\text{dc}^{-1}(z | m) | m) + \text{sc}(\text{dc}^{-1}(z | m) | m))$$

$$\int \text{dn}^{-1}(z | m) dz = \text{dn}^{-1}(z | m) z - i \log(i \text{cn}(\text{dn}^{-1}(z | m) | m) + \text{sn}(\text{dn}^{-1}(z | m) | m))$$

$$\int \text{ds}^{-1}(z | m) dz = z \text{ds}^{-1}(z | m) + \log(\text{cs}(\text{ds}^{-1}(z | m) | m) + \text{ns}(\text{ds}^{-1}(z | m) | m))$$

$$\int \operatorname{nc}^{-1}(z|m) dz = \operatorname{nc}^{-1}(z|m)z - \frac{1}{\sqrt{1-m}} \log\left(\frac{\operatorname{dc}(\operatorname{nc}^{-1}(z|m)|m)}{\sqrt{1-m}} + \operatorname{sc}(\operatorname{nc}^{-1}(z|m)|m)\right)$$

$$\int \operatorname{nd}^{-1}(z|m) dz = \operatorname{nd}^{-1}(z|m)z - \frac{1}{\sqrt{m-1}} \log\left(\frac{\operatorname{cd}(\operatorname{nd}^{-1}(z|m)|m)}{\sqrt{m-1}} + \operatorname{sd}(\operatorname{nd}^{-1}(z|m)|m)\right)$$

$$\int \operatorname{ns}^{-1}(z|m) dz = z \operatorname{ns}^{-1}(z|m) + \log(\operatorname{cs}(\operatorname{ns}^{-1}(z|m)|m) + \operatorname{ds}(\operatorname{ns}^{-1}(z|m)|m))$$

$$\int \operatorname{sc}^{-1}(z|m) dz = \operatorname{sc}^{-1}(z|m)z - \frac{1}{\sqrt{1-m}} \log\left(\frac{\operatorname{dc}(\operatorname{sc}^{-1}(z|m)|m)}{\sqrt{1-m}} + \operatorname{nc}(\operatorname{sc}^{-1}(z|m)|m)\right)$$

$$\int \operatorname{sd}^{-1}(z|m) dz = \operatorname{sd}^{-1}(z|m)z - \frac{1}{\sqrt{m-1}\sqrt{m}} \log\left(\frac{\operatorname{cd}(\operatorname{sd}^{-1}(z|m)|m)}{\sqrt{m-1}} + \frac{\operatorname{nd}(\operatorname{sd}^{-1}(z|m)|m)}{\sqrt{m}}\right)$$

$$\int \operatorname{sn}^{-1}(z|m) dz = \operatorname{sn}^{-1}(z|m)z - \frac{\log(\operatorname{dn}(\operatorname{sn}^{-1}(z|m)|m) - \sqrt{m} \operatorname{cn}(\operatorname{sn}^{-1}(z|m)|m))}{\sqrt{m}}.$$

The indefinite integrals of the twelve inverse Jacobi functions  $\operatorname{cd}^{-1}(z|m)$ ,  $\operatorname{cn}^{-1}(z|m)$ ,  $\operatorname{cs}^{-1}(z|m)$ ,  $\operatorname{dc}^{-1}(z|m)$ ,  $\operatorname{dn}^{-1}(z|m)$ ,  $\operatorname{ds}^{-1}(z|m)$ ,  $\operatorname{nc}^{-1}(z|m)$ ,  $\operatorname{nd}^{-1}(z|m)$ ,  $\operatorname{ns}^{-1}(z|m)$ ,  $\operatorname{sc}^{-1}(z|m)$ ,  $\operatorname{sd}^{-1}(z|m)$ , and  $\operatorname{sn}^{-1}(z|m)$  with respect to variable  $m$  can be expressed through direct and inverse Jacobi and elementary functions by the following formulas:

$$\int \operatorname{cd}^{-1}(z|m) dm = 2\left(E(m) - \frac{1}{z}\left(zE(\sin^{-1}(z)|m) + (m-1)zF(\sin^{-1}(z)|m) + \sqrt{1-z^2}\sqrt{1-mz^2}\right) + (m-1)K(m)\right);$$

$-1 < z < 1 \wedge m < 1$

$$\int \operatorname{cn}^{-1}(z|m) dm = 2\left(\frac{z\sqrt{m(z^2-1)+1} - z}{\sqrt{1-z^2}} + i\sqrt{m}\left(E\left(i\sinh^{-1}\left(\frac{\sqrt{m}}{\sqrt{1-m}}\right)\middle|\frac{m-1}{m}\right) - E\left(i\sinh^{-1}\left(\frac{\sqrt{m}z}{\sqrt{1-m}}\right)\middle|\frac{m-1}{m}\right) - F\left(i\sinh^{-1}\left(\frac{\sqrt{m}}{\sqrt{1-m}}\right)\middle|\frac{m-1}{m}\right) + F\left(i\sinh^{-1}\left(\frac{\sqrt{m}z}{\sqrt{1-m}}\right)\middle|\frac{m-1}{m}\right)\right)\right); z < 1 \wedge 0 < m < 1$$

$$\int \operatorname{dc}^{-1}(z|m) dm = 2\sqrt{m}\left(E\left(\frac{1}{m}\right) - E\left(\sin^{-1}(z)\middle|\frac{1}{m}\right)\right); -1 < z < 1 \wedge m > 1 \vee z > 1 \wedge m < 1$$

$$\int \operatorname{dn}^{-1}(z|m) dm = 2\sqrt{m-1}\left(E\left(\frac{1}{1-m}\right) - E\left(\sin^{-1}(z)\middle|\frac{1}{1-m}\right)\right); z < 1 \wedge m > 1$$

$$\int \operatorname{ds}^{-1}(z|m) dm = 2\left(\frac{\sqrt{z^2+m-1}\sqrt{z^2+m}}{z} - z\left(\log\left(\frac{\sqrt{z^2+m-1} + \sqrt{z^2+m}}{2z}\right) + 1\right) + \sqrt{m-1}i\left(E\left(i\sinh^{-1}\left(\frac{\sqrt{m-1}}{z}\right)\middle|\frac{m}{m-1}\right) - F\left(i\sinh^{-1}\left(\frac{\sqrt{m-1}}{z}\right)\middle|\frac{m}{m-1}\right)\right)\right); z > 0 \wedge m > 0$$

$$\int \text{nc}^{-1}(z | m) dm = -2 \left( \frac{z - z \sqrt{m - m z^2 + z^2}}{\sqrt{z^2 - 1}} + \sqrt{1 - m} \left( E \left( i \sinh^{-1} \left( \frac{\sqrt{1 - m}}{\sqrt{m}} \right) \middle| \frac{m}{m - 1} \right) - E \left( i \sinh^{-1} \left( \frac{\sqrt{1 - m} z}{\sqrt{m}} \right) \middle| \frac{m}{m - 1} \right) - F \left( i \sinh^{-1} \left( \frac{\sqrt{1 - m}}{\sqrt{m}} \right) \middle| \frac{m}{m - 1} \right) + F \left( i \sinh^{-1} \left( \frac{\sqrt{1 - m} z}{\sqrt{m}} \right) \middle| \frac{m}{m - 1} \right) \right) \right) /; z > 1 \wedge m > 0$$

$$\int \text{nd}^{-1}(z | m) dm = \frac{2}{z} \left( -i \sqrt{1 - z^2} \sqrt{(m - 1) z^2 + 1} - \sqrt{m - 1} z E \left( i \sinh^{-1}(\sqrt{m - 1}) \middle| \frac{1}{1 - m} \right) + \sqrt{m - 1} z E \left( i \sinh^{-1}(\sqrt{m - 1} z) \middle| \frac{1}{1 - m} \right) \right) /; z > 1 \wedge m > 1$$

$$\int \text{ns}^{-1}(z | m) dm = 2 \left( -z - \sqrt{m} E \left( \frac{1}{m} \right) + E(m) + \sqrt{m} E \left( \sin^{-1}(z) \middle| \frac{1}{m} \right) + (m - 1) K(m) \right) /; z > 1 \wedge m < 1$$

$$\int \text{sc}^{-1}(z | m) dm = \frac{2}{z} \left( i \sqrt{1 - m} z E \left( i \sinh^{-1}(\sqrt{1 - m} z) \middle| \frac{1}{1 - m} \right) + \sqrt{z^2 + 1} \sqrt{1 - m z^2 + z^2} - 1 \right) /; z \in \mathbb{R} \wedge (1 - m) z^2 > -1$$

$$\int \text{sd}^{-1}(z | m) dm = 2 \sqrt{m - 1} i \left( E \left( i \sinh^{-1}(\sqrt{m - 1} z) \middle| \frac{m}{m - 1} \right) - F \left( i \sinh^{-1}(\sqrt{m - 1} z) \middle| \frac{m}{m - 1} \right) \right) + \frac{1}{z \sqrt{(m - 1) z^2 + 1}} \left( 2(m - 1) \sqrt{m z^2 + 1} z^2 - \sqrt{(m - 1) z^2 + 1} \log \left( \frac{1}{4} \left( (2m - 1) z^2 + 2 \sqrt{(m - 1) z^2 + 1} \sqrt{m z^2 + 1} + 2 \right) \right) - 2 \sqrt{(m - 1) z^2 + 1} + 2 \sqrt{m z^2 + 1} \right) /; z > 0 \wedge m > 0$$

$$\int \text{sn}^{-1}(z | m) dm = 2 \left( \frac{\sqrt{1 - z^2} \sqrt{1 - m z^2} - 1}{z} + E(\sin^{-1}(z) | m) + (m - 1) F(\sin^{-1}(z) | m) \right) /; -1 < z < 1 \wedge m < 1.$$

### Differential equations

The twelve inverse Jacobi functions  $\text{cd}^{-1}(z | m)$ ,  $\text{cn}^{-1}(z | m)$ ,  $\text{cs}^{-1}(z | m)$ ,  $\text{dc}^{-1}(z | m)$ ,  $\text{dn}^{-1}(z | m)$ ,  $\text{ds}^{-1}(z | m)$ ,  $\text{nc}^{-1}(z | m)$ ,  $\text{nd}^{-1}(z | m)$ ,  $\text{ns}^{-1}(z | m)$ ,  $\text{sc}^{-1}(z | m)$ ,  $\text{sd}^{-1}(z | m)$ , and  $\text{sn}^{-1}(z | m)$  are the special solutions of the following second-order ordinary nonlinear differential equations:

$$w''(z) + (2m z^2 - m - 1) z w'(z)^3 = 0 /; w(z) = \text{cd}^{-1}(z | m)$$

$$w''(z) - (2m z^2 - 2m + 1) z w'(z)^3 = 0 /; w(z) = \text{cn}^{-1}(z | m)$$

$$w''(z) + (2z^2 - m + 2) z w'(z)^3 = 0 /; w(z) = \text{cs}^{-1}(z | m)$$

$$w''(z) + (2z^2 - m - 1) z w'(z)^3 = 0 /; w(z) = \text{dc}^{-1}(z | m)$$

$$w''(z) - (2z^2 + m - 2) z w'(z)^3 = 0 /; w(z) = \text{dn}^{-1}(z | m)$$

$$w''(z) + (2z^2 + 2m - 1) z w'(z)^3 = 0 /; w(z) = \text{ds}^{-1}(z | m)$$

$$w''(z) + (2(1-m)z^2 + 2m - 1)zw'(z)^3 = 0 /; w(z) = \text{nc}^{-1}(z | m)$$

$$w''(z) - (2(1-m)z^2 + m - 2)zw'(z)^3 = 0 /; w(z) = \text{nd}^{-1}(z | m)$$

$$w''(z) + (2z^2 - m - 1)zw'(z)^3 = 0 /; w(z) = \text{ns}^{-1}(z | m)$$

$$w''(z) + (2(1-m)z^2 - m + 2)zw'(z)^3 = 0 /; w(z) = \text{sc}^{-1}(z | m)$$

$$w''(z) - (2(1-m)mz^2 - 2m + 1)zw'(z)^3 = 0 /; w(z) = \text{sd}^{-1}(z | m)$$

$$w''(z) + (2mz^2 - m - 1)zw'(z)^3 = 0 /; w(z) = \text{sn}^{-1}(z | m).$$

### Applications of the inverse Jacobi functions

Fields of application of the inverse Jacobi functions include most of the application areas of the direct functions. In many applications, the need for the inversion of the elliptic function fortunately does not arise. In cases where inversion is needed, the inverse Jacobi elliptic functions are very useful tools for calculations.

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