Introductions to Coth

Introduction to the hyperbolic functions

General

The six well-known hyperbolic functions are the hyperbolic sine $\sinh(z)$, hyperbolic cosine $\cosh(z)$, hyperbolic tangent $\tanh(z)$, hyperbolic cotangent $\coth(z)$, hyperbolic cosecant $\operatorname{csch}(z)$, and hyperbolic secant $\operatorname{sech}(z)$. They are among the most used elementary functions. The hyperbolic functions share many common properties and they have many properties and formulas that are similar to those of the trigonometric functions.

Definitions of the hyperbolic functions

All hyperbolic functions can be defined as simple rational functions of the exponential function of z:

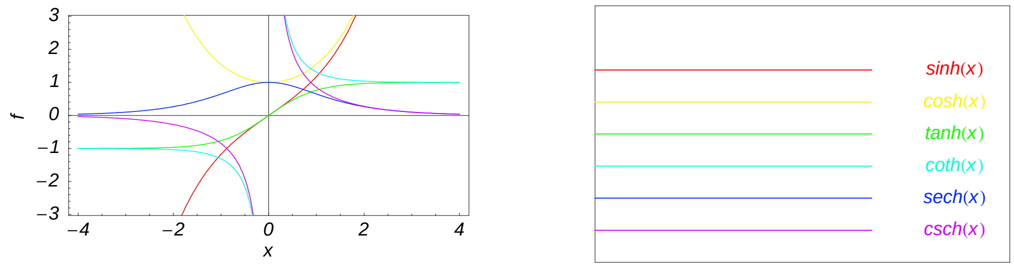
$$\sinh(z) = \frac{e^z - e^{-z}}{2}$$
$$\cosh(z) = \frac{e^z + e^{-z}}{2}$$
$$\tanh(z) = \frac{e^z - e^{-z}}{e^z + e^{-z}}$$
$$\coth(z) = \frac{e^z + e^{-z}}{e^z - e^{-z}}$$
$$\operatorname{csch}(z) = \frac{2}{e^z - e^{-z}}$$
$$\operatorname{sech}(z) = \frac{2}{e^z + e^{-z}}.$$

The functions $\tanh(z)$, $\coth(z)$, $\operatorname{csch}(z)$, and $\operatorname{sech}(z)$ can also be defined through the functions $\sinh(z)$ and $\cosh(z)$ using the following formulas:

$$\tanh(z) = \frac{\sinh(z)}{\cosh(z)}$$
$$\coth(z) = \frac{\cosh(z)}{\sinh(z)}$$
$$\operatorname{csch}(z) = \frac{1}{\sinh(z)}$$
$$\operatorname{sech}(z) = \frac{1}{\cosh(z)}.$$

A quick look at the hyperbolic functions

Here is a quick look at the graphics of the six hyperbolic functions along the real axis.



Connections within the group of hyperbolic functions and with other function groups

Representations through more general functions

The hyperbolic functions are particular cases of more general functions. Among these more general functions, four classes of special functions are of special relevance: Bessel, Jacobi, Mathieu, and hypergeometric functions.

For example, $\sinh(z)$ and $\cosh(z)$ have the following representations through Bessel, Mathieu, and hypergeometric functions:

$$\sinh(z) = -i\sqrt{\frac{\pi i z}{2}}\, J_{1/2}(i z) \quad \sinh(z) = \sqrt{\frac{\pi z}{2}}\, I_{1/2}(z) \quad \sinh(z) = -i\sqrt{\frac{\pi i z}{2}}\, Y_{-1/2}(i z) \quad \sinh(z) = \frac{1}{\sqrt{2\pi}}\left(\sqrt{-z}\, K_{1/2}(-z) - \sqrt{z}\right.$$

$$\cosh(z) = \sqrt{\frac{\pi i z}{2}}\, J_{-1/2}(i z) \quad \cosh(z) = \sqrt{\frac{\pi z}{2}}\, I_{-1/2}(z) \quad \cosh(z) = -\sqrt{\frac{\pi i z}{2}}\, Y_{1/2}(i z) \quad \cosh(z) = \frac{1}{\sqrt{2\pi}}\left(\sqrt{-z}\, K_{1/2}(-z) + \sqrt{z}\right.$$

$$\sinh(z) = -i\,\mathrm{Se}(1, 0, i z) \quad \cosh(z) = \mathrm{Ce}(1, 0, i z)$$

$$\sinh(z) = z\,{}_0F_1\left(;\frac{3}{2};\frac{z^2}{4}\right) \quad \cosh(z) = {}_0F_1\left(;\frac{1}{2};\frac{z^2}{4}\right).$$

All hyperbolic functions can be represented as degenerate cases of the corresponding doubly periodic Jacobi elliptic functions when their second parameter is equal to 0 or 1:

$$\sinh(z) = -i\,\mathrm{sd}(i z \mid 0) = -i\,\mathrm{sn}(i z \mid 0) \quad \sinh(z) = \mathrm{sc}(z \mid 1) = \mathrm{sd}(z \mid 1)$$
$$\cosh(z) = \mathrm{cd}(i z \mid 0) = \mathrm{cn}(i z \mid 0) \quad \cosh(z) = \mathrm{nc}(z \mid 1) = \mathrm{nd}(z \mid 1)$$
$$\tanh(z) = -i\,\mathrm{sc}(i z \mid 0) \quad \tanh(z) = \mathrm{sn}(z \mid 1)$$
$$\coth(z) = i\,\mathrm{cs}(i z \mid 0) \quad \coth(z) = \mathrm{ns}(z \mid 1)$$
$$\mathrm{csch}(z) = i\,\mathrm{ds}(i z \mid 0) = i\,\mathrm{ns}(i z \mid 0) \quad \mathrm{csch}(z) = \mathrm{cs}(z \mid 1) = \mathrm{ds}(z \mid 1)$$
$$\mathrm{sech}(z) = \mathrm{dc}(i z \mid 0) = \mathrm{nc}(i z \mid 0) \quad \mathrm{sech}(z) = \mathrm{cn}(z \mid 1) = \mathrm{dn}(z \mid 1).$$

Representations through related equivalent functions

Each of the six hyperbolic functions can be represented through the corresponding trigonometric function:

$$\sinh(z) = -i\sin(i z) \quad \sinh(i z) = i\sin(z)$$
$$\cosh(z) = \cos(i z) \quad \cosh(i z) = \cos(z)$$
$$\tanh(z) = -i\tan(i z) \quad \tanh(i z) = i\tan(z)$$
$$\coth(z) = i\cot(i z) \quad \coth(i z) = -i\cot(z)$$
$$\mathrm{csch}(z) = i\csc(i z) \quad \mathrm{csch}(i z) = -i\csc(z)$$
$$\mathrm{sech}(z) = \sec(i z) \quad \mathrm{sech}(i z) = \sec(z).$$

Relations to inverse functions

Each of the six hyperbolic functions is connected with a corresponding inverse hyperbolic function by two formulas. One direction can be expressed through a simple formula, but the other direction is much more complicated because of the multivalued nature of the inverse function:

$$\sinh\left(\sinh^{-1}(z)\right) = z \quad \sinh^{-1}(\sinh(z)) = z \;/;\; -\frac{\pi}{2} < \operatorname{Im}(z) < \frac{\pi}{2} \bigvee \left(\operatorname{Im}(z) = -\frac{\pi}{2} \bigwedge \operatorname{Re}(z) \leq 0\right) \bigvee \left(\operatorname{Im}(z) = \frac{\pi}{2} \bigwedge \operatorname{Re}(z) \geq 0\right)$$

$$\cosh\left(\cosh^{-1}(z)\right) = z \quad \cosh^{-1}(\cosh(z)) = z \;/;\; \operatorname{Re}(z) > 0 \land -\pi < \operatorname{Im}(z) \leq \pi \lor \operatorname{Re}(z) = 0 \land \operatorname{Im}(z) \geq 0$$

$$\tanh\left(\tanh^{-1}(z)\right) = z \quad \tanh^{-1}(\tanh(z)) = z \;/;\; -\frac{\pi}{2} < \operatorname{Im}(z) < \frac{\pi}{2} \bigvee \left(\operatorname{Im}(z) = -\frac{\pi}{2} \bigwedge \operatorname{Re}(z) > 0\right) \bigvee \left(\operatorname{Im}(z) = \frac{\pi}{2} \bigwedge \operatorname{Re}(z) < 0\right)$$

$$\coth\left(\coth^{-1}(z)\right) = z \quad \coth^{-1}(\coth(z)) = z \;/;\; -\frac{\pi}{2} < \operatorname{Im}(z) < \frac{\pi}{2} \bigvee \left(\operatorname{Im}(z) = -\frac{\pi}{2} \bigwedge \operatorname{Re}(z) > 0\right) \bigvee \left(\operatorname{Im}(z) = \frac{\pi}{2} \bigwedge \operatorname{Re}(z) \leq 0\right)$$

$$\operatorname{csch}\left(\operatorname{csch}^{-1}(z)\right) = z \quad \operatorname{csch}^{-1}(\operatorname{csch}(z)) = z \;/;\; -\frac{\pi}{2} < \operatorname{Im}(z) < \frac{\pi}{2} \bigvee \left(\operatorname{Im}(z) = -\frac{\pi}{2} \bigwedge \operatorname{Re}(z) \leq 0\right) \bigvee \left(\operatorname{Im}(z) = \frac{\pi}{2} \bigwedge \operatorname{Re}(z) \geq 0\right)$$

$$\operatorname{sech}\left(\operatorname{sech}^{-1}(z)\right) = z \quad \operatorname{sech}^{-1}(\operatorname{sech}(z)) = z \;/;\; -\pi < \operatorname{Im}(z) \leq \pi \land \operatorname{Re}(z) > 0 \lor \operatorname{Re}(z) = 0 \land \operatorname{Im}(z) \geq 0.$$

Representations through other hyperbolic functions

Each of the six hyperbolic functions can be represented through any other function as a rational function of that function with a linear argument. For example, the hyperbolic sine can be representative as a group-defining function because the other five functions can be expressed as:

$$\cosh(z) = -i \sinh\left(\frac{\pi i}{2} - z\right) \qquad \cosh^2(z) = 1 + \sinh^2(z)$$

$$\tanh(z) = \frac{\sinh(z)}{\cosh(z)} = \frac{i \sinh(z)}{\sinh\left(\frac{\pi i}{2} - z\right)} \qquad \tanh^2(z) = \frac{\sinh^2(z)}{1 + \sinh^2(z)}$$

$$\coth(z) = \frac{\cosh(z)}{\sinh(z)} = -\frac{i \sinh\left(\frac{\pi i}{2} - z\right)}{\sinh(z)} \qquad \coth^2(z) = \frac{1 + \sinh^2(z)}{\sinh^2(z)}$$

$$\operatorname{csch}(z) = \frac{1}{\sinh(z)} \qquad \operatorname{csch}^2(z) = \frac{1}{\sinh^2(z)}$$

$$\operatorname{sech}(z) = \frac{1}{\cosh(z)} = \frac{i}{\sinh\left(\frac{\pi i}{2} - z\right)} \qquad \operatorname{sech}^2(z) = \frac{1}{1 + \sinh^2(z)}.$$

All six hyperbolic functions can be transformed into any other function of the group of hyperbolic functions if the argument z is replaced by $p \pi i / 2 + q z$ with $q^2 = 1 \bigwedge p \in \mathbb{Z}$:

$$\sinh(-z - 2 \pi i) = -\sinh(z) \qquad \sinh(z - 2 \pi i) = \sinh(z)$$

$$\sinh\left(-z - \frac{3 \pi i}{2}\right) = i \cosh(z) \qquad \sinh\left(z - \frac{3 \pi i}{2}\right) = i \cosh(z)$$

$$\sinh(-z - \pi i) = \sinh(z) \qquad \sinh(z - \pi i) = -\sinh(z)$$

$$\sinh\left(-z - \frac{\pi i}{2}\right) = -i \cosh(z) \qquad \sinh\left(z - \frac{\pi i}{2}\right) = -i \cosh(z)$$

$$\sinh\left(z + \frac{\pi i}{2}\right) = i \cosh(z) \qquad \sinh\left(\frac{\pi i}{2} - z\right) = i \cosh(z)$$

$$\sinh(z + \pi i) = -\sinh(z) \qquad \sinh(\pi i - z) = \sinh(z)$$

$$\sinh\left(z + \frac{3 \pi i}{2}\right) = -i \cosh(z) \qquad \sinh\left(\frac{3 \pi i}{2} - z\right) = -i \cosh(z)$$

$$\sinh(z + 2 \pi i) = \sinh(z) \qquad \sinh(2 \pi i - z) = -\sinh(z)$$

$\cosh(-z-2\pi i)=\cosh(z)$ $\cosh(z-2\pi i)=\cosh(z)$
$\cosh\left(-z-\frac{3\pi i}{2}\right)=-i\sinh(z)$ $\cosh\left(z-\frac{3\pi i}{2}\right)=i\sinh(z)$
$\cosh(-z-\pi i)=-\cosh(z)$ $\cosh(z-\pi i)=-\cosh(z)$
$\cosh\left(-z-\frac{\pi i}{2}\right)=i\sinh(z)$ $\cosh\left(z-\frac{\pi i}{2}\right)=-i\sinh(z)$
$\cosh\left(z+\frac{\pi i}{2}\right)=i\sinh(z)$ $\cosh\left(\frac{\pi i}{2}-z\right)=-i\sinh(z)$
$\cosh(z+\pi i)=-\cosh(z)$ $\cosh(\pi i-z)=-\cosh(z)$
$\cosh\left(z+\frac{3\pi i}{2}\right)=-i\sinh(z)$ $\cosh\left(\frac{3\pi i}{2}-z\right)=i\sinh(z)$
$\cosh(z+2\pi i)=\cosh(z)$ $\cosh(2\pi i-z)=\cosh(z)$

$\tanh(-z-\pi i)=-\tanh(z)$ $\tanh(z-\pi i)=\tanh(z)$
$\tanh\left(-z-\frac{\pi i}{2}\right)=-\coth(z)$ $\tanh\left(z-\frac{\pi i}{2}\right)=\coth(z)$
$\tanh\left(z+\frac{\pi i}{2}\right)=\coth(z)$ $\tanh\left(\frac{\pi i}{2}-z\right)=-\coth(z)$
$\tanh(z+\pi i)=\tanh(z)$ $\tanh(\pi i-z)=-\tanh(z)$

$\coth(-z-\pi i)=-\coth(z)$ $\coth(z-\pi i)=\coth(z)$
$\coth\left(-z-\frac{\pi i}{2}\right)=-\tanh(z)$ $\coth\left(z-\frac{\pi i}{2}\right)=\tanh(z)$
$\coth\left(z+\frac{\pi i}{2}\right)=\tanh(z)$ $\coth\left(\frac{\pi i}{2}-z\right)=-\tanh(z)$
$\coth(z+\pi i)=\coth(z)$ $\coth(\pi i-z)=-\coth(z)$

$\operatorname{csch}(-z-2\pi i)=-\operatorname{csch}(z)$ $\operatorname{csch}(z-2\pi i)=\operatorname{csch}(z)$
$\operatorname{csch}\left(-z-\frac{3\pi i}{2}\right)=-i\operatorname{sech}(z)$ $\operatorname{csch}\left(z-\frac{3\pi i}{2}\right)=-i\operatorname{sech}(z)$
$\operatorname{csch}(-z-\pi i)=\operatorname{csch}(z)$ $\operatorname{csch}(z-\pi i)=-\operatorname{csch}(z)$
$\operatorname{csch}\left(-z-\frac{\pi i}{2}\right)=i\operatorname{sech}(z)$ $\operatorname{csch}\left(z-\frac{\pi i}{2}\right)=i\operatorname{sech}(z)$
$\operatorname{csch}\left(z+\frac{\pi i}{2}\right)=-i\operatorname{sech}(z)$ $\operatorname{csch}\left(\frac{\pi i}{2}-z\right)=-i\operatorname{sech}(z)$
$\operatorname{csch}(z+\pi i)=-\operatorname{csch}(z)$ $\operatorname{csch}(\pi i-z)=\operatorname{csch}(z)$
$\operatorname{csch}\left(z+\frac{3\pi i}{2}\right)=i\operatorname{sech}(z)$ $\operatorname{csch}\left(\frac{3\pi i}{2}-z\right)=i\operatorname{sech}(z)$
$\operatorname{csch}(z+2\pi i)=\operatorname{csch}(z)$ $\operatorname{csch}(2\pi-z i)=-\operatorname{csch}(z)$

$\operatorname{sech}(-z-2\pi i)=\operatorname{sech}(z)$ $\operatorname{sech}(z-2\pi i)=\operatorname{sech}(z)$
$\operatorname{sech}\left(-z-\frac{3\pi i}{2}\right)=i\operatorname{csch}(z)$ $\operatorname{sech}\left(z-\frac{3\pi i}{2}\right)=-i\operatorname{csch}(z)$
$\operatorname{sech}(-z-\pi i)=-\operatorname{sech}(z)$ $\operatorname{sech}(z-\pi i)=-\operatorname{sech}(z)$
$\operatorname{sech}\left(-z-\frac{\pi i}{2}\right)=-i\operatorname{csch}(z)$ $\operatorname{sech}\left(z-\frac{\pi i}{2}\right)=i\operatorname{csch}(z)$
$\operatorname{sech}\left(z+\frac{\pi i}{2}\right)=-i\operatorname{csch}(z)$ $\operatorname{sech}\left(\frac{\pi i}{2}-z\right)=i\operatorname{csch}(z)$
$\operatorname{sech}(z+\pi i)=-\operatorname{sech}(z)$ $\operatorname{sech}(\pi i-z)=-\operatorname{sech}(z)$
$\operatorname{sech}\left(z+\frac{3\pi i}{2}\right)=i\operatorname{csch}(z)$ $\operatorname{sech}\left(\frac{3\pi i}{2}-z\right)=-i\operatorname{csch}(z)$
$\operatorname{sech}(z+2\pi i)=\operatorname{sech}(z)$ $\operatorname{sech}(2\pi i-z)=\operatorname{sech}(z)$.

The best-known properties and formulas for hyperbolic functions

Real values for real arguments

For real values of argument z, the values of all the hyperbolic functions are real (or infinity).

In the points $z = 2\pi n i/m /; n \in \mathbb{Z} \wedge m \in \mathbb{Z}$, the values of the hyperbolic functions are algebraic. In several cases, they can even be rational numbers, 1, or i (e.g. $\sinh(\pi i/2) = i$, $\text{sech}(0) = 1$, or $\cosh(\pi i/3) = 1/2$). They can be expressed using only square roots if $n \in \mathbb{Z}$ and m is a product of a power of 2 and distinct Fermat primes {3, 5, 17, 257, …}.

Simple values at zero

All hyperbolic functions has rather simple values for arguments $z == 0$ and $z == \pi i/2$:

$$\sinh(0) == 0 \quad \sinh\left(\frac{\pi i}{2}\right) == i$$
$$\cosh(0) == 1 \quad \cosh\left(\frac{\pi i}{2}\right) == 0$$
$$\tanh(0) == 0 \quad \tanh\left(\frac{\pi i}{2}\right) == \tilde{\infty}$$
$$\coth(0) == \tilde{\infty} \quad \coth\left(\frac{\pi i}{2}\right) == 0$$
$$\text{csch}(0) == \tilde{\infty} \quad \text{csch}\left(\frac{\pi i}{2}\right) == -i$$
$$\text{sech}(0) == 1 \quad \text{sech}\left(\frac{\pi i}{2}\right) == \tilde{\infty}.$$

Analyticity

All hyperbolic functions are defined for all complex values of z, and they are analytical functions of z over the whole complex z-plane and do not have branch cuts or branch points. The two functions $\sinh(z)$ and $\cosh(z)$ are entire functions with an essential singular point at $z = \tilde{\infty}$. All other hyperbolic functions are meromorphic functions with simple poles at points $z == \pi k i /; k \in \mathbb{Z}$ (for $\text{csch}(z)$ and $\coth(z)$) and at points $z == \pi i/2 + \pi k i /; k \in \mathbb{Z}$ (for $\text{sech}(z)$ and $\tanh(z)$).

Periodicity

All hyperbolic functions are periodic functions with a real period ($2\pi i$ or πi):

$$\sinh(z) == \sinh(z + 2\pi i) \quad \sinh(z + 2\pi i k) == \sinh(z) /; k \in \mathbb{Z}$$
$$\cosh(z) == \cosh(z + 2\pi i) \quad \cosh(z + 2\pi i k) == \cosh(z) /; k \in \mathbb{Z}$$
$$\tanh(z) == \tanh(z + \pi i) \quad \tanh(z + \pi i k) == \tanh(z) /; k \in \mathbb{Z}$$
$$\coth(z) == \coth(z + \pi i) \quad \coth(z + \pi i k) == \coth(z) /; k \in \mathbb{Z}$$
$$\text{csch}(z) == \text{csch}(z + 2\pi i) \quad \text{csch}(z + 2\pi i k) == \text{csch}(z) /; k \in \mathbb{Z}$$
$$\text{sech}(z) == \text{sech}(z + 2\pi i) \quad \text{sech}(z + 2\pi i k) == \text{sech}(z) /; k \in \mathbb{Z}.$$

Parity and symmetry

All hyperbolic functions have parity (either odd or even) and mirror symmetry:

$$\sinh(-z) == -\sinh(z) \quad \sinh(\bar{z}) == \overline{\sinh(z)}$$
$$\cosh(-z) == \cosh(z) \quad \cosh(\bar{z}) == \overline{\cosh(z)}$$
$$\tanh(-z) == -\tanh(z) \quad \tanh(\bar{z}) == \overline{\tanh(z)}$$
$$\coth(-z) == -\coth(z) \quad \coth(\bar{z}) == \overline{\coth(z)}$$
$$\text{csch}(-z) == -\text{csch}(z) \quad \text{csch}(\bar{z}) == \overline{\text{csch}(z)}$$
$$\text{sech}(-z) == \text{sech}(z) \quad \text{sech}(\bar{z}) == \overline{\text{sech}(z)}.$$

Simple representations of derivatives

The derivatives of all hyperbolic functions have simple representations that can be expressed through other hyperbolic functions:

$$\frac{\partial \sinh(z)}{\partial z} = \cosh(z) \qquad \frac{\partial \cosh(z)}{\partial z} = \sinh(z) \qquad \frac{\partial \tanh(z)}{\partial z} = \operatorname{sech}^2(z)$$
$$\frac{\partial \coth(z)}{\partial z} = -\operatorname{csch}^2(z) \qquad \frac{\partial \operatorname{csch}(z)}{\partial z} = -\coth(z)\operatorname{csch}(z) \qquad \frac{\partial \operatorname{sech}(z)}{\partial z} = -\operatorname{sech}(z)\tanh(z).$$

Simple differential equations

The solutions of the simplest second-order linear ordinary differential equation with constant coefficients can be represented through sinh(z) and cosh(z). The other hyperbolic functions satisfy first-order nonlinear differential equations:

$$w''(z) - w(z) = 0 \;/;\; w(z) = \cosh(z) \land w(0) = 1 \land w'(0) = 0$$
$$w''(z) - w(z) = 0 \;/;\; w(z) = \sinh(z) \land w(0) = 0 \land w'(0) = 1$$
$$w''(z) - w(z) = 0 \;/;\; w(z) = c_1 \cosh(z) + c_2 \sinh(z).$$

All six hyperbolic functions satisfy first-order nonlinear differential equations:

$$w'(z) - \sqrt{1 + (w(z))^2} = 0 \;/;\; w(z) = \sinh(z) \bigwedge w(0) = 0 \bigwedge |\operatorname{Im}(z)| < \frac{\pi}{2}$$
$$w'(z) - \sqrt{-1 + (w(z))^2} = 0 \;/;\; w(z) = \cosh(z) \bigwedge w(0) = 1 \bigwedge |\operatorname{Im}(z)| < \frac{\pi}{2}$$
$$w'(z) + w(z)^2 - 1 = 0 \;/;\; w(z) = \tanh(z) \land w(0) = 0$$
$$w'(z) + w(z)^2 - 1 = 0 \;/;\; w(z) = \coth(z) \bigwedge w\left(\frac{\pi i}{2}\right) = 0$$
$$w'(z)^2 - w(z)^4 - w(z)^2 = 0 \;/;\; w(z) = \operatorname{csch}(z)$$
$$w'(z)^2 + w(z)^4 - w(z)^2 = 0 \;/;\; w(z) = \operatorname{sech}(z).$$

Applications of hyperbolic functions

Trigonometric functions are intimately related to triangle geometry. Functions like sine and cosine are often introduced as edge lengths of right-angled triangles. Hyperbolic functions occur in the theory of triangles in hyperbolic spaces.

Lobachevsky (1829) and J. Bolyai (1832) independently recognized that Euclid's fifth postulate—saying that for a given line and a point not on the line, there is exactly one line parallel to the first—might be changed and still be a consistent geometry. In the hyperbolic geometry it is allowable for more than one line to be parallel to the first (meaning that the parallel lines will never meet the first, however far they are extended). Translated into triangles, this means that the sum of the three angles is always less than π.

A particularly nice representation of the hyperbolic geometry can be realized in the unit disk of complex numbers (the Poincaré disk model). In this model, points are complex numbers in the unit disk, and the lines are either arcs of circles that meet the boundary of the unit circle orthogonal or diameters of the unit circle.

The distance d between two points (meaning complex numbers) A and B in the Poincaré disk is:

$$d(A, B) = 2 \tanh^{-1}\left(\left|\frac{A - B}{1 - \overline{B} A}\right|\right).$$

The attractive feature of the Poincaré disk model is that the hyperbolic angles agree with the Euclidean angles. Formally, the angle α at a point A of two hyperbolic lines $\overline{A\,B}$ and $\overline{A\,C}$ is described by the formula:

$$\cos(\alpha) = \frac{\frac{-A+B}{1-\overline{A}\,B}\ \overline{\frac{-A+C}{1-\overline{A}\,C}}}{\left|\frac{-A+B}{1-\overline{A}\,B}\right|\left|\frac{-A+C}{1-\overline{A}\,C}\right|}.$$

In the following, the values of the three angles of an hyperbolic triangle at the vertices A, B, and C are denoted through α, β, and γ. The hyperbolic length of the three edges opposite to the angles are denoted a, b, and c.

The cosine rule and the second cosine rule for hyperbolic triangles are:

$$\sinh(b)\sinh(c)\cos(\alpha) == \cosh(b)\cosh(c) - \cosh(a)$$
$$\sinh(a)\sinh(c)\cos(\beta) == \cosh(a)\cosh(c) - \cosh(b)$$
$$\sinh(a)\sinh(b)\cos(\gamma) == \cosh(a)\cosh(b) - \cosh(c)$$

$$\sin(\beta)\sin(\gamma)\cosh(a) == \cos(\beta)\cos(\gamma) + \cos(\alpha)$$
$$\sin(\alpha)\sin(\gamma)\cosh(b) == \cos(\alpha)\cos(\alpha) + \cos(\beta)$$
$$\sin(\alpha)\sin(\beta)\cosh(c) == \cos(\alpha)\cos(\beta) + \cos(\gamma).$$

The sine rule for hyperbolic triangles is:

$$\frac{\sin(\alpha)}{\sinh(a)} == \frac{\sin(\beta)}{\sinh(b)} == \frac{\sin(\gamma)}{\sinh(c)}.$$

For a right-angle triangle, the hyperbolic version of the Pythagorean theorem follows from the preceding formulas (the right angle is taken at vertex A):

$$\cosh(a) == \cosh(b)\cosh(c).$$

Using the series expansion $\cosh(x) \approx 1 + x^2/2$ at small scales the hyperbolic geometry is approximated by the familar Euclidean geometry. The cosine formulas and the sine formulas for hyperbolic triangles with a right angle at vertex A become:

$$\cos(\beta) = \frac{\tanh(c)}{\tanh(a)},\ \sin(\beta) = \frac{\sinh(b)}{\sinh(a)}$$
$$\cos(\gamma) = \frac{\tanh(b)}{\tanh(a)},\ \sin(\gamma) = \frac{\text{sinhh}(c)}{\tanh(a)}.$$

The inscribed circle has the radius:

$$\rho = \sqrt{\tanh^{-1}\left(\frac{\cos^2(\alpha) + \cos^2(\beta) + \cos^2(\gamma) + 2\cos(\alpha)\cos(\beta)\cos(\gamma) - 1}{2\,(1+\cos(\alpha))\,(1+\cos(\beta))\,(1+\cos(\gamma))}\right)}\ .$$

The circumscribed circle has the radius:

$$\rho = \tanh^{-1}\left(\frac{4\sinh\left(\frac{a}{2}\right)\sinh\left(\frac{b}{2}\right)\sinh\left(\frac{c}{2}\right)}{\sin(\gamma)\sinh(a)\sinh(b)}\right).$$

Other applications

As rational functions of the exponential function, the hyperbolic functions appear virtually everywhere in quantitative sciences. It is impossible to list their numerous applications in teaching, science, engineering, and art.

Introduction to the Hyperbolic Cotangent Function

Defining the hyperbolic cotangent function

The hyperbolic cotangent function is an old mathematical function. It was first used in the articles by L'Abbe Sauri (1774).

This function is easily defined as the ratio of the hyperbolic sine and cosine functions (or expanded, as the ratio of the half-sum and half-difference of two exponential functions in the points z and $-z$):

$$\coth(z) = \frac{\cosh(z)}{\sinh(z)} = \frac{e^z + e^{-z}}{e^z - e^{-z}}.$$

This function can also be defined as reciprocal to the hyperbolic tangent function:

$$\coth(z) = \frac{1}{\tanh(z)}.$$

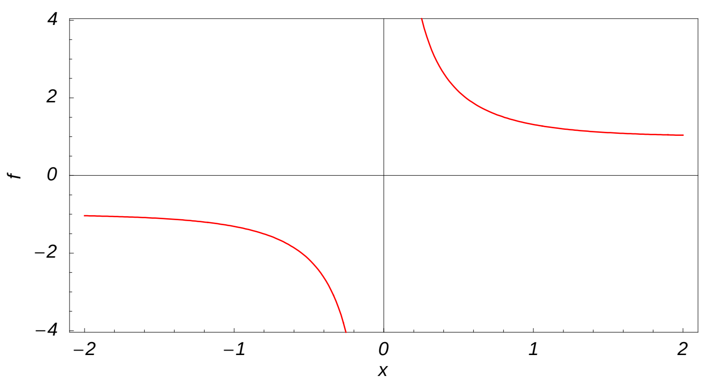
After comparison with the famous Euler formulas for the cosine and sine functions, $\cos(z) = \frac{e^{iz}+e^{-iz}}{2}$ and $\sin(z) = \frac{e^{iz}-e^{-iz}}{2i}$, it is easy to derive the following representation of the hyperbolic cotangent through the circular cotangent:

$$\coth(z) = i \cot(i z).$$

This formula allows for the derivation of all properties and formulas for the hyperbolic cotangent from the corresponding properties and formulas for circular cotangent function.

A quick look at the hyperbolic cotangent function

Here is a graphic of the hyperbolic cotangent function $f(x) = \coth(x)$ for real values of its argument x.



Representation through more general functions

The hyperbolic cotangent function coth(z) can be represented using more general mathematical functions. As the ratio of the hyperbolic cosine and sine functions that are particular cases of the generalized hypergeometric, Bessel, Struve, and Mathieu functions, the hyperbolic cotangent function can also be represented as ratios of those special functions. But these representations are not very useful. It is more useful to write the hyperbolic cotangent function as particular cases of one special function. This can be done using doubly periodic Jacobi elliptic functions that degenerate into the hyperbolic cotangent function when their second parameter is equal to 0 or 1:

$$\coth(z) = \text{ns}(z \mid 1) = -\text{sn}\left(\frac{\pi i}{2} - z \,\middle|\, 1\right) = i\,\text{cs}(i\,z \mid 0) = i\,\text{sc}\left(\frac{\pi}{2} - i\,z \,\middle|\, 0\right).$$

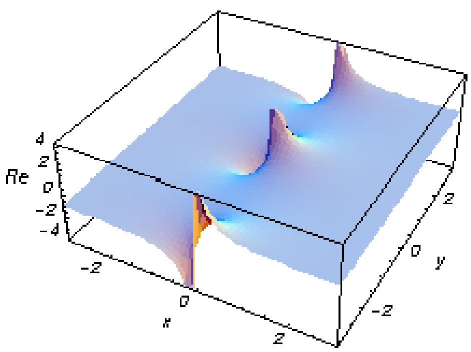
Definition of the hyperbolic cotangent for a complex argument

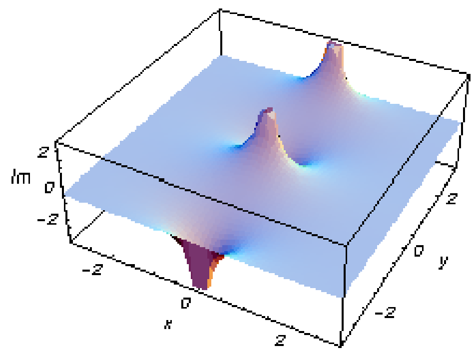
In the complex z-plane, the function coth(z) is defined by the same formula as for real values.

$$\coth(z) = \frac{\cosh(z)}{\sinh(z)} = \frac{e^z + e^{-z}}{e^z - e^{-z}}.$$

In the points $z = \pi k i /; k \in \mathbb{Z}$, where sinh($z$) has zeros, the denominator of the last formula equals zero and coth(z) has singularities (poles of the first order).

Here are two graphics showing the real and imaginary parts of the hyperbolic cotangent function over the complex plane:





The best-known properties and formulas for the hyperbolic cotangent function

Values in points

The values of the hyperbolic cotangent for special values of its argument can be easily derived from the corresponding values of the circular cotangent function in the special points of the circle:

$$\coth(0) = \tilde{\infty} \quad \coth\left(\frac{\pi i}{6}\right) = -i\sqrt{3} \quad \coth\left(\frac{\pi i}{4}\right) = -i \quad \coth\left(\frac{\pi i}{3}\right) = -\frac{i}{\sqrt{3}}$$

$$\coth\left(\frac{\pi i}{2}\right) = 0 \quad \coth\left(\frac{2\pi i}{3}\right) = \frac{i}{\sqrt{3}} \quad \coth\left(\frac{3\pi i}{4}\right) = i \quad \coth\left(\frac{5\pi i}{6}\right) = i\sqrt{3}$$

$$\coth(\pi i) = \tilde{\infty}$$

$$\coth(\pi i m) = \tilde{\infty} /; m \in \mathbb{Z} \quad \coth\left(\pi i \left(\frac{1}{2} + m\right)\right) = 0 /; m \in \mathbb{Z}.$$

The values at infinity can be expressed by the following formulas:

$$\coth(\infty) = 1 \quad \coth(-\infty) = -1.$$

General characteristics

For real values of argument z, the values of $\coth(z)$ are real.

In the points $z = \pi n i / m /; n \in \mathbb{Z} \wedge m \in \mathbb{Z}$, the values of $\coth(z)$ are algebraic. In several cases, they can be $-i$, 0, or i:

$$\coth\left(\frac{\pi i}{4}\right) = -i \quad \coth\left(\frac{\pi i}{2}\right) = 0 \quad \coth\left(-\frac{\pi i}{4}\right) = i.$$

The values of $\coth\left(\frac{n \pi i}{m}\right)$ can be expressed using only square roots if $n \in \mathbb{Z}$ and m is a product of a power of 2 and distinct Fermat primes $\{3, 5, 17, 257, \ldots\}$.

The function $\coth(z)$ is an analytical function of z that is defined over the whole complex z-plane and does not have branch cuts and branch points. It has an infinite set of singular points:

(a) $z = \pi k i /; k \in \mathbb{Z}$ are the simple poles with residues 1.
(b) $z = \tilde{\infty}$ is an essential singular point.

It is a periodic function with period πi:

$$\coth(z + \pi i) = \coth(z)$$
$$\coth(z) = \coth(z + \pi k i) /; k \in \mathbb{Z}.$$

The function $\coth(z)$ is an odd function with mirror symmetry:

$$\coth(-z) = -\coth(z) \quad \coth(\bar{z}) = \overline{\coth(z)}.$$

Differentiation

The first derivative of $\coth(z)$ has simple representations using either the $\sinh(z)$ function or the $\operatorname{csch}(z)$ function:

$$\frac{\partial \coth(z)}{\partial z} = -\frac{1}{\sinh^2(z)} = -\operatorname{csch}^2(z).$$

The n^{th} derivative of $\coth(z)$ has much more complicated representations than symbolic n^{th} derivatives for $\sinh(z)$ and $\cosh(z)$:

$$\frac{\partial^n \coth(z)}{\partial z^n} = \coth(z)\,\delta_n + \operatorname{csch}^2(z)\,\delta_{n-1} - n\,(-i)^n \sum_{k=0}^{n-1} \sum_{j=0}^{k-1} \frac{(-1)^{j+k} (k-j)^{n-1} \sinh^{-2k-2}(z)\, 2^{n-2k}}{k+1} \binom{n-1}{k} \binom{2k}{j} \sinh\left(\frac{i \pi n}{2} + 2(k-j) z\right) /; n \in \mathbb{N},$$

where δ_n is the Kronecker delta symbol: $\delta_0 = 1$ and $\delta_n = 0 /; n \neq 0$.

Ordinary differential equation

The function coth(z) satisfies the following first-order nonlinear differential equation:

$$w'(z) + w(z)^2 - 1 == 0 \;/;\; w(z) == \coth(z) \bigwedge w\left(\frac{\pi i}{2}\right) == 0.$$

Series representation

The function coth(z) has a simple Loran series expansion that converges for all finite values z with $0 < |z| < \pi$:

$$\coth(z) == \frac{1}{z} + \frac{z}{3} - \frac{z^3}{45} + \ldots == \frac{1}{z} + \coth(z) == \frac{1}{z} + \sum_{k=1}^{\infty} \frac{2^{2k} B_{2k} z^{2k-1}}{(2k)!},$$

where the B_{2k} are the Bernoulli numbers.

Integral representation

The function coth(z) has a well-known integral representation through the following definite integral along the positive part of the real axis:

$$\coth(z) == -\frac{2i}{\pi} \int_0^{\infty} \frac{t^{\frac{2iz}{\pi}+1} - 1}{t^2 - 1} \, dt \;/;\; 0 < \mathrm{Im}(z) < \frac{\pi}{2}.$$

Continued fraction representations

The function coth(z) has the following continued fraction representation:

$$\coth(z) == \frac{1}{z} + \cfrac{4\pi^{-2} z}{1 + \cfrac{1 + 4\pi^{-2} z^2}{3 + \cfrac{4\left(4 + 4\pi^{-2} z^2\right)}{5 + \cfrac{9\left(9 + 4\pi^{-2} z^2\right)}{7 + \cfrac{16\left(16 + 4\pi^{-2} z^2\right)}{9 + \cfrac{25\left(25 + 4\pi^{-2} z^2\right)}{11 + \cfrac{36\left(36 + 4\pi^{-2} z^2\right)}{13 + \ldots}}}}}}}.$$

Indefinite integration

Indefinite integrals of expressions involving the hyperbolic cotangent function can sometimes be expressed using elementary functions. However, special functions are frequently needed to express the results even when the integrands have a simple form (if they can be evaluated in closed form). Here are some examples:

$$\int \coth(z) \, dz == \log(\sinh(z))$$

$$\int \sqrt{\coth(z)} \, dz == -\tan^{-1}\left(\coth^{\frac{1}{2}}(z)\right) - \frac{1}{2} \log\left(\coth^{\frac{1}{2}}(z) - 1\right) + \frac{1}{2} \log\left(\coth^{\frac{1}{2}}(z) + 1\right)$$

$$\int \coth^{\nu}(a z) \, dz == \frac{\coth^{\nu+1}(a z)}{a(\nu+1)} \, {}_2F_1\left(\frac{\nu+1}{2}, 1; \frac{\nu+1}{2} + 1; \coth^2(a z)\right).$$

Definite integration

Definite integrals that contain the hyperbolic cotangent function are sometimes simple:

$$\int_0^\infty t\, e^{-t} \coth(t)\, dt = \frac{1}{4}\left(\pi^2 - 4\right).$$

Some special functions can be used to evaluate more complicated definite integrals. For example, the polylogarithm function is needed to express the following integral:

$$\int_0^{\frac{\pi}{4}} \log(\coth(t))\, dt = \frac{1}{12}\left(-3 \log^2\left(\coth\left(\frac{\pi}{4}\right)\right) + 6 \log\left(\coth\left(\frac{\pi}{4}\right) + 1\right) \log\left(\coth\left(\frac{\pi}{4}\right)\right) + \pi^2 + 6\, \mathrm{Li}_2\left(1 - \coth\left(\frac{\pi}{4}\right)\right) - 6\, \mathrm{Li}_2\left(-\tanh\left(\frac{\pi}{4}\right)\right)\right).$$

Finite summation

The following finite sum that contains the hyperbolic cotangent function can be expressed using the hyperbolic cotangent functions:

$$\sum_{k=0}^{n} \frac{1}{2^k \coth\left(\frac{a}{2^k}\right)} = 2 \coth(2a) - \frac{1}{2^n} \coth\left(\frac{a}{2^n}\right).$$

Addition formulas

The hyperbolic cotangent of a sum can be represented by the rule: "the hyperbolic cotangent of a sum is equal to the product of the hyperbolic cotangents plus one divided by the sum of the hyperbolic cotangents." A similar rule is valid for the hyperbolic cotangent of the difference:

$$\coth(a+b) = \frac{\coth(a)\coth(b) + 1}{\coth(a) + \coth(b)}$$

$$\coth(a-b) = \frac{1 - \coth(a)\coth(b)}{\coth(a) - \coth(b)}.$$

Multiple arguments

In the case of multiple arguments $2z$, $3z$, …, the function $\coth(nz)$ can be represented as the ratio of the finite sums containing powers of hyperbolic cotangents:

$$\coth(2z) = \frac{1}{2}(\coth(z) + \tanh(z))$$

$$\coth(3z) = \frac{\coth^3(z) + 3\coth(z)}{3\coth^2(z) + 1}$$

$$\coth(nz) = \frac{1}{\sum_{k=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \binom{n}{2k+1} \coth^{n-(2k+1)}(z)} \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \binom{n}{2k} \coth^{n-2k}(z) \;/;\; n \in \mathbb{N}^+.$$

Half-angle formulas

The hyperbolic cotangent of a half-angle can be represented using two hyperbolic functions by the following simple formulas:

$$\coth\left(\frac{z}{2}\right) = \coth(z) + \operatorname{csch}(z)$$

$$\coth\left(\frac{z}{2}\right) = \frac{\sinh(z)}{\cosh(z) - 1}.$$

The hyperbolic sine function in the last formula can be replaced by a hyperbolic cosine function. But it leads to a more complicated representation that is valid in a horizontal strip:

$$\coth\left(\frac{z}{2}\right) = \frac{\sqrt{-z^2}}{z} \sqrt{\frac{1+\cosh(z)}{1-\cosh(z)}} \;/;\; 0 < |\operatorname{Im}(z)| < \pi \lor \operatorname{Im}(z) = -\pi \land \operatorname{Re}(z) < 0 \lor \operatorname{Im}(z) = \pi \land \operatorname{Re}(z) > 0.$$

The last restrictions can be removed by modifying the last identity (now the identity is valid for all complex z):

$$\coth\left(\frac{z}{2}\right) = z \sqrt{\frac{1}{z^2}} \sqrt{\frac{\cosh(z)+1}{\cosh(z)-1}}\;.$$

Sums of two direct functions

The sum of two hyperbolic cotangent functions can be described by rule: "the sum of the hyperbolic cotangents is equal to the hyperbolic sine of the sum multiplied by the hyperbolic cosecants." A similar rule is valid for the difference of two hyperbolic cotangents:

$$\coth(a) + \coth(b) = \operatorname{csch}(a)\operatorname{csch}(b)\sinh(a+b)$$
$$\coth(a) - \coth(b) = -\operatorname{csch}(a)\operatorname{csch}(b)\sinh(a-b).$$

Products involving the direct function

The product of two hyperbolic cotangents and the product of the hyperbolic cotangent and tangent have the following representations:

$$\coth(a)\coth(b) = \frac{\cosh(a-b)+\cosh(a+b)}{\cosh(a+b)-\cosh(a-b)}$$
$$\coth(a)\tanh(b) = \frac{\sinh(a+b)-\sinh(a-b)}{\sinh(a-b)+\sinh(a+b)}.$$

Inequalities

The most famous inequality for the hyperbolic cotangent function is the following:

$$|\coth(x)| > 1 \;/;\; x \in \mathbb{R}.$$

Relations with its inverse function

There are simple relations between the function $\coth(z)$ and its inverse function $\coth^{-1}(z)$:

$$\coth\left(\coth^{-1}(z)\right) = z \quad \coth^{-1}(\coth(z)) = z \;/;\; -\frac{\pi}{2} < \operatorname{Im}(z) < \frac{\pi}{2} \bigvee \left(\operatorname{Im}(z) = -\frac{\pi}{2} \bigwedge \operatorname{Re}(z) > 0\right) \bigvee \left(\operatorname{Im}(z) = \frac{\pi}{2} \bigwedge \operatorname{Re}(z) \le 0\right).$$

The second formula is valid at least in the horizontal strip $-\frac{\pi}{2} < \mathrm{Im}(z) < \frac{\pi}{2}$. Outside of this strip, a much more complicated relation (containing the unit step, real part, and the floor functions) holds:

$$\coth^{-1}(\coth(z)) == z - i\,\pi \left\lfloor \frac{\mathrm{Im}(z)}{\pi} + \frac{1}{2} \right\rfloor + \frac{\pi\, i}{2} \left(1 + (-1)^{\left\lfloor \frac{\mathrm{Im}(z)}{\pi} - \frac{1}{2} \right\rfloor + \left\lfloor \frac{1}{2} - \frac{\mathrm{Im}(z)}{\pi} \right\rfloor}\right) \theta(-\mathrm{Re}(z)) \;/;\; \frac{i\, z}{\pi} - \frac{1}{2} \notin \mathbb{Z}.$$

Representations through other hyperbolic functions

The hyperbolic cotangent and tangent functions are connected by a very simple formula that contains the linear function in the argument:

$$\coth(z) == \tanh\left(z - \frac{\pi\, i}{2}\right).$$

The hyperbolic cotangent function can also be represented through other hyperbolic functions by the following formulas:

$$\coth(z) == \frac{i \sinh\left(z - \frac{\pi i}{2}\right)}{\sinh(z)} \quad \coth(z) == -\frac{i \cosh(z)}{\cosh\left(z - \frac{\pi i}{2}\right)}$$

$$\coth(z) == \frac{i\,\mathrm{csch}(z)}{\mathrm{csch}\left(z - \frac{\pi i}{2}\right)} \quad \coth(z) == -\frac{i\,\mathrm{sech}\left(z - \frac{\pi i}{2}\right)}{\mathrm{sech}(z)}.$$

Representations through trigonometric functions

The hyperbolic cotangent function has similar representations using related trigonometric functions by the following formulas:

$$\coth(z) == \frac{i \sin\left(\frac{\pi}{2} - i z\right)}{\sin(i z)} \quad \coth(z) == \frac{i \cos(i z)}{\cos\left(\frac{\pi}{2} - i z\right)} \quad \coth(z) == i \tan\left(\frac{\pi}{2} - i z\right) \quad \coth(z) == i \cot(i z)$$

$$\coth(i z) == -i \cot(z) \quad \coth(z) == \frac{i \csc(i z)}{\csc\left(\frac{\pi}{2} - i z\right)} \quad \coth(z) == \frac{i \sec\left(\frac{\pi}{2} - i z\right)}{\sec(i z)}.$$

Applications

The hyperbolic cotangent function is used throughout mathematics, the exact sciences, and engineering.

Introduction to the Hyperbolic Functions in *Mathematica*

Overview

The following shows how the six hyperbolic functions are realized in *Mathematica*. Examples of evaluating *Mathematica* functions applied to various numeric and exact expressions that involve the hyperbolic functions or return them are shown. These involve numeric and symbolic calculations and plots.

Notations

***Mathematica* forms of notations**

All six hyperbolic functions are represented as built-in functions in *Mathematica*. Following *Mathematica*'s general naming convention, the `StandardForm` function names are simply capitalized versions of the traditional mathematics names. Here is a list `hypFunctions` of the six hyperbolic functions in `StandardForm`.

```
hypFunctions = {Sinh[z], Cosh[z], Tanh[z], Coth[z], Sech[z], Cosh[z]}
```

```
{Sinh[z], Cosh[z], Tanh[z], Coth[z], Sech[z], Cosh[z]}
```

Here is a list `hypFunctions` of the six trigonometric functions in `TraditionalForm`.

```
hypFunctions // TraditionalForm
```

$\{\sinh(z), \cosh(z), \tanh(z), \coth(z), \operatorname{sech}(z), \cosh(z)\}$

Additional forms of notations

Mathematica also knows the most popular forms of notations for the hyperbolic functions that are used in other programming languages. Here are three examples: `CForm`, `TeXForm`, and `FortranForm`.

```
hypFunctions /. {z → 2 π z} // CForm
```

```
List(Sinh(2*Pi*z),Cosh(2*Pi*z),Tanh(2*Pi*z),Coth(2*Pi*z),Sech(2*Pi*z),Cosh(2*Pi*z))
```

```
hypFunctions /. {z → 2 π z} // TeXForm
```

```
\{ \sinh (2\,\pi \,z),\cosh (2\,\pi \,z),\tanh (2\,\pi \,z),\coth (2\,\pi \,z),
  \Mfunction{Sech}(2\,\pi \,z),\cosh (2\,\pi \,z)\}
```

```
hypFunctions /. {z → 2 π z} // FortranForm
```

```
List(Sinh(2*Pi*z),Cosh(2*Pi*z),Tanh(2*Pi*z),Coth(2*Pi*z),Sech(2*Pi*z),Cosh(2*Pi*z))
```

Automatic evaluations and transformations

Evaluation for exact, machine-number, and high-precision arguments

For a simple exact argument, *Mathematica* returns an exact result. For instance, for the argument $\pi i/6$, the `Sinh` function evaluates to $i/2$.

$$\text{Sinh}\left[\frac{\pi i}{6}\right]$$

$$\frac{i}{2}$$

$$\{\text{Sinh}[z], \text{Cosh}[z], \text{Tanh}[z], \text{Coth}[z], \text{Csch}[z], \text{Sech}[z]\} \text{ /. } z \to \frac{\pi i}{6}$$

$$\left\{\frac{i}{2}, \frac{\sqrt{3}}{2}, \frac{i}{\sqrt{3}}, -i\sqrt{3}, -2i, \frac{2}{\sqrt{3}}\right\}$$

For a generic machine-number argument (a numerical argument with a decimal point and not too many digits), a machine number is returned.

```
Cosh[3.]
```

```
10.0677
```

```
{Sinh[z], Cosh[z], Tanh[z], Coth[z], Csch[z], Sech[z]} /. z → 2.
```

```
{3.62686, 3.7622, 0.964028, 1.03731, 0.275721, 0.265802}
```

The next inputs calculate 100-digit approximations of the six hyperbolic functions at $z = 1$.

```
N[Tanh[1], 40]
```

```
0.7615941559557648881194582826047935904128
```

```
Coth[1] // N[#, 50] &
```

```
1.3130352854993313036361612469308478329120139412405
```

```
N[{Sinh[z], Cosh[z], Tanh[z], Coth[z], Csch[z], Sech[z]} /. z → 1, 100]
```

```
{1.175201193643801456882381850595600815155717981334095870229565413013307567304323895⋱
  607117452089623392,
 1.543080634815243778477905620757061682601529112365863704737402214710769063049223698⋱
  964264726435543036,
 0.761594155955764888119458282604793590412768597257936551596810500121953244576638483⋱
  4589475216736767144,
 1.313035285499331303636161246930847832912013941240452655543152967567084270461874382⋱
  674679241480856303,
 0.850918128239321545133842763287175284181724660910339616990421151729003364321465103⋱
  8997301773288938124,
 0.648054273663885399574977353226150323108489312071942023037865337318717595646712830⋱
  2808547853078928924}
```

Within a second, it is possible to calculate thousands of digits for the hyperbolic functions. The next input calculates 10000 digits for sinh(1), cosh(1), tanh(1), coth(1), sech(1), and csch(1) and analyzes the frequency of the occurrence of the digit k in the resulting decimal number.

```
Map[Function[w, {First[#], Length[#]} & /@ Split[Sort[First[RealDigits[w]]]]],
 N[{Sinh[z], Cosh[z], Tanh[z], Coth[z], Csch[z], Sech[z]} /. z → 1, 10 000]]
```

```
{{{0, 980}, {1, 994}, {2, 996}, {3, 1014}, {4, 986}, {5, 1001},
  {6, 1017}, {7, 1020}, {8, 981}, {9, 1011}}, {{0, 1015}, {1, 960}, {2, 997},
  {3, 1037}, {4, 1070}, {5, 1018}, {6, 973}, {7, 997}, {8, 963}, {9, 970}},
 {{0, 971}, {1, 1023}, {2, 1016}, {3, 970}, {4, 949}, {5, 1052}, {6, 981},
  {7, 1056}, {8, 1010}, {9, 972}}, {{0, 975}, {1, 986}, {2, 1023},
  {3, 1004}, {4, 1008}, {5, 977}, {6, 977}, {7, 1036}, {8, 1035}, {9, 979}},
 {{0, 979}, {1, 1030}, {2, 987}, {3, 992}, {4, 1016}, {5, 1030}, {6, 1021},
  {7, 969}, {8, 974}, {9, 1002}}, {{0, 1009}, {1, 971}, {2, 1018},
  {3, 994}, {4, 1011}, {5, 1018}, {6, 958}, {7, 1019}, {8, 1016}, {9, 986}}}
```

Here are 50-digit approximations to the six hyperbolic functions at the complex argument $z = 3 + 5\,i$.

```
N[Csch[3 + 5 ⅈ], 100]
```

```
0.0280585164230800759963159842602743697051540123887285931631736730964453318082730911`
  1484269546408531396 +
 0.0953236346741784028519159307062564516454421668787754798038797727933315832622762215`
   3893978444505670174 7 i
```

```
N[{Sinh[z], Cosh[z], Tanh[z], Coth[z], Csch[z], Sech[z]} /. z → 3 + 5 i, 50]
```

```
{2.8416922956063519438168753953062364359281841632360 -
  9.6541254768548391365515436340301659921919691213853 i,
 2.8558150042273872913639018630946098374643609536732 -
  9.6063834484325811198111562160434163877218590394033 i,
 1.0041647106948152119205166259313184311852454735738 -
  0.0027082358362240721322640353684331035927960259125751 i,
 0.99584531857585412976042001587164841711026557204102 +
  0.0026857984057585256446537711012814749378977439361108 i,
 0.028058516423080075996315984260274369705154012388729 +
  0.095323634674178402851915930706256451645442166878775 i,
 0.028433530909971667358833684958646399417265586614624 +
  0.095644640955286344684316595933099452259073530811833 i}
```

Mathematica always evaluates mathematical functions with machine precision, if the arguments are machine numbers. In this case, only six digits after the decimal point are shown in the results. The remaining digits are suppressed, but can be displayed using the function `InputForm`.

```
{Sinh[2.], N[Sinh[2]], N[Sinh[2], 16], N[Sinh[2], 5], N[Sinh[2], 20]}
```

```
{3.62686, 3.62686, 3.62686, 3.62686, 3.6268604078470187677}
```

```
% // InputForm
```

```
{3.6268604078470186, 3.6268604078470186, 3.6268604078470186, 3.6268604078470186,
 3.626860407847018767668213982801262016 44`20}
```

```
Precision[%%]
```

```
16
```

Simplification of the argument

Mathematica uses symmetries and periodicities of all the hyperbolic functions to simplify expressions. Here are some examples.

```
Sinh[-z]
```

```
-Sinh[z]
```

```
Sinh[z + π i]
```

```
-Sinh[z]
```

```
Sinh[z + 2 π i]
```

```
Sinh[z]
```

```
Sinh[z + 34 π i]
```

```
Sinh[z]
```

```
{Sinh[-z], Cosh[-z], Tanh[-z], Coth[-z], Csch[-z], Sech[-z]}
```

$\{-\text{Sinh}[z], \text{Cosh}[z], -\text{Tanh}[z], -\text{Coth}[z], -\text{Csch}[z], \text{Sech}[z]\}$

```
{Sinh[z + π ⅈ], Cosh[z + π ⅈ], Tanh[z + π ⅈ], Coth[z + π ⅈ], Csch[z + π ⅈ], Sech[z + π ⅈ]}
```

$\{-\text{Sinh}[z], -\text{Cosh}[z], \text{Tanh}[z], \text{Coth}[z], -\text{Csch}[z], -\text{Sech}[z]\}$

```
{Sinh[z + 2 π ⅈ], Cosh[z + 2 π ⅈ], Tanh[z + 2 π ⅈ],
 Coth[z + 2 π ⅈ], Csch[z + 2 π ⅈ], Sech[z + 2 π ⅈ]}
```

$\{\text{Sinh}[z], \text{Cosh}[z], \text{Tanh}[z], \text{Coth}[z], \text{Csch}[z], \text{Sech}[z]\}$

```
{Sinh[z + 342 π ⅈ], Cosh[z + 342 π ⅈ], Tanh[z + 342 π ⅈ],
 Coth[z + 342 π ⅈ], Csch[z + 342 π ⅈ], Sech[z + 342 π ⅈ]}
```

$\{\text{Sinh}[z], \text{Cosh}[z], \text{Tanh}[z], \text{Coth}[z], \text{Csch}[z], \text{Sech}[z]\}$

Mathematica automatically simplifies the composition of the direct and the inverse hyperbolic functions into the argument.

```
{Sinh[ArcSinh[z]], Cosh[ArcCosh[z]], Tanh[ArcTanh[z]],
 Coth[ArcCoth[z]], Csch[ArcCsch[z]], Sech[ArcSech[z]]}
```

$\{z, z, z, z, z, z\}$

Mathematica also automatically simplifies the composition of the direct and any of the inverse hyperbolic functions into algebraic functions of the argument.

```
{Sinh[ArcSinh[z]], Sinh[ArcCosh[z]], Sinh[ArcTanh[z]],
 Sinh[ArcCoth[z]], Sinh[ArcCsch[z]], Sinh[ArcSech[z]]}
```

$$\left\{z, \sqrt{\frac{-1+z}{1+z}}\,(1+z), \frac{z}{\sqrt{1-z^2}}, \frac{1}{\sqrt{1-\frac{1}{z^2}}\,z}, \frac{1}{z}, \frac{\sqrt{\frac{1-z}{1+z}}\,(1+z)}{z}\right\}$$

```
{Cosh[ArcSinh[z]], Cosh[ArcCosh[z]], Cosh[ArcTanh[z]],
 Cosh[ArcCoth[z]], Cosh[ArcCsch[z]], Cosh[ArcSech[z]]}
```

$$\left\{\sqrt{1+z^2}, z, \frac{1}{\sqrt{1-z^2}}, \frac{1}{\sqrt{1-\frac{1}{z^2}}}, \sqrt{1+\frac{1}{z^2}}, \frac{1}{z}\right\}$$

```
{Tanh[ArcSinh[z]], Tanh[ArcCosh[z]], Tanh[ArcTanh[z]],
 Tanh[ArcCoth[z]], Tanh[ArcCsch[z]], Tanh[ArcSech[z]]}
```

$$\left\{\frac{z}{\sqrt{1+z^2}},\ \frac{\sqrt{\frac{-1+z}{1+z}}\ (1+z)}{z},\ z,\ \frac{1}{z},\ \frac{1}{\sqrt{1+\frac{1}{z^2}}\ z},\ \sqrt{\frac{1-z}{1+z}}\ (1+z)\right\}$$

{Coth[ArcSinh[z]], Coth[ArcCosh[z]], Coth[ArcTanh[z]], Coth[ArcCoth[z]], Coth[ArcCsch[z]], Coth[ArcSech[z]]}

$$\left\{\frac{\sqrt{1+z^2}}{z},\ \frac{z}{\sqrt{\frac{-1+z}{1+z}}\ (1+z)},\ \frac{1}{z},\ z,\ \sqrt{1+\frac{1}{z^2}}\ z,\ \frac{1}{\sqrt{\frac{1-z}{1+z}}\ (1+z)}\right\}$$

{Csch[ArcSinh[z]], Csch[ArcCosh[z]], Csch[ArcTanh[z]], Csch[ArcCoth[z]], Csch[ArcCsch[z]], Csch[ArcSech[z]]}

$$\left\{\frac{1}{z},\ \frac{1}{\sqrt{\frac{-1+z}{1+z}}\ (1+z)},\ \frac{\sqrt{1-z^2}}{z},\ \sqrt{1-\frac{1}{z^2}}\ z,\ z,\ \frac{z}{\sqrt{\frac{1-z}{1+z}}\ (1+z)}\right\}$$

{Sech[ArcSinh[z]], Sech[ArcCosh[z]], Sech[ArcTanh[z]], Sech[ArcCoth[z]], Sech[ArcCsch[z]], Sech[ArcSech[z]]}

$$\left\{\frac{1}{\sqrt{1+z^2}},\ \frac{1}{z},\ \sqrt{1-z^2},\ \sqrt{1-\frac{1}{z^2}},\ \frac{1}{\sqrt{1+\frac{1}{z^2}}},\ z\right\}$$

In cases where the argument has the structure $\pi k/2+z$ or $\pi k/2-z$, e $\pi k/2+i z$ or $\pi k/2-i z$ with integer k, trigonometric functions can be automatically transformed into other trigonometric or hyperbolic functions. Here are some examples.

$$\mathbf{Tanh}\left[\frac{\pi i}{2}-z\right]$$

-Coth[z]

Csch[ⅈ z]

-ⅈ Csc[z]

$$\left\{\mathbf{Sinh}\left[\frac{\pi i}{2}-z\right],\ \mathbf{Cosh}\left[\frac{\pi i}{2}-z\right],\ \mathbf{Tanh}\left[\frac{\pi i}{2}-z\right],\ \mathbf{Coth}\left[\frac{\pi i}{2}-z\right],\ \mathbf{Csch}\left[\frac{\pi i}{2}-z\right],\ \mathbf{Sech}\left[\frac{\pi i}{2}-z\right]\right\}$$

{ⅈ Cosh[z], -ⅈ Sinh[z], -Coth[z], -Tanh[z], -ⅈ Sech[z], ⅈ Csch[z]}

{Sinh[ⅈ z], Cosh[ⅈ z], Tanh[ⅈ z], Coth[ⅈ z], Csch[ⅈ z], Sech[ⅈ z]}

{ⅈ Sin[z], Cos[z], ⅈ Tan[z], -ⅈ Cot[z], -ⅈ Csc[z], Sec[z]}

Simplification of simple expressions containing hyperbolic functions

Sometimes simple arithmetic operations containing hyperbolic functions can automatically produce other hyperbolic functions.

```
1 / Sech[z]
```

```
Cosh[z]
```

```
{1/Sinh[z], 1/Cosh[z], 1/Tanh[z], 1/Coth[z], 1/Csch[z], 1/Sech[z],
 Sinh[z]/Cosh[z], Cosh[z]/Sinh[z], Sinh[z]/Sinh[π ⅈ/2 - z], Cosh[z]/Sinh[z]^2}
```

```
{Csch[z], Sech[z], Coth[z], Tanh[z], Sinh[z],
 Cosh[z], Tanh[z], Coth[z], -ⅈ Tanh[z], Coth[z] Csch[z]}
```

Hyperbolic functions as special cases of more general functions

All hyperbolic functions can be treated as particular cases of some more advanced special functions. For example, sinh(z) and cosh(z) are sometimes the results of auto-simplifications from Bessel, Mathieu, Jacobi, hypergeometric, and Meijer functions (for appropriate values of their parameters).

$$\text{BesselI}\left[\frac{1}{2}, z\right]$$

$$\frac{\sqrt{\frac{2}{\pi}}\ \text{Sinh}[z]}{\sqrt{z}}$$

```
MathieuC[1, 0, ⅈ z]
```

```
Cosh[z]
```

```
JacobiSN[z, 1]
```

```
Tanh[z]
```

$$\left\{\text{BesselI}\left[\frac{1}{2}, z\right], \text{MathieuS}[1, 0, ⅈ z], \text{JacobiSD}[ⅈ z, 0], \text{HypergeometricPFQ}\left[\{\}, \left\{\frac{3}{2}\right\}, \frac{z^2}{4}\right], \text{MeijerG}\left[\{\{\}, \{\}\}, \left\{\{0\}, \left\{-\frac{1}{2}\right\}\right\}, -\frac{z^2}{4}\right]\right\}$$

$$\left\{\frac{\sqrt{\frac{2}{\pi}}\ \text{Sinh}[z]}{\sqrt{z}}, ⅈ\,\text{Sinh}[z], ⅈ\,\text{Sinh}[z], \frac{\text{Sinh}\left[\sqrt{z^2}\right]}{\sqrt{z^2}}, \frac{2\,\text{Sinh}[z]}{\sqrt{\pi}\ z}\right\}$$

$$\left\{\text{BesselI}\left[-\frac{1}{2}, z\right], \text{MathieuC}[1, 0, ⅈ z], \text{JacobiCD}[ⅈ z, 0], \text{Hypergeometric0F1}\left[\frac{1}{2}, \frac{z^2}{4}\right], \text{MeijerG}\left[\{\{\}, \{\}\}, \left\{\{0\}, \left\{\frac{1}{2}\right\}\right\}, -\frac{z^2}{4}\right]\right\}$$

$$\left\{\frac{\sqrt{\frac{2}{\pi}}\ \text{Cosh}[z]}{\sqrt{z}}, \text{Cosh}[z], \text{Cosh}[z], \text{Cosh}\left[\sqrt{z^2}\right], \frac{\text{Cosh}[z]}{\sqrt{\pi}}\right\}$$

```
{JacobiSC[ⅈ z, 0], JacobiNS[z, 1], JacobiNS[ⅈ z, 0], JacobiDC[ⅈ z, 0]}
```

```
{ⅈ Tanh[z], Coth[z], -ⅈ Csch[z], Sech[z]}
```

Equivalence transformations carried out by specialized *Mathematica* functions

General remarks

Automatic evaluation and transformations can sometimes be inconvenient: They act in only one chosen direction and the result can be overly complicated. For example, the expression $i \cosh(z)/2$ is generally preferable to the more complicated $\sinh(\pi i/2 - z)\cosh(\pi i/3)$. *Mathematica* provides automatic transformation of the second expression into the first one. But compact expressions like $\sinh(2z)\cosh(\pi i/16)$ should not be automatically expanded into the more complicated expression $\sinh(z)\cosh(z)\left(2+\left(2+2^{1/2}\right)^{1/2}\right)^{1/2}$. *Mathematica* has special functions that produce these types of expansions. Some of them are demonstrated in the next section.

TrigExpand

The function `TrigExpand` expands out trigonometric and hyperbolic functions. In more detail, it splits up sums and integer multiples that appear in the arguments of trigonometric and hyperbolic functions, and then expands out the products of the trigonometric and hyperbolic functions into sums of powers, using the trigonometric and hyperbolic identities where possible. Here are some examples.

```
TrigExpand[Sinh[x - y]]
```

```
Cosh[y] Sinh[x] - Cosh[x] Sinh[y]
```

```
Cosh[4 z] // TrigExpand
```

```
Cosh[z]^4 + 6 Cosh[z]^2 Sinh[z]^2 + Sinh[z]^4
```

```
TrigExpand[{{Sinh[x + y], Sinh[3 z]},
            {Cosh[x + y], Cosh[3 z]},
            {Tanh[x + y], Tanh[3 z]},
            {Coth[x + y], Coth[3 z]},
            {Csch[x + y], Csch[3 z]},
            {Sech[x + y], Sech[3 z]}}]
```

$$\{\{\text{Cosh}[y]\,\text{Sinh}[x]+\text{Cosh}[x]\,\text{Sinh}[y],\,3\,\text{Cosh}[z]^2\,\text{Sinh}[z]+\text{Sinh}[z]^3\},$$
$$\{\text{Cosh}[x]\,\text{Cosh}[y]+\text{Sinh}[x]\,\text{Sinh}[y],\,\text{Cosh}[z]^3+3\,\text{Cosh}[z]\,\text{Sinh}[z]^2\},$$
$$\left\{\frac{\text{Cosh}[y]\,\text{Sinh}[x]}{\text{Cosh}[x]\,\text{Cosh}[y]+\text{Sinh}[x]\,\text{Sinh}[y]}+\frac{\text{Cosh}[x]\,\text{Sinh}[y]}{\text{Cosh}[x]\,\text{Cosh}[y]+\text{Sinh}[x]\,\text{Sinh}[y]},\right.$$
$$\left.\frac{3\,\text{Cosh}[z]^2\,\text{Sinh}[z]}{\text{Cosh}[z]^3+3\,\text{Cosh}[z]\,\text{Sinh}[z]^2}+\frac{\text{Sinh}[z]^3}{\text{Cosh}[z]^3+3\,\text{Cosh}[z]\,\text{Sinh}[z]^2}\right\},$$
$$\left\{\frac{\text{Cosh}[x]\,\text{Cosh}[y]}{\text{Cosh}[y]\,\text{Sinh}[x]+\text{Cosh}[x]\,\text{Sinh}[y]}+\frac{\text{Sinh}[x]\,\text{Sinh}[y]}{\text{Cosh}[y]\,\text{Sinh}[x]+\text{Cosh}[x]\,\text{Sinh}[y]},\right.$$
$$\left.\frac{\text{Cosh}[z]^3}{3\,\text{Cosh}[z]^2\,\text{Sinh}[z]+\text{Sinh}[z]^3}+\frac{3\,\text{Cosh}[z]\,\text{Sinh}[z]^2}{3\,\text{Cosh}[z]^2\,\text{Sinh}[z]+\text{Sinh}[z]^3}\right\},$$
$$\left\{\frac{1}{\text{Cosh}[y]\,\text{Sinh}[x]+\text{Cosh}[x]\,\text{Sinh}[y]},\,\frac{1}{3\,\text{Cosh}[z]^2\,\text{Sinh}[z]+\text{Sinh}[z]^3}\right\},$$
$$\left\{\frac{1}{\text{Cosh}[x]\,\text{Cosh}[y]+\text{Sinh}[x]\,\text{Sinh}[y]},\,\frac{1}{\text{Cosh}[z]^3+3\,\text{Cosh}[z]\,\text{Sinh}[z]^2}\right\}\}$$

```
TableForm[(# == TrigExpand[#]) & /@
  Flatten[{{Sinh[x + y], Sinh[3 z]}, {Cosh[x + y], Cosh[3 z]}, {Tanh[x + y], Tanh[3 z]},
    {Coth[x + y], Coth[3 z]}, {Csch[x + y], Csch[3 z]}, {Sech[x + y], Sech[3 z]}}]]
```

$$\text{Sinh}[x+y]==\text{Cosh}[y]\,\text{Sinh}[x]+\text{Cosh}[x]\,\text{Sinh}[y]$$
$$\text{Sinh}[3\,z]==3\,\text{Cosh}[z]^2\,\text{Sinh}[z]+\text{Sinh}[z]^3$$
$$\text{Cosh}[x+y]==\text{Cosh}[x]\,\text{Cosh}[y]+\text{Sinh}[x]\,\text{Sinh}[y]$$
$$\text{Cosh}[3\,z]==\text{Cosh}[z]^3+3\,\text{Cosh}[z]\,\text{Sinh}[z]^2$$
$$\text{Tanh}[x+y]==\frac{\text{Cosh}[y]\,\text{Sinh}[x]}{\text{Cosh}[x]\,\text{Cosh}[y]+\text{Sinh}[x]\,\text{Sinh}[y]}+\frac{\text{Cosh}[x]\,\text{Sinh}[y]}{\text{Cosh}[x]\,\text{Cosh}[y]+\text{Sinh}[x]\,\text{Sinh}[y]}$$
$$\text{Tanh}[3\,z]==\frac{3\,\text{Cosh}[z]^2\,\text{Sinh}[z]}{\text{Cosh}[z]^3+3\,\text{Cosh}[z]\,\text{Sinh}[z]^2}+\frac{\text{Sinh}[z]^3}{\text{Cosh}[z]^3+3\,\text{Cosh}[z]\,\text{Sinh}[z]^2}$$
$$\text{Coth}[x+y]==\frac{\text{Cosh}[x]\,\text{Cosh}[y]}{\text{Cosh}[y]\,\text{Sinh}[x]+\text{Cosh}[x]\,\text{Sinh}[y]}+\frac{\text{Sinh}[x]\,\text{Sinh}[y]}{\text{Cosh}[y]\,\text{Sinh}[x]+\text{Cosh}[x]\,\text{Sinh}[y]}$$
$$\text{Coth}[3\,z]==\frac{\text{Cosh}[z]^3}{3\,\text{Cosh}[z]^2\,\text{Sinh}[z]+\text{Sinh}[z]^3}+\frac{3\,\text{Cosh}[z]\,\text{Sinh}[z]^2}{3\,\text{Cosh}[z]^2\,\text{Sinh}[z]+\text{Sinh}[z]^3}$$
$$\text{Csch}[x+y]==\frac{1}{\text{Cosh}[y]\,\text{Sinh}[x]+\text{Cosh}[x]\,\text{Sinh}[y]}$$
$$\text{Csch}[3\,z]==\frac{1}{3\,\text{Cosh}[z]^2\,\text{Sinh}[z]+\text{Sinh}[z]^3}$$
$$\text{Sech}[x+y]==\frac{1}{\text{Cosh}[x]\,\text{Cosh}[y]+\text{Sinh}[x]\,\text{Sinh}[y]}$$
$$\text{Sech}[3\,z]==\frac{1}{\text{Cosh}[z]^3+3\,\text{Cosh}[z]\,\text{Sinh}[z]^2}$$

TrigFactor

The command `TrigFactor` factors trigonometric and hyperbolic functions. In more detail, it splits up sums and integer multiples that appear in the arguments of trigonometric and hyperbolic functions, and then factors the resulting polynomials in the trigonometric and hyperbolic functions, using the corresponding identities where possible. Here are some examples.

```
TrigFactor[Sinh[x] + ⅈ Cosh[y]]
```

$$\left(i\,\text{Cosh}\left[\frac{x}{2}-\frac{y}{2}\right]+\text{Sinh}\left[\frac{x}{2}-\frac{y}{2}\right]\right)\left(\text{Cosh}\left[\frac{x}{2}+\frac{y}{2}\right]-i\,\text{Sinh}\left[\frac{x}{2}+\frac{y}{2}\right]\right)$$

```
Tanh[x] - Coth[y] // TrigFactor
```

```
-Cosh[x - y] Csch[y] Sech[x]
```

```
TrigFactor[{Sinh[x] + Sinh[y],
            Cosh[x] + Cosh[y],
            Tanh[x] + Tanh[y],
            Coth[x] + Coth[y],
            Csch[x] + Csch[y],
            Sech[x] + Sech[y]}]
```

$$\left\{2\,\text{Cosh}\left[\frac{x}{2}-\frac{y}{2}\right]\text{Sinh}\left[\frac{x}{2}+\frac{y}{2}\right],\ 2\,\text{Cosh}\left[\frac{x}{2}-\frac{y}{2}\right]\text{Cosh}\left[\frac{x}{2}+\frac{y}{2}\right],\ \text{Sech}[x]\,\text{Sech}[y]\,\text{Sinh}[x+y],\right.$$
$$\text{Csch}[x]\,\text{Csch}[y]\,\text{Sinh}[x+y],\ \frac{1}{2}\text{Cosh}\left[\frac{x}{2}-\frac{y}{2}\right]\text{Csch}\left[\frac{x}{2}\right]\text{Csch}\left[\frac{y}{2}\right]\text{Sech}\left[\frac{x}{2}\right]\text{Sech}\left[\frac{y}{2}\right]\text{Sinh}\left[\frac{x}{2}+\frac{y}{2}\right],$$
$$\left.\frac{2\,\text{Cosh}\left[\frac{x}{2}-\frac{y}{2}\right]\text{Cosh}\left[\frac{x}{2}+\frac{y}{2}\right]}{\left(\text{Cosh}\left[\frac{x}{2}\right]-i\,\text{Sinh}\left[\frac{x}{2}\right]\right)\left(\text{Cosh}\left[\frac{x}{2}\right]+i\,\text{Sinh}\left[\frac{x}{2}\right]\right)\left(\text{Cosh}\left[\frac{y}{2}\right]-i\,\text{Sinh}\left[\frac{y}{2}\right]\right)\left(\text{Cosh}\left[\frac{y}{2}\right]+i\,\text{Sinh}\left[\frac{y}{2}\right]\right)}\right\}$$

TrigReduce

The command `TrigReduce` rewrites products and powers of trigonometric and hyperbolic functions in terms of those functions with combined arguments. In more detail, it typically yields a linear expression involving trigonometric and hyperbolic functions with more complicated arguments. `TrigReduce` is approximately opposite to `TrigExpand` and `TrigFactor`. Here are some examples.

```
TrigReduce[Sinh[z]^3]
```

$$\frac{1}{4}\left(-3\,\text{Sinh}[z]+\text{Sinh}[3\,z]\right)$$

```
Sinh[x] Cosh[y] // TrigReduce
```

$$\frac{1}{2}\left(\text{Sinh}[x-y]+\text{Sinh}[x+y]\right)$$

```
TrigReduce[{Sinh[z]^2, Cosh[z]^2, Tanh[z]^2, Coth[z]^2, Csch[z]^2, Sech[z]^2}]
```

$$\left\{\frac{1}{2}\left(-1+\text{Cosh}[2\,z]\right),\ \frac{1}{2}\left(1+\text{Cosh}[2\,z]\right),\right.$$
$$\left.\frac{-1+\text{Cosh}[2\,z]}{1+\text{Cosh}[2\,z]},\ \frac{1+\text{Cosh}[2\,z]}{-1+\text{Cosh}[2\,z]},\ \frac{2}{-1+\text{Cosh}[2\,z]},\ \frac{2}{1+\text{Cosh}[2\,z]}\right\}$$

```
TrigReduce[TrigExpand[{{Sinh[x + y], Sinh[3 z], Sinh[x] Sinh[y]},
                       {Cosh[x + y], Cosh[3 z], Cosh[x] Cosh[y]},
                       {Tanh[x + y], Tanh[3 z], Tanh[x] Tanh[y]},
                       {Coth[x + y], Coth[3 z], Coth[x] Coth[y]},
                       {Csch[x + y], Csch[3 z], Csch[x] Csch[y]},
                       {Sech[x + y], Sech[3 z], Sech[x] Sech[y]}}]]
```

$$\left\{\left\{\text{Sinh}[x+y],\ \text{Sinh}[3z],\ \frac{1}{2}(-\text{Cosh}[x-y]+\text{Cosh}[x+y])\right\},\right.$$
$$\left\{\text{Cosh}[x+y],\ \text{Cosh}[3z],\ \frac{1}{2}(\text{Cosh}[x-y]+\text{Cosh}[x+y])\right\},$$
$$\left\{\text{Tanh}[x+y],\ \text{Tanh}[3z],\ \frac{-\text{Cosh}[x-y]+\text{Cosh}[x+y]}{\text{Cosh}[x-y]+\text{Cosh}[x+y]}\right\},$$
$$\left\{\text{Coth}[x+y],\ \text{Coth}[3z],\ \frac{-\text{Cosh}[x-y]-\text{Cosh}[x+y]}{\text{Cosh}[x-y]-\text{Cosh}[x+y]}\right\},$$
$$\left\{\text{Csch}[x+y],\ \text{Csch}[3z],\ -\frac{2}{\text{Cosh}[x-y]-\text{Cosh}[x+y]}\right\},$$
$$\left.\left\{\text{Sech}[x+y],\ \text{Sech}[3z],\ \frac{2}{\text{Cosh}[x-y]+\text{Cosh}[x+y]}\right\}\right\}$$

```
TrigReduce[TrigFactor[{Sinh[x] + Sinh[y], Cosh[x] + Cosh[y],
   Tanh[x] + Tanh[y], Coth[x] + Coth[y], Csch[x] + Csch[y], Sech[x] + Sech[y]}]]
```

$$\left\{\text{Sinh}[x]+\text{Sinh}[y],\ \text{Cosh}[x]+\text{Cosh}[y],\ \frac{2\,\text{Sinh}[x+y]}{\text{Cosh}[x-y]+\text{Cosh}[x+y]},\right.$$
$$\left.-\frac{2\,\text{Sinh}[x+y]}{\text{Cosh}[x-y]-\text{Cosh}[x+y]},\ -\frac{2\,(\text{Sinh}[x]+\text{Sinh}[y])}{\text{Cosh}[x-y]-\text{Cosh}[x+y]},\ \frac{2\,(\text{Cosh}[x]+\text{Cosh}[y])}{\text{Cosh}[x-y]+\text{Cosh}[x+y]}\right\}$$

TrigToExp

The command `TrigToExp` converts direct and inverse trigonometric and hyperbolic functions to exponentials or logarithmic functions. It tries, where possible, to give results that do not involve explicit complex numbers. Here are some examples.

```
TrigToExp[Sinh[2 z]]
```

$$-\frac{1}{2}e^{-2z}+\frac{e^{2z}}{2}$$

```
Sinh[z] Tanh[2 z] // TrigToExp
```

$$\frac{(-e^{-z}+e^{z})\left(-e^{-2z}+e^{2z}\right)}{2\left(e^{-2z}+e^{2z}\right)}$$

```
TrigToExp[{Sinh[z], Cosh[z], Tanh[z], Coth[z], Csch[z], Sech[z]}]
```

$$\left\{-\frac{e^{-z}}{2}+\frac{e^{z}}{2},\ \frac{e^{-z}}{2}+\frac{e^{z}}{2},\ \frac{-e^{-z}+e^{z}}{e^{-z}+e^{z}},\ \frac{e^{-z}+e^{z}}{-e^{-z}+e^{z}},\ \frac{2}{-e^{-z}+e^{z}},\ \frac{2}{e^{-z}+e^{z}}\right\}$$

ExpToTrig

The command `ExpToTrig` converts exponentials to trigonometric or hyperbolic functions. It tries, where possible, to give results that do not involve explicit complex numbers. It is approximately opposite to `TrigToExp`. Here are some examples.

```
ExpToTrig[e^(x β)]
```

$$\text{Cosh}[x\,\beta]+\text{Sinh}[x\,\beta]$$

$$\frac{e^{x\alpha} - e^{x\beta}}{e^{x\gamma} + e^{x\delta}} \text{ // ExpToTrig}$$

$$\frac{\text{Cosh}[x\,\alpha] - \text{Cosh}[x\,\beta] + \text{Sinh}[x\,\alpha] - \text{Sinh}[x\,\beta]}{\text{Cosh}[x\,\gamma] + \text{Cosh}[x\,\delta] + \text{Sinh}[x\,\gamma] + \text{Sinh}[x\,\delta]}$$

ExpToTrig[TrigToExp[{Sinh[z], Cosh[z], Tanh[z], Coth[z], Csch[z], Sech[z]}]]

{Sinh[z], Cosh[z], Tanh[z], Coth[z], Csch[z], Sech[z]}

ExpToTrig$\left[\left\{\alpha\, e^{-x\beta} + \alpha\, e^{x\beta},\ \alpha\, e^{-x\beta} + \gamma\, e^{i x\beta}\right\}\right]$

{2 α Cosh[x β], γ Cos[x β] + α Cosh[x β] + ⅈ γ Sin[x β] − α Sinh[x β]}

ComplexExpand

The function `ComplexExpand` expands expressions assuming that all the occurring variables are real. The value option `TargetFunctions` is a list of functions from the set `{Re, Im, Abs, Arg, Conjugate, Sign}`. `ComplexExpand` tries to give results in terms of the specified functions. Here are some examples.

ComplexExpand[Sinh[x + ⅈ y] Cosh[x − ⅈ y]]

$\text{Cos}[y]^2\,\text{Cosh}[x]\,\text{Sinh}[x] + \text{Cosh}[x]\,\text{Sin}[y]^2\,\text{Sinh}[x] + i\left(\text{Cos}[y]\,\text{Cosh}[x]^2\,\text{Sin}[y] - \text{Cos}[y]\,\text{Sin}[y]\,\text{Sinh}[x]^2\right)$

Csch[x + ⅈ y] Sech[x − ⅈ y] // ComplexExpand

$$-\frac{4\,\text{Cos}[y]^2\,\text{Cosh}[x]\,\text{Sinh}[x]}{(\text{Cos}[2y] - \text{Cosh}[2x])\,(\text{Cos}[2y] + \text{Cosh}[2x])} - \frac{4\,\text{Cosh}[x]\,\text{Sin}[y]^2\,\text{Sinh}[x]}{(\text{Cos}[2y] - \text{Cosh}[2x])\,(\text{Cos}[2y] + \text{Cosh}[2x])} + i\left(\frac{4\,\text{Cos}[y]\,\text{Cosh}[x]^2\,\text{Sin}[y]}{(\text{Cos}[2y] - \text{Cosh}[2x])\,(\text{Cos}[2y] + \text{Cosh}[2x])} - \frac{4\,\text{Cos}[y]\,\text{Sin}[y]\,\text{Sinh}[x]^2}{(\text{Cos}[2y] - \text{Cosh}[2x])\,(\text{Cos}[2y] + \text{Cosh}[2x])}\right)$$

li1 = {Sinh[x + ⅈ y], Cosh[x + ⅈ y], Tanh[x + ⅈ y], Coth[x + ⅈ y], Csch[x + ⅈ y], Sech[x + ⅈ y]}

{Sinh[x + ⅈ y], Cosh[x + ⅈ y], Tanh[x + ⅈ y], Coth[x + ⅈ y], Csch[x + ⅈ y], Sech[x + ⅈ y]}

ComplexExpand[li1]

$$\left\{i\,\text{Cosh}[x]\,\text{Sin}[y] + \text{Cos}[y]\,\text{Sinh}[x],\ \text{Cos}[y]\,\text{Cosh}[x] + i\,\text{Sin}[y]\,\text{Sinh}[x],\ \frac{i\,\text{Sin}[2y]}{\text{Cos}[2y] + \text{Cosh}[2x]} + \frac{\text{Sinh}[2x]}{\text{Cos}[2y] + \text{Cosh}[2x]},\ \frac{i\,\text{Sin}[2y]}{\text{Cos}[2y] - \text{Cosh}[2x]} - \frac{\text{Sinh}[2x]}{\text{Cos}[2y] - \text{Cosh}[2x]},\ \frac{2\,i\,\text{Cosh}[x]\,\text{Sin}[y]}{\text{Cos}[2y] - \text{Cosh}[2x]} - \frac{2\,\text{Cos}[y]\,\text{Sinh}[x]}{\text{Cos}[2y] - \text{Cosh}[2x]},\ \frac{2\,\text{Cos}[y]\,\text{Cosh}[x]}{\text{Cos}[2y] + \text{Cosh}[2x]} - \frac{2\,i\,\text{Sin}[y]\,\text{Sinh}[x]}{\text{Cos}[2y] + \text{Cosh}[2x]}\right\}$$

ComplexExpand[Re[#] & /@ li1, TargetFunctions → {Re, Im}]

$$\left\{\text{Cos}[y]\,\text{Sinh}[x],\ \text{Cos}[y]\,\text{Cosh}[x],\ \frac{\text{Sinh}[2\,x]}{\text{Cos}[2\,y]+\text{Cosh}[2\,x]},\ -\frac{\text{Sinh}[2\,x]}{\text{Cos}[2\,y]-\text{Cosh}[2\,x]},\ -\frac{2\,\text{Cos}[y]\,\text{Sinh}[x]}{\text{Cos}[2\,y]-\text{Cosh}[2\,x]},\ \frac{2\,\text{Cos}[y]\,\text{Cosh}[x]}{\text{Cos}[2\,y]+\text{Cosh}[2\,x]}\right\}$$

ComplexExpand[Im[#] & /@ li1, TargetFunctions → {Re, Im}]

$$\left\{\text{Cosh}[x]\,\text{Sin}[y],\ \text{Sin}[y]\,\text{Sinh}[x],\ \frac{\text{Sin}[2\,y]}{\text{Cos}[2\,y]+\text{Cosh}[2\,x]},\ \frac{\text{Sin}[2\,y]}{\text{Cos}[2\,y]-\text{Cosh}[2\,x]},\ \frac{2\,\text{Cosh}[x]\,\text{Sin}[y]}{\text{Cos}[2\,y]-\text{Cosh}[2\,x]},\ -\frac{2\,\text{Sin}[y]\,\text{Sinh}[x]}{\text{Cos}[2\,y]+\text{Cosh}[2\,x]}\right\}$$

ComplexExpand[Abs[#] & /@ li1, TargetFunctions → {Re, Im}]

$$\left\{\sqrt{\text{Cosh}[x]^2\,\text{Sin}[y]^2+\text{Cos}[y]^2\,\text{Sinh}[x]^2},\ \sqrt{\text{Cos}[y]^2\,\text{Cosh}[x]^2+\text{Sin}[y]^2\,\text{Sinh}[x]^2},\right.$$
$$\sqrt{\frac{\text{Sin}[2\,y]^2}{(\text{Cos}[2\,y]+\text{Cosh}[2\,x])^2}+\frac{\text{Sinh}[2\,x]^2}{(\text{Cos}[2\,y]+\text{Cosh}[2\,x])^2}},$$
$$\sqrt{\frac{\text{Sin}[2\,y]^2}{(\text{Cos}[2\,y]-\text{Cosh}[2\,x])^2}+\frac{\text{Sinh}[2\,x]^2}{(\text{Cos}[2\,y]-\text{Cosh}[2\,x])^2}},$$
$$\sqrt{\frac{4\,\text{Cosh}[x]^2\,\text{Sin}[y]^2}{(\text{Cos}[2\,y]-\text{Cosh}[2\,x])^2}+\frac{4\,\text{Cos}[y]^2\,\text{Sinh}[x]^2}{(\text{Cos}[2\,y]-\text{Cosh}[2\,x])^2}},$$
$$\left.\sqrt{\frac{4\,\text{Cos}[y]^2\,\text{Cosh}[x]^2}{(\text{Cos}[2\,y]+\text{Cosh}[2\,x])^2}+\frac{4\,\text{Sin}[y]^2\,\text{Sinh}[x]^2}{(\text{Cos}[2\,y]+\text{Cosh}[2\,x])^2}}\right\}$$

% // Simplify[#, {x, y} ∈ Reals] &

$$\left\{\frac{\sqrt{-\text{Cos}[2\,y]+\text{Cosh}[2\,x]}}{\sqrt{2}},\ \frac{\sqrt{\text{Cos}[2\,y]+\text{Cosh}[2\,x]}}{\sqrt{2}},\ \frac{\sqrt{\text{Sin}[2\,y]^2+\text{Sinh}[2\,x]^2}}{\text{Cos}[2\,y]+\text{Cosh}[2\,x]},\ \sqrt{-\frac{\text{Cos}[2\,y]+\text{Cosh}[2\,x]}{\text{Cos}[2\,y]-\text{Cosh}[2\,x]}},\ \frac{\sqrt{2}}{\sqrt{-\text{Cos}[2\,y]+\text{Cosh}[2\,x]}},\ \frac{\sqrt{2}}{\sqrt{\text{Cos}[2\,y]+\text{Cosh}[2\,x]}}\right\}$$

ComplexExpand[Arg[#] & /@ li1, TargetFunctions → {Re, Im}]

```
{ArcTan[Cos[y] Sinh[x], Cosh[x] Sin[y]], ArcTan[Cos[y] Cosh[x], Sin[y] Sinh[x]],
 ArcTan[Sinh[2 x]/(Cos[2 y] + Cosh[2 x]), Sin[2 y]/(Cos[2 y] + Cosh[2 x])],
 ArcTan[-Sinh[2 x]/(Cos[2 y] - Cosh[2 x]), Sin[2 y]/(Cos[2 y] - Cosh[2 x])],
 ArcTan[-(2 Cos[y] Sinh[x])/(Cos[2 y] - Cosh[2 x]), (2 Cosh[x] Sin[y])/(Cos[2 y] - Cosh[2 x])],
 ArcTan[(2 Cos[y] Cosh[x])/(Cos[2 y] + Cosh[2 x]), -(2 Sin[y] Sinh[x])/(Cos[2 y] + Cosh[2 x])]}
```

```
% // Simplify[#, {x, y} ∈ Reals] &
```

```
{ArcTan[Cos[y] Sinh[x], Cosh[x] Sin[y]], ArcTan[Cos[y] Cosh[x], Sin[y] Sinh[x]],
 ArcTan[Sinh[2 x], Sin[2 y]], ArcTan[Cosh[x] Sinh[x], -Cos[y] Sin[y]],
 ArcTan[Cos[y] Sinh[x], -Cosh[x] Sin[y]], ArcTan[Cos[y] Cosh[x], -Sin[y] Sinh[x]]}
```

```
ComplexExpand[Conjugate[#] & /@ li1, TargetFunctions → {Re, Im}] // Simplify
```

```
{-ⅈ Cosh[x] Sin[y] + Cos[y] Sinh[x], Cos[y] Cosh[x] - ⅈ Sin[y] Sinh[x],
 (-ⅈ Sin[2 y] + Sinh[2 x])/(Cos[2 y] + Cosh[2 x]), -(ⅈ Sin[2 y] + Sinh[2 x])/(Cos[2 y] - Cosh[2 x]),
 1/(-ⅈ Cosh[x] Sin[y] + Cos[y] Sinh[x]), 1/(Cos[y] Cosh[x] - ⅈ Sin[y] Sinh[x])}
```

Simplify

The command `Simplify` performs a sequence of algebraic transformations on an expression, and returns the simplest form it finds. Here are two examples.

```
Simplify[Sinh[2 z] / Sinh[z]]
```

```
2 Cosh[z]
```

```
Sinh[2 z] / Cosh[z] // Simplify
```

```
2 Sinh[z]
```

Here is a large collection of hyperbolic identities. All are written as one large logical conjunction.

```
Simplify[#] & /@ (Cosh[z]^2 - Sinh[z]^2 == 1 ∧
  Sinh[z]^2 == (Cosh[2 z] - 1)/2 ∧ Cosh[z]^2 == (1 + Cosh[2 z])/2 ∧
  Tanh[z]^2 == (Cosh[2 z] - 1)/(Cosh[2 z] + 1) ∧ Coth[z]^2 == (Cosh[2 z] + 1)/(Cosh[2 z] - 1) ∧
  Sinh[2 z] == 2 Sinh[z] Cosh[z] ∧ Cosh[2 z] == Cosh[z]^2 + Sinh[z]^2 == 2 Cosh[z]^2 - 1 ∧
  Sinh[a + b] == Sinh[a] Cosh[b] + Cosh[a] Sinh[b] ∧
    Sinh[a - b] == Sinh[a] Cosh[b] - Cosh[a] Sinh[b] ∧
  Cosh[a + b] == Cosh[a] Cosh[b] + Sinh[a] Sinh[b] ∧
    Cosh[a - b] == Cosh[a] Cosh[b] - Sinh[a] Sinh[b] ∧
  Sinh[a] + Sinh[b] == 2 Sinh[(a + b)/2] Cosh[(a - b)/2] ∧
    Sinh[a] - Sinh[b] == 2 Cosh[(a + b)/2] Sinh[(a - b)/2] ∧
  Cosh[a] + Cosh[b] == 2 Cosh[(a + b)/2] Cosh[(a - b)/2] ∧
    Cosh[a] - Cosh[b] == -2 Sinh[(a + b)/2] Sinh[(b - a)/2] ∧
  Tanh[a] + Tanh[b] == Sinh[a + b]/(Cosh[a] Cosh[b]) ∧ Tanh[a] - Tanh[b] == Sinh[a - b]/(Cosh[a] Cosh[b]) ∧
  A Sinh[z] + B Cosh[z] == A Sqrt[1 - B^2/A^2] Sinh[z + ArcTanh[B/A]] ∧
  Sinh[a] Sinh[b] == (Cosh[a + b] - Cosh[a - b])/2 ∧ Cosh[a] Cosh[b] ==
    (Cosh[a - b] + Cosh[a + b])/2 ∧ Sinh[a] Cosh[b] == (Sinh[a + b] + Sinh[a - b])/2 ∧
  Sinh[z/2]^2 == (Cosh[z] - 1)/2 ∧ Cosh[z/2]^2 == (1 + Cosh[z])/2 ∧
  Tanh[z/2] == (Cosh[z] - 1)/Sinh[z] == Sinh[z]/(1 + Cosh[z]) ∧ Coth[z/2] == Sinh[z]/(Cosh[z] - 1) == (1 + Cosh[z])/Sinh[z])
```

```
True
```

The command `Simplify` has the `Assumption` option. For example, *Mathematica* knows that $\sinh(x) > 0$ for all real positive x, and uses the periodicity of hyperbolic functions for the symbolic integer coefficient k of $k \pi i$.

```
Simplify[Abs[Sinh[x]] > 0, x > 0]
```

```
True
```

```
Abs[Sinh[x]] > 0 // Simplify[#, x > 0] &
```

```
True
```

```
Simplify[{Sinh[z + 2 k π ⅈ], Cosh[z + 2 k π ⅈ], Tanh[z + k π ⅈ],
  Coth[z + k π ⅈ], Csch[z + 2 k π ⅈ], Sech[z + 2 k π ⅈ]}, k ∈ Integers]
```

```
{Sinh[z], Cosh[z], Tanh[z], Coth[z], Csch[z], Sech[z]}
```

```
Simplify[{Sinh[z + k π ⅈ] / Sinh[z], Cosh[z + k π ⅈ] / Cosh[z], Tanh[z + k π ⅈ] / Tanh[z],
  Coth[z + k π ⅈ] / Coth[z], Csch[z + k π ⅈ] / Csch[z], Sech[z + k π ⅈ] / Sech[z]}, k ∈ Integers]
```

$$\{(-1)^k, (-1)^k, 1, 1, (-1)^k, (-1)^k\}$$

Mathematica also knows that the composition of inverse and direct hyperbolic functions produces the value of the inner argument under the appropriate restriction. Here are some examples.

```
Simplify[{ArcSinh[Sinh[z]], ArcTanh[Tanh[z]],
  ArcCoth[Coth[z]], ArcCsch[Csch[z]]}, -π / 2 < Im[z] < π / 2]
```

```
{z, z, z, z}
```

```
Simplify[{ArcCosh[Cosh[z]], ArcSech[Sech[z]]}, -π < Im[z] < π ∧ Re[z] > 0]
```

```
{z, z}
```

FunctionExpand (and Together)

While the hyperbolic functions auto-evaluate for simple fractions of πi, for more complicated cases they stay as hyperbolic functions to avoid the build up of large expressions. Using the function `FunctionExpand`, such expressions can be transformed into explicit radicals.

$$\text{Cosh}\left[\frac{\pi i}{32}\right]$$

$$\text{Cos}\left[\frac{\pi}{32}\right]$$

$$\text{FunctionExpand}\left[\text{Cosh}\left[\frac{\pi i}{32}\right]\right]$$

$$\frac{1}{2}\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}$$

$$\text{Coth}\left[\frac{\pi i}{24}\right] \text{ // FunctionExpand}$$

$$-\frac{i\left(\frac{\sqrt{2-\sqrt{2}}}{4}+\frac{1}{4}\sqrt{3\left(2+\sqrt{2}\right)}\right)}{-\frac{1}{4}\sqrt{3\left(2-\sqrt{2}\right)}+\frac{\sqrt{2+\sqrt{2}}}{4}}$$

{Sinh[πⅈ/16], Cosh[πⅈ/16], Tanh[πⅈ/16], Coth[πⅈ/16], Csch[πⅈ/16], Sech[πⅈ/16]}

$$\left\{i \operatorname{Sin}\left[\frac{\pi}{16}\right], \operatorname{Cos}\left[\frac{\pi}{16}\right], i \operatorname{Tan}\left[\frac{\pi}{16}\right], -i \operatorname{Cot}\left[\frac{\pi}{16}\right], -i \operatorname{Csc}\left[\frac{\pi}{16}\right], \operatorname{Sec}\left[\frac{\pi}{16}\right]\right\}$$

FunctionExpand[%]

$$\left\{\frac{1}{2} i \sqrt{2-\sqrt{2+\sqrt{2}}}, \frac{1}{2} \sqrt{2+\sqrt{2+\sqrt{2}}}, i \sqrt{\frac{2-\sqrt{2+\sqrt{2}}}{2+\sqrt{2+\sqrt{2}}}}, -i \sqrt{\frac{2+\sqrt{2+\sqrt{2}}}{2-\sqrt{2+\sqrt{2}}}}, -\frac{2 i}{\sqrt{2-\sqrt{2+\sqrt{2}}}}, \frac{2}{\sqrt{2+\sqrt{2+\sqrt{2}}}}\right\}$$

{Sinh[πⅈ/60], Cosh[πⅈ/60], Tanh[πⅈ/60], Coth[πⅈ/60], Csch[πⅈ/60], Sech[πⅈ/60]}

$$\left\{i \operatorname{Sin}\left[\frac{\pi}{60}\right], \operatorname{Cos}\left[\frac{\pi}{60}\right], i \operatorname{Tan}\left[\frac{\pi}{60}\right], -i \operatorname{Cot}\left[\frac{\pi}{60}\right], -i \operatorname{Csc}\left[\frac{\pi}{60}\right], \operatorname{Sec}\left[\frac{\pi}{60}\right]\right\}$$

Together[FunctionExpand[%]]

$$\left\{\frac{1}{16} i \left(-\sqrt{2}-\sqrt{6}+\sqrt{10}+\sqrt{30}+2 \sqrt{5+\sqrt{5}}-2 \sqrt{3 \left(5+\sqrt{5}\right)}\right),\right.$$

$$\frac{1}{16} \left(\sqrt{2}-\sqrt{6}-\sqrt{10}+\sqrt{30}+2 \sqrt{5+\sqrt{5}}+2 \sqrt{3 \left(5+\sqrt{5}\right)}\right),$$

$$-\frac{i \left(1+\sqrt{3}-\sqrt{5}-\sqrt{15}-\sqrt{2 \left(5+\sqrt{5}\right)}+\sqrt{6 \left(5+\sqrt{5}\right)}\right)}{1-\sqrt{3}-\sqrt{5}+\sqrt{15}+\sqrt{2 \left(5+\sqrt{5}\right)}+\sqrt{6 \left(5+\sqrt{5}\right)}},$$

$$\frac{i \left(1-\sqrt{3}-\sqrt{5}+\sqrt{15}+\sqrt{2 \left(5+\sqrt{5}\right)}+\sqrt{6 \left(5+\sqrt{5}\right)}\right)}{1+\sqrt{3}-\sqrt{5}-\sqrt{15}-\sqrt{2 \left(5+\sqrt{5}\right)}+\sqrt{6 \left(5+\sqrt{5}\right)}},$$

$$-\frac{16 i}{-\sqrt{2}-\sqrt{6}+\sqrt{10}+\sqrt{30}+2 \sqrt{5+\sqrt{5}}-2 \sqrt{3 \left(5+\sqrt{5}\right)}},$$

$$\left.\frac{16}{\sqrt{2}-\sqrt{6}-\sqrt{10}+\sqrt{30}+2 \sqrt{5+\sqrt{5}}+2 \sqrt{3 \left(5+\sqrt{5}\right)}}\right\}$$

If the denominator contains squares of integers other than 2, the results always contain complex numbers (meaning that the imaginary number $i = \sqrt{-1}$ appears unavoidably).

```
{Sinh[π ⅈ/9], Cosh[π ⅈ/9], Tanh[π ⅈ/9], Coth[π ⅈ/9], Csch[π ⅈ/9], Sech[π ⅈ/9]}
```

$$\left\{i \operatorname{Sin}\left[\frac{\pi}{9}\right], \operatorname{Cos}\left[\frac{\pi}{9}\right], i \operatorname{Tan}\left[\frac{\pi}{9}\right], -i \operatorname{Cot}\left[\frac{\pi}{9}\right], -i \operatorname{Csc}\left[\frac{\pi}{9}\right], \operatorname{Sec}\left[\frac{\pi}{9}\right]\right\}$$

```
FunctionExpand[%] // Together
```

$$\Bigg\{\frac{1}{8}\left(2^{2/3}\left(-1-i\sqrt{3}\right)^{1/3} + i\,2^{2/3}\sqrt{3}\left(-1-i\sqrt{3}\right)^{1/3} - 2^{2/3}\left(-1+i\sqrt{3}\right)^{1/3} + i\,2^{2/3}\sqrt{3}\left(-1+i\sqrt{3}\right)^{1/3}\right),$$

$$\frac{1}{8}\left(2^{2/3}\left(-1-i\sqrt{3}\right)^{1/3} + i\,2^{2/3}\sqrt{3}\left(-1-i\sqrt{3}\right)^{1/3} + 2^{2/3}\left(-1+i\sqrt{3}\right)^{1/3} - i\,2^{2/3}\sqrt{3}\left(-1+i\sqrt{3}\right)^{1/3}\right),$$

$$\frac{-i\left(-1-i\sqrt{3}\right)^{1/3} + \sqrt{3}\left(-1-i\sqrt{3}\right)^{1/3} + i\left(-1+i\sqrt{3}\right)^{1/3} + \sqrt{3}\left(-1+i\sqrt{3}\right)^{1/3}}{-i\left(-1-i\sqrt{3}\right)^{1/3} + \sqrt{3}\left(-1-i\sqrt{3}\right)^{1/3} - i\left(-1+i\sqrt{3}\right)^{1/3} - \sqrt{3}\left(-1+i\sqrt{3}\right)^{1/3}},$$

$$\frac{-i\left(-1-i\sqrt{3}\right)^{1/3} + \sqrt{3}\left(-1-i\sqrt{3}\right)^{1/3} - i\left(-1+i\sqrt{3}\right)^{1/3} - \sqrt{3}\left(-1+i\sqrt{3}\right)^{1/3}}{-i\left(-1-i\sqrt{3}\right)^{1/3} + \sqrt{3}\left(-1-i\sqrt{3}\right)^{1/3} + i\left(-1+i\sqrt{3}\right)^{1/3} + \sqrt{3}\left(-1+i\sqrt{3}\right)^{1/3}},$$

$$-(8\,i)\Big/\left(-i\,2^{2/3}\left(-1-i\sqrt{3}\right)^{1/3} + 2^{2/3}\sqrt{3}\left(-1-i\sqrt{3}\right)^{1/3} + i\,2^{2/3}\left(-1+i\sqrt{3}\right)^{1/3} + 2^{2/3}\sqrt{3}\left(-1+i\sqrt{3}\right)^{1/3}\right),$$

$$-(8\,i)\Big/\left(-i\,2^{2/3}\left(-1-i\sqrt{3}\right)^{1/3} + 2^{2/3}\sqrt{3}\left(-1-i\sqrt{3}\right)^{1/3} - i\,2^{2/3}\left(-1+i\sqrt{3}\right)^{1/3} - 2^{2/3}\sqrt{3}\left(-1+i\sqrt{3}\right)^{1/3}\right)\Bigg\}$$

Here the function `RootReduce` is used to express the previous algebraic numbers as numbered roots of polynomial equations.

```
RootReduce[Simplify[%]]
```

```
{Root[3 + 36 #1^2 + 96 #1^4 + 64 #1^6 &, 4], Root[-1 - 6 #1 + 8 #1^3 &, 3],
 Root[3 + 27 #1^2 + 33 #1^4 + #1^6 &, 4], Root[1 + 33 #1^2 + 27 #1^4 + 3 #1^6 &, 3],
 Root[64 + 96 #1^2 + 36 #1^4 + 3 #1^6 &, 5], Root[-8 + 6 #1^2 + #1^3 &, 3]}
```

The function `FunctionExpand` also reduces hyperbolic expressions with compound arguments or compositions, including hyperbolic functions, to simpler forms. Here are some examples.

```
FunctionExpand[Coth[Sqrt[-z^2]]]
```

$$-\frac{\sqrt{-z}\ \mathrm{Cot}[z]}{\sqrt{z}}$$

Tanh$\left[\sqrt{i\, z^2}\right]$ **// FunctionExpand**

$$-\frac{(-1)^{3/4}\sqrt{-(-1)^{3/4} z}\ \sqrt{(-1)^{3/4} z}\ \mathrm{Tanh}\left[(-1)^{1/4} z\right]}{z}$$

$\left\{\mathbf{Sinh}\left[\sqrt{z^2}\right], \mathbf{Cosh}\left[\sqrt{z^2}\right], \mathbf{Tanh}\left[\sqrt{z^2}\right], \mathbf{Coth}\left[\sqrt{z^2}\right], \mathbf{Csch}\left[\sqrt{z^2}\right], \mathbf{Sech}\left[\sqrt{z^2}\right]\right\}$ **// FunctionExpand**

$$\left\{\frac{\sqrt{-i z}\ \sqrt{i z}\ \mathrm{Sinh}[z]}{z}, \mathrm{Cosh}[z], \frac{\sqrt{-i z}\ \sqrt{i z}\ \mathrm{Tanh}[z]}{z}, \frac{\sqrt{-i z}\ \sqrt{i z}\ \mathrm{Coth}[z]}{z}, \frac{\sqrt{-i z}\ \sqrt{i z}\ \mathrm{Csch}[z]}{z}, \mathrm{Sech}[z]\right\}$$

Applying `Simplify` to the last expression gives a more compact result.

Simplify[%]

$$\left\{\frac{\sqrt{z^2}\ \mathrm{Sinh}[z]}{z}, \mathrm{Cosh}[z], \frac{\sqrt{z^2}\ \mathrm{Tanh}[z]}{z}, \frac{\sqrt{z^2}\ \mathrm{Coth}[z]}{z}, \frac{\sqrt{z^2}\ \mathrm{Csch}[z]}{z}, \mathrm{Sech}[z]\right\}$$

Here are some similar examples.

Sinh[2 ArcTanh[z]] // FunctionExpand

$$\frac{2 z}{1 - z^2}$$

Cosh$\left[\frac{\mathbf{ArcCoth[z]}}{\mathbf{2}}\right]$ **// FunctionExpand**

$$\frac{\sqrt{1 + \frac{\sqrt{-i z}\ \sqrt{i z}}{\sqrt{(-1+z)(1+z)}}}}{\sqrt{2}}$$

{Sinh[2 ArcSinh[z]], Cosh[2 ArcCosh[z]], Tanh[2 ArcTanh[z]], Coth[2 ArcCoth[z]], Csch[2 ArcCsch[z]], Sech[2 ArcSech[z]]} // FunctionExpand

$$\left\{2 z \sqrt{i(-i + z)}\ \sqrt{-i(i + z)}, z^2 + (-1 + z)(1 + z), -\frac{2(-1 + z) z (1 + z)}{\left(1 - z^2\right)\left(1 + z^2\right)}, \frac{1}{2}\left(1 - \frac{1}{z^2}\right) z \left(\frac{1}{(-1 + z)(1 + z)} + \frac{z^2}{(-1 + z)(1 + z)}\right), \frac{\sqrt{-z}\ z^{3/2}}{2\sqrt{-1 - z^2}}, \frac{z^2}{2 - z^2}\right\}$$

$$\left\{\text{Sinh}\left[\frac{\text{ArcSinh}[z]}{2}\right], \text{Cosh}\left[\frac{\text{ArcCosh}[z]}{2}\right], \text{Tanh}\left[\frac{\text{ArcTanh}[z]}{2}\right], \text{Coth}\left[\frac{\text{ArcCoth}[z]}{2}\right], \text{Csch}\left[\frac{\text{ArcCsch}[z]}{2}\right], \text{Sech}\left[\frac{\text{ArcSech}[z]}{2}\right]\right\} \text{ // FunctionExpand}$$

$$\left\{\frac{z\sqrt{-1+\sqrt{i(-i+z)}\sqrt{-i(i+z)}}}{\sqrt{2}\sqrt{-iz}\sqrt{iz}}, \frac{\sqrt{1+z}}{\sqrt{2}}, \frac{z}{1+\sqrt{1-z}\sqrt{1+z}}, z\left(1+\frac{\sqrt{(-1+z)(1+z)}}{\sqrt{-iz}\sqrt{iz}}\right), \frac{\sqrt{2}\sqrt{-\frac{i}{z}}\sqrt{\frac{i}{z}}\,z}{\sqrt{-1+\frac{\sqrt{-1-z^2}}{\sqrt{-z}\sqrt{z}}}}, \frac{\sqrt{2}\sqrt{-z}}{\sqrt{-1-z}}\right\}$$

$$\text{Simplify}[\%]$$

$$\left\{\frac{z\sqrt{-1+\sqrt{1+z^2}}}{\sqrt{2}\sqrt{z^2}}, \frac{\sqrt{1+z}}{\sqrt{2}}, \frac{z}{1+\sqrt{1-z^2}}, z+\frac{\sqrt{z^2}\sqrt{-1+z^2}}{z}, \frac{\sqrt{2}\sqrt{\frac{1}{z^2}}\,z}{\sqrt{-1+\sqrt{1+\frac{1}{z^2}}}}, \frac{\sqrt{2}}{\sqrt{1+\frac{1}{z}}}\right\}$$

FullSimplify

The function `FullSimplify` tries a wider range of transformations than the function `Simplify` and returns the simplest form it finds. Here are some examples that contrast the results of applying these functions to the same expressions.

$$\text{Cosh}\left[\frac{1}{2}\text{Log}[1-iz]-\frac{1}{2}\text{Log}[1+iz]\right] \text{ // Simplify}$$

$$\text{Cosh}\left[\frac{1}{2}(\text{Log}[1-iz]-\text{Log}[1+iz])\right]$$

$$\% \text{ // FullSimplify}$$

$$\frac{1}{\sqrt{1+z^2}}$$

$$\left\{\text{Sinh}\left[-\text{Log}\left[iz+\sqrt{1-z^2}\right]\right], \text{Cosh}\left[-\text{Log}\left[iz+\sqrt{1-z^2}\right]\right], \text{Tanh}\left[-\text{Log}\left[iz+\sqrt{1-z^2}\right]\right], \text{Coth}\left[-\text{Log}\left[iz+\sqrt{1-z^2}\right]\right], \text{Csch}\left[-\text{Log}\left[iz+\sqrt{1-z^2}\right]\right], \text{Sech}\left[-\text{Log}\left[iz+\sqrt{1-z^2}\right]\right]\right\} \text{ // Simplify}$$

$$\left\{-i z,\ \frac{1-z^2+i z\sqrt{1-z^2}}{i z+\sqrt{1-z^2}},\ -\frac{-1+\left(i z+\sqrt{1-z^2}\right)^2}{1+\left(i z+\sqrt{1-z^2}\right)^2},\ -\frac{1+\left(i z+\sqrt{1-z^2}\right)^2}{-1+\left(i z+\sqrt{1-z^2}\right)^2},\ \frac{i}{z},\ \frac{2\left(i z+\sqrt{1-z^2}\right)}{1+\left(i z+\sqrt{1-z^2}\right)^2}\right\}$$

```
{Sinh[-Log[ⅈ z + Sqrt[1 - z^2]]], Cosh[-Log[ⅈ z + Sqrt[1 - z^2]]],
  Tanh[-Log[ⅈ z + Sqrt[1 - z^2]]], Coth[-Log[ⅈ z + Sqrt[1 - z^2]]],
  Csch[-Log[ⅈ z + Sqrt[1 - z^2]]], Sech[-Log[ⅈ z + Sqrt[1 - z^2]]]} // FullSimplify
```

$$\left\{-i z,\ \sqrt{1-z^2},\ -\frac{i z}{\sqrt{1-z^2}},\ \frac{i\sqrt{1-z^2}}{z},\ \frac{i}{z},\ \frac{1}{\sqrt{1-z^2}}\right\}$$

Operations performed by specialized *Mathematica* functions

Series expansions

Calculating the series expansion of hyperbolic functions to hundreds of terms can be done in seconds. Here are some examples.

```
Series[Sinh[z], {z, 0, 5}]
```

$$z+\frac{z^3}{6}+\frac{z^5}{120}+O[z]^6$$

```
Normal[%]
```

$$z+\frac{z^3}{6}+\frac{z^5}{120}$$

```
Series[{Sinh[z], Cosh[z], Tanh[z], Coth[z], Csch[z], Sech[z]}, {z, 0, 3}]
```

$$\left\{z+\frac{z^3}{6}+O[z]^4,\ 1+\frac{z^2}{2}+O[z]^4,\ z-\frac{z^3}{3}+O[z]^4,\ \frac{1}{z}+\frac{z}{3}-\frac{z^3}{45}+O[z]^4,\ \frac{1}{z}-\frac{z}{6}+\frac{7 z^3}{360}+O[z]^4,\ 1-\frac{z^2}{2}+O[z]^4\right\}$$

```
Series[Coth[z], {z, 0, 100}] // Timing
```

$$\left\{0.79\ \text{Second},\ \frac{1}{z}+\frac{z}{3}-\frac{z^3}{45}+\frac{2 z^5}{945}-\frac{z^7}{4725}+\frac{2 z^9}{93\,555}-\frac{1382 z^{11}}{638\,512\,875}+\frac{4 z^{13}}{18\,243\,225}-\frac{3617 z^{15}}{162\,820\,783\,125}+\frac{87\,734 z^{17}}{38\,979\,295\,480\,125}-\frac{349\,222 z^{19}}{1\,531\,329\,465\,290\,625}+\right.$$

$$\frac{310\,732\,z^{21}}{13\,447\,856\,940\,643\,125} - \frac{472\,728\,182\,z^{23}}{201\,919\,571\,963\,756\,521\,875} + \frac{2\,631\,724\,z^{25}}{11\,094\,481\,976\,030\,578\,125} -$$

$$\frac{13\,571\,120\,588\,z^{27}}{564\,653\,660\,170\,076\,273\,671\,875} + \frac{13\,785\,346\,041\,608\,z^{29}}{5\,660\,878\,804\,669\,082\,674\,070\,015\,625} -$$

$$\frac{7\,709\,321\,041\,217\,z^{31}}{31\,245\,110\,285\,511\,170\,603\,633\,203\,125} + \frac{303\,257\,395\,102\,z^{33}}{12\,130\,454\,581\,433\,748\,587\,292\,890\,625} -$$

$$\frac{52\,630\,543\,106\,106\,954\,746\,z^{35}}{20\,777\,977\,561\,866\,588\,586\,487\,628\,662\,044\,921\,875} + \frac{616\,840\,823\,966\,644\,z^{37}}{2\,403\,467\,618\,492\,375\,776\,343\,276\,883\,984\,375} -$$

$$\frac{522\,165\,436\,992\,898\,244\,102\,z^{39}}{20\,080\,431\,172\,289\,638\,826\,798\,401\,128\,390\,556\,640\,625} +$$

$$\frac{6\,080\,390\,575\,672\,283\,210\,764\,z^{41}}{2\,307\,789\,189\,818\,960\,127\,712\,594\,427\,864\,667\,427\,734\,375} -$$

$$\frac{10\,121\,188\,937\,927\,645\,176\,372\,z^{43}}{37\,913\,679\,547\,025\,773\,526\,706\,908\,457\,776\,679\,169\,921\,875} +$$

$$\frac{207\,461\,256\,206\,578\,143\,748\,856\,z^{45}}{7\,670\,102\,214\,448\,301\,053\,033\,358\,480\,610\,212\,529\,462\,890\,625} -$$

$$\frac{11\,218\,806\,737\,995\,635\,372\,498\,255\,094\,z^{47}}{4\,093\,648\,603\,384\,274\,996\,519\,698\,921\,478\,879\,580\,162\,286\,669\,921\,875} +$$

$$\frac{79\,209\,152\,838\,572\,743\,713\,996\,404\,z^{49}}{285\,258\,771\,457\,546\,764\,463\,363\,635\,252\,374\,414\,183\,254\,365\,234\,375} -$$

$$\frac{246\,512\,528\,657\,073\,833\,030\,130\,766\,724\,z^{51}}{8\,761\,982\,491\,474\,419\,367\,550\,817\,114\,626\,909\,562\,924\,278\,968\,505\,859\,375} +$$

$$\frac{233\,199\,709\,079\,078\,899\,371\,344\,990\,501\,528\,z^{53}}{81\,807\,125\,729\,900\,063\,867\,074\,959\,072\,425\,603\,825\,198\,823\,017\,351\,806\,640\,625} -$$

$$\frac{1\,416\,795\,959\,607\,558\,144\,963\,094\,708\,378\,988\,z^{55}}{4\,905\,352\,087\,939\,496\,310\,826\,487\,207\,538\,302\,184\,255\,342\,959\,123\,162\,841\,796\,875} +$$

$$\frac{23\,305\,824\,372\,104\,839\,134\,357\,731\,308\,699\,592\,z^{57}}{796\,392\,368\,980\,577\,121\,745\,974\,726\,570\,063\,253\,238\,310\,542\,073\,919\,837\,646\,484\,375} -$$

$$\frac{9\,721\,865\,123\,870\,044\,576\,322\,439\,952\,638\,561\,968\,331\,928\,z^{59}}{3\,278\,777\,586\,273\,629\,598\,615\,520\,165\,380\,455\,583\,231\,003\,564\,645\,636\,125\,000\,418\,914\,794\,921\,875} +$$

$$\frac{6\,348\,689\,256\,302\,894\,731\,330\,601\,216\,724\,328\,336\,z^{61}}{21\,132\,271\,510\,899\,613\,925\,529\,439\,369\,536\,628\,424\,678\,570\,233\,931\,462\,891\,949\,462\,890\,625} -$$

$$\frac{106\,783\,830\,147\,866\,529\,886\,385\,444\,979\,142\,647\,942\,017\,z^{63}}{3\,508\,062\,732\,166\,890\,409\,707\,514\,582\,539\,928\,001\,638\,766\,051\,683\,792\,497\,378\,070\,587\,158\,203\,125} +$$

$$\left(267\,745\,458\,568\,424\,664\,373\,021\,714\,282\,169\,516\,771\,254\,382\,z^{65}\right) /$$
$$86\,812\,790\,293\,146\,213\,360\,651\,966\,604\,262\,937\,105\,495\,141\,563\,588\,806\,888\,204\,273\,501\,373\,291\,015\,625 - \left(250\,471\,004\,320\,250\,327\,955\,196\,022\,920\,428\,000\,776\,938\,z^{67}\right) /$$
$$801\,528\,196\,428\,242\,695\,121\,010\,267\,455\,843\,804\,062\,822\,357\,897\,831\,858\,125\,102\,407\,684\,326\,171\,875 + \left(172\,043\,582\,552\,384\,800\,434\,637\,321\,986\,040\,823\,829\,878\,646\,884\,z^{69}\right) /$$
$$5\,433\,748\,964\,547\,053\,581\,149\,916\,185\,708\,338\,218\,048\,392\,402\,830\,337\,634\,114\,958\,370\,880\,742\,156\,982\,421\,875 - \left(11\,655\,909\,923\,339\,888\,220\,876\,554\,489\,282\,134\,730\,564\,976\,603\,688\,520\,858\,z^{71}\right) /$$
$$3\,633\,348\,205\,269\,879\,230\,856\,840\,004\,304\,821\,536\,968\,049\,780\,112\,803\,650\,817\,771\,432\,558\,560\,793$$

458 452 606 201 171 875 +
(3 692 153 220 456 342 488 035 683 646 645 690 290 452 790 030 604 z^{73}) /
11 359 005 221 796 317 918 049 302 062 760 294 302 183 889 391 189 419 445 133 951 612 582 060 536
346 435 546 875 − (5 190 545 015 986 394 254 249 936 008 544 252 611 445 319 542 919 116 z^{75}) /
157 606 197 452 423 911 112 934 066 120 799 083 442 801 465 302 753 194 801 233 578 624 576 089
941 806 793 212 890 625 +
(255 290 071 123 323 586 643 187 098 799 718 199 072 122 692 536 861 835 992 z^{77}) /
76 505 736 228 426 953 173 738 238 352 183 101 801 688 392 812 244 485 181 277 127 930 109 049 138
257 655 704 498 291 015 625 −
(9 207 568 598 958 915 293 871 149 938 038 093 699 588 515 745 502 577 839 313 734 z^{79}) /
27 233 582 984 369 795 892 070 228 410 001 578 355 986 013 571 390 071 723 225 259 349 721 067 988
068 852 863 296 604 156 494 140 625 +
(163 611 136 505 867 886 519 332 147 296 221 453 678 803 514 884 902 772 183 572 z^{81}) /
4 776 089 171 877 348 057 451 105 924 101 750 653 118 402 745 283 825 543 113 171 217 116 857 704
024 700 607 798 175 811 767 578 125 −
(8 098 304 783 741 161 440 924 524 640 446 924 039 959 669 564 792 363 509 124 335 729 908 z^{83}) /
2 333 207 846 470 426 678 843 707 227 616 712 214 909 162 634 745 895 349 325 948 586 531 533 393
530 725 143 500 144 033 328 342 437 744 140 625 +
(122 923 650 124 219 284 385 832 157 660 699 813 260 991 755 656 444 452 420 836 648 z^{85}) /
349 538 086 043 843 717 584 559 187 055 386 621 548 470 304 913 596 772 372 737 435 524 697 231
069 047 713 981 709 496 784 210 205 078 125 −
(476 882 359 517 824 548 362 004 154 188 840 670 307 545 554 753 464 961 562 516 323 845 108 z^{87}) /
13 383 510 964 174 348 021 497 060 628 653 950 829 663 288 548 327 870 152 944 013 988 358 928 114
528 962 242 087 062 453 152 690 410 614 013 671 875 +
(1 886 491 646 433 732 479 814 597 361 998 744 134 040 407 919 471 435 385 970 472 345 164 676 056
z^{89}) /
522 532 651 330 971 490 226 753 590 247 329 744 050 384 290 675 644 135 735 656 667 608 610 471
400 391 047 234 539 824 350 830 981 313 610 076 904 296 875 −
(450 638 590 680 882 618 431 105 331 665 591 912 924 988 342 163 281 788 877 675 244 114 763 912
z^{91}) /
1 231 931 818 039 911 948 327 467 370 123 161 265 684 460 571 086 659 079 080 437 659 781 065 743
269 173 212 919 832 661 978 537 311 246 395 111 083 984 375 +
(415 596 189 473 955 564 121 634 614 268 323 814 113 534 779 643 471 190 276 158 333 713 923 216
z^{93}) /
11 213 200 675 690 943 223 287 032 785 929 540 201 272 600 687 465 377 745 332 153 847 964 679 254
692 602 138 023 498 144 562 090 675 557 613 372 802 734 375 −
(423 200 899 194 533 026 195 195 456 219 648 467 346 087 908 778 120 468 301 277 466 840 101 336
699 974 518 z^{95}) /
112 694 926 530 960 148 011 367 752 417 874 063 473 378 698 369 880 587 800 838 274 234 349 237
591 647 453 413 782 021 538 312 594 164 677 406 144 702 434 539 794 921 875 +
(5 543 531 483 502 489 438 698 050 411 951 314 743 456 505 773 755 468 368 087 670 306 121 873 229
244 z^{97}) /
14 569 479 835 935 377 894 165 191 004 250 040 526 616 509 162 234 077 285 176 247 476 968 227 225
810 918 346 966 001 491 701 692 846 112 140 419 483 184 814 453 125 −
(378 392 151 276 488 501 180 909 732 277 974 887 490 811 366 132 267 744 533 542 784 817 245 581
660 788 990 844 z^{99}) /
9 815 205 420 757 514 710 108 178 059 369 553 458 327 392 260 750 404 049 930 407 987 933 582 359
080 767 225 644 716 670 683 512 153 512 547 802 166 033 089 160 919 189 453 125 + $O[z]^{101}$)

Mathematica comes with the add-on package `DiscreteMath`RSolve`` that allows finding the general terms of series for many functions. After loading this package, and using the package function `SeriesTerm`, the following n^{th} term for odd hyperbolic functions can be evaluated.

```
<< DiscreteMath`RSolve`
```

```
SeriesTerm[{Sinh[z], Cosh[z], Tanh[z], Coth[z], Csch[z], Sech[z]}, {z, 0, n}] z^n
```

$$\left\{z^n \operatorname{If}\left[\operatorname{Odd}[n], \frac{1}{n!}, 0\right], z^n \operatorname{If}\left[\operatorname{Even}[n], \frac{1}{n!}, 0\right], z^n \operatorname{If}\left[\operatorname{Odd}[n], \frac{2^{1+n}\left(-1+2^{1+n}\right) \operatorname{BernoulliB}[1+n]}{(1+n)!}, 0\right], \frac{2^{1+n} z^n \operatorname{BernoulliB}[1+n]}{(1+n)!}, \frac{2^{1+n} z^n \operatorname{BernoulliB}\left[1+n, \frac{1}{2}\right]}{(1+n)!}, \frac{z^n \operatorname{EulerE}[n]}{n!}\right\}$$

Here is a quick check of the last result.

This series should be evaluated to {sinh(z), cosh(z), tanh(z), coth(z), csch(z), sech(z)}, which can be concluded from the following relation.

```
Sum[#, {n, 0, 100}] & /@ % -
 Series[{Sinh[z], Cosh[z], Tanh[z], Coth[z], Csch[z], Sech[z]}, {z, 0, 100}]
```

$$\left\{O[z]^{101}, O[z]^{101}, O[z]^{101}, -\frac{1}{z} - 1 + O[z]^{101}, -\frac{1}{z} + O[z]^{101}, O[z]^{101}\right\}$$

Differentiation

Mathematica can evaluate derivatives of hyperbolic functions of an arbitrary positive integer order.

```
D[Sinh[z], z]
```

```
Cosh[z]
```

```
Sinh[z] // D[#, z] &
```

```
Cosh[z]
```

∂_z `{Sinh[z], Cosh[z], Tanh[z], Coth[z], Csch[z], Sech[z]}`

$$\left\{\operatorname{Cosh}[z], \operatorname{Sinh}[z], \operatorname{Sech}[z]^2, -\operatorname{Csch}[z]^2, -\operatorname{Coth}[z] \operatorname{Csch}[z], -\operatorname{Sech}[z] \operatorname{Tanh}[z]\right\}$$

$\partial_{\{z,2\}}$ `{Sinh[z], Cosh[z], Tanh[z], Coth[z], Csch[z], Sech[z]}`

$$\left\{\operatorname{Sinh}[z], \operatorname{Cosh}[z], -2 \operatorname{Sech}[z]^2 \operatorname{Tanh}[z], 2 \operatorname{Coth}[z] \operatorname{Csch}[z]^2, \operatorname{Coth}[z]^2 \operatorname{Csch}[z] + \operatorname{Csch}[z]^3, -\operatorname{Sech}[z]^3 + \operatorname{Sech}[z] \operatorname{Tanh}[z]^2\right\}$$

```
Table[D[{Sinh[z], Cosh[z], Tanh[z], Coth[z], Csch[z], Sech[z]}, {z, n}], {n, 4}]
```

$$\{\{\text{Cosh}[z], \text{Sinh}[z], \text{Sech}[z]^2, -\text{Csch}[z]^2, -\text{Coth}[z]\,\text{Csch}[z], -\text{Sech}[z]\,\text{Tanh}[z]\},$$
$$\{\text{Sinh}[z], \text{Cosh}[z], -2\,\text{Sech}[z]^2\,\text{Tanh}[z], 2\,\text{Coth}[z]\,\text{Csch}[z]^2,$$
$$\text{Coth}[z]^2\,\text{Csch}[z] + \text{Csch}[z]^3, -\text{Sech}[z]^3 + \text{Sech}[z]\,\text{Tanh}[z]^2\},$$
$$\{\text{Cosh}[z], \text{Sinh}[z], -2\,\text{Sech}[z]^4 + 4\,\text{Sech}[z]^2\,\text{Tanh}[z]^2, -4\,\text{Coth}[z]^2\,\text{Csch}[z]^2 - 2\,\text{Csch}[z]^4,$$
$$-\text{Coth}[z]^3\,\text{Csch}[z] - 5\,\text{Coth}[z]\,\text{Csch}[z]^3, 5\,\text{Sech}[z]^3\,\text{Tanh}[z] - \text{Sech}[z]\,\text{Tanh}[z]^3\},$$
$$\{\text{Sinh}[z], \text{Cosh}[z], 16\,\text{Sech}[z]^4\,\text{Tanh}[z] - 8\,\text{Sech}[z]^2\,\text{Tanh}[z]^3,$$
$$8\,\text{Coth}[z]^3\,\text{Csch}[z]^2 + 16\,\text{Coth}[z]\,\text{Csch}[z]^4, \text{Coth}[z]^4\,\text{Csch}[z] + 18\,\text{Coth}[z]^2\,\text{Csch}[z]^3 +$$
$$5\,\text{Csch}[z]^5, 5\,\text{Sech}[z]^5 - 18\,\text{Sech}[z]^3\,\text{Tanh}[z]^2 + \text{Sech}[z]\,\text{Tanh}[z]^4\}\}$$

Finite summation

Mathematica can calculate finite sums that contain hyperbolic functions. Here are two examples.

```
Sum[Sinh[a k], {k, 0, n}]
```

$$\frac{-1 + e^{a+a\,n}}{2\,(-1 + e^a)} - \frac{e^{-a\,n}\,(-1 + e^{a+a\,n})}{2\,(-1 + e^a)}$$

$$\sum_{k=0}^{n} (-1)^k\, \text{Sinh}[a\,k]$$

$$-\frac{e^a + (-e^{-a})^n}{2\,(1 + e^a)} + \frac{1 + e^a\,(-e^a)^n}{2\,(1 + e^a)}$$

Infinite summation

Mathematica can calculate infinite sums that contain hyperbolic functions. Here are some examples.

$$\sum_{k=1}^{\infty} z^k\, \text{Sinh}[k\,x]$$

$$-\frac{z}{2\,(e^x - z)} - \frac{e^x\,z}{2\,(-1 + e^x\,z)}$$

$$\sum_{k=1}^{\infty} \frac{\text{Sinh}[k\,x]}{k\,!}$$

$$\frac{1}{2}\left(1 - e^{e^{-x}}\right) + \frac{1}{2}\left(-1 + e^{e^x}\right)$$

$$\sum_{k=1}^{\infty} \frac{\text{Cosh}[k\,x]}{k}$$

$$-\frac{1}{2}\,\text{Log}[1 - e^{-x}] - \frac{1}{2}\,\text{Log}[1 - e^x]$$

Finite products

Mathematica can calculate some finite symbolic products that contain the hyperbolic functions. Here are two examples.

$$\prod_{k=1}^{n-1} \text{Sinh}\left[\frac{\pi k i}{n}\right]$$

$$\left(\frac{i}{2}\right)^{-1+n} n$$

$$\prod_{k=1}^{n-1} \text{Cosh}\left[z + \frac{\pi k i}{n}\right]$$

$$-(-1)^n 2^{1-n} \text{Sech}[z] \text{Sin}\left[\frac{1}{2} n (\pi + 2 i z)\right]$$

Infinite products

Mathematica can calculate infinite products that contain hyperbolic functions. Here are some examples.

$$\prod_{k=1}^{\infty} \text{Exp}\left[z^k \text{Sinh}[k x]\right]$$

$$e^{-\frac{\left(-1+e^{2 x}\right) z}{2 \left(e^x-z\right) \left(-1+e^x z\right)}}$$

$$\prod_{k=1}^{\infty} \text{Exp}\left[\frac{\text{Cosh}[k x]}{k!}\right]$$

$$e^{\frac{1}{2} \left(-2+e^{e^{-x}}+e^{e^x}\right)}$$

Indefinite integration

Mathematica can calculate a huge set of doable indefinite integrals that contain hyperbolic functions. Here are some examples.

$$\int \text{Sinh}[7 z] \, dz$$

$$\frac{1}{7} \text{Cosh}[7 z]$$

$$\int \left\{\left\{\text{Sinh}[z], \text{Sinh}[z]^a\right\}, \{\text{Cosh}[z], \text{Cosh}[z]^a\}, \{\text{Tanh}[z], \text{Tanh}[z]^a\}, \{\text{Coth}[z], \text{Coth}[z]^a\}, \{\text{Csch}[z], \text{Csch}[z]^a\}, \{\text{Sech}[z], \text{Sech}[z]^a\}\right\} dz$$

$$\Big\{\Big\{\text{Cosh}[z],\; -\text{Cosh}[z]\,\text{Hypergeometric2F1}\Big[\frac{1}{2},\frac{1-a}{2},\frac{3}{2},\text{Cosh}[z]^2\Big]\,\text{Sinh}[z]^{1+a}\left(-\text{Sinh}[z]^2\right)^{\frac{1}{2}(-1-a)}\Big\},$$

$$\Big\{\text{Sinh}[z],\; -\frac{\text{Cosh}[z]^{1+a}\,\text{Hypergeometric2F1}\Big[\frac{1+a}{2},\frac{1}{2},\frac{3+a}{2},\text{Cosh}[z]^2\Big]\,\text{Sinh}[z]}{(1+a)\sqrt{-\text{Sinh}[z]^2}}\Big\},$$

$$\Big\{\text{Log}[\text{Cosh}[z]],\; \frac{\text{Hypergeometric2F1}\Big[\frac{1+a}{2},1,1+\frac{1+a}{2},\text{Tanh}[z]^2\Big]\,\text{Tanh}[z]^{1+a}}{1+a}\Big\},$$

$$\Big\{\text{Log}[\text{Sinh}[z]],\; \frac{\text{Coth}[z]^{1+a}\,\text{Hypergeometric2F1}\Big[\frac{1+a}{2},1,1+\frac{1+a}{2},\text{Coth}[z]^2\Big]}{1+a}\Big\},$$

$$\Big\{-\text{Log}\Big[\text{Cosh}\Big[\frac{z}{2}\Big]\Big]+\text{Log}\Big[\text{Sinh}\Big[\frac{z}{2}\Big]\Big],\; -\text{Cosh}[z]\,\text{Csch}[z]^{-1+a}\,\text{Hypergeometric2F1}\Big[\frac{1}{2},\frac{1+a}{2},\frac{3}{2},\text{Cosh}[z]^2\Big]\left(-\text{Sinh}[z]^2\right)^{\frac{1}{2}(-1+a)}\Big\},$$

$$\Big\{2\,\text{ArcTan}\Big[\text{Tanh}\Big[\frac{z}{2}\Big]\Big],\; -\frac{\text{Hypergeometric2F1}\Big[\frac{1-a}{2},\frac{1}{2},\frac{3-a}{2},\text{Cosh}[z]^2\Big]\,\text{Sech}[z]^{-1+a}\,\text{Sinh}[z]}{(1-a)\sqrt{-\text{Sinh}[z]^2}}\Big\}\Big\}$$

Definite integration

Mathematica can calculate wide classes of definite integrals that contain hyperbolic functions. Here are some examples.

$$\int_0^{\pi/2}\sqrt[3]{\text{Sinh}[z]}\,dz$$

$$-\frac{(-1)^{1/3}\sqrt{\pi}\,\text{Gamma}\left[\frac{2}{3}\right]}{2\,\text{Gamma}\left[\frac{7}{6}\right]}+(-1)^{1/3}\,\text{Cosh}\Big[\frac{\pi}{2}\Big]\,\text{Hypergeometric2F1}\Big[\frac{1}{2},\frac{1}{3},\frac{3}{2},\text{Cosh}\Big[\frac{\pi}{2}\Big]^2\Big]$$

$$\int_1^{\pi/2}\left\{\sqrt{\text{Sinh}[z]},\sqrt{\text{Cosh}[z]},\sqrt{\text{Tanh}[z]},\sqrt{\text{Coth}[z]},\sqrt{\text{Csch}[z]},\sqrt{\text{Sech}[z]}\right\}dz$$

$$\Big\{2\,(-1)^{1/4}\,\text{EllipticE}\Big[\Big(\frac{1}{4}-\frac{i}{4}\Big)\pi,\,2\Big]-2\,(-1)^{1/4}\,\text{EllipticE}\Big[\frac{1}{4}\,(-2\,i+\pi),\,2\Big],$$

$$2\,i\,\text{EllipticE}\Big[\frac{i}{2},\,2\Big]-2\,i\,\text{EllipticE}\Big[\frac{i\,\pi}{4},\,2\Big],$$

$$\frac{1}{2}\Bigg(i\,\text{Log}\Big[1-i\,\sqrt{\frac{-1+e^2}{1+e^2}}\Big]-i\,\text{Log}\Big[1+i\,\sqrt{\frac{-1+e^2}{1+e^2}}\Big]-$$
$$i\,\text{Log}\Big[1-i\,\sqrt{\frac{-1+e^\pi}{1+e^\pi}}\Big]+i\,\text{Log}\Big[1+i\,\sqrt{\frac{-1+e^\pi}{1+e^\pi}}\Big]+\text{Log}\Big[1-\sqrt{\text{Tanh}[1]}\Big]-$$
$$\text{Log}\Big[1+\sqrt{\text{Tanh}[1]}\Big]-\text{Log}\Big[1-\sqrt{\text{Tanh}\Big[\frac{\pi}{2}\Big]}\Big]+\text{Log}\Big[1+\sqrt{\text{Tanh}\Big[\frac{\pi}{2}\Big]}\Big]\Bigg),$$

$$\frac{1}{2}\,i\,\Bigg(\text{Log}\Big[1-i\,\sqrt{\frac{1+e^2}{-1+e^2}}\Big]-\text{Log}\Big[1+i\,\sqrt{\frac{1+e^2}{-1+e^2}}\Big]-\text{Log}\Big[1-i\,\sqrt{\frac{1+e^\pi}{-1+e^\pi}}\Big]+$$
$$\text{Log}\Big[1+i\,\sqrt{\frac{1+e^\pi}{-1+e^\pi}}\Big]-i\,\text{Log}\Big[-1+\sqrt{\text{Coth}[1]}\Big]+i\,\text{Log}\Big[1+\sqrt{\text{Coth}[1]}\Big]+$$
$$i\,\text{Log}\Big[-1+\sqrt{\text{Coth}\Big[\frac{\pi}{2}\Big]}\Big]-i\,\text{Log}\Big[1+\sqrt{\text{Coth}\Big[\frac{\pi}{2}\Big]}\Big]\Bigg),$$

$$2\,(-1)^{3/4}\,\text{EllipticF}\Big[\Big(\frac{1}{4}-\frac{i}{4}\Big)\pi,\,2\Big]-2\,(-1)^{3/4}\,\text{EllipticF}\Big[\frac{1}{4}\,(-2\,i+\pi),\,2\Big],$$

$$2\,i\,\text{EllipticF}\Big[\frac{i}{2},\,2\Big]-2\,i\,\text{EllipticF}\Big[\frac{i\,\pi}{4},\,2\Big]\Big\}$$

$$\int_1^{\frac{\pi}{4}}\Big\{\{\textbf{Sinh}[z],\,\textbf{Sinh}[z]^{a}\},\,\{\textbf{Cosh}[z],\,\textbf{Cosh}[z]^{a}\},\,\{\textbf{Tanh}[z],\,\textbf{Tanh}[z]^{a}\},$$
$$\{\textbf{Coth}[z],\,\textbf{Coth}[z]^{a}\},\,\{\textbf{Csch}[z],\,\textbf{Csch}[z]^{a}\},\,\{\textbf{Sech}[z],\,\textbf{Sech}[z]^{a}\}\Big\}\,dz$$

$$\Big\{\Big\{-\text{Cosh}[1]+\text{Cosh}\Big[\frac{\pi}{4}\Big],$$
$$(-1)^{-\frac{1}{2}-\frac{a}{2}}\,\text{Cosh}[1]\,\text{Hypergeometric2F1}\Big[\frac{1}{2},\frac{1-a}{2},\frac{3}{2},\text{Cosh}[1]^2\Big]\,\text{Sinh}[1]^{1+2\left(-\frac{1}{2}-\frac{a}{2}\right)+a}-$$
$$(-1)^{-\frac{1}{2}-\frac{a}{2}}\,\text{Cosh}\Big[\frac{\pi}{4}\Big]\,\text{Hypergeometric2F1}\Big[\frac{1}{2},\frac{1-a}{2},\frac{3}{2},\text{Cosh}\Big[\frac{\pi}{4}\Big]^2\Big]\,\text{Sinh}\Big[\frac{\pi}{4}\Big]^{1+2\left(-\frac{1}{2}-\frac{a}{2}\right)+a}\Big\},$$
$$\Big\{-\text{Sinh}[1]+\text{Sinh}\Big[\frac{\pi}{4}\Big],\ -\frac{i\,\text{Cosh}[1]^{1+a}\,\text{Hypergeometric2F1}\Big[\frac{1+a}{2},\frac{1}{2},\frac{3+a}{2},\text{Cosh}[1]^2\Big]}{1+a}+$$
$$\frac{i\,\text{Cosh}\big[\frac{\pi}{4}\big]^{1+a}\,\text{Hypergeometric2F1}\Big[\frac{1+a}{2},\frac{1}{2},\frac{3+a}{2},\text{Cosh}\big[\frac{\pi}{4}\big]^2\Big]}{1+a}\Big\},$$
$$\Big\{-\text{Log}[\text{Cosh}[1]]+\text{Log}\Big[\text{Cosh}\Big[\frac{\pi}{4}\Big]\Big],\ -\frac{\text{Hypergeometric2F1}\Big[\frac{1+a}{2},1,1+\frac{1+a}{2},\text{Tanh}[1]^2\Big]\,\text{Tanh}[1]^{1+a}}{1+a}+$$
$$\frac{\text{Hypergeometric2F1}\Big[\frac{1+a}{2},1,1+\frac{1+a}{2},\text{Tanh}\big[\frac{\pi}{4}\big]^2\Big]\,\text{Tanh}\big[\frac{\pi}{4}\big]^{1+a}}{1+a}\Big\},$$
$$\Big\{-\text{Log}[\text{Sinh}[1]]+\text{Log}\Big[\text{Sinh}\Big[\frac{\pi}{4}\Big]\Big],\ -\frac{\text{Coth}[1]^{1+a}\,\text{Hypergeometric2F1}\Big[\frac{1+a}{2},1,1+\frac{1+a}{2},\text{Coth}[1]^2\Big]}{1+a}+$$
$$\frac{\text{Coth}\big[\frac{\pi}{4}\big]^{1+a}\,\text{Hypergeometric2F1}\Big[\frac{1+a}{2},1,1+\frac{1+a}{2},\text{Coth}\big[\frac{\pi}{4}\big]^2\Big]}{1+a}\Big\},$$
$$\Big\{\text{Log}\Big[\text{Cosh}\Big[\frac{1}{2}\Big]\Big]-\text{Log}\Big[\text{Cosh}\Big[\frac{\pi}{8}\Big]\Big]-\text{Log}\Big[\text{Sinh}\Big[\frac{1}{2}\Big]\Big]+\text{Log}\Big[\text{Sinh}\Big[\frac{\pi}{8}\Big]\Big],$$
$$(-1)^{-\frac{1}{2}+\frac{a}{2}}\,\text{Cosh}[1]\,\text{Csch}[1]^{-1+a}\,\text{Hypergeometric2F1}\Big[\frac{1}{2},\frac{1+a}{2},\frac{3}{2},\text{Cosh}[1]^2\Big]\,\text{Sinh}[1]^{2\left(-\frac{1}{2}+\frac{a}{2}\right)}-$$
$$(-1)^{-\frac{1}{2}+\frac{a}{2}}\,\text{Cosh}\Big[\frac{\pi}{4}\Big]\,\text{Csch}\Big[\frac{\pi}{4}\Big]^{-1+a}\,\text{Hypergeometric2F1}\Big[\frac{1}{2},\frac{1+a}{2},\frac{3}{2},\text{Cosh}\Big[\frac{\pi}{4}\Big]^2\Big]\,\text{Sinh}\Big[\frac{\pi}{4}\Big]^{2\left(-\frac{1}{2}+\frac{a}{2}\right)}\Big\},$$
$$\Big\{-2\,\text{ArcTan}\Big[\text{Tanh}\Big[\frac{1}{2}\Big]\Big]+2\,\text{ArcTan}\Big[\text{Tanh}\Big[\frac{\pi}{8}\Big]\Big],$$
$$\frac{i\,\text{Hypergeometric2F1}\Big[\frac{1-a}{2},\frac{1}{2},\frac{3-a}{2},\text{Cosh}[1]^2\Big]\,\text{Sech}[1]^{-1+a}}{-1+a}-$$
$$\frac{i\,\text{Hypergeometric2F1}\Big[\frac{1-a}{2},\frac{1}{2},\frac{3-a}{2},\text{Cosh}\big[\frac{\pi}{4}\big]^2\Big]\,\text{Sech}\big[\frac{\pi}{4}\big]^{-1+a}}{-1+a}\Big\}\Big\}$$

$$\int_0^\infty\Big\{\frac{1}{a+b\,\text{Sinh}[z]},\ \frac{1}{a+b\,\text{Cosh}[z]},$$
$$\frac{1}{a+b\,\text{Tanh}[z]},\ \frac{1}{a+b\,\text{Coth}[z]},\ \frac{1}{a+b\,\text{Csch}[z]},\ \frac{1}{a+b\,\text{Sech}[z]}\Big\}\,dz$$

$$\Big\{-\frac{1}{\sqrt{-a^2-b^2}}\Big(i\Big(\text{Log}\Big[1-\frac{i\,a}{\sqrt{-a^2-b^2}}\Big]-\text{Log}\Big[1+\frac{i\,a}{\sqrt{-a^2-b^2}}\Big]+\text{Log}\Big[\frac{i\,a-i\,b+\sqrt{-a^2-b^2}}{\sqrt{-a^2-b^2}}\Big]-\text{Log}\Big[\frac{-i\,a+i\,b+\sqrt{-a^2-b^2}}{\sqrt{-a^2-b^2}}\Big]\Big)\Big),$$

$$-\frac{1}{\sqrt{-a^2+b^2}}\Big(i\Big(\text{Log}\Big[1-\frac{i\,a}{\sqrt{-a^2+b^2}}\Big]-\text{Log}\Big[1+\frac{i\,a}{\sqrt{-a^2+b^2}}\Big]-\text{Log}\Big[\frac{-i\,a-i\,b+\sqrt{-a^2+b^2}}{\sqrt{-a^2+b^2}}\Big]+\text{Log}\Big[\frac{i\,a+i\,b+\sqrt{-a^2+b^2}}{\sqrt{-a^2+b^2}}\Big]\Big)\Big),$$

$$\frac{b\,(\text{Log}[2\,a]-\text{Log}[a+b])}{a^2-b^2},\ \frac{b\,(\text{Log}[-a-b]-\text{Log}[-2\,b])}{-a^2+b^2},$$

$$\frac{1}{a\sqrt{-a^2-b^2}}\Big(i\,b\Big(\text{Log}\Big[\frac{-i\,b+\sqrt{-a^2-b^2}}{\sqrt{-a^2-b^2}}\Big]-\text{Log}\Big[\frac{i\,a-i\,b+\sqrt{-a^2-b^2}}{\sqrt{-a^2-b^2}}\Big]-\text{Log}\Big[\frac{i\,b+\sqrt{-a^2-b^2}}{\sqrt{-a^2-b^2}}\Big]+\text{Log}\Big[\frac{-i\,a+i\,b+\sqrt{-a^2-b^2}}{\sqrt{-a^2-b^2}}\Big]\Big)\Big),$$

$$\frac{1}{a\sqrt{a^2-b^2}}\Big(i\,b\Big(\text{Log}\Big[1-\frac{i\,b}{\sqrt{a^2-b^2}}\Big]-\text{Log}\Big[1+\frac{i\,b}{\sqrt{a^2-b^2}}\Big]-\text{Log}\Big[\frac{-i\,a-i\,b+\sqrt{a^2-b^2}}{\sqrt{a^2-b^2}}\Big]+\text{Log}\Big[\frac{i\,a+i\,b+\sqrt{a^2-b^2}}{\sqrt{a^2-b^2}}\Big]\Big)\Big)\Big\}$$

Limit operation

Mathematica can calculate limits that contain hyperbolic functions. Here are some examples.

$$\textbf{Limit}\Big[\frac{\textbf{Sinh}[\textbf{z}]}{\textbf{z}}+\textbf{Cosh}[\textbf{z}]^3,\ \textbf{z}\to\textbf{0}\Big]$$

2

$$\textbf{Limit}\Big[\Big(\frac{\textbf{Tanh}[\textbf{x}]}{\textbf{x}}\Big)^{\frac{1}{x^2}},\ \textbf{x}\to\textbf{0}\Big]$$

$$\frac{1}{e^{1/3}}$$

```
Limit[Sinh[Sqrt[z^2]]/z, z -> 0, Direction -> 1]
```

−1

```
Limit[Sinh[Sqrt[z^2]]/z, z -> 0, Direction -> -1]
```

1

Solving equations

The next input solves equations that contain hyperbolic functions. The message indicates that the multivalued functions are used to express the result and that some solutions might be absent.

```
Solve[Tanh[z]^2 + 3 Sinh[z + Pi/6] == 4, z]
```

Solve::ifun : Inverse functions are being used by Solve, so some solutions may not be found.

```
{{z -> -ArcSech[
    3 (-2 Root[1 + 6 #1^2 - 6 Sinh[π/6] #1^3 + 9 #1^4 + 9 Cosh[π/6]^2 #1^4 - 18 Sinh[π/6] #1^5 - 9 Cosh[π/6]^2
          #1^6 + 9 Sinh[π/6]^2 #1^6 &, 1] - 3 Root[1 + 6 #1^2 - 6 Sinh[π/6] #1^3 + 9 #1^4 +
          9 Cosh[π/6]^2 #1^4 - 18 Sinh[π/6] #1^5 - 9 Cosh[π/6]^2 #1^6 + 9 Sinh[π/6]^2 #1^6 &, 1]^3 -
       3 Cosh[π/6]^2 Root[1 + 6 #1^2 - 6 Sinh[π/6] #1^3 + 9 #1^4 + 9 Cosh[π/6]^2 #1^4 -
          18 Sinh[π/6] #1^5 - 9 Cosh[π/6]^2 #1^6 + 9 Sinh[π/6]^2 #1^6 &, 1]^3 +
       3 Cosh[π/6]^2 Root[1 + 6 #1^2 - 6 Sinh[π/6] #1^3 + 9 #1^4 + 9 Cosh[π/6]^2 #1^4 -
          18 Sinh[π/6] #1^5 - 9 Cosh[π/6]^2 #1^6 + 9 Sinh[π/6]^2 #1^6 &, 1]^5 +
       2 Root[1 + 6 #1^2 - 6 Sinh[π/6] #1^3 + 9 #1^4 + 9 Cosh[π/6]^2 #1^4 - 18 Sinh[π/6] #1^5 -
          9 Cosh[π/6]^2 #1^6 + 9 Sinh[π/6]^2 #1^6 &, 1]^2 Sinh[π/6] +
       6 Root[1 + 6 #1^2 - 6 Sinh[π/6] #1^3 + 9 #1^4 + 9 Cosh[π/6]^2 #1^4 - 18 Sinh[π/6] #1^5 -
          9 Cosh[π/6]^2 #1^6 + 9 Sinh[π/6]^2 #1^6 &, 1]^4 Sinh[π/6] -
       3 Root[1 + 6 #1^2 - 6 Sinh[π/6] #1^3 + 9 #1^4 + 9 Cosh[π/6]^2 #1^4 - 18 Sinh[π/6] #1^5 -
```

$$9\,\text{Cosh}\left[\frac{\pi}{6}\right]^2 \#1^6 + 9\,\text{Sinh}\left[\frac{\pi}{6}\right]^2 \#1^6\ \&,\ 1\right]^5 \text{Sinh}\left[\frac{\pi}{6}\right]^2\Big)\Big]\Big\},$$

$$\Big\{z \to \text{ArcSech}\Big[3\ \Big(-2\,\text{Root}\Big[1 + 6\,\#1^2 - 6\,\text{Sinh}\left[\frac{\pi}{6}\right] \#1^3 + 9\,\#1^4 + 9\,\text{Cosh}\left[\frac{\pi}{6}\right]^2 \#1^4 -$$

$$18\,\text{Sinh}\left[\frac{\pi}{6}\right] \#1^5 - 9\,\text{Cosh}\left[\frac{\pi}{6}\right]^2 \#1^6 + 9\,\text{Sinh}\left[\frac{\pi}{6}\right]^2 \#1^6\ \&,\ 2\Big] -$$

$$3\,\text{Root}\Big[1 + 6\,\#1^2 - 6\,\text{Sinh}\left[\frac{\pi}{6}\right] \#1^3 + 9\,\#1^4 + 9\,\text{Cosh}\left[\frac{\pi}{6}\right]^2 \#1^4 - 18\,\text{Sinh}\left[\frac{\pi}{6}\right] \#1^5 -$$

$$9\,\text{Cosh}\left[\frac{\pi}{6}\right]^2 \#1^6 + 9\,\text{Sinh}\left[\frac{\pi}{6}\right]^2 \#1^6\ \&,\ 2\Big]^3 -$$

$$3\,\text{Cosh}\left[\frac{\pi}{6}\right]^2 \text{Root}\Big[1 + 6\,\#1^2 - 6\,\text{Sinh}\left[\frac{\pi}{6}\right] \#1^3 + 9\,\#1^4 + 9\,\text{Cosh}\left[\frac{\pi}{6}\right]^2 \#1^4 -$$

$$18\,\text{Sinh}\left[\frac{\pi}{6}\right] \#1^5 - 9\,\text{Cosh}\left[\frac{\pi}{6}\right]^2 \#1^6 + 9\,\text{Sinh}\left[\frac{\pi}{6}\right]^2 \#1^6\ \&,\ 2\Big]^3 +$$

$$3\,\text{Cosh}\left[\frac{\pi}{6}\right]^2 \text{Root}\Big[1 + 6\,\#1^2 - 6\,\text{Sinh}\left[\frac{\pi}{6}\right] \#1^3 + 9\,\#1^4 + 9\,\text{Cosh}\left[\frac{\pi}{6}\right]^2 \#1^4 -$$

$$18\,\text{Sinh}\left[\frac{\pi}{6}\right] \#1^5 - 9\,\text{Cosh}\left[\frac{\pi}{6}\right]^2 \#1^6 + 9\,\text{Sinh}\left[\frac{\pi}{6}\right]^2 \#1^6\ \&,\ 2\Big]^5 +$$

$$2\,\text{Root}\Big[1 + 6\,\#1^2 - 6\,\text{Sinh}\left[\frac{\pi}{6}\right] \#1^3 + 9\,\#1^4 + 9\,\text{Cosh}\left[\frac{\pi}{6}\right]^2 \#1^4 - 18\,\text{Sinh}\left[\frac{\pi}{6}\right] \#1^5 -$$

$$9\,\text{Cosh}\left[\frac{\pi}{6}\right]^2 \#1^6 + 9\,\text{Sinh}\left[\frac{\pi}{6}\right]^2 \#1^6\ \&,\ 2\Big]^2 \text{Sinh}\left[\frac{\pi}{6}\right] +$$

$$6\,\text{Root}\Big[1 + 6\,\#1^2 - 6\,\text{Sinh}\left[\frac{\pi}{6}\right] \#1^3 + 9\,\#1^4 + 9\,\text{Cosh}\left[\frac{\pi}{6}\right]^2 \#1^4 - 18\,\text{Sinh}\left[\frac{\pi}{6}\right] \#1^5 -$$

$$9\,\text{Cosh}\left[\frac{\pi}{6}\right]^2 \#1^6 + 9\,\text{Sinh}\left[\frac{\pi}{6}\right]^2 \#1^6\ \&,\ 2\Big]^4 \text{Sinh}\left[\frac{\pi}{6}\right] -$$

$$3\,\text{Root}\Big[1 + 6\,\#1^2 - 6\,\text{Sinh}\left[\frac{\pi}{6}\right] \#1^3 + 9\,\#1^4 + 9\,\text{Cosh}\left[\frac{\pi}{6}\right]^2 \#1^4 - 18\,\text{Sinh}\left[\frac{\pi}{6}\right] \#1^5 -$$

$$9\,\text{Cosh}\left[\frac{\pi}{6}\right]^2 \#1^6 + 9\,\text{Sinh}\left[\frac{\pi}{6}\right]^2 \#1^6\ \&,\ 2\Big]^5 \text{Sinh}\left[\frac{\pi}{6}\right]^2\Big)\Big]\Big\},$$

$$\Big\{z \to \text{ArcSech}\Big[3\ \Big(-2\,\text{Root}\Big[1 + 6\,\#1^2 - 6\,\text{Sinh}\left[\frac{\pi}{6}\right] \#1^3 + 9\,\#1^4 + 9\,\text{Cosh}\left[\frac{\pi}{6}\right]^2 \#1^4 -$$

$$18\,\text{Sinh}\left[\frac{\pi}{6}\right] \#1^5 - 9\,\text{Cosh}\left[\frac{\pi}{6}\right]^2 \#1^6 + 9\,\text{Sinh}\left[\frac{\pi}{6}\right]^2 \#1^6\ \&,\ 3\Big] -$$

$$3\,\text{Root}\Big[1 + 6\,\#1^2 - 6\,\text{Sinh}\left[\frac{\pi}{6}\right] \#1^3 + 9\,\#1^4 + 9\,\text{Cosh}\left[\frac{\pi}{6}\right]^2 \#1^4 - 18\,\text{Sinh}\left[\frac{\pi}{6}\right] \#1^5 -$$

$$9\,\text{Cosh}\left[\frac{\pi}{6}\right]^2 \#1^6 + 9\,\text{Sinh}\left[\frac{\pi}{6}\right]^2 \#1^6\ \&,\ 3\Big]^3 -$$

$$3\,\text{Cosh}\left[\frac{\pi}{6}\right]^2 \text{Root}\Big[1 + 6\,\#1^2 - 6\,\text{Sinh}\left[\frac{\pi}{6}\right] \#1^3 + 9\,\#1^4 + 9\,\text{Cosh}\left[\frac{\pi}{6}\right]^2 \#1^4 -$$

$$18\,\text{Sinh}[\tfrac{\pi}{6}]\,\#1^5 - 9\,\text{Cosh}[\tfrac{\pi}{6}]^2\,\#1^6 + 9\,\text{Sinh}[\tfrac{\pi}{6}]^2\,\#1^6\,\&,\,3]^3 +$$

$$3\,\text{Cosh}[\tfrac{\pi}{6}]^2\,\text{Root}[1 + 6\,\#1^2 - 6\,\text{Sinh}[\tfrac{\pi}{6}]\,\#1^3 + 9\,\#1^4 + 9\,\text{Cosh}[\tfrac{\pi}{6}]^2\,\#1^4 -$$

$$18\,\text{Sinh}[\tfrac{\pi}{6}]\,\#1^5 - 9\,\text{Cosh}[\tfrac{\pi}{6}]^2\,\#1^6 + 9\,\text{Sinh}[\tfrac{\pi}{6}]^2\,\#1^6\,\&,\,3]^5 +$$

$$2\,\text{Root}[1 + 6\,\#1^2 - 6\,\text{Sinh}[\tfrac{\pi}{6}]\,\#1^3 + 9\,\#1^4 + 9\,\text{Cosh}[\tfrac{\pi}{6}]^2\,\#1^4 - 18\,\text{Sinh}[\tfrac{\pi}{6}]\,\#1^5 -$$

$$9\,\text{Cosh}[\tfrac{\pi}{6}]^2\,\#1^6 + 9\,\text{Sinh}[\tfrac{\pi}{6}]^2\,\#1^6\,\&,\,3]^2\,\text{Sinh}[\tfrac{\pi}{6}] +$$

$$6\,\text{Root}[1 + 6\,\#1^2 - 6\,\text{Sinh}[\tfrac{\pi}{6}]\,\#1^3 + 9\,\#1^4 + 9\,\text{Cosh}[\tfrac{\pi}{6}]^2\,\#1^4 - 18\,\text{Sinh}[\tfrac{\pi}{6}]\,\#1^5 -$$

$$9\,\text{Cosh}[\tfrac{\pi}{6}]^2\,\#1^6 + 9\,\text{Sinh}[\tfrac{\pi}{6}]^2\,\#1^6\,\&,\,3]^4\,\text{Sinh}[\tfrac{\pi}{6}] -$$

$$3\,\text{Root}[1 + 6\,\#1^2 - 6\,\text{Sinh}[\tfrac{\pi}{6}]\,\#1^3 + 9\,\#1^4 + 9\,\text{Cosh}[\tfrac{\pi}{6}]^2\,\#1^4 - 18\,\text{Sinh}[\tfrac{\pi}{6}]\,\#1^5 -$$

$$9\,\text{Cosh}[\tfrac{\pi}{6}]^2\,\#1^6 + 9\,\text{Sinh}[\tfrac{\pi}{6}]^2\,\#1^6\,\&,\,3]^5\,\text{Sinh}[\tfrac{\pi}{6}]^2\Big)\Big]\Big]\Big\},$$

$$\Big\{z \to \text{ArcSech}\Big[3\,\Big(-2\,\text{Root}[1 + 6\,\#1^2 - 6\,\text{Sinh}[\tfrac{\pi}{6}]\,\#1^3 + 9\,\#1^4 + 9\,\text{Cosh}[\tfrac{\pi}{6}]^2\,\#1^4 -$$

$$18\,\text{Sinh}[\tfrac{\pi}{6}]\,\#1^5 - 9\,\text{Cosh}[\tfrac{\pi}{6}]^2\,\#1^6 + 9\,\text{Sinh}[\tfrac{\pi}{6}]^2\,\#1^6\,\&,\,4] -$$

$$3\,\text{Root}[1 + 6\,\#1^2 - 6\,\text{Sinh}[\tfrac{\pi}{6}]\,\#1^3 + 9\,\#1^4 + 9\,\text{Cosh}[\tfrac{\pi}{6}]^2\,\#1^4 - 18\,\text{Sinh}[\tfrac{\pi}{6}]\,\#1^5 -$$

$$9\,\text{Cosh}[\tfrac{\pi}{6}]^2\,\#1^6 + 9\,\text{Sinh}[\tfrac{\pi}{6}]^2\,\#1^6\,\&,\,4]^3 -$$

$$3\,\text{Cosh}[\tfrac{\pi}{6}]^2\,\text{Root}[1 + 6\,\#1^2 - 6\,\text{Sinh}[\tfrac{\pi}{6}]\,\#1^3 + 9\,\#1^4 + 9\,\text{Cosh}[\tfrac{\pi}{6}]^2\,\#1^4 -$$

$$18\,\text{Sinh}[\tfrac{\pi}{6}]\,\#1^5 - 9\,\text{Cosh}[\tfrac{\pi}{6}]^2\,\#1^6 + 9\,\text{Sinh}[\tfrac{\pi}{6}]^2\,\#1^6\,\&,\,4]^3 +$$

$$3\,\text{Cosh}[\tfrac{\pi}{6}]^2\,\text{Root}[1 + 6\,\#1^2 - 6\,\text{Sinh}[\tfrac{\pi}{6}]\,\#1^3 + 9\,\#1^4 + 9\,\text{Cosh}[\tfrac{\pi}{6}]^2\,\#1^4 -$$

$$18\,\text{Sinh}[\tfrac{\pi}{6}]\,\#1^5 - 9\,\text{Cosh}[\tfrac{\pi}{6}]^2\,\#1^6 + 9\,\text{Sinh}[\tfrac{\pi}{6}]^2\,\#1^6\,\&,\,4]^5 +$$

$$2\,\text{Root}[1 + 6\,\#1^2 - 6\,\text{Sinh}[\tfrac{\pi}{6}]\,\#1^3 + 9\,\#1^4 + 9\,\text{Cosh}[\tfrac{\pi}{6}]^2\,\#1^4 - 18\,\text{Sinh}[\tfrac{\pi}{6}]\,\#1^5 -$$

$$9\,\text{Cosh}[\tfrac{\pi}{6}]^2\,\#1^6 + 9\,\text{Sinh}[\tfrac{\pi}{6}]^2\,\#1^6\,\&,\,4]^2\,\text{Sinh}[\tfrac{\pi}{6}] +$$

$$6\,\text{Root}[1 + 6\,\#1^2 - 6\,\text{Sinh}[\tfrac{\pi}{6}]\,\#1^3 + 9\,\#1^4 + 9\,\text{Cosh}[\tfrac{\pi}{6}]^2\,\#1^4 - 18\,\text{Sinh}[\tfrac{\pi}{6}]\,\#1^5 -$$

$$9\,\mathrm{Cosh}\left[\frac{\pi}{6}\right]^2 \sharp 1^6 + 9\,\mathrm{Sinh}\left[\frac{\pi}{6}\right]^2 \sharp 1^6\,\&,\,4\right]^4 \mathrm{Sinh}\left[\frac{\pi}{6}\right] -$$

$$3\,\mathrm{Root}\left[1 + 6\,\sharp 1^2 - 6\,\mathrm{Sinh}\left[\frac{\pi}{6}\right] \sharp 1^3 + 9\,\sharp 1^4 + 9\,\mathrm{Cosh}\left[\frac{\pi}{6}\right]^2 \sharp 1^4 - 18\,\mathrm{Sinh}\left[\frac{\pi}{6}\right] \sharp 1^5 -\right.$$

$$\left.9\,\mathrm{Cosh}\left[\frac{\pi}{6}\right]^2 \sharp 1^6 + 9\,\mathrm{Sinh}\left[\frac{\pi}{6}\right]^2 \sharp 1^6\,\&,\,4\right]^5 \mathrm{Sinh}\left[\frac{\pi}{6}\right]^2\Bigg)\Bigg]\Bigg\},$$

$$\Bigg\{z \to -\mathrm{ArcSech}\Bigg[3\,\Bigg(-2\,\mathrm{Root}\left[1 + 6\,\sharp 1^2 - 6\,\mathrm{Sinh}\left[\frac{\pi}{6}\right] \sharp 1^3 + 9\,\sharp 1^4 + 9\,\mathrm{Cosh}\left[\frac{\pi}{6}\right]^2 \sharp 1^4 -\right.$$

$$\left.18\,\mathrm{Sinh}\left[\frac{\pi}{6}\right] \sharp 1^5 - 9\,\mathrm{Cosh}\left[\frac{\pi}{6}\right]^2 \sharp 1^6 + 9\,\mathrm{Sinh}\left[\frac{\pi}{6}\right]^2 \sharp 1^6\,\&,\,5\right] -$$

$$3\,\mathrm{Root}\left[1 + 6\,\sharp 1^2 - 6\,\mathrm{Sinh}\left[\frac{\pi}{6}\right] \sharp 1^3 + 9\,\sharp 1^4 + 9\,\mathrm{Cosh}\left[\frac{\pi}{6}\right]^2 \sharp 1^4 - 18\,\mathrm{Sinh}\left[\frac{\pi}{6}\right] \sharp 1^5 -\right.$$

$$\left.9\,\mathrm{Cosh}\left[\frac{\pi}{6}\right]^2 \sharp 1^6 + 9\,\mathrm{Sinh}\left[\frac{\pi}{6}\right]^2 \sharp 1^6\,\&,\,5\right]^3 -$$

$$3\,\mathrm{Cosh}\left[\frac{\pi}{6}\right]^2 \mathrm{Root}\left[1 + 6\,\sharp 1^2 - 6\,\mathrm{Sinh}\left[\frac{\pi}{6}\right] \sharp 1^3 + 9\,\sharp 1^4 + 9\,\mathrm{Cosh}\left[\frac{\pi}{6}\right]^2 \sharp 1^4 -\right.$$

$$\left.18\,\mathrm{Sinh}\left[\frac{\pi}{6}\right] \sharp 1^5 - 9\,\mathrm{Cosh}\left[\frac{\pi}{6}\right]^2 \sharp 1^6 + 9\,\mathrm{Sinh}\left[\frac{\pi}{6}\right]^2 \sharp 1^6\,\&,\,5\right]^3 +$$

$$3\,\mathrm{Cosh}\left[\frac{\pi}{6}\right]^2 \mathrm{Root}\left[1 + 6\,\sharp 1^2 - 6\,\mathrm{Sinh}\left[\frac{\pi}{6}\right] \sharp 1^3 + 9\,\sharp 1^4 + 9\,\mathrm{Cosh}\left[\frac{\pi}{6}\right]^2 \sharp 1^4 -\right.$$

$$\left.18\,\mathrm{Sinh}\left[\frac{\pi}{6}\right] \sharp 1^5 - 9\,\mathrm{Cosh}\left[\frac{\pi}{6}\right]^2 \sharp 1^6 + 9\,\mathrm{Sinh}\left[\frac{\pi}{6}\right]^2 \sharp 1^6\,\&,\,5\right]^5 +$$

$$2\,\mathrm{Root}\left[1 + 6\,\sharp 1^2 - 6\,\mathrm{Sinh}\left[\frac{\pi}{6}\right] \sharp 1^3 + 9\,\sharp 1^4 + 9\,\mathrm{Cosh}\left[\frac{\pi}{6}\right]^2 \sharp 1^4 - 18\,\mathrm{Sinh}\left[\frac{\pi}{6}\right] \sharp 1^5 -\right.$$

$$\left.9\,\mathrm{Cosh}\left[\frac{\pi}{6}\right]^2 \sharp 1^6 + 9\,\mathrm{Sinh}\left[\frac{\pi}{6}\right]^2 \sharp 1^6\,\&,\,5\right]^2 \mathrm{Sinh}\left[\frac{\pi}{6}\right] +$$

$$6\,\mathrm{Root}\left[1 + 6\,\sharp 1^2 - 6\,\mathrm{Sinh}\left[\frac{\pi}{6}\right] \sharp 1^3 + 9\,\sharp 1^4 + 9\,\mathrm{Cosh}\left[\frac{\pi}{6}\right]^2 \sharp 1^4 - 18\,\mathrm{Sinh}\left[\frac{\pi}{6}\right] \sharp 1^5 -\right.$$

$$\left.9\,\mathrm{Cosh}\left[\frac{\pi}{6}\right]^2 \sharp 1^6 + 9\,\mathrm{Sinh}\left[\frac{\pi}{6}\right]^2 \sharp 1^6\,\&,\,5\right]^4 \mathrm{Sinh}\left[\frac{\pi}{6}\right] -$$

$$3\,\mathrm{Root}\left[1 + 6\,\sharp 1^2 - 6\,\mathrm{Sinh}\left[\frac{\pi}{6}\right] \sharp 1^3 + 9\,\sharp 1^4 + 9\,\mathrm{Cosh}\left[\frac{\pi}{6}\right]^2 \sharp 1^4 - 18\,\mathrm{Sinh}\left[\frac{\pi}{6}\right] \sharp 1^5 -\right.$$

$$\left.9\,\mathrm{Cosh}\left[\frac{\pi}{6}\right]^2 \sharp 1^6 + 9\,\mathrm{Sinh}\left[\frac{\pi}{6}\right]^2 \sharp 1^6\,\&,\,5\right]^5 \mathrm{Sinh}\left[\frac{\pi}{6}\right]^2\Bigg)\Bigg]\Bigg\},$$

$$\Bigg\{z \to -\mathrm{ArcSech}\Bigg[3\,\Bigg(-2\,\mathrm{Root}\left[1 + 6\,\sharp 1^2 - 6\,\mathrm{Sinh}\left[\frac{\pi}{6}\right] \sharp 1^3 + 9\,\sharp 1^4 + 9\,\mathrm{Cosh}\left[\frac{\pi}{6}\right]^2 \sharp 1^4 -\right.$$

$$\left.18\,\mathrm{Sinh}\left[\frac{\pi}{6}\right] \sharp 1^5 - 9\,\mathrm{Cosh}\left[\frac{\pi}{6}\right]^2 \sharp 1^6 + 9\,\mathrm{Sinh}\left[\frac{\pi}{6}\right]^2 \sharp 1^6\,\&,\,6\right] -$$

$$3\,\mathrm{Root}\left[1 + 6\,\sharp 1^2 - 6\,\mathrm{Sinh}\left[\frac{\pi}{6}\right] \sharp 1^3 + 9\,\sharp 1^4 + 9\,\mathrm{Cosh}\left[\frac{\pi}{6}\right]^2 \sharp 1^4 - 18\,\mathrm{Sinh}\left[\frac{\pi}{6}\right] \sharp 1^5 -\right.$$

```
      9 Cosh[π/6]^2 #1^6 + 9 Sinh[π/6]^2 #1^6 &, 6]^3 -
   3 Cosh[π/6]^2 Root[1 + 6 #1^2 - 6 Sinh[π/6] #1^3 + 9 #1^4 + 9 Cosh[π/6]^2 #1^4 -
      18 Sinh[π/6] #1^5 - 9 Cosh[π/6]^2 #1^6 + 9 Sinh[π/6]^2 #1^6 &, 6]^3 +
   3 Cosh[π/6]^2 Root[1 + 6 #1^2 - 6 Sinh[π/6] #1^3 + 9 #1^4 + 9 Cosh[π/6]^2 #1^4 -
      18 Sinh[π/6] #1^5 - 9 Cosh[π/6]^2 #1^6 + 9 Sinh[π/6]^2 #1^6 &, 6]^5 +
   2 Root[1 + 6 #1^2 - 6 Sinh[π/6] #1^3 + 9 #1^4 + 9 Cosh[π/6]^2 #1^4 - 18 Sinh[π/6] #1^5 -
      9 Cosh[π/6]^2 #1^6 + 9 Sinh[π/6]^2 #1^6 &, 6]^2 Sinh[π/6] +
   6 Root[1 + 6 #1^2 - 6 Sinh[π/6] #1^3 + 9 #1^4 + 9 Cosh[π/6]^2 #1^4 - 18 Sinh[π/6] #1^5 -
      9 Cosh[π/6]^2 #1^6 + 9 Sinh[π/6]^2 #1^6 &, 6]^4 Sinh[π/6] -
   3 Root[1 + 6 #1^2 - 6 Sinh[π/6] #1^3 + 9 #1^4 + 9 Cosh[π/6]^2 #1^4 - 18 Sinh[π/6] #1^5 -
      9 Cosh[π/6]^2 #1^6 + 9 Sinh[π/6]^2 #1^6 &, 6]^5 Sinh[π/6]^2)]}}
```

Complete solutions can be obtained by using the function Reduce.

```
Reduce[Sinh[x] == a, x] // InputForm

// InputForm = C[1] ∈ Integers &&
  (x == I * Pi - ArcSinh[a] + (2 * I) * Pi * C[1] || x == ArcSinh[a] + (2 * I) * Pi * C[1])

Reduce[Cosh[x] == a, x] // InputForm

// InputForm =
 C[1] ∈ Integers && (x == -ArcCosh[a] + (2 * I) * Pi * C[1] || x == ArcCosh[a] + (2 * I) * Pi * C[1])

Reduce[Tanh[x] == a, x] // InputForm

// InputForm = C[1] ∈ Integers && -1 + a^2 ≠ 0 && x == ArcTanh[a] + I * Pi * C[1]

Reduce[Coth[x] == a, x] // InputForm

// InputForm = C[1] ∈ Integers && -1 + a^2 ≠ 0 && x == ArcCoth[a] + I * Pi * C[1]

Reduce[Csch[x] == a, x] // InputForm

// InputForm = C[1] ∈ Integers && a ≠ 0 &&
  (x == I * Pi - ArcSinh[a^(-1)] + (2 * I) * Pi * C[1] || x == ArcSinh[a^(-1)] + (2 * I) * Pi * C[1])
```

```
Reduce[Sech[x] == a, x] // InputForm
```

```
// InputForm = C[1] ∈ Integers && a ≠ 0 &&
  (x == -ArcCosh[a^(-1)] + (2*I)*Pi*C[1] || x == ArcCosh[a^(-1)] + (2*I)*Pi*C[1])
```

Solving differential equations

Here are differential equations whose linear-independent solutions are hyperbolic functions. The solutions of the simplest second-order linear ordinary differential equation with constant coefficients can be represented through sinh(z) and cosh(z).

```
{DSolve[w''[z] - w[z] == 0, w[z], z],
  DSolve[w'[z] + w[z]^2 - 1 == 0, w[z], z]} // (ExpToTrig //@ #) &
```

```
{{{w[z] → C[1] Cosh[z] + C[2] Cosh[z] + C[1] Sinh[z] - C[2] Sinh[z]}},
 {{w[z] → (Cosh[2 z] + Cosh[2 C[1]] + Sinh[2 z] + Sinh[2 C[1]]) / (Cosh[2 z] - Cosh[2 C[1]] + Sinh[2 z] - Sinh[2 C[1]])}}}
```

All hyperbolic functions satisfy first-order nonlinear differential equations. In carrying out the algorithm to solve the nonlinear differential equation, *Mathematica* has to solve a transcendental equation. In doing so, the generically multivariate inverse of a function is encountered, and a message is issued that a solution branch is potentially missed.

```
DSolve[{w′[z] == Sqrt[1 + w[z]^2], w[0] == 0}, w[z], z]
```

```
Solve::ifun : Inverse functions are being used by Solve, so some solutions may not be found.
```

```
{{w[z] → Sinh[z]}}
```

```
DSolve[{w′[z] == Sqrt[-1 + w[z]^2], w[0] == 1}, w[z], z] // FullSimplify
```

```
Solve::ifun : Inverse functions are being used by Solve, so some
   solutions may not be found; use Reduce for complete solution information. More…
```

```
{{w[z] → Cosh[z]}}
```

```
DSolve[{w'[z] + w[z]^2 - 1 == 0, w[0] == 0}, w[z], z] // FullSimplify
```

```
Solve::ifun : Inverse functions are being used by Solve, so some
   solutions may not be found; use Reduce for complete solution information. More…
```

```
{{w[z] → Tanh[z]}}
```

```
DSolve[{w′[z] - w[z]^2 + 1 == 0, w[π ⅈ/2] == 0}, w[z], z] // FullSimplify
```

```
Solve::ifun : Inverse functions are being used by Solve, so some
   solutions may not be found; use Reduce for complete solution information. More…
```

```
{{w[z] → -Coth[z]}}
```

Integral transforms

Mathematica supports the main integral transforms like direct and inverse Fourier, Laplace, and Z transforms that can give results containing classical or generalized functions. Here are some transforms of hyperbolic functions.

```
LaplaceTransform[Sinh[t], t, s]
```

$$\frac{1}{-1+s^2}$$

```
LaplaceTransform[Cosh[t], t, s]
```

$$\frac{s}{-1+s^2}$$

```
FourierTransform[Csch[t], t, s]
```

$$i\sqrt{\frac{\pi}{2}}\ \text{Tanh}\left[\frac{\pi s}{2}\right]$$

```
FourierTransform[Sech[t], t, s]
```

$$\sqrt{\frac{\pi}{2}}\ \text{Sech}\left[\frac{\pi s}{2}\right]$$

Plotting

Mathematica has built-in functions for 2D and 3D graphics. Here are some examples.

$$\text{Plot}\left[\text{Sin}\left[\text{Sinh}\left[\sum_{k=0}^{5} z^k\right]\right], \{z, -3/2, 4/5\}, \text{PlotRange} \to \text{All}, \text{PlotPoints} \to 120\right];$$

```
Plot3D[Re[Tanh[x + i y]], {x, -2, 2}, {y, -2, 2},
  PlotPoints → 240, PlotRange → {-5, 5},
  ClipFill → None, Mesh → False, AxesLabel → {"x", "y", None}];
```

$$\text{ContourPlot}\left[\text{Arg}\left[\text{Sech}\left[\frac{1}{x+i\,y}\right]\right], \left\{x, -\frac{1}{4}, \frac{1}{4}\right\}, \left\{y, -\frac{1}{3}, \frac{1}{3}\right\},\right.$$

```
  PlotPoints → 400, PlotRange → {-π, π}, FrameLabel → {"x", "y", None, None},
  ColorFunction → Hue, ContourLines → False, Contours → 200];
```

Introduction to the Hyperbolic Cotangent Function in *Mathematica*

Overview

The following shows how the hyperbolic cotangent function is realized in *Mathematica*. Examples of evaluating *Mathematica* functions applied to various numeric and exact expressions that involve the hyperbolic cotangent function or return it are shown. These involve numeric and symbolic calculations and plots.

Notations

***Mathematica* forms of notations**

Following *Mathematica*'s general naming convention, function names in `StandardForm` are just the capitalized versions of their traditional mathematics names. This shows the hyperbolic cotangent function in `StandardForm`.

```
Coth[z]
```

```
Coth[z]
```

This shows the hyperbolic cotangent function in `TraditionalForm`.

```
% // TraditionalForm
```

coth(z)

Additional forms of notations

Mathematica also knows the most popular forms of notations for the hyperbolic cotangent function that are used in other programming languages. Here are three examples: `CForm`, `TeXForm`, and `FortranForm`.

```
{CForm[Coth[2 π z]], TeXForm[Coth[2 π z]], FortranForm[Coth[2 π z]]}
```

```
{Coth(2*Pi*z), \coth(2\, \pi \, z), Coth(2*Pi*z)}
```

Automatic evaluations and transformations

Evaluation for exact and machine-number values of arguments

For the exact argument $z = \pi i / 4$, *Mathematica* returns exact result.

```
Coth[π ⅈ / 4]
```

```
-ⅈ
```

```
Coth[z] /. z → π ⅈ / 4
```

```
-ⅈ
```

For a machine-number argument (numerical argument with a decimal point), a machine number is also returned.

```
Coth[3.]
```

```
1.00497
```

```
Coth[z] /. z → 2.
```

```
1.03731
```

The next inputs calculate 100-digit approximations at $z = 1$ and $z = 2$.

```
N[Coth[z] /. z → 1, 100]
```

```
1.3130352854993313036361612469308478329120139412404526555431529675670842704618743826
  74679241480856303
```

```
N[Coth[2], 100]
```

```
1.037314720727548095877809764767820711662391269249194603569981733844518757519256433O
 66813381577266509
```

```
Coth[2] // N[#, 100] &
```

```
1.037314720727548095877809764767820711662391269249194603569981733844518757519256433O
 66813381577266509
```

It is possible to calculate thousands of digits for the hyperbolic cotangent function within a second. The next input calculates 10000 digits for coth(1) and analyzes the frequency of the digit *k* in the resulting decimal number.

```
Map[Function[w, {First[#], Length[#]} & /@ Split[Sort[First[RealDigits[w]]]]] ,
 N[{Coth[z]} /. z → 1, 10 000]]
```

```
{{{0, 975}, {1, 986}, {2, 1023}, {3, 1004},
  {4, 1008}, {5, 977}, {6, 977}, {7, 1036}, {8, 1035}, {9, 979}}}
```

Here is a 50-digit approximation to the hyperbolic cotangent function at the complex argument $z = 3 - 2\,i$.

```
N[Coth[3 - 2 ⅈ], 50]
```

```
0.99675779656935831046096879711747071833201292579034 -
 0.0037397103763369566601174086919025762400058903825788 ⅈ
```

```
{N[Coth[z] /. z → 3 - 2 ⅈ, 50], Coth[3 - 2 ⅈ] // N[#, 50] &}
```

```
{0.99675779656935831046096879711747071833201292579034 -
  0.0037397103763369566601174086919025762400058903825788 ⅈ,
 0.99675779656935831046096879711747071833201292579034 -
  0.0037397103763369566601174086919025762400058903825788 ⅈ}
```

Mathematica automatically evaluates mathematical functions with machine precision, if the arguments of the function are numerical values and include machine-number elements. In this case only six digits after the decimal point are shown. The remaining digits are suppressed, but can be displayed using the function `InputForm`.

```
{Coth[3.], N[Coth[3]], N[Coth[3], 16], N[Coth[3], 5], N[Coth[3], 20]}
```

```
{1.00497, 1.00497, 1.00497, 1.00497, 1.0049698233136891711}
```

```
% // InputForm
```

```
{1.0049698233136892, 1.0049698233136892, 1.0049698233136892, 1.0049698233136892,
 1.004969823313689171093151242828005`20}
```

Simplification of the argument

Mathematica knows the symmetry and periodicity of the hyperbolic cotangent function. Here are some examples.

```
Coth[-3]
```

```
-Coth[3]
```

```
{Coth[-z], Coth[z + π ⅈ], Coth[z + 2 π ⅈ], Coth[-z + 21 π ⅈ]}
```

```
{-Coth[z], Coth[z], Coth[z], -Coth[z]}
```

Mathematica automatically simplifies the composition of the direct and the inverse hyperbolic cotangent functions into its argument.

```
Coth[ArcCoth[z]]
```

z

Mathematica also automatically simplifies the composition of the direct and any of the inverse hyperbolic functions into algebraic functions of the argument.

```
{Coth[ArcSinh[z]], Coth[ArcCosh[z]], Coth[ArcTanh[z]],
 Coth[ArcCoth[z]], Coth[ArcCsch[z]], Coth[ArcSech[z]]}
```

$$\left\{\frac{\sqrt{1+z^2}}{z},\ \frac{z}{\sqrt{\frac{-1+z}{1+z}}\,(1+z)},\ \frac{1}{z},\ z,\ \sqrt{1+\frac{1}{z^2}}\,z,\ \frac{1}{\sqrt{\frac{1-z}{1+z}}\,(1+z)}\right\}$$

In the cases where the argument has the structure $\pi k i/2+z$ or $\pi k i/2-z$, and $\pi k i/2+i z$ or $\pi k i/2-i z$ with integer k, the hyperbolic cotangent function can be automatically transformed into hyperbolic or trigonometric cotangent or tangent functions.

$$\text{Coth}\left[\frac{\pi i}{2}-4\right]$$

$-\text{Tanh}[4]$

$$\left\{\text{Coth}\left[\frac{\pi i}{2}-z\right],\ \text{Coth}\left[\frac{\pi i}{2}+z\right],\ \text{Coth}\left[-\frac{\pi i}{2}-z\right],\ \text{Coth}\left[-\frac{\pi i}{2}+z\right],\ \text{Coth}[\pi i-z],\ \text{Coth}[\pi i+z]\right\}$$

{-Tanh[z], Tanh[z], -Tanh[z], Tanh[z], -Coth[z], Coth[z]}

$\text{Coth}[i\,5]$

$-i\,\text{Cot}[5]$

$$\left\{\text{Coth}[i z],\ \text{Coth}\left[\frac{\pi i}{2}-i z\right],\ \text{Coth}\left[\frac{\pi i}{2}+i z\right],\ \text{Coth}[\pi i-i z],\ \text{Coth}[\pi i+i z]\right\}$$

$\{-i\,\text{Cot}[z],\ -i\,\text{Tan}[z],\ i\,\text{Tan}[z],\ i\,\text{Cot}[z],\ -i\,\text{Cot}[z]\}$

Simplification of combinations of hyperbolic cotangent functions

Sometimes simple arithmetic operations containing the hyperbolic cotangent function can automatically generate other equal hyperbolic functions.

```
1 / Coth[4]
```

Tanh[4]

```
{1 / Coth[z], 1 / Coth[π ⅈ / 2 - z], Coth[π ⅈ / 2 - z] / Coth[z],
 Coth[z] / Coth[π ⅈ / 2 - z], 1 / Coth[π ⅈ / 2 - z], Coth[π ⅈ / 2 - z] / Coth[z]^2}
```

$\{\text{Tanh}[z],\ -\text{Coth}[z],\ -\text{Tanh}[z]^2,\ -\text{Coth}[z]^2,\ -\text{Coth}[z],\ -\text{Tanh}[z]^3\}$

The hyperbolic cotangent function arising as special cases from more general functions

The hyperbolic cotangent function can be treated as a particular case of some more general special functions. For example, $\coth(z)$ appears automatically from Bessel, Mathieu, Jacobi, hypergeometric, and Meijer functions or their ratios for appropriate parameters.

$$\left\{\text{BesselI}\left[-\frac{1}{2}, z\right] \Big/ \text{BesselI}\left[\frac{1}{2}, z\right], \frac{\text{MathieuC}[1, 0, i z]}{\text{MathieuS}[1, 0, i z]}, \text{JacobiCS}[i z, 0],\right.$$
$$\text{JacobiSC}\left[\frac{\pi}{2} - i z, 0\right], -i\,\text{JacobiNS}[z, 1], i\,\text{JacobiSN}\left[\frac{\pi i}{2} - z, 1\right],$$
$$\text{HypergeometricPFQ}\left[\{\}, \left\{\frac{1}{2}\right\}, \frac{z^2}{4}\right] \Big/ \text{HypergeometricPFQ}\left[\{\}, \left\{\frac{3}{2}\right\}, \frac{z^2}{4}\right],$$
$$\left.\text{MeijerG}\left[\{\{\}, \{\}\}, \left\{\left\{-\frac{1}{2}\right\}, \{0\}\right\}, -\frac{z^2}{4}\right] \Big/ \text{MeijerG}\left[\{\{\}, \{\}\}, \left\{\left\{\frac{1}{2}\right\}, \{0\}\right\}, -\frac{z^2}{4}\right]\right\}$$

$$\left\{\text{Coth}[z], -i\,\text{Coth}[z], -i\,\text{Coth}[z], -i\,\text{Coth}[z],\right.$$
$$\left.-i\,\text{Coth}[z], -i\,\text{Coth}[z], \sqrt{z^2}\,\text{Coth}\left[\sqrt{z^2}\right], -\frac{2\,\text{Coth}[z]}{z}\right\}$$

Equivalence transformations using specialized *Mathematica* functions

General remarks

Almost everybody prefers using $\coth(z) - i$ instead of $\coth(z - \pi i) + \coth(\pi i / 4)$. *Mathematica* automatically transforms the second expression into the first one. The automatic application of transformation rules to mathematical expressions can give overly complicated results. Compact expressions like $\coth(\pi i / 16)$ should not be automatically expanded into the more complicated expression $-i\left(\left(2 + \left(2 + 2^{1/2}\right)^{1/2}\right) \Big/ \left(2 - \left(2 + 2^{1/2}\right)\right)\right)^{1/2}$. *Mathematica* has special functions that produce such expansions. Some are demonstrated in the next section.

TrigExpand

The function `TrigExpand` expands out trigonometric and hyperbolic functions. In more detail, it splits up sums and integer multiples that appear in the arguments of trigonometric and hyperbolic functions, and then expands out the products of trigonometric and hyperbolic functions into sums of powers, using trigonometric and hyperbolic identities where possible. Here are some examples.

```
TrigExpand[Coth[x - y]]
```

$$-\frac{\text{Cosh}[x]\,\text{Cosh}[y]}{-\text{Cosh}[y]\,\text{Sinh}[x] + \text{Cosh}[x]\,\text{Sinh}[y]} + \frac{\text{Sinh}[x]\,\text{Sinh}[y]}{-\text{Cosh}[y]\,\text{Sinh}[x] + \text{Cosh}[x]\,\text{Sinh}[y]}$$

```
Coth[4 z] // TrigExpand
```

$$\frac{\text{Cosh}[z]^4}{4\,\text{Cosh}[z]^3\,\text{Sinh}[z] + 4\,\text{Cosh}[z]\,\text{Sinh}[z]^3} + \frac{6\,\text{Cosh}[z]^2\,\text{Sinh}[z]^2}{4\,\text{Cosh}[z]^3\,\text{Sinh}[z] + 4\,\text{Cosh}[z]\,\text{Sinh}[z]^3} + \frac{\text{Sinh}[z]^4}{4\,\text{Cosh}[z]^3\,\text{Sinh}[z] + 4\,\text{Cosh}[z]\,\text{Sinh}[z]^3}$$

```
Coth[2 z]^2 // TrigExpand
```

$$\frac{3}{4}+\frac{\text{Coth}[z]^2}{8}+\frac{1}{8}\text{Csch}[z]^2\,\text{Sech}[z]^2+\frac{\text{Tanh}[z]^2}{8}$$

```
TrigExpand[{Coth[x + y + z], Coth[3 z]}]
```

```
{(Cosh[x] Cosh[y] Cosh[z]) / (Cosh[y] Cosh[z] Sinh[x] +
     Cosh[x] Cosh[z] Sinh[y] + Cosh[x] Cosh[y] Sinh[z] + Sinh[x] Sinh[y] Sinh[z]) +
  (Cosh[z] Sinh[x] Sinh[y]) / (Cosh[y] Cosh[z] Sinh[x] + Cosh[x] Cosh[z] Sinh[y] +
     Cosh[x] Cosh[y] Sinh[z] + Sinh[x] Sinh[y] Sinh[z]) +
  (Cosh[y] Sinh[x] Sinh[z]) / (Cosh[y] Cosh[z] Sinh[x] + Cosh[x] Cosh[z] Sinh[y] +
     Cosh[x] Cosh[y] Sinh[z] + Sinh[x] Sinh[y] Sinh[z]) +
  (Cosh[x] Sinh[y] Sinh[z]) / (Cosh[y] Cosh[z] Sinh[x] + Cosh[x] Cosh[z] Sinh[y] +
     Cosh[x] Cosh[y] Sinh[z] + Sinh[x] Sinh[y] Sinh[z]),
```

$$\frac{\text{Cosh}[z]^3}{3\,\text{Cosh}[z]^2\,\text{Sinh}[z]+\text{Sinh}[z]^3}+\frac{3\,\text{Cosh}[z]\,\text{Sinh}[z]^2}{3\,\text{Cosh}[z]^2\,\text{Sinh}[z]+\text{Sinh}[z]^3}\}$$

TrigFactor

The function `TrigFactor` factors trigonometric and hyperbolic functions. In more detail, it splits up sums and integer multiples that appear in the arguments of trigonometric and hyperbolic functions, and then factors the resulting polynomials into trigonometric and hyperbolic functions, using trigonometric and hyperbolic identities where possible. Here are some examples.

```
TrigFactor[Coth[x] + Coth[y]]
```

```
Csch[x] Csch[y] Sinh[x + y]
```

```
Coth[x] - Tanh[y] // TrigFactor
```

```
Cosh[x - y] Csch[x] Sech[y]
```

TrigReduce

The function `TrigReduce` rewrites the products and powers of trigonometric and hyperbolic functions in terms of trigonometric and hyperbolic functions with combined arguments. In more detail, it typically yields a linear expression involving trigonometric and hyperbolic functions with more complicated arguments. `TrigReduce` is approximately opposite to `TrigExpand` and `TrigFactor`. Here are some examples.

```
TrigReduce[Coth[x] Coth[y]]
```

$$\frac{-\text{Cosh}[x-y]-\text{Cosh}[x+y]}{\text{Cosh}[x-y]-\text{Cosh}[x+y]}$$

```
Coth[x] Tanh[y] // TrigReduce
```

$$\frac{-\text{Sinh}[x-y]+\text{Sinh}[x+y]}{\text{Sinh}[x-y]+\text{Sinh}[x+y]}$$

```
Table[TrigReduce[Coth[z]^n], {n, 2, 5}]
```

$$\left\{\frac{1+\text{Cosh}[2z]}{-1+\text{Cosh}[2z]},\ \frac{-3\,\text{Cosh}[z]-\text{Cosh}[3z]}{3\,\text{Sinh}[z]-\text{Sinh}[3z]},\ \frac{-3-4\,\text{Cosh}[2z]-\text{Cosh}[4z]}{-3+4\,\text{Cosh}[2z]-\text{Cosh}[4z]},\ \frac{10\,\text{Cosh}[z]+5\,\text{Cosh}[3z]+\text{Cosh}[5z]}{10\,\text{Sinh}[z]-5\,\text{Sinh}[3z]+\text{Sinh}[5z]}\right\}$$

```
TrigReduce[TrigExpand[{Coth[x + y + z], Coth[3 z], Coth[x] Coth[y]}]]
```

$$\left\{\text{Coth}[x+y+z],\ \text{Coth}[3z],\ \frac{-\text{Cosh}[x-y]-\text{Cosh}[x+y]}{\text{Cosh}[x-y]-\text{Cosh}[x+y]}\right\}$$

```
TrigFactor[Coth[x] + Coth[y]] // TrigReduce
```

$$-\frac{2\,\text{Sinh}[x+y]}{\text{Cosh}[x-y]-\text{Cosh}[x+y]}$$

TrigToExp

The function `TrigToExp` converts trigonometric and hyperbolic functions to exponentials. It tries, where possible, to give results that do not involve explicit complex numbers. Here are some examples.

```
TrigToExp[Coth[z]]
```

$$\frac{e^{-z}+e^{z}}{-e^{-z}+e^{z}}$$

```
Coth[a z] + Coth[b z] // TrigToExp
```

$$\frac{e^{-a z}+e^{a z}}{-e^{-a z}+e^{a z}}+\frac{e^{-b z}+e^{b z}}{-e^{-b z}+e^{b z}}$$

ExpToTrig

The function `ExpToTrig` converts exponentials to trigonometric and hyperbolic functions. It is approximately opposite to `TrigToExp`. Here are some examples.

```
ExpToTrig[TrigToExp[Coth[z]]]
```

Coth[z]

$$\left\{\alpha\, e^{-x\beta}+\alpha\, e^{x\beta}\Big/\left(\alpha\, e^{-x\beta}+\gamma\, e^{x\beta}\right)\right\}\ //\ \text{ExpToTrig}$$

$$\left\{\alpha\,\text{Cosh}[x\beta]-\alpha\,\text{Sinh}[x\beta]+\frac{\alpha\,(\text{Cosh}[x\beta]+\text{Sinh}[x\beta])}{\alpha\,\text{Cosh}[x\beta]+\gamma\,\text{Cosh}[x\beta]-\alpha\,\text{Sinh}[x\beta]+\gamma\,\text{Sinh}[x\beta]}\right\}$$

ComplexExpand

The function `ComplexExpand` expands expressions assuming that all the variables are real. The option `TargetFunctions` can be given as a list of functions from the set `{Re, Im, Abs, Arg, Conjugate, Sign}`. `ComplexExpand` tries to give results in terms of the functions specified. Here are some examples.

```
ComplexExpand[Coth[x + ⅈ y]]
```

$$\frac{i\,\text{Sin}[2y]}{\text{Cos}[2y]-\text{Cosh}[2x]}-\frac{\text{Sinh}[2x]}{\text{Cos}[2y]-\text{Cosh}[2x]}$$

```
Coth[x + ⅈ y] + Coth[x - ⅈ y] // ComplexExpand
```

$$-\frac{2\,\text{Sinh}[2\,x]}{\text{Cos}[2\,y]-\text{Cosh}[2\,x]}$$

```
ComplexExpand[Re[Coth[x + ⅈ y]], TargetFunctions → {Re, Im}]
```

$$-\frac{\text{Sinh}[2\,x]}{\text{Cos}[2\,y]-\text{Cosh}[2\,x]}$$

```
ComplexExpand[Im[Coth[x + ⅈ y]], TargetFunctions → {Re, Im}]
```

$$\frac{\text{Sin}[2\,y]}{\text{Cos}[2\,y]-\text{Cosh}[2\,x]}$$

```
ComplexExpand[Abs[Coth[x + ⅈ y]], TargetFunctions → {Re, Im}]
```

$$\sqrt{\frac{\text{Sin}[2\,y]^2}{(\text{Cos}[2\,y]-\text{Cosh}[2\,x])^2}+\frac{\text{Sinh}[2\,x]^2}{(\text{Cos}[2\,y]-\text{Cosh}[2\,x])^2}}$$

```
ComplexExpand[Abs[Coth[x + ⅈ y]], TargetFunctions → {Re, Im}] //
 Simplify[#, {x, y} ∈ Reals] &
```

$$\sqrt{-\frac{\text{Cos}[2\,y]+\text{Cosh}[2\,x]}{\text{Cos}[2\,y]-\text{Cosh}[2\,x]}}$$

```
ComplexExpand[Re[Coth[x + ⅈ y]] + Im[Coth[x + ⅈ y]], TargetFunctions → {Re, Im}]
```

$$\frac{\text{Sin}[2\,y]}{\text{Cos}[2\,y]-\text{Cosh}[2\,x]}-\frac{\text{Sinh}[2\,x]}{\text{Cos}[2\,y]-\text{Cosh}[2\,x]}$$

```
ComplexExpand[Arg[Coth[x + ⅈ y]], TargetFunctions → {Re, Im}]
```

$$\text{ArcTan}\left[-\frac{\text{Sinh}[2\,x]}{\text{Cos}[2\,y]-\text{Cosh}[2\,x]},\ \frac{\text{Sin}[2\,y]}{\text{Cos}[2\,y]-\text{Cosh}[2\,x]}\right]$$

```
ComplexExpand[Arg[Coth[x + ⅈ y]], TargetFunctions → {Re, Im}] //
 Simplify[#, {x, y} ∈ Reals] &
```

$$\text{ArcTan}[\text{Cosh}[x]\,\text{Sinh}[x],\ -\text{Cos}[y]\,\text{Sin}[y]]$$

```
ComplexExpand[Conjugate[Coth[x + ⅈ y]], TargetFunctions → {Re, Im}] // Simplify
```

$$-\frac{i\,\text{Sin}[2\,y]+\text{Sinh}[2\,x]}{\text{Cos}[2\,y]-\text{Cosh}[2\,x]}$$

Simplify

The function `Simplify` performs a sequence of algebraic transformations on the expression, and returns the simplest form it finds. Here are some examples.

```
(Coth[z1] + Coth[z2] + Coth[z3] + Coth[z1] Coth[z2] Coth[z3]) /
  (1 + Coth[z1] Coth[z2] + Coth[z1] Coth[z3] + Coth[z2] Coth[z3]) // Simplify
```

$\text{Coth}[z_1 + z_2 + z_3]$

```
Simplify[Coth[z - π ⅈ/3] Coth[π ⅈ/3 + z] + Coth[z - π ⅈ/3] Coth[z] + Coth[z] Coth[π ⅈ/3 + z]]
```

3

Here is a collection of hyperbolic identities. Each is written as a logical conjunction.

```
Simplify[#] & /@
 (Coth[z/2] == Coth[z] + Csch[z] ∧ Coth[z]^2 == (Cosh[2 z] + 1)/(Cosh[2 z] - 1) ∧ Coth[z]^2 == 1/(1 - Sech[z]^2) ∧
   Coth[z/2] == Sinh[z]/(Cosh[z] - 1) == (1 + Cosh[z])/Sinh[z] ∧ Coth[z] Coth[2 z] == 1/2 (Coth[z]^2 + 1) ∧
   Coth[a]^2 - Coth[b]^2 == -Csch[a]^2 Csch[b]^2 Sinh[a - b] Sinh[a + b] ∧
   Coth[z]^3 == -(3 Cosh[z] + Cosh[3 z])/(3 Sinh[z] - Sinh[3 z]) ∧ Coth[3 z] == (Coth[z]^3 + 3 Coth[z])/(3 Coth[z]^2 + 1))
```

True

The function `Simplify` has the `Assumption` option. For example, *Mathematica* treats the periodicity of hyperbolic functions for the symbolic integer coefficient k of $k \pi i$.

```
Simplify[{Coth[z + 2 k π ⅈ], Coth[z + k π ⅈ] / Coth[z]}, k ∈ Integers]
```

{Coth[z], 1}

Mathematica also knows that the composition of the inverse and direct hyperbolic functions produces the value of the inner argument under the corresponding restriction.

```
ArcCoth[Coth[z]]
```

ArcCoth[Coth[z]]

```
Simplify[ArcCoth[Coth[z]], -π / 2 < Im[z] < π / 2]
```

z

FunctionExpand (and Together)

While the hyperbolic cotangent function auto-evaluates for simple fractions of πi, for more complicated cases it stays as a hyperbolic cotangent function to avoid the build up of large expressions. Using the function `FunctionExpand`, the hyperbolic cotangent function can sometimes be transformed into explicit radicals. Here are some examples.

```
{Coth[π ⅈ/16], Coth[π ⅈ/60]}
```

$\left\{-i\,\mathrm{Cot}\left[\frac{\pi}{16}\right],\ -i\,\mathrm{Cot}\left[\frac{\pi}{60}\right]\right\}$

FunctionExpand[%]

$$\left\{-i\sqrt{\frac{2+\sqrt{2+\sqrt{2}}}{2-\sqrt{2+\sqrt{2}}}},\ -\frac{i\left(-\frac{-\frac{1}{8}\sqrt{3}\left(-1+\sqrt{5}\right)-\frac{1}{4}\sqrt{\frac{1}{2}\left(5+\sqrt{5}\right)}}{\sqrt{2}}-\frac{\frac{1}{8}\left(-1+\sqrt{5}\right)-\frac{1}{4}\sqrt{\frac{3}{2}\left(5+\sqrt{5}\right)}}{\sqrt{2}}\right)}{-\frac{-\frac{1}{8}\sqrt{3}\left(-1+\sqrt{5}\right)-\frac{1}{4}\sqrt{\frac{1}{2}\left(5+\sqrt{5}\right)}}{\sqrt{2}}+\frac{\frac{1}{8}\left(-1+\sqrt{5}\right)-\frac{1}{4}\sqrt{\frac{3}{2}\left(5+\sqrt{5}\right)}}{\sqrt{2}}}\right\}$$

Together[%]

$$\left\{-i\sqrt{\frac{2+\sqrt{2+\sqrt{2}}}{2-\sqrt{2+\sqrt{2}}}},\ \frac{i\left(1-\sqrt{3}-\sqrt{5}+\sqrt{15}+\sqrt{2\left(5+\sqrt{5}\right)}+\sqrt{6\left(5+\sqrt{5}\right)}\right)}{1+\sqrt{3}-\sqrt{5}-\sqrt{15}-\sqrt{2\left(5+\sqrt{5}\right)}+\sqrt{6\left(5+\sqrt{5}\right)}}\right\}$$

If the denominator contains squares of integers other than 2, the results always contain complex numbers deeply inside the expression (meaning that the imaginary number $i = \sqrt{-1}$ appears unavoidably).

$\left\{\mathbf{Coth}\left[\frac{\pi\, i}{9}\right]\right\}$

$\left\{-i\,\mathrm{Cot}\left[\frac{\pi}{9}\right]\right\}$

FunctionExpand[%] // Together

$$\left\{\frac{-i\left(-1-i\sqrt{3}\right)^{1/3}+\sqrt{3}\left(-1-i\sqrt{3}\right)^{1/3}-i\left(-1+i\sqrt{3}\right)^{1/3}-\sqrt{3}\left(-1+i\sqrt{3}\right)^{1/3}}{-i\left(-1-i\sqrt{3}\right)^{1/3}+\sqrt{3}\left(-1-i\sqrt{3}\right)^{1/3}+i\left(-1+i\sqrt{3}\right)^{1/3}+\sqrt{3}\left(-1+i\sqrt{3}\right)^{1/3}}\right\}$$

Here the function `RootReduce` is used to express the previous algebraic numbers as roots of polynomial equations.

RootReduce[**Simplify**[%]]

$\left\{\mathrm{Root}\left[1+33\,\#1^2+27\,\#1^4+3\,\#1^6\,\&,\ 3\right]\right\}$

The function `FunctionExpand` also reduces hyperbolic expressions with compound arguments or compositions, including inverse hyperbolic functions, to simpler ones. Here are some examples.

$\left\{\mathbf{Coth}\left[\sqrt{z^2}\right],\ \mathbf{Coth}\left[\frac{\mathbf{ArcCoth}[z]}{2}\right],\ \mathbf{Coth}[2\,\mathbf{ArcCoth}[z]],\ \mathbf{Coth}[3\,\mathbf{ArcSinh}[z]]\right\}$ // **FunctionExpand**

$$\left\{\frac{\sqrt{-i z}\,\sqrt{i z}\,\text{Coth}[z]}{z},\ z\left(1+\frac{\sqrt{(-1+z)\,(1+z)}}{\sqrt{-i z}\,\sqrt{i z}}\right),\right.$$
$$\frac{1}{2}\left(1-\frac{1}{z^2}\right) z\left(\frac{1}{(-1+z)\,(1+z)}+\frac{z^2}{(-1+z)\,(1+z)}\right),$$
$$\left.\frac{i\left(3 z^2\sqrt{i\,(-i+z)}\,\sqrt{-i\,(i+z)}+(i\,(-i+z))^{3/2}\,(-i\,(i+z))^{3/2}\right)}{i z^3+3 i z\left(1+z^2\right)}\right\}$$

Applying `Simplify` to the last expression gives a more compact result.

```
Simplify[%]
```

$$\left\{\frac{\sqrt{z^2}\,\text{Coth}[z]}{z},\ z+\frac{\sqrt{z^2}\,\sqrt{-1+z^2}}{z},\ \frac{1+z^2}{2 z},\ \frac{\sqrt{1+z^2}\left(1+4 z^2\right)}{z\left(3+4 z^2\right)}\right\}$$

FullSimplify

The function `FullSimplify` tries a wider range of transformations than `Simplify` and returns the simplest form it finds. Here are some examples that compare the results of applying these functions to the same expressions.

$$\text{set1}=\left\{\text{Coth}\left[\text{Log}\left[z+\sqrt{1+z^2}\right]\right],\ \text{Coth}\left[\frac{\pi i}{2}+\text{Log}\left[z+\sqrt{1+z^2}\right]\right],\right.$$
$$\text{Coth}\left[\frac{1}{2}\text{Log}[1-z]-\frac{1}{2}\text{Log}[1+z]\right],\ \text{Coth}\left[\frac{1}{2}\text{Log}\left[1-\frac{1}{z}\right]-\frac{1}{2}\text{Log}\left[1+\frac{1}{z}\right]\right],$$
$$\left.\text{Coth}\left[\text{Log}\left[\sqrt{1+\frac{1}{z^2}}+\frac{1}{z}\right]\right],\ \text{Coth}\left[\frac{\pi i}{2}+\text{Log}\left[\sqrt{1+\frac{1}{z^2}}+\frac{1}{z}\right]\right]\right\}$$

$$\left\{\frac{1+\left(z+\sqrt{1+z^2}\right)^2}{-1+\left(z+\sqrt{1+z^2}\right)^2},\ \frac{-1+\left(z+\sqrt{1+z^2}\right)^2}{1+\left(z+\sqrt{1+z^2}\right)^2},\ \text{Coth}\left[\frac{1}{2}\text{Log}[1-z]-\frac{1}{2}\text{Log}[1+z]\right],\right.$$

$$\left.\text{Coth}\left[\frac{1}{2}\text{Log}\left[1-\frac{1}{z}\right]-\frac{1}{2}\text{Log}\left[1+\frac{1}{z}\right]\right],\ \frac{1+\left(\sqrt{1+\frac{1}{z^2}}+\frac{1}{z}\right)^2}{-1+\left(\sqrt{1+\frac{1}{z^2}}+\frac{1}{z}\right)^2},\ \frac{-1+\left(\sqrt{1+\frac{1}{z^2}}+\frac{1}{z}\right)^2}{1+\left(\sqrt{1+\frac{1}{z^2}}+\frac{1}{z}\right)^2}\right\}$$

```
set1 // Simplify
```

$$\left\{\frac{1+z^2+z\sqrt{1+z^2}}{z^2+z\sqrt{1+z^2}},\ \frac{z\left(z+\sqrt{1+z^2}\right)}{1+z^2+z\sqrt{1+z^2}},\ \text{Coth}\left[\frac{1}{2}\left(\text{Log}[1-z]-\text{Log}[1+z]\right)\right],\right.$$

$$\left.\text{Coth}\left[\frac{1}{2}\left(-\text{Log}\left[1+\frac{1}{z}\right]+\text{Log}\left[\frac{-1+z}{z}\right]\right)\right],\ \frac{1+\sqrt{1+\frac{1}{z^2}}\,z+z^2}{1+\sqrt{1+\frac{1}{z^2}}\,z},\ \frac{1+\sqrt{1+\frac{1}{z^2}}\,z}{1+\sqrt{1+\frac{1}{z^2}}\,z+z^2}\right\}$$

```
set1 // FullSimplify
```

$$\left\{\frac{\sqrt{1+z^2}}{z},\ \frac{z}{\sqrt{1+z^2}},\ -\frac{1}{z},\ -z,\ \sqrt{1+\frac{1}{z^2}}\,z,\ \frac{1}{\sqrt{1+\frac{1}{z^2}}\,z}\right\}$$

Operations under special *Mathematica* functions

Series expansions

Calculating the series expansion of a hyperbolic cotangent function to hundreds of terms can be done in seconds.

```
Series[Coth[z], {z, 0, 3}]
```

$$\frac{1}{z}+\frac{z}{3}-\frac{z^3}{45}+O[z]^4$$

```
Normal[%]
```

$$\frac{1}{z}+\frac{z}{3}-\frac{z^3}{45}$$

```
Series[Coth[z], {z, 0, 100}] // Timing
```

$$\left\{1.482\ \text{Second},\ \frac{1}{z}+\frac{z}{3}-\frac{z^3}{45}+\frac{2z^5}{945}-\frac{z^7}{4725}+\frac{2z^9}{93\,555}-\frac{1382\,z^{11}}{638\,512\,875}+\right.$$
$$\frac{4z^{13}}{18\,243\,225}-\frac{3617\,z^{15}}{162\,820\,783\,125}+\frac{87\,734\,z^{17}}{38\,979\,295\,480\,125}-\frac{349\,222\,z^{19}}{1\,531\,329\,465\,290\,625}+$$
$$\frac{310\,732\,z^{21}}{13\,447\,856\,940\,643\,125}-\frac{472\,728\,182\,z^{23}}{201\,919\,571\,963\,756\,521\,875}+\frac{2\,631\,724\,z^{25}}{11\,094\,481\,976\,030\,578\,125}-$$
$$\frac{13\,571\,120\,588\,z^{27}}{564\,653\,660\,170\,076\,273\,671\,875}+\frac{13\,785\,346\,041\,608\,z^{29}}{5\,660\,878\,804\,669\,082\,674\,070\,015\,625}-$$
$$\frac{7\,709\,321\,041\,217\,z^{31}}{31\,245\,110\,285\,511\,170\,603\,633\,203\,125}+\frac{303\,257\,395\,102\,z^{33}}{12\,130\,454\,581\,433\,748\,587\,292\,890\,625}-$$
$$\frac{52\,630\,543\,106\,106\,954\,746\,z^{35}}{20\,777\,977\,561\,866\,588\,586\,487\,628\,662\,044\,921\,875}+\frac{616\,840\,823\,966\,644\,z^{37}}{2\,403\,467\,618\,492\,375\,776\,343\,276\,883\,984\,375}-$$
$$\frac{522\,165\,436\,992\,898\,244\,102\,z^{39}}{20\,080\,431\,172\,289\,638\,826\,798\,401\,128\,390\,556\,640\,625}+$$

$$\frac{6\,080\,390\,575\,672\,283\,210\,764\, z^{41}}{2\,307\,789\,189\,818\,960\,127\,712\,594\,427\,864\,667\,427\,734\,375} -$$

$$\frac{10\,121\,188\,937\,927\,645\,176\,372\, z^{43}}{37\,913\,679\,547\,025\,773\,526\,706\,908\,457\,776\,679\,169\,921\,875} +$$

$$\frac{207\,461\,256\,206\,578\,143\,748\,856\, z^{45}}{7\,670\,102\,214\,448\,301\,053\,033\,358\,480\,610\,212\,529\,462\,890\,625} -$$

$$\frac{11\,218\,806\,737\,995\,635\,372\,498\,255\,094\, z^{47}}{4\,093\,648\,603\,384\,274\,996\,519\,698\,921\,478\,879\,580\,162\,286\,669\,921\,875} +$$

$$\frac{79\,209\,152\,838\,572\,743\,713\,996\,404\, z^{49}}{285\,258\,771\,457\,546\,764\,463\,363\,635\,252\,374\,414\,183\,254\,365\,234\,375} -$$

$$\frac{246\,512\,528\,657\,073\,833\,030\,130\,766\,724\, z^{51}}{8\,761\,982\,491\,474\,419\,367\,550\,817\,114\,626\,909\,562\,924\,278\,968\,505\,859\,375} +$$

$$\frac{233\,199\,709\,079\,078\,899\,371\,344\,990\,501\,528\, z^{53}}{81\,807\,125\,729\,900\,063\,867\,074\,959\,072\,425\,603\,825\,198\,823\,017\,351\,806\,640\,625} -$$

$$\frac{1\,416\,795\,959\,607\,558\,144\,963\,094\,708\,378\,988\, z^{55}}{4\,905\,352\,087\,939\,496\,310\,826\,487\,207\,538\,302\,184\,255\,342\,959\,123\,162\,841\,796\,875} +$$

$$\frac{23\,305\,824\,372\,104\,839\,134\,357\,731\,308\,699\,592\, z^{57}}{796\,392\,368\,980\,577\,121\,745\,974\,726\,570\,063\,253\,238\,310\,542\,073\,919\,837\,646\,484\,375} -$$

$$\frac{9\,721\,865\,123\,870\,044\,576\,322\,439\,952\,638\,561\,968\,331\,928\, z^{59}}{3\,278\,777\,586\,273\,629\,598\,615\,520\,165\,380\,455\,583\,231\,003\,564\,645\,636\,125\,000\,418\,914\,794\,921\,875} +$$

$$\frac{6\,348\,689\,256\,302\,894\,731\,330\,601\,216\,724\,328\,336\, z^{61}}{21\,132\,271\,510\,899\,613\,925\,529\,439\,369\,536\,628\,424\,678\,570\,233\,931\,462\,891\,949\,462\,890\,625} -$$

$$\frac{106\,783\,830\,147\,866\,529\,886\,385\,444\,979\,142\,647\,942\,017\, z^{63}}{3\,508\,062\,732\,166\,890\,409\,707\,514\,582\,539\,928\,001\,638\,766\,051\,683\,792\,497\,378\,070\,587\,158\,203\,125} +$$

$$\left(267\,745\,458\,568\,424\,664\,373\,021\,714\,282\,169\,516\,771\,254\,382\, z^{65}\right) / 86\,812\,790\,293\,146\,213\,360\,651\,966\,604\,262\,937\,105\,495\,141\,563\,588\,806\,888\,204\,273\,501\,373\,291\,015\,625 -$$

$$\left(250\,471\,004\,320\,250\,327\,955\,196\,022\,920\,428\,000\,776\,938\, z^{67}\right) / 801\,528\,196\,428\,242\,695\,121\,010\,267\,455\,843\,804\,062\,822\,357\,897\,831\,858\,125\,102\,407\,684\,326\,171\,875 +$$

$$\left(172\,043\,582\,552\,384\,800\,434\,637\,321\,986\,040\,823\,829\,878\,646\,884\, z^{69}\right) / 5\,433\,748\,964\,547\,053\,581\,149\,916\,185\,708\,338\,218\,048\,392\,402\,830\,337\,634\,114\,958\,370\,880\,742\,156\,982\,421\,875 -$$

$$\left(11\,655\,909\,923\,339\,888\,220\,876\,554\,489\,282\,134\,730\,564\,976\,603\,688\,520\,858\, z^{71}\right) / 3\,633\,348\,205\,269\,879\,230\,856\,840\,004\,304\,821\,536\,968\,049\,780\,112\,803\,650\,817\,771\,432\,558\,560\,793\,458\,452\,606\,201\,171\,875 +$$

$$\left(3\,692\,153\,220\,456\,342\,488\,035\,683\,646\,645\,690\,290\,452\,790\,030\,604\, z^{73}\right) / 11\,359\,005\,221\,796\,317\,918\,049\,302\,062\,760\,294\,302\,183\,889\,391\,189\,419\,445\,133\,951\,612\,582\,060\,536\,346\,435\,546\,875 -$$

$$\left(5\,190\,545\,015\,986\,394\,254\,249\,936\,008\,544\,252\,611\,445\,319\,542\,919\,116\, z^{75}\right) / 157\,606\,197\,452\,423\,911\,112\,934\,066\,120\,799\,083\,442\,801\,465\,302\,753\,194\,801\,233\,578\,624\,576\,089\,941\,806\,793\,212\,890\,625 +$$

$$\left(255\,290\,071\,123\,323\,586\,643\,187\,098\,799\,718\,199\,072\,122\,692\,536\,861\,835\,992\, z^{77}\right) / 76\,505\,736\,228\,426\,953\,173\,738\,238\,352\,183\,101\,801\,688\,392\,812\,244\,485\,181\,277\,127\,930\,109\,049\,138\,257\,655\,704\,498\,291\,015\,625 -$$

$$\left(9\,207\,568\,598\,958\,915\,293\,871\,149\,938\,038\,093\,699\,588\,515\,745\,502\,577\,839\,313\,734\, z^{79}\right) / 27\,233\,582\,984\,369\,795\,892\,070\,228\,410\,001\,578\,355\,986\,013\,571\,390\,071\,723\,225\,259\,349\,721\,067\,988\,068\,852\,863\,296\,604\,156\,494\,140\,625 +$$

```
(163 611 136 505 867 886 519 332 147 296 221 453 678 803 514 884 902 772 183 572 z^81) /
 4 776 089 171 877 348 057 451 105 924 101 750 653 118 402 745 283 825 543 113 171 217 116 857 704 \
  024 700 607 798 175 811 767 578 125 -
(8 098 304 783 741 161 440 924 524 640 446 924 039 959 669 564 792 363 509 124 335 729 908 z^83) /
 2 333 207 846 470 426 678 843 707 227 616 712 214 909 162 634 745 895 349 325 948 586 531 533 393 \
  530 725 143 500 144 033 328 342 437 744 140 625 +
(122 923 650 124 219 284 385 832 157 660 699 813 260 991 755 656 444 452 420 836 648 z^85) /
 349 538 086 043 843 717 584 559 187 055 386 621 548 470 304 913 596 772 372 737 435 524 697 231 \
  069 047 713 981 709 496 784 210 205 078 125 -
(476 882 359 517 824 548 362 004 154 188 840 670 307 545 554 753 464 961 562 516 323 845 108 z^87) /
 13 383 510 964 174 348 021 497 060 628 653 950 829 663 288 548 327 870 152 944 013 988 358 928 114 \
  528 962 242 087 062 453 152 690 410 614 013 671 875 +
(1 886 491 646 433 732 479 814 597 361 998 744 134 040 407 919 471 435 385 970 472 345 164 676 056
   z^89) /
 522 532 651 330 971 490 226 753 590 247 329 744 050 384 290 675 644 135 735 656 667 608 610 471 \
  400 391 047 234 539 824 350 830 981 313 610 076 904 296 875 -
(450 638 590 680 882 618 431 105 331 665 591 912 924 988 342 163 281 788 877 675 244 114 763 912
   z^91) /
 1 231 931 818 039 911 948 327 467 370 123 161 265 684 460 571 086 659 079 080 437 659 781 065 743 \
  269 173 212 919 832 661 978 537 311 246 395 111 083 984 375 +
(415 596 189 473 955 564 121 634 614 268 323 814 113 534 779 643 471 190 276 158 333 713 923 216
   z^93) /
 11 213 200 675 690 943 223 287 032 785 929 540 201 272 600 687 465 377 745 332 153 847 964 679 254 \
  692 602 138 023 498 144 562 090 675 557 613 372 802 734 375 -
(423 200 899 194 533 026 195 195 456 219 648 467 346 087 908 778 120 468 301 277 466 840 101 336 \
   699 974 518 z^95) /
 112 694 926 530 960 148 011 367 752 417 874 063 473 378 698 369 880 587 800 838 274 234 349 237 \
  591 647 453 413 782 021 538 312 594 164 677 406 144 702 434 539 794 921 875 +
(5 543 531 483 502 489 438 698 050 411 951 314 743 456 505 773 755 468 368 087 670 306 121 873 229 \
   244 z^97) /
 14 569 479 835 935 377 894 165 191 004 250 040 526 616 509 162 234 077 285 176 247 476 968 227 225 \
  810 918 346 966 001 491 701 692 846 112 140 419 483 184 814 453 125 -
(378 392 151 276 488 501 180 909 732 277 974 887 490 811 366 132 267 744 533 542 784 817 245 581 \
   660 788 990 844 z^99) /
 9 815 205 420 757 514 710 108 178 059 369 553 458 327 392 260 750 404 049 930 407 987 933 582 359 \
  080 767 225 644 716 670 683 512 153 512 547 802 166 033 089 160 919 189 453 125 + O[z]^101}
```

Mathematica comes with the add-on package `DiscreteMath`RSolve`` that allows finding the general terms of the series for many functions. After loading this package, and using the package function `SeriesTerm`, the following n^{th} term of $\coth(z)$ can be evaluated.

```
<< DiscreteMath`RSolve`
```

```
SeriesTerm[Coth[z], {z, 0, n}] z^n
```

$$\frac{2^{1+n} z^n \text{BernoulliB}[1+n]}{(1+n)!}$$

This result can be easily verified.

$$\text{Sum}\left[\frac{2^{1+n} z^n \text{BernoulliB}[1+n]}{(1+n)!}, \{n, 1, \infty\}\right] \text{ // FullSimplify}$$

$$-\frac{1}{z} + \text{Coth}[z]$$

Differentiation

Mathematica can evaluate derivatives of the hyperbolic cotangent function of an arbitrary positive integer order.

$$\partial_z \text{Coth}[z]$$

$$-\text{Csch}[z]^2$$

$$\partial_{\{z,2\}} \text{Coth}[z]$$

$$2\,\text{Coth}[z]\,\text{Csch}[z]^2$$

```
Table[D[Coth[z], {z, n}], {n, 10}]
```

$$\{-\text{Csch}[z]^2,\ 2\,\text{Coth}[z]\,\text{Csch}[z]^2,\ -4\,\text{Coth}[z]^2\,\text{Csch}[z]^2 - 2\,\text{Csch}[z]^4,\ 8\,\text{Coth}[z]^3\,\text{Csch}[z]^2 + 16\,\text{Coth}[z]\,\text{Csch}[z]^4,\ -16\,\text{Coth}[z]^4\,\text{Csch}[z]^2 - 88\,\text{Coth}[z]^2\,\text{Csch}[z]^4 - 16\,\text{Csch}[z]^6,\ 32\,\text{Coth}[z]^5\,\text{Csch}[z]^2 + 416\,\text{Coth}[z]^3\,\text{Csch}[z]^4 + 272\,\text{Coth}[z]\,\text{Csch}[z]^6,\ -64\,\text{Coth}[z]^6\,\text{Csch}[z]^2 - 1824\,\text{Coth}[z]^4\,\text{Csch}[z]^4 - 2880\,\text{Coth}[z]^2\,\text{Csch}[z]^6 - 272\,\text{Csch}[z]^8,\ 128\,\text{Coth}[z]^7\,\text{Csch}[z]^2 + 7680\,\text{Coth}[z]^5\,\text{Csch}[z]^4 + 24\,576\,\text{Coth}[z]^3\,\text{Csch}[z]^6 + 7936\,\text{Coth}[z]\,\text{Csch}[z]^8,\ -256\,\text{Coth}[z]^8\,\text{Csch}[z]^2 - 31\,616\,\text{Coth}[z]^6\,\text{Csch}[z]^4 - 185\,856\,\text{Coth}[z]^4\,\text{Csch}[z]^6 - 137\,216\,\text{Coth}[z]^2\,\text{Csch}[z]^8 - 7936\,\text{Csch}[z]^{10},\ 512\,\text{Coth}[z]^9\,\text{Csch}[z]^2 + 128\,512\,\text{Coth}[z]^7\,\text{Csch}[z]^4 + 1\,304\,832\,\text{Coth}[z]^5\,\text{Csch}[z]^6 + 1\,841\,152\,\text{Coth}[z]^3\,\text{Csch}[z]^8 + 353\,792\,\text{Coth}[z]\,\text{Csch}[z]^{10}\}$$

Indefinite integration

Mathematica can calculate a huge set of doable indefinite integrals that contain the hyperbolic cotangent function. The results can contain special functions. Here are some examples.

$$\int \text{Coth}[z]\, dz$$

$$\text{Log}[\text{Sinh}[z]]$$

$$\int \text{Coth}[z]^a\, dz$$

$$\frac{\text{Coth}[z]^{1+a}\,\text{Hypergeometric2F1}\left[\frac{1+a}{2}, 1, 1+\frac{1+a}{2}, \text{Coth}[z]^2\right]}{1+a}$$

Definite integration

Mathematica can calculate wide classes of definite integrals that contain the hyperbolic cotangent function. Here are some examples.

$$\int_0^{\infty} t\, e^{-t}\, \mathrm{Coth}[t]\, dt$$

$$\frac{1}{4}\left(-4+\pi^2\right)$$

$$\int_0^{\pi/2} \sqrt{\mathrm{Coth}[z]}\, dz$$

$$\frac{1}{2}\left(\pi - i\,\mathrm{Log}\left[1 - i\sqrt{\frac{1+e^{\pi}}{-1+e^{\pi}}}\right] + i\,\mathrm{Log}\left[1 + i\sqrt{\frac{1+e^{\pi}}{-1+e^{\pi}}}\right] - \mathrm{Log}\left[-1+\sqrt{\mathrm{Coth}\left[\frac{\pi}{2}\right]}\right] + \mathrm{Log}\left[1+\sqrt{\mathrm{Coth}\left[\frac{\pi}{2}\right]}\right]\right)$$

$$\int_0^{\frac{\pi}{4}} \mathrm{Log}[\mathrm{Coth}[t]]\, dt$$

$$\frac{1}{8}\left(\pi^2 + 4\,\mathrm{PolyLog}\left[2, -e^{-\pi/2}\right] - 4\,\mathrm{PolyLog}\left[2, e^{-\pi/2}\right]\right)$$

Limit operation

Mathematica can calculate limits that contain the hyperbolic cotangent function. Here are some examples.

$$\mathrm{Limit}\left[z^2\,\mathrm{Coth}[3\,z]^2,\ z \to 0\right]$$

$$\frac{1}{9}$$

$$\mathrm{Limit}\left[z\,\mathrm{Coth}\left[2\sqrt{z^2}\right],\ z \to 0,\ \mathrm{Direction} \to 1\right]$$

$$-\frac{1}{2}$$

$$\mathrm{Limit}\left[z\,\mathrm{Coth}\left[2\sqrt{z^2}\right],\ z \to 0,\ \mathrm{Direction} \to -1\right]$$

$$\frac{1}{2}$$

Solving equations

The next inputs solve two equations that contain the hyperbolic cotangent function. Because of the multivalued nature of the inverse hyperbolic cotangent function, a message is printed indicating that only some of the possible solutions are returned.

$$\mathrm{Solve}\left[\mathrm{Coth}[z]^2 + 3\,\mathrm{Coth}\left[z + \mathrm{Pi}/3\right] == 4,\ z\right]$$

```
Solve::ifun : Inverse functions are being used by Solve, so some solutions may not be found.
```

$$\Bigg\{\Bigg\{z\to -\mathrm{ArcCosh}\Bigg[-\sqrt{\Bigg(\frac{31\,\mathrm{Cosh}\left[\frac{\pi}{3}\right]^2}{2\left(15\,\mathrm{Cosh}\left[\frac{\pi}{3}\right]^2-18\,\mathrm{Cosh}\left[\frac{\pi}{3}\right]\mathrm{Sinh}\left[\frac{\pi}{3}\right]+3\,\mathrm{Sinh}\left[\frac{\pi}{3}\right]^2\right)}-\frac{21\,\mathrm{Cosh}\left[\frac{\pi}{3}\right]\mathrm{Sinh}\left[\frac{\pi}{3}\right]}{15\,\mathrm{Cosh}\left[\frac{\pi}{3}\right]^2-18\,\mathrm{Cosh}\left[\frac{\pi}{3}\right]\mathrm{Sinh}\left[\frac{\pi}{3}\right]+3\,\mathrm{Sinh}\left[\frac{\pi}{3}\right]^2}+\frac{11\,\mathrm{Sinh}\left[\frac{\pi}{3}\right]^2}{2\left(15\,\mathrm{Cosh}\left[\frac{\pi}{3}\right]^2-18\,\mathrm{Cosh}\left[\frac{\pi}{3}\right]\mathrm{Sinh}\left[\frac{\pi}{3}\right]+3\,\mathrm{Sinh}\left[\frac{\pi}{3}\right]^2\right)}-\left(\sqrt{\left(\mathrm{Cosh}\left[\frac{\pi}{3}\right]^4-12\,\mathrm{Cosh}\left[\frac{\pi}{3}\right]^3\mathrm{Sinh}\left[\frac{\pi}{3}\right]-14\,\mathrm{Cosh}\left[\frac{\pi}{3}\right]^2\mathrm{Sinh}\left[\frac{\pi}{3}\right]^2+12\,\mathrm{Cosh}\left[\frac{\pi}{3}\right]\mathrm{Sinh}\left[\frac{\pi}{3}\right]^3+13\,\mathrm{Sinh}\left[\frac{\pi}{3}\right]^4\right)}\right)\Big/\left(2\left(15\,\mathrm{Cosh}\left[\frac{\pi}{3}\right]^2-18\,\mathrm{Cosh}\left[\frac{\pi}{3}\right]\mathrm{Sinh}\left[\frac{\pi}{3}\right]+3\,\mathrm{Sinh}\left[\frac{\pi}{3}\right]^2\right)\right)\Bigg)}\Bigg]\Bigg\},$$

$$\Bigg\{z\to -\mathrm{ArcCosh}\Bigg[\sqrt{\Bigg(\frac{31\,\mathrm{Cosh}\left[\frac{\pi}{3}\right]^2}{2\left(15\,\mathrm{Cosh}\left[\frac{\pi}{3}\right]^2-18\,\mathrm{Cosh}\left[\frac{\pi}{3}\right]\mathrm{Sinh}\left[\frac{\pi}{3}\right]+3\,\mathrm{Sinh}\left[\frac{\pi}{3}\right]^2\right)}-\frac{21\,\mathrm{Cosh}\left[\frac{\pi}{3}\right]\mathrm{Sinh}\left[\frac{\pi}{3}\right]}{15\,\mathrm{Cosh}\left[\frac{\pi}{3}\right]^2-18\,\mathrm{Cosh}\left[\frac{\pi}{3}\right]\mathrm{Sinh}\left[\frac{\pi}{3}\right]+3\,\mathrm{Sinh}\left[\frac{\pi}{3}\right]^2}+\frac{11\,\mathrm{Sinh}\left[\frac{\pi}{3}\right]^2}{2\left(15\,\mathrm{Cosh}\left[\frac{\pi}{3}\right]^2-18\,\mathrm{Cosh}\left[\frac{\pi}{3}\right]\mathrm{Sinh}\left[\frac{\pi}{3}\right]+3\,\mathrm{Sinh}\left[\frac{\pi}{3}\right]^2\right)}-\left(\sqrt{\left(\mathrm{Cosh}\left[\frac{\pi}{3}\right]^4-12\,\mathrm{Cosh}\left[\frac{\pi}{3}\right]^3\mathrm{Sinh}\left[\frac{\pi}{3}\right]-14\,\mathrm{Cosh}\left[\frac{\pi}{3}\right]^2\mathrm{Sinh}\left[\frac{\pi}{3}\right]^2+12\,\mathrm{Cosh}\left[\frac{\pi}{3}\right]\mathrm{Sinh}\left[\frac{\pi}{3}\right]^3+13\,\mathrm{Sinh}\left[\frac{\pi}{3}\right]^4\right)}\right)\Big/\left(2\left(15\,\mathrm{Cosh}\left[\frac{\pi}{3}\right]^2-18\,\mathrm{Cosh}\left[\frac{\pi}{3}\right]\mathrm{Sinh}\left[\frac{\pi}{3}\right]+3\,\mathrm{Sinh}\left[\frac{\pi}{3}\right]^2\right)\right)\Bigg)}\Bigg]\Bigg\},$$

$$\Bigg\{z\to -\mathrm{ArcCosh}\Bigg[-\sqrt{\Bigg(\frac{31\,\mathrm{Cosh}\left[\frac{\pi}{3}\right]^2}{2\left(15\,\mathrm{Cosh}\left[\frac{\pi}{3}\right]^2-18\,\mathrm{Cosh}\left[\frac{\pi}{3}\right]\mathrm{Sinh}\left[\frac{\pi}{3}\right]+3\,\mathrm{Sinh}\left[\frac{\pi}{3}\right]^2\right)}-\frac{21\,\mathrm{Cosh}\left[\frac{\pi}{3}\right]\mathrm{Sinh}\left[\frac{\pi}{3}\right]}{15\,\mathrm{Cosh}\left[\frac{\pi}{3}\right]^2-18\,\mathrm{Cosh}\left[\frac{\pi}{3}\right]\mathrm{Sinh}\left[\frac{\pi}{3}\right]+3\,\mathrm{Sinh}\left[\frac{\pi}{3}\right]^2}+$$

$$\frac{11\,\text{Sinh}\left[\frac{\pi}{3}\right]^2}{2\left(15\,\text{Cosh}\left[\frac{\pi}{3}\right]^2-18\,\text{Cosh}\left[\frac{\pi}{3}\right]\text{Sinh}\left[\frac{\pi}{3}\right]+3\,\text{Sinh}\left[\frac{\pi}{3}\right]^2\right)}+$$

$$\left(\sqrt{\left(\text{Cosh}\left[\frac{\pi}{3}\right]^4-12\,\text{Cosh}\left[\frac{\pi}{3}\right]^3\text{Sinh}\left[\frac{\pi}{3}\right]-14\,\text{Cosh}\left[\frac{\pi}{3}\right]^2\text{Sinh}\left[\frac{\pi}{3}\right]^2+12\,\text{Cosh}\left[\frac{\pi}{3}\right]\text{Sinh}\left[\frac{\pi}{3}\right]^3+\right.}\right.$$

$$\left.\left.13\,\text{Sinh}\left[\frac{\pi}{3}\right]^4\right)\right)\Big/\left(2\left(15\,\text{Cosh}\left[\frac{\pi}{3}\right]^2-18\,\text{Cosh}\left[\frac{\pi}{3}\right]\text{Sinh}\left[\frac{\pi}{3}\right]+3\,\text{Sinh}\left[\frac{\pi}{3}\right]^2\right)\right)\Bigg)\Bigg]\Bigg\},$$

$$\left\{z\to-\text{ArcCosh}\left[\sqrt{\left(\frac{31\,\text{Cosh}\left[\frac{\pi}{3}\right]^2}{2\left(15\,\text{Cosh}\left[\frac{\pi}{3}\right]^2-18\,\text{Cosh}\left[\frac{\pi}{3}\right]\text{Sinh}\left[\frac{\pi}{3}\right]+3\,\text{Sinh}\left[\frac{\pi}{3}\right]^2\right)}-\right.}\right.\right.$$

$$\frac{21\,\text{Cosh}\left[\frac{\pi}{3}\right]\text{Sinh}\left[\frac{\pi}{3}\right]}{15\,\text{Cosh}\left[\frac{\pi}{3}\right]^2-18\,\text{Cosh}\left[\frac{\pi}{3}\right]\text{Sinh}\left[\frac{\pi}{3}\right]+3\,\text{Sinh}\left[\frac{\pi}{3}\right]^2}+$$

$$\frac{11\,\text{Sinh}\left[\frac{\pi}{3}\right]^2}{2\left(15\,\text{Cosh}\left[\frac{\pi}{3}\right]^2-18\,\text{Cosh}\left[\frac{\pi}{3}\right]\text{Sinh}\left[\frac{\pi}{3}\right]+3\,\text{Sinh}\left[\frac{\pi}{3}\right]^2\right)}+$$

$$\left(\sqrt{\left(\text{Cosh}\left[\frac{\pi}{3}\right]^4-12\,\text{Cosh}\left[\frac{\pi}{3}\right]^3\text{Sinh}\left[\frac{\pi}{3}\right]-14\,\text{Cosh}\left[\frac{\pi}{3}\right]^2\text{Sinh}\left[\frac{\pi}{3}\right]^2+12\,\text{Cosh}\left[\frac{\pi}{3}\right]\text{Sinh}\left[\frac{\pi}{3}\right]^3+\right.}\right.$$

$$\left.\left.13\,\text{Sinh}\left[\frac{\pi}{3}\right]^4\right)\right)\Big/\left(2\left(15\,\text{Cosh}\left[\frac{\pi}{3}\right]^2-18\,\text{Cosh}\left[\frac{\pi}{3}\right]\text{Sinh}\left[\frac{\pi}{3}\right]+3\,\text{Sinh}\left[\frac{\pi}{3}\right]^2\right)\right)\Bigg)\Bigg]\Bigg\}\Bigg\}$$

```
Solve[Coth[x] == a, x]
```

```
Solve::ifun : Inverse functions are being used by Solve, so some solutions may not be found.
```

```
{{x → ArcCoth[a]}}
```

A complete solution of the previous equation can be obtained using the function `Reduce`.

```
Reduce[Coth[x] == a, x] // InputForm
```

```
C[1] ∈ Integers && -1 + a^2 != 0 && x == ArcCoth[a] + I*Pi*C[1]
```

Solving differential equations

Here is a linear inhomogeneous differential equation whose independent solution includes the hyperbolic tangent function.

```
DSolve[w′[z] == -Csch[z]^2, w[z], z]
```

```
{{w[z] → C[1] + Coth[z]}}
```

Here is a nonlinear differential equation whose solution is the hyperbolic tangent function with a shifted argument.

```
DSolve[{w′[z] + w[z]^2 - 1 == 0}, w[z], z] // FullSimplify
```

```
{{w[z] → Coth[z - C[1]]}}
```

Plotting

Mathematica has built-in functions for 2D and 3D graphics. Here are some examples.

```
Plot[Coth[Sin[z]], {z, -2, 2}];
```

```
Plot3D[Re[Coth[x + ⅈ y]], {x, -3, 3}, {y, 0, π},
  PlotPoints → 240, PlotRange → {-5, 5},
  ClipFill → None, Mesh → False, AxesLabel → {"x", "y", None}];
```

```
ContourPlot[Arg[Coth[1/(x + ⅈ y)]], {x, -1/2, 1/2}, {y, -1/2, 1/2},
  PlotPoints → 400, PlotRange → {-π, π}, FrameLabel → {"x", "y", None, None},
  ColorFunction → (Hue[0.78 #] &), ContourLines → False, Contours → 200];
```

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