Introductions to Gamma

Introduction to the gamma functions

General

The gamma function $\Gamma(z)$ is applied in exact sciences almost as often as the well-known factorial symbol $n!$. It was introduced by the famous mathematician L. Euler (1729) as a natural extension of the factorial operation $n!$ from positive integers n to real and even complex values of this argument. This relation is described by the formula:

$$\Gamma(n) = (n-1)!.$$

Euler derived some basic properties and formulas for the gamma function. He started investigations of $n!$ from the infinite product:

$$\frac{1}{\Gamma(z)} = z\, e^{z\gamma} \prod_{k=1}^{\infty} \left(1+\frac{z}{k}\right) e^{-\frac{z}{k}}.$$

The gamma function $\Gamma(z)$ has a long history of development and numerous applications since 1729 when Euler derived his famous integral representation of the factorial function. In modern notation it can be rewritten as the following:

$$n! = \Gamma(n+1) = \int_0^1 \left(\log\left(\frac{1}{t}\right)\right)^n dt = \int_0^\infty \tau^n e^{-\tau} d\tau \;/;\; \operatorname{Re}(n) > -1.$$

The history of the gamma function is described in the subsection "General" of the section "Gamma function." Since the famous work of J. Stirling (1730) who first used series for $\log(n!)$ to derive the asymptotic formula for $n!$, mathematicians have used the logarithm of the gamma function $\log(\Gamma(z))$ for their investigations of the gamma function $\Gamma(z)$. Investigators of mention include: C. Siegel, A. M. Legendre, K. F. Gauss, C. J. Malmstén, O. Schlömilch, J. P. M. Binet (1843), E. E. Kummer (1847), and G. Plana (1847). M. A. Stern (1847) proved convergence of the Stirling's series for the derivative of $\log(\Gamma(z))$. C. Hermite (1900) proved convergence of the Stirling's series for $\log(\Gamma(z+1))$ if z is a complex number.

During the twentieth century, the function $\log(\Gamma(z))$ was used in many works where the gamma function was applied or investigated. The appearance of computer systems at the end of the twentieth century demanded more careful attention to the structure of branch cuts for basic mathematical functions to support the validity of the mathematical relations everywhere in the complex plane. This lead to the appearance of a special log-gamma function $\log\Gamma(z)$, which is equivalent to the logarithm of the gamma function $\log(\Gamma(z))$ as a multivalued analytic function, except that it is conventionally defined with a different branch cut structure and principal sheet. The log-gamma function $\log\Gamma(z)$ was introduced by J. Keiper (1990) for *Mathematica*. It allows a concise formulation of many identities related to the Riemann zeta function $\zeta(z)$.

The importance of the gamma function and its Euler integral stimulated some mathematicians to study the incomplete Euler integrals, which are actually equal to the indefinite integral of the expression $\tau^n e^{-\tau}$. They were introduced in an article by A. M. Legendre (1811). Later, P. Schlömilch (1871) introduced the name "incomplete gamma function" for such an integral. These functions were investigated by J. Tannery (1882), F. E. Prym (1877), and M. Lerch (1905) (who gave a series representation for the incomplete gamma function). N. Nielsen (1906) and other mathematicians also had special interests in these functions, which were included in the main handbooks of special functions and current computer systems like *Mathematica*.

The needs of computer systems lead to the implementation of slightly more general incomplete gamma functions and their regularized and inverse versions. In addition to the classical gamma function $\Gamma(z)$, *Mathematica* includes the following related set of gamma functions: incomplete gamma function $\Gamma(a, z)$, generalized incomplete gamma function $\Gamma(a, z_1, z_2)$, regularized incomplete gamma function $Q(a, z)$, generalized regularized incomplete gamma function $Q(a, z_1, z_2)$, log-gamma function $\log\Gamma(z)$, inverse of the regularized incomplete gamma function $Q^{-1}(a, z)$, and inverse of the generalized regularized incomplete gamma function $Q^{-1}(a, z_1, z_2)$.

Definitions of gamma functions

The gamma function $\Gamma(z)$, the incomplete gamma function $\Gamma(a, z)$, the generalized incomplete gamma function $\Gamma(a, z_1, z_2)$, the regularized incomplete gamma function $Q(a, z)$, the generalized regularized incomplete gamma function $Q(a, z_1, z_2)$, the log-gamma function (almost equal to the logarithm of the gamma function) $\log\Gamma(z)$, the inverse of the regularized incomplete gamma function $Q^{-1}(a, z)$, and the inverse of the generalized regularized incomplete gamma function $Q^{-1}(a, z_1, z_2)$ are defined by the following formulas:

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} \, dt \;/;\; \mathrm{Re}(z) > 0$$

$$\Gamma(z) = \int_1^\infty t^{z-1} e^{-t} \, dt + \sum_{k=0}^{\infty} \frac{(-1)^k}{k! \, (k+z)}$$

$$\Gamma(a, z) = \int_z^\infty t^{a-1} e^{-t} \, dt$$

$$\Gamma(a, z_1, z_2) = \int_{z_1}^{z_2} t^{a-1} e^{-t} \, dt$$

$$Q(a, z) = \frac{\Gamma(a, z)}{\Gamma(a)}$$

$$Q(a, z_1, z_2) = \frac{\Gamma(a, z_1, z_2)}{\Gamma(a)}$$

$$\log\Gamma(z) = \sum_{k=1}^{\infty} \left(\frac{z}{k} - \log\left(1 + \frac{z}{k}\right) \right) - \gamma z - \log(z).$$

The function $\log\Gamma(z)$ is equivalent to $\log(\Gamma(z))$ as a multivalued analytic function, except that it is conventionally defined with a different branch cut structure and principal sheet. The function $\log\Gamma(z)$ allows a concise formulation of many identities related to the Riemann zeta function $\zeta(z)$:

$z == Q(a, w)$ /; $w == Q^{-1}(a, z)$

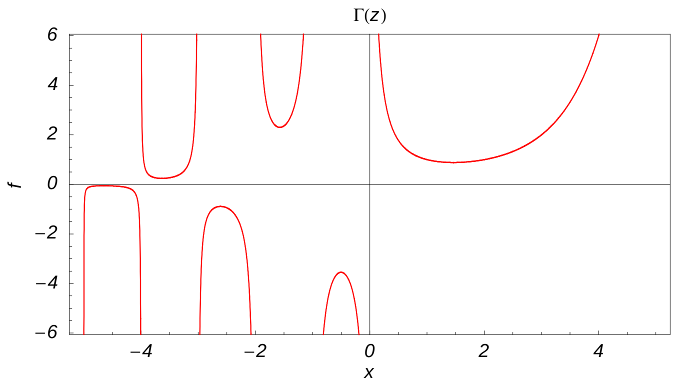
$z_2 == Q(a, z_1, w)$ /; $w == Q^{-1}(a, z_1, z_2)$.

The previous functions comprise the interconnected group called the gamma functions.

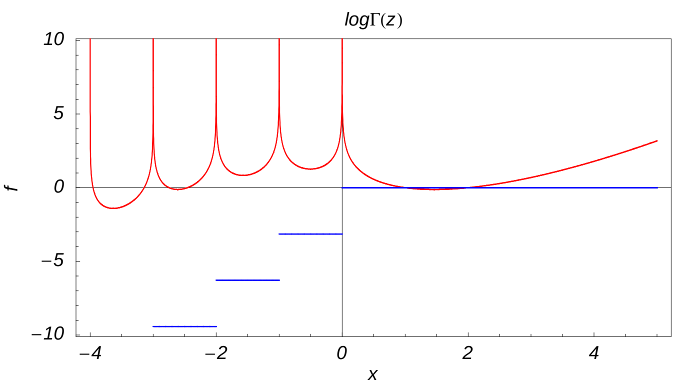
Instead of the first three previous classical definitions using definite integrals, the other equivalent definitions with infinite series can be used.

A quick look at the gamma functions

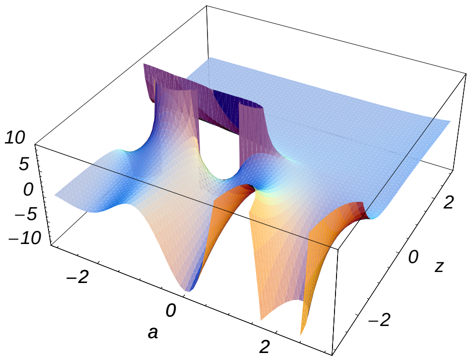
Here is a quick look at graphics for the gamma function and the function $\log\Gamma(z)$ along the real axis. The real parts are shown in red and the imaginary parts are shown in blue.

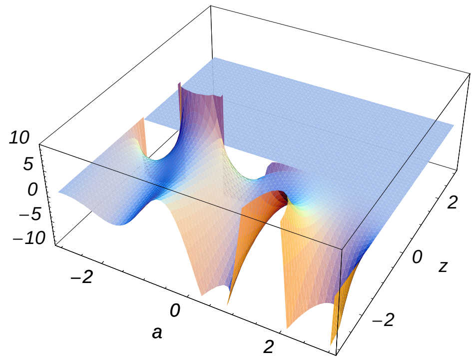


Here is a quick look at the graphics for the gamma function and the function $\log\Gamma(z)$ along the real axis.

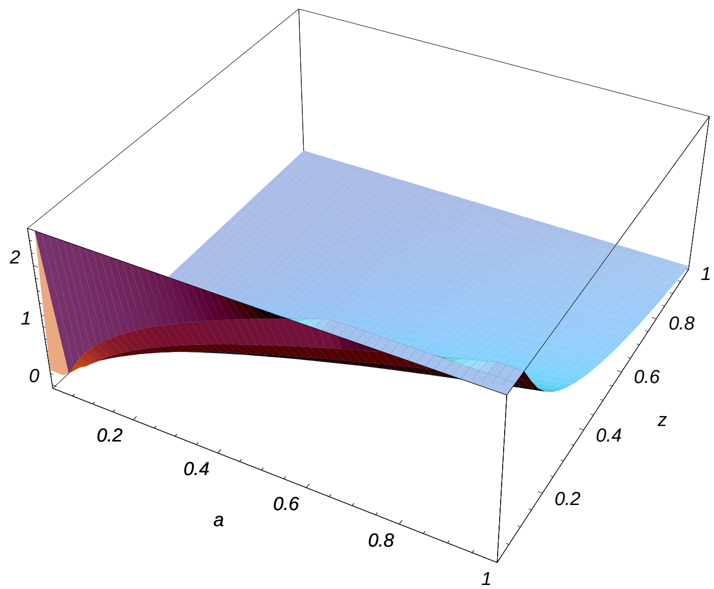


These two graphics show the real part (left) and imaginary part (right) of $\Gamma(a, z)$ over the a-z–plane.





The next graphic shows the regularized incomplete gamma function $Q(a, z)$ over the a-z-plane.



Connections within the group of gamma functions and with other function groups

Representations through more general functions

The incomplete gamma functions $\Gamma(a, z)$, $\Gamma(a, z_1, z_2)$, $Q(a, z)$, and $Q(a, z_1, z_2)$ are particular cases of the more general hypergeometric and Meijer G functions.

For example, they can be represented through hypergeometric functions ${}_1F_1$ and ${}_1\tilde{F}_1$ or the Tricomi confluent hypergeometric function U:

$$\Gamma(a, z) = \Gamma(a)\left(1 - z^a\, {}_1\tilde{F}_1(a; a+1; -z)\right) /; -a \notin \mathbb{N}$$

$$\Gamma(a, z) = \Gamma(a) - \frac{z^a}{a}\, {}_1F_1(a; a+1; -z) /; -a \notin \mathbb{N}$$

$$\Gamma(a, z) = e^{-z}\, U(1-a, 1-a, z)$$

$$\Gamma(a, z_1, z_2) = \Gamma(a)\left(z_2^a\, {}_1\tilde{F}_1(a; a+1; -z_2) - z_1^a\, {}_1\tilde{F}_1(a; a+1; -z_1)\right) /; -a \notin \mathbb{N}$$

$$\Gamma(a, z_1, z_2) = \frac{z_2^a}{a} \, {}_1F_1(a; a+1; -z_2) - \frac{z_1^a}{a} \, {}_1F_1(a; a+1; -z_1) \,/;\, -a \notin \mathbb{N}$$

$$\Gamma(a, z_1, z_2) = e^{-z_1} \, U(1-a, 1-a, z_1) - e^{-z_2} \, U(1-a, 1-a, z_2)$$

$$Q(a, z) = 1 - z^a \, {}_1\tilde{F}_1(a; a+1; -z) \,/;\, -a \notin \mathbb{N}^+$$

$$Q(a, z) = 1 - \frac{z^a}{\Gamma(a+1)} \, {}_1F_1(a; a+1; -z) \,/;\, -a \notin \mathbb{N}^+$$

$$Q(a, z) = \frac{1}{\Gamma(a)} \, e^{-z} \, U(1-a, 1-a, z)$$

$$Q(a, z_1, z_2) = z_2^a \, {}_1\tilde{F}_1(a; a+1; -z_2) - z_1^a \, {}_1\tilde{F}_1(a; a+1; -z_1)$$

$$Q(a, z_1, z_2) = \frac{z_2^a}{\Gamma(a+1)} \, {}_1F_1(a; a+1; -z_2) - \frac{z_1^a}{\Gamma(a+1)} \, {}_1F_1(a; a+1; -z_1) \,/;\, -a \notin \mathbb{N}$$

$$Q(a, z_1, z_2) = \frac{1}{\Gamma(a)} \left(e^{-z_1} \, U(1-a, 1-a, z_1) - e^{-z_2} \, U(1-a, 1-a, z_2)\right).$$

These functions also have rather simple representations in terms of classical Meijer G functions:

$$\Gamma(a, z) = G_{1,2}^{2,0}\left(z \,\middle|\, \begin{matrix} 1 \\ 0, a \end{matrix}\right)$$

$$\Gamma(a, z_1, z_2) = G_{1,2}^{1,1}\left(z_2 \,\middle|\, \begin{matrix} 1 \\ a, 0 \end{matrix}\right) - G_{1,2}^{1,1}\left(z_1 \,\middle|\, \begin{matrix} 1 \\ a, 0 \end{matrix}\right)$$

$$Q(a, z) = \frac{1}{\Gamma(a)} \, G_{1,2}^{2,0}\left(z \,\middle|\, \begin{matrix} 1 \\ 0, a \end{matrix}\right)$$

$$Q(a, z_1, z_2) = \frac{1}{\Gamma(a)} \left(G_{1,2}^{1,1}\left(z_2 \,\middle|\, \begin{matrix} 1 \\ a, 0 \end{matrix}\right) - G_{1,2}^{1,1}\left(z_1 \,\middle|\, \begin{matrix} 1 \\ a, 0 \end{matrix}\right)\right).$$

The log-gamma function $\log\Gamma(z)$ can be expressed through polygamma and zeta functions by the following formulas:

$$\log\Gamma(z) = \int_1^z \psi(t) \, dt$$

$$\log\Gamma(z) = \frac{\partial^{-\nu-1} \psi^{(\nu)}(z)}{\partial z^{-\nu-1}}$$

$$\log\Gamma(z) = \frac{1}{2} \log(2\pi) + \zeta^{(1,0)}(0, z) \,/;\, \mathrm{Re}(z) > 0.$$

Representations through related equivalent functions

The gamma functions $\Gamma(a, z)$, $\Gamma(a, z_1, z_2)$, $Q(a, z)$, and $Q(a, z_1, z_2)$ can be represented using the related exponential integral $E_\nu(z)$ by the following formulas:

$$\Gamma(a, z) = z^a \, E_{1-a}(z)$$

$$\Gamma(a, z_1, z_2) = z_1^a E_{1-a}(z_1) - z_2^a E_{1-a}(z_2)$$

$$Q(a, z) = \frac{z^a E_{1-a}(z)}{\Gamma(a)}$$

$$Q(a, z_1, z_2) = \frac{1}{\Gamma(a)} (z_1^a E_{1-a}(z_1) - z_2^a E_{1-a}(z_2)).$$

Relations to inverse functions

The gamma functions $\Gamma(a, z)$, $\Gamma(a, z_1, z_2)$, $Q(a, z)$, and $Q(a, z_1, z_2)$ are connected with the inverse of the regularized incomplete gamma function $Q^{-1}(a, z)$ and the inverse of the generalized regularized incomplete gamma function $Q^{-1}(a, z_1, z_2)$ by the following formulas:

$$\Gamma\left(a, Q^{-1}(a, z)\right) = \Gamma(a)\, z$$

$$\Gamma\left(a, z_1, Q^{-1}(a, z_1, z_2)\right) = \Gamma(a)\, z_2$$

$$Q\left(a, Q^{-1}(a, z)\right) = z$$

$$Q\left(a, z_1, Q^{-1}(a, z_1, z_2)\right) = z_2$$

$$Q^{-1}(a, Q(a, z_1) - z_2) = Q^{-1}(a, z_1, z_2)$$

$$Q^{-1}(a, z_1, z_2) = Q^{-1}(a, Q(a, z_1) - z_2).$$

Representations through other gamma functions

The gamma functions $\Gamma(a)$, $\Gamma(a, z)$, $\Gamma(a, z_1, z_2)$, $Q(a, z)$, $Q(a, z_1, z_2)$, and $\log\Gamma(z)$ are connected with each other by the formulas:

$$\Gamma(a) = \Gamma(a, 0) \;/;\; \mathrm{Re}(a) > 0$$

$$\Gamma(a, z) = \Gamma(a) + \Gamma(a, z, 0) \;/;\; \mathrm{Re}(a) > 0$$

$$\Gamma(a, z) = \Gamma(a)\,(Q(a, z, 0) + 1) \;/;\; \mathrm{Re}(a) > 0$$

$$\Gamma(a, z) = \Gamma(a)\, Q(a, z)$$

$$\Gamma(a, z_1, z_2) = \Gamma(a, z_1) - \Gamma(a, z_2)$$

$$\Gamma(a, z_1, z_2) = \Gamma(a)\, Q(a, z_1, z_2)$$

$$Q(a, z) = \frac{\Gamma(a, z, 0)}{\Gamma(a)} + 1 \;/;\; \mathrm{Re}(a) > 0$$

$$Q(a, z) = Q(a, z, 0) + 1 \;/;\; \mathrm{Re}(a) > 0$$

$$Q(a, z) = \frac{\Gamma(a, z)}{\Gamma(a)}$$

$$Q(a, z_1, z_2) = Q(a, z_1) - Q(a, z_2)$$

$$Q(a, z_1, z_2) = \frac{\Gamma(a, z_1, z_2)}{\Gamma(a)}$$

$$\log\Gamma(z) = \log(\Gamma(z)) \;/;\; 0 < \mathrm{Re}(z) \le 2 \bigwedge |\mathrm{Im}(z)| \le \frac{7}{2}$$

$$\log\Gamma(z) = 2\, i\, \pi\, k(z) + \log(\Gamma(z)) \;/;\; k(z) = \int_0^z \theta(-\mathrm{Re}(\Gamma(t)))\, |\mathrm{Im}(\Gamma(t)\, \psi(t))|\, \delta(\mathrm{Im}(\Gamma(t)))\, dt \in \mathbb{Z}.$$

The best-known properties and formulas for exponential integrals

Real values for real arguments

For real values of z, the values of the gamma function $\Gamma(z)$ are real (or infinity). For real values of the parameter a and positive arguments z, z_1, z_2, the values of the gamma functions $\Gamma(a, z)$, $\Gamma(a, z_1, z_2)$, $Q(a, z)$, $Q(a, z_1, z_2)$, and $\log\Gamma(z)$ are real (or infinity).

Simple values at zero

The gamma functions $\Gamma(z)$, $\Gamma(a, z)$, $\Gamma(a, z_1, z_2)$, $Q(a, z)$, $Q(a, z_1, z_2)$, $\log\Gamma(z)$, $Q^{-1}(a, z)$, and $Q^{-1}(a, z_1, z_2)$ have the following values at zero arguments:

$$\Gamma(0) = \tilde{\infty}$$

$$\Gamma(0, 0) = \tilde{\infty}$$

$$\Gamma(0, 0, 0) = ¿$$

$$Q(0, 0) = 0$$

$$Q(0, 0, 0) = 0$$

$$\log\Gamma(0) = \infty$$

$$Q^{-1}(0, 0) = 0$$

$$Q^{-1}(0, 0, 0) = 0.$$

Specific values for specialized variables

If the variable z is equal to 0 and $\mathrm{Re}(a) > 0$, the incomplete gamma function $\Gamma(a, z)$ coincides with the gamma function $\Gamma(a)$ and the corresponding regularized gamma function $Q(a, z)$ is equal to 1:

$$\Gamma(a, 0) = \Gamma(a) \;/;\; \mathrm{Re}(a) > 0 \quad Q(a, 0) = 1 \;/;\; \mathrm{Re}(a) > 0.$$

In cases when the parameter a equals 1, 2, 3, …, the incomplete gamma functions $\Gamma(a, z)$ and $Q(a, z)$ can be expressed as an exponential function multiplied by a polynomial. In cases when the parameter a equals 0, −1, −2, …, the incomplete gamma function $\Gamma(a, z)$ can be expressed with the exponential integral $\mathrm{Ei}(z)$, exponential, and logarithmic functions, but the regularized incomplete gamma function $Q(a, z)$ is equal to 0. In cases when the parameter a equals $\pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{5}{2}, \ldots$, the incomplete gamma functions $\Gamma(a, z)$ and $Q(a, z)$ can be expressed through the complementary error function $\mathrm{erfc}(z)$ and the exponential function, for example:

$$\Gamma(1, z) = e^{-z} \qquad Q(1, z) = e^{-z}$$

$$\Gamma(0, z) = -\text{Ei}(-z) + \frac{1}{2}\left(\log(-z) - \log\left(-\frac{1}{z}\right)\right) - \log(z) \qquad Q(0, z) = 0$$

$$\Gamma(-1, z) = \text{Ei}(-z) + \frac{1}{2}\left(\log\left(-\frac{1}{z}\right) - \log(-z)\right) + \log(z) + \frac{e^{-z}}{z} \qquad Q(-1, z) = 0$$

$$\Gamma\left(\frac{1}{2}, z\right) = \sqrt{\pi}\ \text{erfc}\left(\sqrt{z}\right) \qquad Q\left(\frac{1}{2}, z\right) = \text{erfc}\left(\sqrt{z}\right)$$

$$\Gamma\left(-\frac{1}{2}, z\right) = \frac{2 e^{-z}}{\sqrt{z}} - 2\sqrt{\pi}\ \text{erfc}\left(\sqrt{z}\right) \qquad Q\left(-\frac{1}{2}, z\right) = \text{erfc}\left(\sqrt{z}\right) - \frac{e^{-z}}{\sqrt{\pi}\sqrt{z}}.$$

These formulas are particular cases of the following general formulas:

$$\Gamma(n, z) = \frac{(-1)^{n-1}}{(-n)!}\left(\text{Ei}(-z) - \frac{1}{2}\left(\log(-z) - \log\left(-\frac{1}{z}\right)\right) + \log(z)\right) + e^{-z}\sum_{k=0}^{n-1}\frac{z^k}{(n)_{k-n+1}} - e^{-z}\sum_{k=n}^{-1}\frac{z^k}{(n)_{k-n+1}} \text{ /; } n \in \mathbb{Z}$$

$$\Gamma\left(n + \frac{1}{2}, z\right) = \text{erfc}\left(\sqrt{z}\right)\Gamma\left(n + \frac{1}{2}\right) + e^{-z}\sum_{k=0}^{n-1}\frac{z^{k+\frac{1}{2}}}{\left(n + \frac{1}{2}\right)_{k-n+1}} - e^{-z}\sum_{k=n}^{-1}\frac{z^{k+\frac{1}{2}}}{\left(n + \frac{1}{2}\right)_{k-n+1}} \text{ /; } n \in \mathbb{Z}$$

$$Q(n, z) = e^{-z}\sum_{k=0}^{n-1}\frac{z^k}{k!} \text{ /; } n \in \mathbb{N}^+$$

$$Q(-n, z) = 0 \text{ /; } n \in \mathbb{N}$$

$$Q\left(n + \frac{1}{2}, z\right) = \text{erfc}\left(\sqrt{z}\right) + \frac{1}{\Gamma\left(n + \frac{1}{2}\right)}\left(e^{-z}\sum_{k=0}^{n-1}\frac{z^{k+\frac{1}{2}}}{\left(n + \frac{1}{2}\right)_{k-n+1}} - e^{-z}\sum_{k=n}^{-1}\frac{z^{k+\frac{1}{2}}}{\left(n + \frac{1}{2}\right)_{k-n+1}}\right) \text{ /; } n \in \mathbb{Z}.$$

If the argument $z > 0$, the log-gamma function $\log\Gamma(z)$ can be evaluated at these points where the gamma function can be evaluated in closed form. The log-gamma function $\log\Gamma(z)$ can also be represented recursively in terms of $\Gamma(z)$ for $0 < \text{Re}(z) < 1$:

$$\log\Gamma(1) = 0$$

$$\log\Gamma(n) = \log((n-1)!) \text{ /; } n \in \mathbb{N}^+$$

$$\log\Gamma\left(\frac{n}{2}\right) = \log\left(\frac{2^{1-n}\sqrt{\pi}\ (n-1)!}{\frac{n-1}{2}!}\right) \text{ /; } n \in \mathbb{N}$$

$$\log\Gamma(-n) = \infty \text{ /; } n \in \mathbb{N}$$

$$\log\Gamma\left(\frac{p}{q} + n\right) = \log\left(\Gamma\left(\frac{p}{q}\right)\right) - n\log(q) + \sum_{k=1}^{n}\log(p + k\,q - q) \text{ /; } n \in \mathbb{N} \land p \in \mathbb{N}^+ \land q \in \mathbb{N}^+ \land p < q$$

$$\log\Gamma\left(\frac{p}{q} - n\right) = \log\left(\Gamma\left(\frac{p}{q}\right)\right) + \log(q)\,n - \pi\,i\,n - \sum_{k=1}^{n}\log(q\,k - p) \text{ /; } n \in \mathbb{N} \land p \in \mathbb{N}^+ \land q \in \mathbb{N}^+ \land p < q.$$

The generalized incomplete gamma functions $\Gamma(a, z_1, z_2)$ and $Q(a, z_1, z_2)$ in particular cases can be represented through incomplete gamma functions $\Gamma(a, z)$ and $Q(a, z)$ and the gamma function $\Gamma(a)$:

$$\Gamma(a, z_1, 0) = \Gamma(a, z_1) - \Gamma(a) \text{ /; } \text{Re}(a) > 0$$

$\Gamma(a, 0, z_2) == \Gamma(a) - \Gamma(a, z_2)$ /; $\mathrm{Re}(a) > 0$

$\Gamma(a, z_1, \infty) == \Gamma(a, z_1)$

$\Gamma(a, \infty, z_2) == -\Gamma(a, z_2)$

$\Gamma(a, 0, \infty) == \Gamma(a)$ /; $\mathrm{Re}(a) > 0$

$Q(-n, z_1, z_2) == 0$ /; $n \in \mathbb{N}$

$Q(a, z_1, \infty) == Q(a, z_1)$

$Q(a, 0, \infty) == 1$ /; $\mathrm{Re}(a) > 0$

$Q(a, z_1, 0) == Q(a, z_1) - 1$ /; $\mathrm{Re}(a) > 0$

$Q(a, 0, z_2) == 1 - Q(a, z_2)$ /; $\mathrm{Re}(a) > 0$.

The inverse of the regularized incomplete gamma functions $Q^{-1}(a, z)$ and $Q^{-1}(a, z_1, z_2)$ for particular values of arguments satisfy the following relations:

$Q^{-1}(a, 0) == \infty$ /; $a > 0$

$Q^{-1}(a, 1) == 0$ /; $a > 0$

$Q^{-1}(a, \infty, z) == Q^{-1}(a, -z)$.

Analyticity

The gamma functions $\Gamma(z)$, $\Gamma(a, z)$, $\Gamma(a, z_1, z_2)$, $Q(a, z)$, $Q(a, z_1, z_2)$, and $\log\Gamma(z)$ are defined for all complex values of their arguments.

The functions $\Gamma(a, z)$ and $Q(a, z)$ are analytic functions of a and z over the whole complex a- and z-planes excluding the branch cut on the z-plane. For fixed z, they are entire functions of a. The functions $\Gamma(a, z_1, z_2)$ and $Q(a, z_1, z_2)$ are analytic functions of a, z_1, and z_2 over the whole complex a-, z_1-, and z_2-planes excluding the branch cuts on the z_1- and z_2-planes. For fixed z_1 and z_2, they are entire functions of a.

The function $\log\Gamma(z)$ is an analytical function of z over the whole complex z-plane excluding the branch cut.

Poles and essential singularities

For fixed a, the functions $\Gamma(a, z)$ and $Q(a, z)$ have an essential singularity at $z == \tilde{\infty}$. At the same time, the point $z == \tilde{\infty}$ is a branch point for generic a. For fixed z, the functions $\Gamma(a, z)$ and $Q(a, z)$ have only one singular point at $a = \tilde{\infty}$. It is an essential singularity.

For fixed a, the functions $\Gamma(a, z_1, z_2)$ and $Q(a, z_1, z_2)$ have an essential singularity at $z_1 == \tilde{\infty}$ (for fixed z_2) and at $z_2 == \tilde{\infty}$ (for fixed z_1). At the same time, the points $z_k == \tilde{\infty}$ /; $k == 1, 2$ are branch points for generic a. For fixed z_1 and z_2, the functions $\Gamma(a, z_1, z_2)$ and $Q(a, z_1, z_2)$ have only one singular point at $a = \tilde{\infty}$. It is an essential singularity.

The function $\log\Gamma(z)$ does not have poles or essential singularities.

Branch points and branch cuts

For fixed a, not a positive integer, the functions $\Gamma(a, z)$ and $Q(a, z)$ have two branch points: $z == 0$ and $z == \tilde{\infty}$.

For fixed a, not a positive integer, the functions $\Gamma(a, z)$ and $Q(a, z)$ are single-valued functions on the z-plane cut along the interval $(-\infty, 0)$, where they are continuous from above:

$$\lim_{\epsilon \to +0} \Gamma(a, x + i\epsilon) == \Gamma(a, x) /; x < 0$$

$$\lim_{\epsilon \to +0} \Gamma(a, x - i\epsilon) == \Gamma(a) - e^{-2 i \pi a} (\Gamma(a) - \Gamma(a, x)) /; x < 0$$

$$\lim_{\epsilon \to +0} Q(a, x + i\epsilon) == Q(a, x) /; x < 0$$

$$\lim_{\epsilon \to +0} Q(a, x - i\epsilon) == 1 - e^{-2 i \pi a} (1 - Q(a, x)) /; x < 0.$$

For fixed z, the functions $\Gamma(a, z)$ and $Q(a, z)$ do not have branch points and branch cuts.

For fixed a, z_1 or fixed a, z_2 (with $a \notin \mathbb{N}^+$), the functions $\Gamma(a, z_1, z_2)$ and $Q(a, z_1, z_2)$ have two branch points with respect to z_2 or z_1: $z_k == 0$, $z_k == \tilde{\infty}$, $k = 1, 2$.

For fixed z_1 and $a \notin \mathbb{N}^+$, the functions $\Gamma(a, z_1, z_2)$ and $Q(a, z_1, z_2)$ are single-valued functions on the z_2-plane cut along the interval $(-\infty, 0)$, where they are continuous from above:

$$\lim_{\epsilon \to +0} \Gamma(a, z_1, x_2 + i\epsilon) == \Gamma(a, z_1, x_2) /; x_2 < 0$$

$$\lim_{\epsilon \to +0} \Gamma(a, z_1, x_2 - i\epsilon) == \Gamma(a, z_1, x_2) + \left(1 - e^{-2 i \pi a}\right) \Gamma(a, x_2, 0) /; x_2 < 0$$

$$\lim_{\epsilon \to +0} Q(a, z_1, x_2 + i\epsilon) == Q(a, z_1, x_2) /; x_2 < 0$$

$$\lim_{\epsilon \to +0} Q(a, z_1, x_2 - i\epsilon) == Q(a, z_1, x_2) + \left(1 - e^{-2 i \pi a}\right) Q(a, x_2, 0) /; x_2 < 0.$$

For fixed z_2 and $a \notin \mathbb{N}^+$, the functions $\Gamma(a, z_1, z_2)$ and $Q(a, z_1, z_2)$ are single-valued functions on the z_1-plane cut along the interval $(-\infty, 0)$, where they are continuous from above:

$$\lim_{\epsilon \to +0} \Gamma(a, x_1 + i\epsilon, z_2) == \Gamma(a, x_1, z_2) /; x_1 < 0$$

$$\lim_{\epsilon \to +0} \Gamma(a, x_1 - i\epsilon, z_2) == \left(1 - e^{-2 i \pi a}\right) \Gamma(a, 0, x_1) + \Gamma(a, x_1, z_2) /; x_1 < 0$$

$$\lim_{\epsilon \to +0} Q(a, x_1 + i\epsilon, z_2) == Q(a, x_1, z_2) /; x_1 < 0$$

$$\lim_{\epsilon \to +0} Q(a, x_1 - i\epsilon, z_2) == \left(1 - e^{-2 i \pi a}\right) Q(a, 0, x_1) + Q(a, x_1, z_2) /; x_1 < 0.$$

For fixed z_1 and z_2, the functions $\Gamma(a, z_1, z_2)$ and $Q(a, z_1, z_2)$ do not have branch points and branch cuts.

The function $\log\Gamma(z)$ has two branch points: $z == 0$ and $z = \tilde{\infty}$.

The function $\log\Gamma(z)$ is a single-valued function on the z-plane cut along the interval $(-\infty, 0)$, where it is continuous from above:

$$\lim_{\epsilon\to+0} \log\Gamma(x + i\,\epsilon) == \log\Gamma(x) /; x < 0$$

$$\lim_{\epsilon\to+0} \log\Gamma(x - i\,\epsilon) == \log\Gamma(x) - 2\,i\,\pi\,\lfloor x\rfloor /; x < 0.$$

Periodicity

The gamma functions $\Gamma(z)$, $\Gamma(a, z)$, $\Gamma(a, z_1, z_2)$, $Q(a, z)$, $Q(a, z_1, z_2)$, the log-gamma function $\log\Gamma(z)$, and their inverses $Q^{-1}(a, z)$ and $Q^{-1}(a, z_1, z_2)$ do not have periodicity.

Parity and symmetry

The gamma functions $\Gamma(a, z)$, $\Gamma(a, z_1, z_2)$, $Q(a, z)$, $Q(a, z_1, z_2)$, and the log-gamma function $\log\Gamma(z)$ have mirror symmetry (except on the branch cut intervals):

$$\Gamma(\overline{a}, \overline{z}) == \overline{\Gamma(a, z)} /; z \notin (-\infty, 0)$$

$$\Gamma(\overline{a}, \overline{z_1}, \overline{z_2}) == \overline{\Gamma(a, z_1, z_2)} /; z_1 \notin (-\infty, 0) \bigwedge z_2 \notin (-\infty, 0)$$

$$Q(\overline{a}, \overline{z}) == \overline{Q(a, z)} /; z \notin (-\infty, 0)$$

$$Q(\overline{a}, \overline{z_1}, \overline{z_2}) == \overline{Q(a, z_1, z_2)} /; z_1 \notin (-\infty, 0) \bigwedge z_2 \notin (-\infty, 0)$$

$$\log\Gamma(\overline{z}) == \overline{\log\Gamma(z)} /; z \notin (-\infty, 0).$$

Two of the gamma functions have the following permutation symmetry:

$$\Gamma(a, z_1, z_2) == -\Gamma(a, z_2, z_1)$$

$$Q(a, z_1, z_2) == -Q(a, z_2, z_1).$$

Series representations

The gamma functions $\Gamma(a, z)$, $\Gamma(a, z_1, z_2)$, $Q(a, z)$, $Q(a, z_1, z_2)$, the log-gamma function $\log\Gamma(z)$, and the inverse $Q^{-1}(a, z)$ have the following series expansions:

$$\Gamma(a, z) \propto \Gamma(a) - \frac{z^a}{a}\left(1 - \frac{a\,z}{a+1} + \frac{a\,z^2}{2\,(a+2)} - \ldots\right) /; (z \to 0)$$

$$\Gamma(a, z) == \Gamma(a) - z^a \sum_{k=0}^{\infty} \frac{(-z)^k}{(a+k)\,k!}$$

$$\Gamma(n, z) == (n-1)!\,e^{-z} \sum_{k=0}^{n-1} \frac{z^k}{k!} /; n \in \mathbb{N}^+$$

$$\Gamma(-n, z) == \frac{(-1)^n}{n!}\,(\psi(n+1) - \log(z)) - z^{-n} \sum_{\substack{k=0 \\ k\neq n}}^{\infty} \frac{(-z)^k}{(k-n)\,k!} /; n \in \mathbb{N}$$

$$\Gamma(a, z_1, z_2) \propto z_2^a \left(\frac{1}{a} - \frac{z_2}{a+1} + \frac{z_2^2}{2\,(a+2)} + \ldots\right) - z_1^a \left(\frac{1}{a} - \frac{z_1}{a+1} + \frac{z_1^2}{2\,(a+2)} + \ldots\right) /;\ (z_1 \to 0) \bigwedge (z_2 \to 0)$$

$$\Gamma(a, z_1, z_2) = z_2^a \sum_{k=0}^{\infty} \frac{(-z_2)^k}{(a+k)\,k!} - z_1^a \sum_{k=0}^{\infty} \frac{(-z_1)^k}{(a+k)\,k!}$$

$$\Gamma(n, z_1, z_2) = (n-1)! \left(e^{-z_1} \sum_{k=0}^{n-1} \frac{z_1^k}{k!} - e^{-z_2} \sum_{k=0}^{n-1} \frac{z_2^k}{k!}\right) /;\ n \in \mathbb{N}^+$$

$$\Gamma(-n, z_1, z_2) = \frac{(-1)^{n-1}}{n!} (\log(z_1) - \log(z_2)) + \sum_{\substack{k=0 \\ k \neq n}}^{\infty} \frac{(-1)^k \left(z_2^{k-n} - z_1^{k-n}\right)}{(k-n)\,k!} /;\ n \in \mathbb{N}$$

$$Q(a, z) \propto 1 - z^a \left(\frac{1}{\Gamma(a+1)} - \frac{a\,z}{\Gamma(a+2)} + \frac{a\,(a+1)\,z^2}{2\,\Gamma(a+3)} - \ldots\right) /;\ (z \to 0)$$

$$Q(a, z) = 1 - z^a \sum_{k=0}^{\infty} \frac{(a)_k\,(-z)^k}{\Gamma(a+k+1)\,k!}$$

$$Q(n, z) = e^{-z} \sum_{k=0}^{n-1} \frac{z^k}{k!} /;\ n \in \mathbb{N}^+$$

$$Q(a, z_1, z_2) \propto z_2^a \left(\frac{1}{\Gamma(a+1)} - \frac{a\,z_2}{\Gamma(a+2)} + \frac{a\,(a+1)\,z_2^2}{2\,\Gamma(a+3)} - \ldots\right) - z_1^a \left(\frac{1}{\Gamma(a+1)} - \frac{a\,z_1}{\Gamma(a+2)} + \frac{a\,(a+1)\,z_1^2}{2\,\Gamma(a+3)} - \ldots\right) /;\ (z_1 \to 0) \bigwedge (z_2 \to 0)$$

$$Q(a, z_1, z_2) = z_2^a \sum_{k=0}^{\infty} \frac{(a)_k\,(-z_2)^k}{\Gamma(a+k+1)\,k!} - z_1^a \sum_{k=0}^{\infty} \frac{(a)_k\,(-z_1)^k}{\Gamma(a+k+1)\,k!}$$

$$Q(n, z_1, z_2) = e^{-z_1} \sum_{k=0}^{n-1} \frac{z_1^k}{k!} - e^{-z_2} \sum_{k=0}^{n-1} \frac{z_2^k}{k!} /;\ n \in \mathbb{N}^+$$

$$\log\Gamma(z) \propto -\log(z) - \gamma\,z + \frac{\pi^2\,z^2}{12} - \frac{\zeta(3)\,z^3}{3} + \frac{\pi^4\,z^4}{360} - \ldots /;\ (z \to 0)$$

$$\log\Gamma(z) = -\log(z) - \gamma\,z + \sum_{j=0}^{\infty} \frac{(-1)^j\,\zeta(j+2)\,z^{j+2}}{j+2} /;\ |z| < 1$$

$$\log\Gamma(z) \propto \log\Gamma(z_0) + \psi(z_0)\,(z - z_0) + \frac{\zeta(2, z_0)}{2}\,(z - z_0)^2 - \frac{\zeta(3, z_0)}{3}\,(z - z_0)^3 + \ldots /;\ (z \to z_0) \bigwedge \neg\,(z_0 \in \mathbb{Z} \bigwedge z_0 \le 0)$$

$$\log\Gamma(z) = \log\Gamma(z_0) + \psi(z_0)\,(z - z_0) + \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{(-1)^j\,(z - z_0)^{j+2}}{(j+2)\,(k+z_0)^{j+2}} /;\ \neg\,(z_0 \in \mathbb{Z} \bigwedge z_0 \le 0)$$

$$\log\Gamma(z) = -\log(z+n) + \log\Gamma\,(z+n+1) - \sum_{k=0}^{n-1} \log(z+k) /;\ (z \to -n) \land n \in \mathbb{N}$$

$$\log\Gamma(z) \propto -\log(z+n) - \sum_{k=0}^{n-1} \log(z+k)\,(1+O(z+n)) \;/;\; (z \to -n) \land n \in \mathbb{N}$$

$$Q^{-1}(a, z) = (-(z-1)\,\Gamma(a+1))^{1/a} + \frac{\left((-(z-1)\,\Gamma(a+1))^{1/a}\right)^2}{a+1} + \frac{(3\,a+5)\left((-(z-1)\,\Gamma(a+1))^{1/a}\right)^3}{2\,(a+1)^2\,(a+2)} + O\left((z-1)^{4/a}\right).$$

Asymptotic series expansions

The asymptotic behavior of the gamma functions $\Gamma(a, z)$ and $Q(a, z)$, the log-gamma function $\log\Gamma(z)$, and the inverse $Q^{-1}(a, z)$ can be described by the following formulas (only the main terms of asymptotic expansion are given):

$$\Gamma(a, z) \propto e^{-z}\, z^{a-1} \left(1 + O\left(\frac{1}{z}\right)\right) \;/;\; (|z| \to \infty)$$

$$Q(a, z) \propto \frac{e^{-z}\, z^{a-1}}{\Gamma(a)} \left(1 + O\left(\frac{1}{z}\right)\right) \;/;\; (|z| \to \infty)$$

$$\log\Gamma(z) \propto \left(z - \frac{1}{2}\right)\log(z) - z + \frac{\log(2\,\pi)}{2} + \frac{1}{12\,z}\left(1 + O\left(\frac{1}{z^2}\right)\right) \;/;\; |\mathrm{Arg}(z)| < \pi \bigwedge (|z| \to \infty)$$

$$Q^{-1}(a, z) \propto -(a-1)\, W_{-1}\left(-\frac{z^{\frac{1}{a-1}}\,\Gamma(a)^{\frac{1}{a-1}}}{a-1}\right) \;/;\; (z \to 0).$$

Integral representations

The gamma functions $\Gamma(a, z)$, $\Gamma(a, z_1, z_2)$, $Q(a, z)$, $Q(a, z_1, z_2)$, and the log-gamma function $\log\Gamma(z)$ can also be represented through the following integrals:

$$\Gamma(a, z) = \int_z^\infty t^{a-1}\, e^{-t}\, dt$$

$$\Gamma(a, z_1, z_2) = \int_{z_1}^{z_2} t^{a-1}\, e^{-t}\, dt$$

$$Q(a, z) = \frac{1}{\Gamma(a)} \int_z^\infty t^{a-1}\, e^{-t}\, dt$$

$$Q(a, z_1, z_2) = \frac{1}{\Gamma(a)} \int_{z_1}^{z_2} t^{a-1}\, e^{-t}\, dt$$

$$\log\Gamma(z) = -\int_0^\infty \frac{e^{-t}}{t}\left(\frac{e^{t\,z}-1}{1-e^{-t}} - z\right) dt + \log(\pi) - \log(\sin(\pi\, z)) \;/;\; \mathrm{Re}(z) < 1$$

$$\log\Gamma(z) = \int_0^\infty \frac{1}{t}\left((z-1)\, e^{-t} + \frac{e^{-t\,z} - e^{-t}}{1-e^{-t}}\right) dt \;/;\; \mathrm{Re}(z) > 0$$

$$\log\Gamma(z) = 2\int_0^\infty \frac{\tan^{-1}\left(\frac{t}{z}\right)}{e^{2\,\pi\,t}-1}\, dt + \frac{\log(2\,\pi)}{2} + \left(z - \frac{1}{2}\right)\log(z) - z \;/;\; \mathrm{Re}(z) > 0.$$

Transformations

The argument of the log-gamma function $\log\Gamma(a - z)$ can be simplified if $a == 1$ or 0:

$$\log\Gamma(1-z) == \log(\pi) - \log(\sin(\pi z)) - \log\Gamma(z) \;/;\; -\frac{1}{2} \leq \mathrm{Re}(z) < \frac{\pi}{2}$$

$$\log\Gamma(1-z) == \log(\pi) - \log(\sin(\pi z)) - \log\Gamma(z) + 2 i \pi \left\lfloor \frac{2\,\mathrm{Re}(z)+1}{4} \right\rfloor \left(\mathrm{sgn}(\mathrm{Im}(z)) + \left(\mathrm{sgn}(\mathrm{Im}(z))^2 - 1\right) \mathrm{sgn}(\mathrm{Re}(z))\right) \;/;\; \frac{2\,\mathrm{Re}(z)+1}{4} \notin \mathbb{Z}$$

$$\log\Gamma(-z) == \log(\pi) - \log(-z) - \log(\sin(\pi z)) - \log\Gamma(z) \;/;\; -\frac{1}{2} \leq \mathrm{Re}(z) < \frac{\pi}{2}$$

$$\log\Gamma(-z) == \log(\pi) - \log(-z) - \log(\sin(\pi z)) - \log\Gamma(z) + 2 i \pi \left\lfloor \frac{2\,\mathrm{Re}(z)+1}{4} \right\rfloor \left(\mathrm{sgn}(\mathrm{Im}(z)) + \left(\mathrm{sgn}(\mathrm{Im}(z))^2 - 1\right) \mathrm{sgn}(\mathrm{Re}(z))\right) \;/;\; \frac{2\,\mathrm{Re}(z)+1}{4} \notin \mathbb{Z}.$$

Multiple arguments

The log-gamma function $\log\Gamma(m z)$ with $m == 2, 3, \ldots$ can be represented by a formula that follows from the corresponding multiplication formula for the gamma function $\Gamma(z)$:

$$\log\Gamma(2 z) == \log\Gamma\left(z + \frac{1}{2}\right) + \log\Gamma(z) + (2 z - 1) \log(2) - \frac{\log(\pi)}{2}$$

$$\log\Gamma(m z) == \sum_{k=0}^{m-1} \log\Gamma\left(z + \frac{k}{m}\right) + m z \log(m) - \frac{1}{2} (\log(m) + (m-1) \log(2 \pi)) \;/;\; m \in \mathbb{N}^+.$$

Identities

The gamma functions $\Gamma(a, z)$, $\Gamma(a, z_1, z_2)$, $Q(a, z)$, $Q(a, z_1, z_2)$, and the log-gamma function $\log\Gamma(z)$ satisfy the following recurrence identities:

$$\Gamma(a, z) == \frac{1}{a} \left(\Gamma(a+1, z) - e^{-z} z^a\right)$$

$$\Gamma(a, z) == (a-1)\, \Gamma(a-1, z) + e^{-z} z^{a-1}$$

$$\Gamma(a, z_1, z_2) == \frac{1}{a} \left(\Gamma(a+1, z_1, z_2) - e^{-z_1} z_1^a + e^{-z_2} z_2^a\right)$$

$$\Gamma(a, z_1, z_2) == (a-1)\, \Gamma(a-1, z_1, z_2) + e^{-z_1} z_1^{a-1} - z_2^{a-1} e^{-z_2}$$

$$Q(a, z) == Q(a+1, z) - \frac{e^{-z} z^a}{\Gamma(a+1)}$$

$$Q(a, z) == Q(a-1, z) + \frac{e^{-z} z^{a-1}}{\Gamma(a)}$$

$$Q(a, z_1, z_2) == Q(a+1, z_1, z_2) + \frac{e^{-z_2} z_2^a - e^{-z_1} z_1^a}{\Gamma(a+1)}$$

$$Q(a, z_1, z_2) = Q(a-1, z_1, z_2) + \frac{z_1^{a-1} e^{-z_1} - z_2^{a-1} e^{-z_2}}{\Gamma(a)}$$

$$\log\Gamma(z) = \log\Gamma(z+1) - \log(z)$$

$$\log\Gamma(z) = \log\Gamma(z-1) + \log(z-1).$$

The previous formulas can be generalized to the following recurrence identities with a jump of length n:

$$\Gamma(a, z) = \frac{1}{(a)_n} \Gamma(a+n, z) - z^{a-1} e^{-z} \sum_{k=1}^{n} \frac{z^k}{(a)_k} /; n \in \mathbb{N}$$

$$\Gamma(a, z) = (-1)^n (1-a)_n \left(\Gamma(a-n, z) + z^{a-n-1} e^{-z} \sum_{k=1}^{n} \frac{z^k}{(a-n)_k} \right) /; n \in \mathbb{N}$$

$$\Gamma(a, z_1, z_2) = \frac{1}{(a)_n} \Gamma(a+n, z_1, z_2) - e^{-z_1} \sum_{k=1}^{n} \frac{z_1^{a+k-1}}{(a)_k} + e^{-z_2} \sum_{k=1}^{n} \frac{z_2^{a+k-1}}{(a)_k} /; n \in \mathbb{N}$$

$$\Gamma(a, z_1, z_2) = (-1)^n (1-a)_n \left(\Gamma(a-n, z_1, z_2) + e^{-z_1} \sum_{k=1}^{n} \frac{z_1^{a+k-n-1}}{(a-n)_k} - e^{-z_2} \sum_{k=1}^{n} \frac{z_2^{a+k-n-1}}{(a-n)_k} \right) /; n \in \mathbb{N}$$

$$Q(a, z) = Q(a+n, z) - z^{a-1} e^{-z} \sum_{k=1}^{n} \frac{z^k}{\Gamma(a+k)} /; n \in \mathbb{N}$$

$$Q(a, z) = Q(a-n, z) + z^{a-1} e^{-z} \sum_{k=0}^{n-1} \frac{z^{-k}}{\Gamma(a-k)} /; n \in \mathbb{N}$$

$$Q(a, z_1, z_2) = Q(a+n, z_1, z_2) - e^{-z_1} \sum_{k=1}^{n} \frac{z_1^{a+k-1}}{\Gamma(a+k)} + e^{-z_2} \sum_{k=1}^{n} \frac{z_2^{a+k-1}}{\Gamma(a+k)} /; n \in \mathbb{N}$$

$$Q(a, z_1, z_2) = Q(a-n, z_1, z_2) + e^{-z_1} \sum_{k=0}^{n-1} \frac{z_1^{a-k-1}}{\Gamma(a-k)} - e^{-z_2} \sum_{k=0}^{n-1} \frac{z_2^{a-k-1}}{\Gamma(a-k)} /; n \in \mathbb{N}$$

$$\log\Gamma(z) = \log\Gamma(z+n) - \sum_{k=0}^{n-1} \log(z+k) /; n \in \mathbb{N}$$

$$\log\Gamma(z) = \log\Gamma(z-n) + \sum_{k=1}^{n} \log(z-k) /; n \in \mathbb{N}.$$

Representations of derivatives

The derivatives of the gamma functions $\Gamma(a, z)$, $\Gamma(a, z_1, z_2)$, $Q(a, z)$, and $Q(a, z_1, z_2)$ with respect to the variables z, z_1, and z_2 have simple representations in terms of elementary functions:

$$\frac{\partial \Gamma(a, z)}{\partial z} = -e^{-z} z^{a-1}$$

$$\frac{\partial \Gamma(a, z_1, z_2)}{\partial z_1} = -e^{-z_1} z_1^{a-1}$$

$$\frac{\partial \Gamma(a, z_1, z_2)}{\partial z_2} = e^{-z_2} z_2^{a-1}$$

$$\frac{\partial Q(a, z)}{\partial z} = -\frac{e^{-z} z^{a-1}}{\Gamma(a)}$$

$$\frac{\partial Q(a, z_1, z_2)}{\partial z_1} = -\frac{e^{-z_1} z_1^{a-1}}{\Gamma(a)}$$

$$\frac{\partial Q(a, z_1, z_2)}{\partial z_2} = \frac{e^{-z_2} z_2^{a-1}}{\Gamma(a)}.$$

The derivatives of the log-gamma function $\log\Gamma(z)$ and the inverses of the regularized incomplete gamma functions $Q^{-1}(a, z)$, and $Q^{-1}(a, z_1, z_2)$ with respect to the variables z, z_1, and z_2 have more complicated representations by the formulas:

$$\frac{\partial \log\Gamma(z)}{\partial z} = \psi(z)$$

$$\frac{\partial Q^{-1}(a, z)}{\partial z} = -e^{Q^{-1}(a,z)} Q^{-1}(a, z)^{1-a} \Gamma(a)$$

$$\frac{\partial Q^{-1}(a, z_1, z_2)}{\partial z_1} = e^{Q^{-1}(a,z_1,z_2)-z_1} \left(\frac{Q^{-1}(a, z_1, z_2)}{z_1}\right)^{1-a}$$

$$\frac{\partial Q^{-1}(a, z_1, z_2)}{\partial z_2} = e^{Q^{-1}(a,z_1,z_2)} \Gamma(a) Q^{-1}(a, z_1, z_2)^{1-a}.$$

The derivative of the exponential integral $E_\nu(z)$ by its parameter ν can be represented in terms of the regularized hypergeometric function ${}_2\tilde{F}_2$:

$$\frac{\partial E_\nu(z)}{\partial \nu} = z^{\nu-1} \Gamma(1-\nu) (\log(z) - \psi(1-\nu)) - \Gamma(1-\nu)^2 \, {}_2\tilde{F}_2(1-\nu, 1-\nu; 2-\nu, 2-\nu; -z).$$

The derivatives of the gamma functions $\Gamma(a, z)$, $\Gamma(a, z_1, z_2)$, $Q(a, z)$, and $Q(a, z_1, z_2)$, and their inverses $Q^{-1}(a, z)$ and $Q^{-1}(a, z_1, z_2)$ with respect to the parameter a can be represented in terms of the regularized hypergeometric function ${}_2\tilde{F}_2$:

$$\frac{\partial \Gamma(a, z)}{\partial a} = \Gamma(a)^2 z^a \, {}_2\tilde{F}_2(a, a; a+1, a+1; -z) - \Gamma(a, 0, z) \log(z) + \Gamma(a) \psi(a)$$

$$\frac{\partial \Gamma(a, z_1, z_2)}{\partial a} =$$
$$\Gamma(a)^2 \, {}_2\tilde{F}_2(a, a; a+1, a+1; -z_1) z_1^a - \Gamma(a)^2 \, {}_2\tilde{F}_2(a, a; a+1, a+1; -z_2) z_2^a - \Gamma(a, 0, z_1) \log(z_1) + \Gamma(a, 0, z_2) \log(z_2)$$

$$\frac{\partial Q(a, z)}{\partial a} = \Gamma(a)\, z^a\, {}_2\tilde{F}_2(a, a; a+1, a+1; -z) + Q(a, z, 0)\,(\log(z) - \psi(a))$$

$$\frac{\partial Q(a, z_1, z_2)}{\partial a} = \Gamma(a)\, z_1^a\, {}_2\tilde{F}_2(a, a; a+1, a+1; -z_1) - \Gamma(a)\, z_2^a\, {}_2\tilde{F}_2(a, a; a+1, a+1; -z_2) + Q(a, z_1, 0)\log(z_1) - Q(a, z_2, 0)\log(z_2) - \psi(a)\, Q(a, z_1, z_2)$$

$$\frac{\partial Q^{-1}(a, z)}{\partial a} = e^w\, w^{1-a}\left(\Gamma(a)^2\, {}_2\tilde{F}_2(a, a; a+1, a+1; -w)\, w^a + (z-1)\,\Gamma(a)\log(w) + (\Gamma(a) - \Gamma(a, w))\,\psi(a)\right) /; w = Q^{-1}(a, z)$$

$$\frac{\partial Q^{-1}(a, z_1, z_2)}{\partial a} = e^w\, w^{1-a}\left(\frac{1}{a^2}\left(w^a\, {}_2F_2(a, a; a+1, a+1; -w) - z_1^a\, {}_2F_2(a, a; a+1, a+1; -z_1)\right) + \Gamma(a, w, 0)\log(w) + \Gamma(a, 0, z_1)\log(z_1) + \Gamma(a, z_1, w)\,\psi(a)\right) /; w = Q^{-1}(a, z_1, z_2).$$

The symbolic n^{th}-order derivatives of all gamma functions $\Gamma(a, z)$, $\Gamma(a, z_1, z_2)$, $Q(a, z)$, $Q(a, z_1, z_2)$, and their inverses $Q^{-1}(a, z)$, and $Q^{-1}(a, z_1, z_2)$ have the following representations:

$$\frac{\partial^n \Gamma(a, z)}{\partial z^n} = z^{-n} \sum_{k=0}^{n} (-1)^n \binom{n}{k} (-a)_k\, \Gamma(a-k+n, z) /; n \in \mathbb{N}$$

$$\frac{\partial^n \Gamma(a, z)}{\partial a^n} = \Gamma^{(n)}(a) - z^a \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} (n-j)!\, \Gamma(a)^{n-j+1} \log^j(z)\, {}_{n-j+1}\tilde{F}_{n-j+1}\left(a_1, a_2, \ldots, a_{n-j+1}; a_1+1, a_2+1, \ldots, a_{n-j+1}+1; -z\right) /; a_1 = a_2 = \ldots = a_{n+1} = a \land n \in \mathbb{N}$$

$$\frac{\partial^n \Gamma(a, z_1, z_2)}{\partial z_1^n} = z_1^{-n} \sum_{k=0}^{n} (-1)^n \binom{n}{k} (-a)_k\, \Gamma(a-k+n, z_1) /; n \in \mathbb{N}^+$$

$$\frac{\partial^n \Gamma(a, z_1, z_2)}{\partial z_2^n} = -z_2^{-n} \sum_{k=0}^{n} (-1)^n \binom{n}{k} (-a)_k\, \Gamma(a-k+n, z_2) /; n \in \mathbb{N}^+$$

$$\frac{\partial^n \Gamma(a, z_1, z_2)}{\partial a^n} = z_2^a \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} (n-j)!\, \Gamma(a)^{n-j+1} \log^j(z_2)\, {}_{n-j+1}\tilde{F}_{n-j+1}\left(a_1, a_2, \ldots, a_{n-j+1}; a_1+1, a_2+1, \ldots, a_{n-j+1}+1; -z_2\right) - z_1^a \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} (n-j)!\, \Gamma(a)^{n-j+1} \log^j(z_1)\, {}_{n-j+1}\tilde{F}_{n-j+1}\left(a_1, a_2, \ldots, a_{n-j+1}; a_1+1, a_2+1, \ldots, a_{n-j+1}+1; -z_1\right) /; a_1 = a_2 = \ldots = a_{n+1} = a \land n \in \mathbb{N}$$

$$\frac{\partial^n Q(a, z)}{\partial z^n} = -a\, z^{-n} \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} (1-a-k)_{n-1}\, Q(a+k, z) /; n \in \mathbb{N}$$

$$\frac{\partial^n Q(a, z)}{\partial a^n} = \frac{\Gamma^{(n)}(a)}{\Gamma(a)} - \frac{1}{\Gamma(a)} \sum_{k=0}^{\infty} \frac{(-1)^{n-k}\, \Gamma(n+1, -(a+k)\log(z))}{(a+k)^{n+1}\, k!} \;/;\; n \in \mathbb{N}$$

$$\frac{\partial^n Q(a, z_1, z_2)}{\partial z_1^n} = -a\, z_1^{-n} \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} (1-a-k)_{n-1}\, Q(a+k, z_1) \;/;\; n \in \mathbb{N}^+$$

$$\frac{\partial^n Q(a, z_1, z_2)}{\partial z_2^n} = a\, z_2^{-n} \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} (1-a-k)_{n-1}\, Q(a+k, z_2) \;/;\; n \in \mathbb{N}^+$$

$$\frac{\partial^n Q(a, z_1, z_2)}{\partial a^n} = n! \sum_{k=0}^{n} \Bigg(z_2^a \sum_{i=0}^{n-k} (-1)^{n-i-k} \binom{n-k}{i} (n-i-k)!\, \Gamma(a)^{n-i-k+1} \log^i(z_2)\, {}_{n-k-i+1}\tilde{F}_{n-k-i+1}(a_1, a_2, \ldots, a_{n-k-i+1}; a_1+1, a_2+1, \ldots, a_{n-k-i+1}+1; -z_2) - z_1^a \sum_{i=0}^{n-k} (-1)^{n-i-k} \binom{n-k}{i} (n-i-k)!\, \Gamma(a)^{n-i-k+1} \log^i(z_1)\, {}_{n-k-i+1}\tilde{F}_{n-k-i+1}(a_1, a_2, \ldots, a_{n-k-i+1}; a_1+1, a_2+1, \ldots, a_{n-k-i+1}+1; -z_1) \Bigg) \sum_{j=0}^{k} \frac{(-1)^j (k+1)\, \Gamma(a)^{-j-1}}{(j+1)!\,(n-k)!\,(k-j)!} \frac{\partial^k \Gamma(a)^j}{\partial a^k} \;/;\; a_1 = a_2 = \ldots = a_{n+1} = a \bigwedge n \in \mathbb{N}$$

$$\frac{\partial^\alpha \log\Gamma(z)}{\partial z^\alpha} = \psi^{(\alpha-1)}(z)$$

$$\frac{\partial^n Q^{-1}(a, z)}{\partial z^n} = w\, \delta_n + \left(-\frac{\Gamma(a)\, e^w}{w^{a-1}}\right)^n \sum_{j_2=0}^{n} \cdots \sum_{j_n=0}^{n} \delta_{\sum_{i=2}^n (i-1)\, j_i, n-1} (-1)^{\sum_{i=2}^n j_i} \left(n + \sum_{i=2}^{n} j_i - 1\right)! \prod_{i=2}^{n} \frac{1}{j_i!} \left(\frac{\Gamma(a+1)\, e^w\, w^{-a-i+1}}{i!}\right)^{j_i} \left(\sum_{k=0}^{i} (-1)^{i-k} \binom{i}{k} (-a-k+1)_{i-1}\, Q(a+k, w)\right)^{j_i} \;/;\; w = Q^{-1}(a, z) \bigwedge n \in \mathbb{N}$$

$$\frac{\partial^n Q^{-1}(a, z_1, z_2)}{\partial z_2^n} = w\, \delta_n + \grave{}\left(\frac{\Gamma(a)\, e^w}{w^{a-1}}\right)^n \sum_{j_2=0}^{n} \cdots \sum_{j_n=0}^{n} \delta_{\sum_{i=2}^n (i-1)\, j_i, n-1} (-1)^{\sum_{i=2}^n j_i} \left(n + \sum_{i=2}^{n} j_i - 1\right)! \prod_{i=2}^{n} \frac{1}{j_i!} \left(\frac{\Gamma(a+1)\, e^w\, w^{1-a}}{i!}\right)^{j_i} \left(a\, w^{-i} \sum_{k=0}^{i} (-1)^{i-k} \binom{i}{k} (1-a-k)_{i-1}\, Q(a+k, w) + Q(a, z_1)\, \delta_i\right)^{j_i} \;/;\; w = Q^{-1}(a, z_1, z_2) \bigwedge n \in \mathbb{N}.$$

Differential equations

The gamma functions $\Gamma(a, z)$, $\Gamma(a, z_1, z_2)$, $Q(a, z)$, and $Q(a, z_1, z_2)$ satisfy the following second-order linear differential equations:

$$z\, w''(z) + (1-a+z)\, w'(z) = 0 \;/;\; w(z) = c_1\, \Gamma(a, z) + c_2$$

$$z_1\, w''(z_1) + (1-a+z_1)\, w'(z_1) = 0 \;/;\; w(z_1) = c_1\, \Gamma(a, z_1, z_2) + c_2$$

$$z_2\, w''(z_2) + (1-a+z_2)\, w'(z_2) = 0 \;/;\; w(z_2) = c_1\, \Gamma(a, z_1, z_2) + c_2$$

$$z\, w''(z) + (1 - a + z)\, w'(z) == 0 \;/;\; w(z) == c_1\, Q(a, z) + c_2$$

$$z_1\, w''(z_1) + (1 - a + z_1)\, w'(z_1) == 0 \;/;\; w(z_1) == c_1\, Q(a, z_1, z_2) + c_2$$

$$z_2\, w''(z_2) + (1 - a + z_2)\, w'(z_2) == 0 \;/;\; w(z_2) == c_1\, Q(a, z_1, z_2) + c_2,$$

where c_1 and c_2 are arbitrary constants.

The log-gamma function $\log\Gamma(z)$ satisfies the following simple first-order linear differential equation:

$$\frac{\partial w(z)}{\partial z} == \psi(z) \;/;\; w(z) == \log\Gamma(z).$$

The inverses of the regularized incomplete gamma functions $Q^{-1}(a, z)$ and $Q^{-1}(a, z_1, z_2)$ satisfy the following ordinary nonlinear second-order differential equation:

$$w(z)\, w''(z) - w'(z)^2\, (w(z) + 1 - a) == 0 \;/;\; w(z) == Q^{-1}(a, z)$$

$$w(z_2)\, w''(z_2) - w'(z_2)^2\, (-a + w(z_2) + 1) == 0 \;/;\; w(z_2) == Q^{-1}(a, z_1, z_2).$$

Applications of gamma functions

The gamma functions are used throughout mathematics, the exact sciences, and engineering. In particular, the incomplete gamma function is used in solid state physics and statistics, and the logarithm of the gamma function is used in discrete mathematics, number theory, and other fields of sciences.

Introduction to the Gamma Function

General

The gamma function $\Gamma(z)$ is used in the mathematical and applied sciences almost as often as the well-known factorial symbol $n!$. It was introduced by the famous mathematician L. Euler (1729) as a natural extension of the factorial operation $n!$ from positive integers n to real and even complex values of the argument n. This relation is described by the following formula:

$$\Gamma(n) = (n - 1)!.$$

L. Euler derived some basic properties and formulas for the gamma function. He started investigations of $n!$ from the infinite product

$$n! == \lim_{m \to \infty} \frac{m!\, (m + 1)^n}{\prod_{k=1}^{m} (k + n)}$$

and the integral

$$\int_0^1 t^{a-1}\, (1 - t)^{b-1}\, dt,$$

which is currently known as the beta function integral. As a result, Euler derived the following integral representation for factorial $n!$:

$$n! == \int_0^1 (-\log(t))^n \, dt,$$

which can be easily converted into the well-known Euler integral for the gamma function:

$$.\Gamma(n+1) == n! == \int_0^\infty \tau^n \, e^{-\tau} \, d\tau.$$

Also, during his research, Euler closely approached the famous reflection formula:

$$\Gamma(z)\,\Gamma(z) == \frac{\pi}{\sin(\pi z)},$$

which later got his name.

At the same time, J. Stirling (1730) found the famous asymptotic formula for the factorial, which bears his name. This formula was also naturally applied to the gamma function resulting in the following asymptotic relation:

$$\Gamma(x) \propto \sqrt{2\pi} \; x^{x-\frac{1}{2}} \, e^{-x} \; /; \; (x \to \infty).$$

Later, A. M. Legendre (1808, 1814) suggested the current symbol Γ for the gamma function and discovered the duplication formula:

$$\Gamma(2z) == \frac{2^{2z-1}}{\sqrt{\pi}} \, \Gamma(z)\,\Gamma\!\left(z + \frac{1}{2}\right).$$

It was generalized by C. F. Gauss (1812) to the multiplication formula:

$$\Gamma(n z) == n^{n z - \frac{1}{2}} \, (2\pi)^{\frac{1-n}{2}} \prod_{k=0}^{n-1} \Gamma\!\left(z + \frac{k}{n}\right) /; \; n \in \mathbb{N}^+.$$

F. W. Newman (1848) studied the reciprocal of the gamma function and found that it is an entire function and has the following product representation valid for the whole complex plane:

$$\frac{1}{\Gamma(z)} == z \, e^{z\gamma} \prod_{k=1}^{\infty} \left(1 + \frac{z}{k}\right) e^{-\frac{z}{k}},$$

where $\gamma = 0.5772156 \ldots$ is the Euler–Mascheroni gamma constant.

B. Riemann (1856) proved an important relation between the gamma and zeta functions:

$$\zeta(s) == \zeta(1-s)\,\Gamma(1-s)\,2^s\,\pi^{s-1} \; \sin\!\left(\frac{\pi s}{2}\right),$$

which was mentioned centuries ago in an article by Euler (1749) for particular values of the argument s.

K. Weierstrass (1856) and other nineteenth century mathematicians widely used the gamma function in their investigations and discovered many more complicated properties and formulas for it. In particular, H. Hankel (1864, 1880) derived its contour integral representation for complex arguments, and O. Hölder (1887) proved that the gamma function does not satisfy any algebraic differential equation. This result was subsequently re-proved by A. Ostrowski (1925).

Many mathematicians devote special attention to the question of the uniqueness of extending the factorial operation $n!$ from positive integers to arbitrary real or complex values. Evidently this question is connected to the solutions of the functional equation:

$$\Phi(z) = z\,\Phi(z-1).$$

J. Hadamard (1894) found that the function $y(z) == 1/\Gamma(1-z)\,\frac{\partial}{\partial z}\log(\Gamma((1-z)/2)/\Gamma(1-z/2))$ is an entire analytic function that coincides with $(z-1)!$ for $z == 1, 2, 3, \ldots$. But this function satisfies the more complicated functional equation $\phi(z+1) == z\,\phi(z) + 1/\Gamma(1-z)$ and has a more complicated integral representation than the classical gamma function defined by the Euler integral.

H. Bohr and J. Mollerup (1922) proved that the gamma function $\Phi(z) = \Gamma(z)$ is the only function that satisfies the recurrence relationship $\phi(z+1) == z\,\phi(z)$, is positive for $z > 0$, equals one at $z == 1$, and is logarithmically convex (that is, $\log(\Gamma(z))$ is convex). If the restriction on convexity is absent, then the recurrence relationship has an infinite set of solutions in the form $\phi(z) == \theta(z)\,\Gamma(z)$, where $\theta(z)$ is an arbitrary periodic function with period 1.

Definition of gamma function

The gamma function $\Gamma(z)$ in the half-plane $\mathrm{Re}(z) > 0$ is defined as the value of the following definite integral:

$$\Gamma(z) == \int_0^\infty t^{z-1}\,e^{-t}\,dt\ /;\ \mathrm{Re}(z) > 0.$$

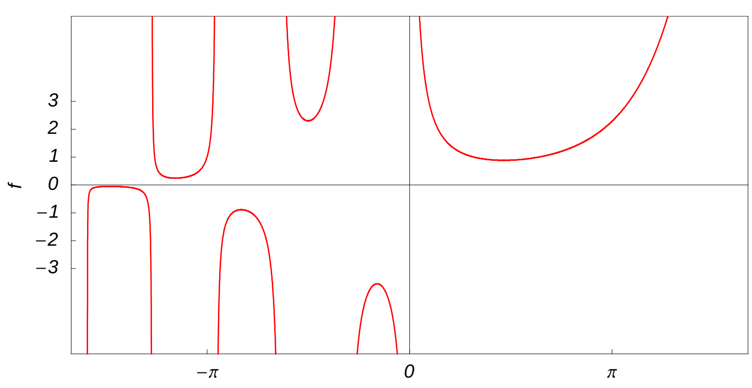
This integral is an analytic function that can be represented in different forms; for example, as the following sum of an integral and a series without any restrictions on the argument:

$$\Gamma(z) == \int_1^\infty t^{z-1}\,e^{-t}\,dt + \sum_{k=0}^{\infty}\frac{(-1)^k}{k!\,(k+z)}.$$

The last formula can also be used as an equivalent definition of the gamma function.

A quick look at the gamma function

Here is a quick look at the graphics for the gamma function along the real axis.



Connections within the group of gamma functions and with other function groups

Representations through more general functions

The gamma function $\Gamma(z)$ is the main example of a group of functions collectively referred to as gamma functions. For example, it can be written in terms of the incomplete gamma function:

$$\Gamma(z) = \Gamma(z, 0) \;/;\; \mathrm{Re}(z) > 0.$$

All four incomplete gamma functions $\Gamma(a, z)$, $\Gamma(a, z_1, z_2)$, $Q(a, z)$, and $Q(a, z_1, z_2)$ can be represented as cases of the hypergeometric function ${}_1F_1$. Further, the gamma function $\Gamma(z)$ is the special degenerate case of the hypergeometric function ${}_1F_1$.

Representations through related equivalent functions

The gamma function $\Gamma(z)$ and two factorial functions are connected by the formulas:

$$\Gamma(z) = (z-1)!$$

$$\Gamma(z) = 2^{\frac{1}{4}(3-4z+\cos(2\pi z))}\, \pi^{\frac{1}{2}\sin^2(\pi z)}\, (2z-2)!!.$$

The best-known properties and formulas for the gamma function

Values at points

The gamma function $\Gamma(z)$ can be exactly evaluated in the points $z = \frac{n}{2} \;/;\; n \in \mathbb{Z}$. Here are examples:

$$\begin{array}{llll}
\Gamma(-3) = \tilde{\infty} & \Gamma\left(-\frac{5}{2}\right) = -\frac{8}{15}\sqrt{\pi} & \Gamma(-2) = \tilde{\infty} & \Gamma\left(-\frac{3}{2}\right) = \frac{4\sqrt{\pi}}{3} \\
\Gamma(-1) = \tilde{\infty} & \Gamma\left(-\frac{1}{2}\right) = -2\sqrt{\pi} & \Gamma(0) = \tilde{\infty} & \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \\
\Gamma(1) = 1 & \Gamma\left(\frac{3}{2}\right) = \frac{\sqrt{\pi}}{2} & \Gamma(2) = 1 & \Gamma\left(\frac{5}{2}\right) = \frac{3\sqrt{\pi}}{4} \\
\Gamma(3) = 2 & \Gamma\left(\frac{7}{2}\right) = \frac{15\sqrt{\pi}}{8} & \Gamma(4) = 6 & \Gamma\left(\frac{9}{2}\right) = \frac{105\sqrt{\pi}}{16}.
\end{array}$$

Specific values for specialized variables

The preceding evaluations can be provided by the formulas:

$$\Gamma(n) = (n-1)! \;/;\; n \in \mathbb{N}^+$$

$$\Gamma(-n) = \tilde{\infty} \;/;\; n \in \mathbb{N}$$

$$\Gamma\left(\frac{n}{2}\right) = \frac{2^{1-n}\sqrt{\pi}\,(n-1)!}{\frac{n-1}{2}!} \;/;\; n \in \mathbb{N}^+$$

$$\Gamma\left(-\frac{n}{2}\right) = \frac{(-1)^{\frac{n+1}{2}}\, 2^n \sqrt{\pi}\, \frac{n-1}{2}!}{n!} \;/;\; n \in \mathbb{N}^+.$$

At the points $z = n + \frac{p}{q} \;/;\; n \in \mathbb{Z} \land p \in \mathbb{N}^+ \land q \in \mathbb{N}^+ \land p < q$, the values of the gamma function $\Gamma(z)$ can be represented through values of $\Gamma\left(\frac{p}{q}\right)$:

$$\Gamma\left(\frac{p}{q}+n\right) == \frac{1}{q^n}\,\Gamma\left(\frac{p}{q}\right)\prod_{k=1}^{n}(p+k\,q-q) \;/;\; n\in\mathbb{N} \bigwedge p\in\mathbb{N}^+ \bigwedge q\in\mathbb{N}^+ \bigwedge p<q$$

$$\Gamma\left(\frac{p}{q}-n\right) == \frac{(-1)^n\,q^n}{\prod_{k=1}^{n}(q\,k-p)}\,\Gamma\left(\frac{p}{q}\right) \;/;\; n\in\mathbb{N} \bigwedge p\in\mathbb{N}^+ \bigwedge q\in\mathbb{N}^+ \bigwedge p<q.$$

Real values for real arguments

For real values of argument z, the values of the gamma function $\Gamma(z)$ are real (or infinity). The gamma function is not equal to zero:

$$\Gamma(z) \neq 0 \;/;\; \forall\, z.$$

Analyticity

The gamma function $\Gamma(z)$ is an analytical function of z, which is defined over the whole complex z-plane with the exception of countably many points $z = -k \;/;\; k\in\mathbb{N}$. The reciprocal of the gamma function $1/\Gamma(z)$ is an entire function.

Poles and essential singularities

The function $\Gamma(z)$ has an infinite set of singular points $z == -k \;/;\; k\in\mathbb{N}$, which are the simple poles with residues $\frac{(-1)^k}{k!}$. The point $z == \tilde{\infty}$ is the accumulation point of the poles, which means that $\tilde{\infty}$ is an essential singular point.

Branch points and branch cuts

The function $\Gamma(z)$ does not have branch points and branch cuts.

Periodicity

The function $\Gamma(z)$ does not have periodicity.

Parity and symmetry

The function $\Gamma(z)$ has mirror symmetry:

$$\Gamma(\bar{z}) == \overline{\Gamma(z)}.$$

Differentiation

The derivatives of $\Gamma(z)$ can be represented through gamma and polygamma functions:

$$\frac{\partial \Gamma(z)}{\partial z} == \Gamma(z)\,\psi(z)$$

$$\frac{\partial^2 \Gamma(z)}{\partial z^2} == \Gamma(z)\,\psi(z)^2 + \Gamma(z)\,\psi^{(1)}(z).$$

Ordinary differential equation

The gamma function $\Gamma(z)$ does not satisfy any algebraic differential equation (O. Hölder, 1887). But it is the solution of the following nonalgebraic equation:

$$\frac{\partial w(z)}{\partial z} == w(z)\,\psi(z) \text{ /; } w(z) == \Gamma(z).$$

Series representations

Series representations of the gamma function $\Gamma(z)$ near the poles $z == 0, -1, -2, \ldots$ are of great interest for applications in the theory of generalized hypergeometric, Meijer G, and Fox H functions. These representations can be described by the formulas:

$$\Gamma(z) \propto \frac{(-1)^n}{n!\,(z+n)} + \frac{(-1)^n\,\psi(n+1)}{n!} + O(z+n) \text{ /; } (z \to -n) \land n \in \mathbb{N}$$

$$\Gamma(z) \propto \frac{(-1)^n}{n!\,(z+n)} \sum_{v=0}^{\infty} \sum_{s=0}^{v} \frac{(-1)^{\frac{s+v}{2}-1}\,(2^{v-s}-2)\,B_{v-s}\,\pi^{v-s}\,\Gamma(n+1)}{s!\,(v-s)!} \left(\frac{\partial^s \frac{1}{\Gamma(z)}}{\partial z^s} \text{ /. } \{z \to n+1\} \right) (z+n)^v \text{ /; } (z \to -n) \land n \in \mathbb{N},$$

where B_v are the Bernoulli numbers.

Asymptotic series expansions

Asymptotic behavior of the gamma function $\Gamma(z)$ is described by the famous Stirling formula:

$$\Gamma(z) \propto \sqrt{2\pi}\; z^{z-\frac{1}{2}}\, e^{-z} \left(1 + O\!\left(\frac{1}{z}\right)\right) \text{ /; } |\mathrm{Arg}(z)| < \pi \bigwedge (|z| \to \infty).$$

This formula allows derivation of the following asymptotic expansion for the ratio of gamma functions:

$$\frac{\Gamma(z+a)}{\Gamma(z+b)} \propto z^{a-b} \left(1 + O\!\left(\frac{1}{z}\right)\right) \text{ /; } |\mathrm{Arg}(a+z)| < \pi \bigwedge (|z| \to \infty).$$

Integral representations

The gamma function $\Gamma(z)$ has several integral representations that are different from the Euler integral:

$$\Gamma(z) == \int_0^{\infty} t^{z-1}\, e^{-t}\, dt \text{ /; } \mathrm{Re}(z) > 0$$

and related integral

$$\Gamma(z) == \int_1^{\infty} t^{z-1}\, e^{-t}\, dt + \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,(k+z)},$$

which can be used for defining the gamma function over the whole complex plane.

Some of the integral representations are the following:

$$\Gamma(z) == \int_0^{\infty} \left(e^{-t} - \sum_{k=0}^{n} \frac{(-t)^k}{k!} \right) t^{z-1}\, dt \text{ /; } n \in \mathbb{N} \land -n-1 < \mathrm{Re}(z) < -n$$

$$\Gamma(z) == \int_0^{1} \log^{z-1}\!\left(\frac{1}{t}\right) dt \text{ /; } \mathrm{Re}(z) > 0$$

$$\Gamma(z) = e^{-\gamma z}\, e^{\int_0^1 \frac{x^z - \log(x^z) - 1}{(x-1)\log(x)}\, dx} \,/;\, \mathrm{Re}\,(z) > 0$$

$$\Gamma(z) = e^{\int_0^1 \frac{x^z - z\,(x-1) - 1}{(x-1)\log(x)}\, dx} \,/;\, \mathrm{Re}(z) > 0$$

$$\Gamma(z) = s^z \int_0^{e^{i\delta}\infty} t^{z-1}\, e^{-s\,t}\, dt \,/;\, \mathrm{Re}(z) > 0 \bigwedge |\delta + \mathrm{Arg}(s)| < \frac{\pi}{2} \bigvee 0 < \mathrm{Re}(z) < 1 \bigwedge |\delta + \mathrm{Arg}(s)| = \frac{\pi}{2}$$

$$\Gamma(z) = \frac{1}{e^{2\pi i z} - 1} \int_L e^{-t}\, t^{z-1}\, dt.$$

This final formula is known as Hankel's contour integral. The path of integration L starts at $\infty + i\,0$ on the real axis, goes to $\epsilon + i\,0$, circles the origin in the counterclockwise direction with radius ϵ to the point $\epsilon - i\,0$, and returns to the point $\infty - i\,0$.

Product representations

The following infinite product representation for $\Gamma(z)$ clearly illustrates that $\Gamma(z) = \tilde{\infty}$ at $z = -k \bigwedge k \in \mathbb{N}$:

$$\Gamma(z) = \frac{1}{z} \prod_{k=1}^{\infty} \frac{\left(1 + \frac{1}{k}\right)^z}{1 + \frac{z}{k}}.$$

The similar product representation for $1/\Gamma(z)$ illustrates that $\Gamma(z)$ is an entire function:

$$\frac{1}{\Gamma(z)} = z\, e^{z\gamma} \prod_{k=1}^{\infty} \left(1 + \frac{z}{k}\right) e^{-\frac{z}{k}}.$$

Limit representations

The following famous limit representation for $\Gamma(z)$ was known to L. Euler:

$$\Gamma(z) = \lim_{n\to\infty} \frac{(n+1)^{z-1}\, n!}{(z)_n}.$$

It can be modified to the following related limit representations:

$$\Gamma(z) = \lim_{n\to\infty} \frac{(n+1)^z\, n!}{(z)_{n+1}}$$

$$\Gamma(z) = \lim_{n\to\infty} \frac{n!\, n^z}{(z)_{n+1}}$$

$$\Gamma(z) = \lim_{n\to\infty} \frac{(1)_n\, n^{z-1}}{(z)_n}.$$

The gamma function can be evaluated as the limit of the following definite integral:

$$\Gamma(z) = \lim_{n\to\infty} \int_0^n \left(1 - \frac{t}{n}\right)^n t^{z-1}\, dt \,/;\, \mathrm{Re}(z) > 0.$$

Definite integration

The most famous definite integrals, including the gamma function, belong to the class of Mellin–Barnes integrals. They are used to provide a uniform representation of all generalized hypergeometric, Meijer G, and Fox H functions. For example, the Meijer G function is defined as the value of the following Mellin–Barnes integral:

$$G_{p,q}^{m,n}\left(z \,\middle|\, \begin{matrix} a_1, \ldots, a_n, a_{n+1}, \ldots, a_p \\ b_1, \ldots, b_m, b_{m+1}, \ldots, b_q \end{matrix}\right) = \frac{1}{2\pi i} \int_{\mathcal{L}} \frac{\left(\prod_{k=1}^{m} \Gamma(s+b_k)\right) \prod_{k=1}^{n} \Gamma(1-a_k-s)}{\left(\prod_{k=n+1}^{p} \Gamma(s+a_k)\right) \prod_{k=m+1}^{q} \Gamma(1-b_k-s)} z^{-s}\, ds \;/;$$
$$m \in \mathbb{N} \land n \in \mathbb{N} \land p \in \mathbb{N} \land q \in \mathbb{N} \land m \leq q \land n \leq p.$$

The infinite contour of integration $\mathcal{L}$ separates the poles of $\Gamma(1-a_k-s)$ at $s = 1-a_k+j,\ j \in \mathbb{N}$ from the poles of $\Gamma(b_i + s)$ at $s = -b_i - l,\ l \in \mathbb{N}$. Such a contour always exists in the cases $a_k - b_i - 1 \notin \mathbb{N}$.

There are three possibilities for the contour $\mathcal{L}$:

(i) $\mathcal{L}$ runs from γ-$i\infty$ to γ+$i\infty$ (where $\mathrm{Im}(\gamma) = 0$) so that all poles of $\Gamma(b_i + s)$, $i = 1, \ldots, m$, are to the left of $\mathcal{L}$, and all the poles of $\Gamma(1-a_i - s)$, $i = 1, \ldots, n$ are to the right of $\mathcal{L}$. This contour can be a straight line $(\gamma - i\infty,\ \gamma + i\infty)$ if $\mathrm{Re}(b_i - a_k) > -1$ (then $-\mathrm{Re}(b_i) < \gamma < 1 - \mathrm{Re}(a_k)$). In this case, the integral converges if $p+q < 2(m+n)$, $|\mathrm{Arg}(z)| < \left(m+n-\frac{p+q}{2}\right)\pi$. If $m+n-\frac{p+q}{2} = 0$, then z must be real and positive, and the additional condition $(q-p)\gamma + \mathrm{Re}(\mu) < 0$, $\mu = \sum_{l=1}^{q} b_l - \sum_{k=1}^{p} a_k + \frac{p-q}{2} + 1$, should be added.

(ii) $\mathcal{L}$ is a left loop, starting and ending at $-\infty$ and encircling all poles of $\Gamma(b_i + s)$, $i = 1, \ldots, m$, once in the positive direction, but none of the poles of $\Gamma(1-a_i - s)$, $i = 1, \ldots, n$. In this case, the integral converges if $q \geq 1$ and either $q > p$ or $q = p$ and $|z| < 1$ or $q = p$ and $|z| = 1$ and $m+n-\frac{p+q}{2} \geq 0$ and $\mathrm{Re}(\mu) < 0$.

(iii) $\mathcal{L}$ is a right loop, starting and ending at $+\infty$ and encircling all poles of $\Gamma(1-a_i-s)$, $i = 1, \ldots, n$, once in the negative direction, but none of the poles of $\Gamma(b_i+s)$, $i = 1, \ldots, m$. In this case, the integral converges if $p \geq 1$, and either $p > q$ or $p = q$ and $|z| > 1$ or $q = p$ and $|z| = 1$ and $m+n-\frac{p+q}{2} \geq 0$ and $\mathrm{Re}(\mu) < 0$.

In particular cases, the last integral can be evaluated using simpler elementary and special functions:

$$\frac{1}{2\pi i} \int_{\mathcal{L}} \Gamma(s+b)\, z^{-s}\, ds = e^{-z} z^b$$

$$\frac{1}{2\pi i} \int_{\mathcal{L}} \Gamma(s+b)\, \Gamma(1-a-s)\, z^{-s}\, ds = \Gamma(1-a+b)\, z^b\, (z+1)^{a-b-1}$$

$$\frac{1}{2\pi i} \int_{\mathcal{L}} \frac{\Gamma(s)\,(1-x)^{-s}}{\Gamma(s+1)}\, ds = \theta(x) \;/;\; x < 2$$

$$\frac{1}{4\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \Gamma(s)\, \Gamma(s-\nu) \left(\frac{z}{2}\right)^{\nu-2s} = K_\nu(z) \;/;\; \gamma > \max(\mathrm{Re}(\nu), 0)$$

$$\frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\Gamma(s)}{\Gamma(1+\nu-s)} \left(\frac{x}{2}\right)^{\nu-2s} ds = J_\nu(x) \;/;\; x > 0 \bigwedge 0 < \gamma < \frac{3}{4} + \frac{\mathrm{Re}(\nu)}{2}$$

$$\int_{\gamma-i\infty}^{\gamma+i\infty}\Gamma(a+t)\,\Gamma(b+t)\,\Gamma(c-t)\,\Gamma(d-t)\,dt == \frac{2\pi i\,\Gamma(a+c)\,\Gamma(a+d)\,\Gamma(b+c)\,\Gamma(b+d)}{\Gamma(a+b+c+d)}\ /;$$
$$-\min(\operatorname{Re}(a),\,\operatorname{Re}(b)) < \gamma < \min(\operatorname{Re}(c),\,\operatorname{Re}(d))$$

$$\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{\Gamma(a+t)\,\Gamma(b-t)}{\Gamma(c+t)\,\Gamma(d-t)}\,dt == \frac{2\pi i\,\Gamma(a+b)\,\Gamma(c+d-a-b-1)}{\Gamma(c+d-1)\,\Gamma(c-a)\,\Gamma(d-b)}\ /;\ -\operatorname{Re}(a) < \gamma < \operatorname{Re}(b) \bigwedge \operatorname{Re}(a+b-c-d) < -1.$$

Integral transforms

The definition of the Meijer G function through a Mellin-Barnes integral realizes the inverse Mellin integral transform of ratios of gamma functions:

$$\frac{\left(\prod_{k=1}^{m}\Gamma(s+b_k)\right)\prod_{k=1}^{n}\Gamma(1-a_k-s)}{\left(\prod_{k=n+1}^{p}\Gamma(s+a_k)\right)\prod_{k=m+1}^{q}\Gamma(1-b_k-s)}.$$

The contour $\mathcal{L}$ is the vertical straight line $(\gamma - i\infty,\ \gamma + i\infty)$. It allows the writing of the following rather general formula for the inverse Mellin integral transform:

$$\mathcal{M}_s^{-1}\left[\frac{\prod_{k=1}^{A}\Gamma(s+a_k)\prod_{k=1}^{B}\Gamma(b_k-s)}{\prod_{k=1}^{C}\Gamma(s+c_k)\prod_{k=1}^{D}\Gamma(d_k-s)}\right](t) == G_{B+C,A+D}^{A,B}\left(t\,\middle|\,\begin{matrix}1-b_1,\,\ldots,\,1-b_B,\,c_1,\,\ldots,\,c_C\\ a_1,\,\ldots,\,a_A,\,1-d_1,\,\ldots,\,1-d_D\end{matrix}\right)/;$$
$$\Delta == A-B-C+D \bigwedge \mathrm{E} == A+B-C-D \bigwedge \nu == \sum_{k=1}^{A}a_k+\sum_{k=1}^{B}b_k-\sum_{k=1}^{C}c_k-\sum_{k=1}^{D}d_k \bigwedge$$
$$-\min(\operatorname{Re}(a_1),\,\ldots,\,\operatorname{Re}(a_A)) < \operatorname{Re}(s) < \min(\operatorname{Re}(b_1),\,\ldots,\,\operatorname{Re}(b_B)) \bigwedge$$
$$\left(|\operatorname{Arg}(t)| < \frac{\pi\,\mathrm{E}}{2} \bigwedge \mathrm{E} > 0 \bigvee |\operatorname{Arg}(t)| == \frac{\pi\,\mathrm{E}}{2} \bigwedge \mathrm{E} > 0 \bigwedge -\frac{\mathrm{E}}{2}+\Delta\operatorname{Re}(s)+\operatorname{Re}(\nu) < -1 \bigvee t > 0 \bigwedge \mathrm{E} == 0 \bigwedge\right.$$
$$\left.\Delta \neq 0 \bigwedge \Delta\operatorname{Re}(s)+\operatorname{Re}(\nu) < \frac{1}{2} \bigvee t > 0 \wedge \mathrm{E} == 0 \wedge \Delta == 0 \wedge (\operatorname{Re}(\nu) < 0 \wedge t \neq 1 \vee \operatorname{Re}(\nu) < -1 \wedge t == 1)\right).$$

In particular cases, it gives the following representations:

$$\mathcal{M}_s^{-1}[\Gamma(s)](t) == e^{-t}\ /;\ \operatorname{Re}(s) > 0$$

$$\mathcal{M}_s^{-1}[\Gamma(s)\,\Gamma(a-s)](t) == (t+1)^{-a}\,\Gamma(a)\ /;\ 0 < \operatorname{Re}(s) < \operatorname{Re}(a)$$

$$\mathcal{M}_s^{-1}\left[\frac{\Gamma(s)}{\Gamma(a-s)}\right](t) == t^{\frac{1-a}{2}}\,J_{a-1}\!\left(2\sqrt{t}\right)/;\ 0 < \operatorname{Re}(s) < \frac{2\operatorname{Re}(a)+1}{4}$$

$$\mathcal{M}_s^{-1}\left[\frac{\Gamma(s)}{\Gamma(a+s)}\right](t) == \frac{(1-t)^{a-1}\,\theta(1-t)}{\Gamma(a)}\ /;\ \operatorname{Re}(a) > 0 \wedge \operatorname{Re}(s) > 0.$$

Transformations

The following formulas describe some transformations of the gamma functions with linear arguments into expressions that contain the gamma function with the simplest argument:

$$\Gamma(-z) == -\frac{\pi\csc(\pi z)}{z\,\Gamma(z)}$$

$$\Gamma(z+1) == z\,\Gamma(z)$$

$$\Gamma(z+n) = (z)_n\, \Gamma(z)$$

$$\Gamma(z-1) = \frac{\Gamma(z)}{z-1}$$

$$\Gamma(z-n) = \frac{(-1)^n\, \Gamma(z)}{(1-z)_n} \;/;\; n \in \mathbb{Z}.$$

Multiple arguments

In the case of multiple arguments $2\,z$, $3\,z, \ldots, n\,z$, the gamma function $\Gamma(z)$ can be represented by the following duplication and multiplication formulas, derived by A. M. Legendre and C. F. Gauss:

$$\Gamma(2\,z) = \frac{2^{2\,z-1}}{\sqrt{\pi}}\, \Gamma(z)\, \Gamma\!\left(z+\frac{1}{2}\right)$$

$$\Gamma(n\,z) = n^{n\,z-\frac{1}{2}}\, (2\,\pi)^{\frac{1-n}{2}} \prod_{k=0}^{n-1} \Gamma\!\left(z+\frac{k}{n}\right) \;/;\; n \in \mathbb{N}^+.$$

Products involving the direct function

The product of two gamma functions $\Gamma(z)$ and $\Gamma(w)$, with arguments satisfying the condition that $z+w$ is an integer, can be represented through elementary functions:

$$\Gamma(z)\, \Gamma(n-z) = \frac{\pi}{\sin(\pi\, z)}\, (1-z)_{n-1} \;/;\; n \in \mathbb{Z}.$$

The preceding formula transforms into the following formula and its relatives:

$$\Gamma(z)\, \Gamma(1-z) = \frac{\pi}{\sin(\pi\, z)}$$

$$\Gamma\!\left(z+\frac{1}{2}\right) \Gamma\!\left(\frac{1}{2}-z\right) = \frac{\pi}{\cos(\pi\, z)}$$

$$\Gamma(z)\, \Gamma(-z) = -\frac{\pi}{z \sin(\pi\, z)}.$$

The ratio of two gamma functions $\Gamma(w)$ and $\Gamma(z)$, with arguments satisfying the condition that $z+w$ is integer, can be represented through a polynomial or rational function:

$$\frac{\Gamma(z+n)}{\Gamma(z)} = (z)_n = \prod_{k=0}^{n-1} (z+k) \;/;\; n \in \mathbb{N}$$

$$\frac{\Gamma(z-n)}{\Gamma(z)} = \frac{(-1)^n}{(1-z)_n} = \prod_{k=1}^{n} \frac{1}{z-k} \;/;\; n \in \mathbb{N}.$$

Identities

The gamma function $\Gamma(z)$ satisfies the following recurrence identities:

$$\Gamma(z) == \frac{\Gamma(z+1)}{z}$$

$$\Gamma(z) == (z-1)\,\Gamma(z-1).$$

These formulas can be generalized to the following recurrence identities with a jump of length *n*:

$$\Gamma(z) == \frac{\Gamma(z+n)}{(z)_n}$$

$$\Gamma(z) == (-1)^n\,(1-z)_n\,\Gamma(z-n)\;/;\;n \in \mathbb{Z}.$$

Inequalities

The most famous inequalities for the gamma function can be described by the following formulas:

$$|\Gamma(z)| \leq |\Gamma(\mathrm{Re}(z))|$$

$$\Gamma(x) \leq x^x\,e^{1-x}\;/;\;x \in \mathbb{R} \land x \geq 1$$

$$\left(\frac{x}{e}\right)^{x-1} \leq \Gamma(x) \leq \;\leq \left(\frac{x}{2}\right)^{x-1}\;/;\;x \in \mathbb{R} \land x \geq 2.$$

Applications

The gamma function is used throughout mathematics, the exact sciences, and engineering.

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