

# Introductions to DirectedInfinity

## Introduction to the symbols

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### General

The concepts of indeterminate, infinity, and directed infinity surfaced in mathematics on an intuitive level many centuries ago. For example, it was clear that it was not possible to find the largest integer. Euclid already proved that the largest prime number ( $\lim_{k \rightarrow \infty} p_k = \infty$  in modern notations) does not exist.

The modern mathematical symbol arose with the development of calculus. J. Wallis (1655) introduced the sign  $\infty$  to signify infinite numbers. Subsequently many mathematicians started to use this or similar symbols. In the twentieth century, K. Weierstrass (1876) used the symbol  $\infty$  to represent an actual infinite quantity.

The mathematical symbols used to designate an indeterminate quantity also came from calculus. L'Hospital (1696) treated the sign  $0/0$  as an indeterminate value. Later, J. Bernoulli (1704, 1730), G. Cramer (1732), J. D'Alembert (1754), and others extensively discussed the symbol  $0/0$  and tried to introduce special notation for it. The appearance of the modern definition of a limit allowed for an evolving understanding of indeterminate quantities, like  $0/0$ , to have all possible values of the double limit of  $x/y$ , when variables  $x$  and  $y$  tend to be 0 independently (at different rates).

### Definitions of symbols

There are four symbols discussed here—an indeterminate numerical quantity  $\iota$ , infinity  $\infty$ , complex infinity  $\tilde{\infty}$ , and directed infinity in the complex plane  $z\infty$ . They are defined as follows:

Indeterminate  $\iota$  is a symbol that represents a numerical quantity whose magnitude cannot be determined. In particular, arbitrary functions with any argument being  $\iota$  also becomes  $\iota$ :  $f(\dots, \iota, \dots) = \iota$ .

$\infty$  is a symbol that represents a positive infinite quantity.

$\tilde{\infty}$  represents an infinite numerical quantity whose direction in the complex plane is unknown (undetermined).

$z\infty$  represents an infinite numerical quantity that is a positive real multiple of the complex number  $z$ .

### Connections within the group of symbols and with other function groups

#### Representations through related functions

The symbols  $\iota$ ,  $\infty$ , and  $\tilde{\infty}$  are connected to each other by the following formulas, including the operations plus, subtract, times, divide, and power:

Plus	1	0	$\infty$	$-\infty$	$\tilde{\infty}$	$\zeta$	$z \infty$
1	2	1	$\infty$	$-\infty$	$\tilde{\infty}$	$\zeta$	$\text{sgn}(z) \infty$
0	1	0	$\infty$	$-\infty$	$\tilde{\infty}$	$\zeta$	$\text{sgn}(z) \infty$
$\infty$	$\infty$	$\infty$	$\infty$	$\zeta$	$\zeta$	$\zeta$	$\zeta$
$-\infty$	$-\infty$	$-\infty$	$\zeta$	$-\infty$	$\zeta$	$\zeta$	$\zeta$
$\tilde{\infty}$	$\tilde{\infty}$	$\tilde{\infty}$	$\zeta$	$\zeta$	$\zeta$	$\zeta$	$\zeta$
$\zeta$	$\zeta$	$\zeta$	$\zeta$	$\zeta$	$\zeta$	$\zeta$	$\zeta$
$w \infty$	$w \infty$	$w \infty$	$\zeta$	$\zeta$	$\zeta$	$\zeta$	$\zeta$

Subtract	1	0	$\infty$	$-\infty$	$\tilde{\infty}$	$\zeta$	$z \infty$
1	0	1	$-\infty$	$\infty$	$\tilde{\infty}$	$\zeta$	$-\text{sgn}(z) \infty$
0	-1	0	$-\infty$	$\infty$	$\tilde{\infty}$	$\zeta$	$-\text{sgn}(z) \infty$
$\infty$	$\infty$	$\infty$	$\zeta$	$\infty$	$\zeta$	$\zeta$	$\zeta$
$-\infty$	$-\infty$	$-\infty$	$-\infty$	$\zeta$	$\zeta$	$\zeta$	$\zeta$
$\tilde{\infty}$	$\tilde{\infty}$	$\tilde{\infty}$	$\zeta$	$\zeta$	$\zeta$	$\zeta$	$\zeta$
$\zeta$	$\zeta$	$\zeta$	$\zeta$	$\zeta$	$\zeta$	$\zeta$	$\zeta$
$w \infty$	$w \infty$	$w \infty$	$\zeta$	$\zeta$	$\zeta$	$\zeta$	$\zeta$

Times	1	0	$\infty$	$-\infty$	$\tilde{\infty}$	$\zeta$	$z \infty$
1	1	0	$\infty$	$-\infty$	$\tilde{\infty}$	$\zeta$	$\text{sgn}(z) \infty$
0	0	0	$\zeta$	$\zeta$	$\zeta$	$\zeta$	$\zeta$
$\infty$	$\infty$	$\zeta$	$\infty$	$-\infty$	$\tilde{\infty}$	$\zeta$	$\text{sgn}(z) \infty$
$-\infty$	$-\infty$	$\zeta$	$-\infty$	$\infty$	$\tilde{\infty}$	$\zeta$	$-\text{sgn}(z) \infty$
$\tilde{\infty}$	$\tilde{\infty}$	$\zeta$	$\tilde{\infty}$	$\tilde{\infty}$	$\tilde{\infty}$	$\zeta$	$\tilde{\infty}$
$\zeta$	$\zeta$	$\zeta$	$\zeta$	$\zeta$	$\zeta$	$\zeta$	$\zeta$
$w \infty$	$w \infty$	$\zeta$	$w \infty$	$-\text{sgn}(w) \infty$	$\tilde{\infty}$	$\zeta$	$\text{sgn}(wz) \infty$

Divide	1	0	$\infty$	$-\infty$	$\tilde{\infty}$	$\zeta$	$z \infty$
1	1	$\tilde{\infty}$	0	0	0	$\zeta$	0
0	0	$\zeta$	0	0	0	$\zeta$	0
$\infty$	$\infty$	$\tilde{\infty}$	$\zeta$	$\zeta$	$\zeta$	$\zeta$	$\zeta$
$-\infty$	$-\infty$	$\tilde{\infty}$	$\zeta$	$\zeta$	$\zeta$	$\zeta$	$\zeta$
$\tilde{\infty}$	$\tilde{\infty}$	$\zeta$	$\zeta$	$\zeta$	$\zeta$	$\zeta$	$\zeta$
$\zeta$	$\zeta$	$\zeta$	$\zeta$	$\zeta$	$\zeta$	$\zeta$	$\zeta$
$w \infty$	$w \infty$	$\tilde{\infty}$	$\zeta$	$\zeta$	$\zeta$	$\zeta$	$\zeta$

Power	1	0	$\infty$	$-\infty$	$\tilde{\infty}$	$\zeta$	$z \infty$
1	1	1	$\zeta$	$\zeta$	$\zeta$	$\zeta$	$\zeta$
0	0	$\zeta$	0	$\tilde{\infty}$	$\zeta$	$\zeta$	$\zeta$
$\infty$	$\infty$	$\zeta$	$\tilde{\infty}$	0	$\zeta$	$\zeta$	$\zeta$
$-\infty$	$-\infty$	$\zeta$	$\tilde{\infty}$	0	$\zeta$	$\zeta$	$\zeta$
$\tilde{\infty}$	$\tilde{\infty}$	$\zeta$	$\tilde{\infty}$	0	$\zeta$	$\zeta$	$\zeta$
$\zeta$	$\zeta$	$\zeta$	$\zeta$	$\zeta$	$\zeta$	$\zeta$	$\zeta$
$w \infty$	$w \infty$	$\zeta$	$\tilde{\infty}$	0	$\zeta$	$\zeta$	$\zeta$

### Representations through other functions

Some of the symbols  $\zeta$ ,  $\infty$ ,  $\tilde{\infty}$ , and  $z \infty$  can be represented as values of basic arithmetic operations, for example:

$$i = \frac{0}{0}$$

$$\infty = \left| \frac{1}{0} \right|$$

$$\tilde{\infty} = \frac{1}{0}$$

Some of the symbols  $i$ ,  $\infty$ ,  $\tilde{\infty}$ , and  $z \infty$  can be represented as values of different mathematical functions, for example:

$$i = e^{\frac{1}{0}}$$

$$\infty = -\log(0)$$

$$\tilde{\infty} = \cot(0)$$

$$\tilde{\infty} = \Gamma(0).$$

## The best-known properties and formulas for symbols

### Specific values

The symbol  $z \infty$  has the following values at some finite points  $z$ :

$$0 \infty = \tilde{\infty}$$

$$1 \infty = \infty$$

$$-1 \infty = -\infty$$

$$i \infty = i \infty$$

$$-i \infty = -i \infty$$

$$(1 + i) \infty = \frac{1 + i}{\sqrt{2}} \infty.$$

The symbol  $z \infty$  has the following values at some infinite points  $z$ :

$$\infty \infty = \infty$$

$$-\infty \infty = -\infty$$

$$i \infty \infty = i \infty$$

$$-i \infty \infty = -i \infty$$

$$\tilde{\infty} \infty = \tilde{\infty}.$$

The symbol  $z \infty$  has the following value at point  $z = i$ :

$$i \infty = \tilde{\infty}.$$

**General characteristics**

$\zeta$  is a symbol. It represents an unknown or not exactly determined point (potentially with infinite magnitude) of the complex plane. Often it results from a double limit where two infinitesimal parameters approach zero at different speeds (e.g.  $\lim_{x \rightarrow 0, y \rightarrow 0} x/y$ ).

$\infty$  is a symbol. On the Riemann sphere, it is the north pole approached from exactly east. In the projective complex plane, it is a point at the line at infinity.

$\tilde{\infty}$  is a symbol. On the Riemann sphere, it is the north pole. In the projective complex plane, it is the line at infinity.

$z \infty$  is a symbol. On the Riemann sphere, it is the north pole together with the direction  $z$  how to approach it. In the projective complex plane, it is a point at the line at infinity.

**Complex characteristics**

The symbols  $\zeta$ ,  $\infty$ ,  $\tilde{\infty}$ , and  $z \infty$  have the following complex characteristics:

$z$	Abs	Arg	Re	Im	Conjugate	Sign
$\zeta$	$ \zeta  = \zeta$	$\text{Arg}(\zeta) = \zeta$	$\text{Re}(\zeta) = \zeta$	$\text{Im}(\zeta) = \zeta$	$\bar{\zeta} = \zeta$	$\text{sgn}(\zeta)$
$\infty$	$ \infty  = \infty$	$\text{Arg}(\infty) = 0$	$\text{Re}(\infty) = \infty$	$\text{Im}(\infty) = 0$	$\bar{\infty} = \infty$	$\text{sgn}(\infty)$
$\tilde{\infty}$	$ \tilde{\infty}  = \infty$	$\text{Arg}(\tilde{\infty}) \in (-\pi, \pi]$	$\text{Re}(\tilde{\infty}) = \zeta$	$\text{Im}(\tilde{\infty}) = \zeta$	$\bar{\tilde{\infty}} = \tilde{\infty}$	$\text{sgn}(\tilde{\infty})$
$z \infty$	$ z \infty  = \infty$	$\text{Arg}(z \infty) = \text{Arg}(z)$ $\text{Arg}(e^{i \cdot x} \infty) = x + 2\pi \left\lfloor \frac{\pi - x}{2\pi} \right\rfloor /;$ $x \in \mathbb{R}$	$\text{Re}(z \infty) = 0 /;$ $\text{Re}(z) = 0$ $\text{Re}(z \infty) = (\text{sgn}(\text{Re}(z)) \infty) /;$ $\text{Re}(z) \neq 0$	$\text{Im}(z \infty) = 0 /;$ $\text{Im}(z) = 0$ $\text{Im}(z \infty) = (\text{sgn}(\text{Im}(z)) \infty) /;$ $\text{Im}(z) \neq 0$	$\bar{z \infty} = \bar{z} \infty$	$\text{sgn}(z \infty)$

**Differentiation**

Derivatives of the symbols  $\zeta$ ,  $\infty$ ,  $\tilde{\infty}$ , and  $z \infty$  satisfy the following relations:

$z$	$\frac{\partial f(z)}{\partial z}$
$\zeta$	$\frac{\partial \zeta}{\partial z} = \zeta$
$\infty$	$\frac{\partial \infty}{\partial z} = 0$
$\tilde{\infty}$	$\frac{\partial \tilde{\infty}}{\partial z} = 0$

**Integration**

Simple indefinite integrals of the symbols  $\zeta$ ,  $\infty$ , and  $\tilde{\infty}$  have the following representations:

$z$	$\int f(z) dz$
$\zeta$	$\int \zeta dz = \zeta$
$\infty$	$\int \infty dz = z \infty$
$\tilde{\infty}$	$\int \tilde{\infty} dz = z \infty$

**Integral transforms**

All Fourier integral transforms of the symbols  $\zeta$ ,  $\infty$ ,  $\tilde{\infty}$ , and  $z \infty$  can be evaluated using the following formal rules:

$z$	$\mathcal{F}_t[f(t)](z)$	$\mathcal{F}_t^{-1}[f(t)](z)$	$\mathcal{F}_{c_t}[f(t)](z)$	$\mathcal{F}_{s_t}[f(t)](z)$
$i$	$\mathcal{F}_t[i](z) = i$	$\mathcal{F}_t^{-1}[i](z) = i$	$\mathcal{F}_{c_t}[i](z) = i$	$\mathcal{F}_{s_t}[i](z) = i$
$\infty$	$\mathcal{F}_t[\infty](z) = \delta(z) \infty$	$\mathcal{F}_t^{-1}[\infty](z) = \delta(z) \infty$	$\mathcal{F}_{c_t}[\infty](z) = \delta(z) \infty$	$\mathcal{F}_{s_t}[\infty](z) = \frac{\infty}{\text{sgn}(z)}$
$\tilde{\infty}$	$\mathcal{F}_t[\tilde{\infty}](z) = \tilde{\infty}$	$\mathcal{F}_t^{-1}[\tilde{\infty}](z) = \tilde{\infty}$	$\mathcal{F}_{c_t}[\tilde{\infty}](z) = \tilde{\infty}$	$\mathcal{F}_{s_t}[\tilde{\infty}](z) = \tilde{\infty}$

Laplace direct and inverse integral transforms of the symbols  $i$ ,  $\infty$ ,  $\tilde{\infty}$ , and  $z \infty$  can be evaluated using the following formal rules:

$z$	$\mathcal{L}_t[f(t)](z)$	$\mathcal{L}_t^{-1}[f(t)](z)$
$i$	$\mathcal{L}_t[i](z) = i$	$\mathcal{L}_t^{-1}[i](z) = i$
$\infty$	$\mathcal{L}_t[\infty](z) = \frac{\infty}{z}$	$\mathcal{L}_t^{-1}[\infty](z) = \delta(z) \infty$
$\tilde{\infty}$	$\mathcal{L}_t[\tilde{\infty}](z) = \tilde{\infty}$	$\mathcal{L}_t^{-1}[\tilde{\infty}](z) = \tilde{\infty}$

### Inequalities

The symbols  $\infty$ ,  $\tilde{\infty}$ , and  $z \infty$  satisfy some obvious inequalities, for example:

$$-\infty < z < \infty \text{ ; } z \in \mathbb{R}$$

$$|z| \leq \infty$$

$$|\tilde{\infty}| > z \text{ ; } z \in \mathbb{R}$$

$$|z| \leq \infty$$

$$|z \infty| > x \text{ ; } x \in \mathbb{R}.$$

### Applications of the symbols

All symbols are used throughout mathematics, the exact sciences, and engineering.

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