WhittakerW

Notations

Traditional name

Whittaker hypergeometric function W

Traditional notation

$W_{\nu,\mu}(z)$

Mathematica StandardForm notation

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WhittakerW[ν, μ, z]
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Primary definition

07.45.02.0001.01

$$W_{\nu,\mu}(z) = z^{\mu+\frac{1}{2}}\, e^{-\frac{z}{2}}\, U\left(\mu-\nu+\frac{1}{2}, 2\,\mu+1, z\right)$$

Specific values

Specialized values

For fixed ν, μ

07.45.03.0001.01

$$W_{\nu,\mu}(0) = 0 \;/;\; -\frac{1}{2} < \mathrm{Re}(\mu) < \frac{1}{2}$$

07.45.03.0002.01

$$W_{\nu,\mu}(0) = \tilde{\infty} \;/;\; |\mathrm{Re}(\mu)| > \frac{1}{2}$$

For fixed μ, z

07.45.03.0003.01

$$W_{-\mu-\frac{3}{2},\mu}(z) = \frac{z^{\frac{1}{2}-\mu}}{2\,\mu+1}\, e^{-\frac{z}{2}} \left(e^z\,(z+2\,\mu+1)\,E_{2\,\mu+1}(z)-1\right)$$

07.45.03.0004.01

$$W_{-\mu-\frac{1}{2},\mu}(z) = e^{z/2}\, z^{\mu+\frac{1}{2}}\, \Gamma(-2\,\mu, z)$$

07.45.03.0005.01

$$W_{-\mu-\frac{1}{2},\mu}(z) = z^{\frac{1}{2}-\mu} e^{z/2} E_{2\mu+1}(z)$$

07.45.03.0006.01

$$W_{\frac{1}{2}-\mu,\mu}(z) = z^{\frac{1}{2}-\mu} e^{-\frac{z}{2}}$$

07.45.03.0007.01

$$W_{\frac{3}{2}-\mu,\mu}(z) = z^{\frac{1}{2}-\mu} (z+2\mu-1) e^{-\frac{z}{2}}$$

07.45.03.0008.01

$$W_{-\mu-n-\frac{1}{2},\mu}(z) = (-1)^n e^{z/2} z^{\mu+\frac{1}{2}} \Gamma(-n-2\mu) L_n^{2\mu}(-z) - \frac{e^{-\frac{z}{2}} z^{\frac{1}{2}-\mu}}{n!\,(2\mu+1)_n} \frac{\partial^n \left(e^z z^{n+2\mu} (\Gamma(-2\mu) - \Gamma(-2\mu, z))\right)}{\partial z^n} /; n \in \mathbb{N}$$

07.45.03.0009.01

$$W_{-\frac{1}{2},\mu}(z) = \frac{z}{2\sqrt{\pi}\,\mu} \left(K_{\mu+\frac{1}{2}}\left(\frac{z}{2}\right) - K_{\mu-\frac{1}{2}}\left(\frac{z}{2}\right)\right)$$

07.45.03.0010.01

$$W_{0,\mu}(z) = \frac{\sqrt{z}}{\sqrt{\pi}} K_\mu\left(\frac{z}{2}\right)$$

07.45.03.0011.01

$$W_{\frac{1}{2},\mu}(z) = \frac{z}{2\sqrt{\pi}} \left(K_{\frac{1}{2}-\mu}\left(\frac{z}{2}\right) + K_{-\mu-\frac{1}{2}}\left(\frac{z}{2}\right)\right)$$

07.45.03.0012.01

$$W_{-\frac{n}{2},\mu}(z) = \frac{z^{\frac{n+1}{2}}}{\sqrt{\pi}\,(-n-2\mu)_{n+1}} \sum_{k=0}^{n} \frac{(2k-n-2\mu)\,(-n)_{n-k}\,(-n-2\mu)_k}{(n-k)!\,(1-2\mu)_k} K_{-k+\frac{n}{2}+\mu}\left(\frac{z}{2}\right) /; n \in \mathbb{N}$$

07.45.03.0013.01

$$W_{\frac{n}{2},\mu}(z) = \frac{z^{\frac{n+1}{2}}}{\sqrt{\pi}} \left(\frac{1-n}{2}+\mu\right)_n \sum_{k=0}^{n} \frac{(-1)^{k+n}\,(2k-n+2\mu)\,(-n)_{n-k}}{(n-k)!\,(k-n+2\mu)_{n+1}} K_{-k+\frac{n}{2}-\mu}\left(\frac{z}{2}\right) /; n \in \mathbb{N}$$

For fixed ν, z

07.45.03.0014.01

$$W_{\nu,\nu-\frac{5}{2}}(z) = z^{\nu-2} \left(z^2 + (6-4\nu)z + 4\nu^2 - 14\nu + 12\right) e^{-\frac{z}{2}}$$

07.45.03.0015.01

$$W_{\nu,\nu-\frac{3}{2}}(z) = z^{\nu-1} (z-2\nu+2) e^{-\frac{z}{2}}$$

07.45.03.0016.01

$$W_{\nu,\nu-\frac{1}{2}}(z) = e^{-\frac{z}{2}} z^\nu$$

07.45.03.0017.01

$$W_{\nu,\nu+\frac{1}{2}}(z) = e^{z/2} z^{-\nu} \Gamma(2\nu+1, z)$$

07.45.03.0018.01

$$W_{\nu,\nu+\frac{3}{2}}(z) = \frac{z^{-\nu-1} e^{-\frac{z}{2}}}{2(\nu+1)} \left(z^{2\nu+3} - e^z (z - 2(\nu+1)) \Gamma(2\nu+3, z)\right)$$

07.45.03.0019.01

$$W_{\nu,\nu-n-\frac{1}{2}}(z) = (-1)^n n! z^{\nu-n} e^{-\frac{z}{2}} L_n^{-2n+2\nu-1}(z) /; n \in \mathbb{N}$$

07.45.03.0020.01

$$W_{\frac{2n+1}{4},-\frac{1}{4}}(z) = 2^{-n} \sqrt[4]{z}\, e^{-\frac{z}{2}} H_n\left(\sqrt{z}\right) /; n \in \mathbb{N}$$

07.45.03.0021.01

$$W_{\frac{2n+1}{4},\frac{1}{4}}(z) = 2^{-n} \sqrt[4]{z}\, e^{-\frac{z}{2}} H_n\left(\sqrt{z}\right) /; n \in \mathbb{N}$$

07.45.03.0022.01

$$W_{\nu,n+\nu+\frac{1}{2}}(z) = \frac{e^{-\frac{z}{2}} z^{n+\nu+1}}{n! (-2(n+\nu))_n} \frac{\partial^n \left(e^z z^{-n-2\nu-1} \Gamma(2n+2\nu+1, z)\right)}{\partial z^n} /; n \in \mathbb{N}$$

07.45.03.0023.01

$$W_{\nu,\nu+n+\frac{1}{2}}(z) = -(2n+2\nu+1) z^{n+\nu+1} e^{z/2} \sum_{p=0}^{n} \sum_{k=0}^{n} \sum_{q=k}^{n} \frac{(-1)^q z^{-n-p-q-2\nu-1} \Gamma(k-2n-2\nu-1)}{k!\, p!\, (n-p-q)!\, \Gamma(-k+q+1)\, \Gamma(-n-p-2\nu)} \Gamma(-k+2n+q+2\nu+1, z) /; n \in \mathbb{N}$$

For fixed z and integer parameters

07.45.03.0024.01

$$W_{\frac{m}{2},\frac{m+1}{2}}(z) = m!\, z^{-\frac{m}{2}} e^{-\frac{z}{2}} \sum_{k=0}^{m} \frac{z^k}{k!} /; m \in \mathbb{N}$$

07.45.03.0025.01

$$W_{\frac{m}{2},\frac{m+3}{2}}(z) = \frac{z^{-\frac{m}{2}-1}}{m+2} \left(z^{m+3} + (m-z+2)(m+2)! \sum_{k=0}^{m+2} \frac{z^k}{k!}\right) e^{-\frac{z}{2}} /; m \in \mathbb{N}$$

07.45.03.0026.01

$$W_{\frac{n}{2},-\frac{n+1}{2}}(z) = n!\, z^{-\frac{n}{2}} e^{-\frac{z}{2}} \sum_{k=0}^{n} \frac{z^k}{k!} /; n \in \mathbb{N}$$

07.45.03.0027.01

$$W_{0,-n-\frac{1}{2}}(z) = \frac{\sqrt{z}}{\sqrt{\pi}} K_{n+\frac{1}{2}}\left(\frac{z}{2}\right) /; n \in \mathbb{N}$$

07.45.03.0028.01

$$W_{0,-n-\frac{1}{2}}(z) = \frac{\sqrt{z}}{\sqrt{\pi}} K_{n+\frac{1}{2}}\left(\frac{z}{2}\right) /; n \in \mathbb{N}$$

07.45.03.0029.01

$$W_{n-\frac{m}{2},-\frac{m+1}{2}}(z) = \frac{m!\, e^{-\frac{z}{2}} z^{-\frac{m}{2}}}{(m-n)!\, (-m)_{m-n}} \sum_{k=0}^{m} \frac{(k-m)_{m-n} z^k}{k!} /; m \in \mathbb{N} \land n \in \mathbb{Z} \land 0 \leq n \leq m$$

07.45.03.0030.01

$$W_{\frac{m}{2}+n,\frac{m-1}{2}}(z) = (-1)^n (m+n-1)!\, z^{m/2}\, e^{-\frac{z}{2}} \sum_{k=0}^{n} \frac{(-n)_k\, z^k}{(k+m-1)!\, k!} \;/;\; n \in \mathbb{N} \land m \in \mathbb{Z} \land m+n > 0$$

07.45.03.0031.01

$$W_{-\frac{m}{2}-n,-\frac{m+1}{2}}(z) = (m+1)\, z^{-\frac{m}{2}}\, e^{z/2} \sum_{p=0}^{n-1} \sum_{q=0}^{n-1} \sum_{k=0}^{q} \frac{(-1)^q\, z^{m+n-p-q}\, (k+m)!}{p!\, k!\, (n-p-q-1)!\, (m+n-p)!\, (q-k)!} \left(\frac{(-1)^{q-k+m}}{(k+m-q+1)!} \left(\mathrm{Ei}(-z) - \frac{1}{2} \left(\log(-z) - \log\left(-\frac{1}{z}\right) \right) + \log(z) \right) + e^{-z} \sum_{j=0}^{-k-m+q-2} \frac{z^j}{(q-k-m-1)_{j+k+m-q+2}} - e^{-z} \sum_{j=-k-m+q-1}^{-1} \frac{z^j}{(q-k-m-1)_{j+k+m-q+2}} \right) \;/;\; n \in \mathbb{N}^+ \bigwedge m \in \mathbb{N}$$

07.45.03.0032.01

$$W_{\frac{m}{2}-n,\frac{m-1}{2}}(z) = (m-1)\, z^{m/2}\, e^{z/2} \sum_{p=0}^{n-m} \sum_{q=0}^{n-m} \sum_{k=0}^{q} \frac{(-1)^q\, z^{n-m-p-q}\, (k+m-2)!}{p!\, k!\, (n-m-p-q)!\, (n-p-1)!\, (q-k)!} \left(\frac{(-1)^{q-k+m}}{(k+m-q-1)!} \left(\mathrm{Ei}(-z) - \frac{1}{2} \left(\log(-z) - \log\left(-\frac{1}{z}\right) \right) + \log(z) \right) + e^{-z} \sum_{j=0}^{q-k-m} \frac{z^j}{(q-k-m+1)_{j+k+m-q}} - e^{-z} \sum_{j=q-k-m+1}^{-1} \frac{z^j}{(q-k-m+1)_{j+k+m-q}} \right) \;/;\; n \in \mathbb{Z} \land n > 1 \land m \in \mathbb{Z} \land 1 < m \le n$$

07.45.03.0033.01

$$W_{\frac{m}{2}-n,\frac{m-1}{2}}(z) = \frac{(m-2)!\, z^{m/2}}{(n-1)!\, (2-m)_{n-1}\, e^{z/2}} \sum_{k=0}^{m-2} \frac{(k-m+2)_{n-1}\, z^{k-m+1}}{k!} \;/;\; n \in \mathbb{N}^+ \bigwedge m \in \mathbb{Z} \bigwedge m > n$$

General characteristics

Domain and analyticity

$W_{\nu,\mu}(z)$ is an analytical function of ν, μ and z which is defined in $\mathbb{C}^3$.

07.45.04.0001.01

$$(\nu * \mu * z) \longrightarrow W_{\nu,\mu}(z) :: (\mathbb{C} \otimes \mathbb{C} \otimes \mathbb{C}) \longrightarrow \mathbb{C}$$

Symmetries and periodicities

Mirror symmetry

07.45.04.0002.01

$$W_{\overline{\nu},\overline{\mu}}(\overline{z}) = \overline{W_{\nu,\mu}(z)} \;/;\; z \notin (-\infty, 0)$$

Periodicity

No periodicity

Poles and essential singularities

With respect to z

For fixed ν, μ, the function $W_{\nu,\mu}(z)$ has an essential singular point at $z = \tilde{\infty}$. At the same time, the point $z == \tilde{\infty}$ is a branch point.

07.45.04.0003.01

$$\mathcal{S}ing_z\left(W_{\nu,\mu}(z)\right) == \{\{\tilde{\infty}, \infty\}\}$$

With respect to μ

For fixed ν, z, the function $W_{\nu,\mu}(z)$ has only one singular point at $\mu = \tilde{\infty}$. It is an essential singular point.

07.45.04.0004.01

$$\mathcal{S}ing_\mu\left(W_{\nu,\mu}(z)\right) == \{\{\tilde{\infty}, \infty\}\}$$

With respect to ν

For fixed μ, z, the function $W_{\nu,\mu}(z)$ has only one singular point at $\nu = \tilde{\infty}$. It is an essential singular point.

07.45.04.0005.01

$$\mathcal{S}ing_\nu\left(W_{\nu,\mu}(z)\right) == \{\{\tilde{\infty}, \infty\}\}$$

Branch points

With respect to z

For fixed ν, μ, the function $W_{\nu,\mu}(z)$ has two branch points: $z == 0$, $z == \tilde{\infty}$. At the same time, the point $z == \tilde{\infty}$ is an essential singularity.

07.45.04.0006.01

$$\mathcal{BP}_z\left(W_{\nu,\mu}(z)\right) == \{0, \tilde{\infty}\}$$

07.45.04.0007.01

$$\mathcal{R}_z\left(W_{\nu,\mu}(z), 0\right) == \log \text{ /; } \mu \notin \mathbb{Q} \lor \mu \in \mathbb{Z}$$

07.45.04.0008.01

$$\mathcal{R}_z\left(M_{\nu,\frac{p}{q}}(z), 0\right) == 2\, q \text{ /; } p \in \mathbb{Z} \land q - 1 \in \mathbb{N}^+ \land \gcd(2\, p + q, 2\, q) == 1$$

07.45.04.0009.01

$$\mathcal{R}_z\left(W_{\nu,\mu}(z), \tilde{\infty}\right) == \log \text{ /; } \nu \notin \mathbb{Q}$$

07.45.04.0010.01

$$\mathcal{R}_z\left(W_{\frac{p}{q},\mu}(z), \tilde{\infty}\right) == q \text{ /; } p \in \mathbb{Z} \land q - 1 \in \mathbb{N}^+ \land \gcd(p, q) == 1$$

With respect to μ

The function $W_{\nu,\mu}(z)$ does not have branch points with respect to μ.

07.45.04.0011.01

$$\mathcal{BP}_\mu\left(W_{\nu,\mu}(z)\right) == \{\}$$

With respect to ν

The function $W_{\nu,\mu}(z)$ does not have branch points with respect to ν.

07.45.04.0012.01

$$\mathcal{BP}_\nu\left(W_{\nu,\mu}(z)\right) == \{\}$$

Branch cuts

With respect to z

For fixed ν, μ /; $\nu - \mu - \frac{1}{2} \notin \mathbb{N} \bigwedge \nu \notin \mathbb{Z}$, the function $W_{\nu,\mu}(x)$ is a single-valued function on the z-plane cut along the interval $(-\infty,\ 0)$, where it is continuous from above.

07.45.04.0013.01

$$\mathcal{BC}_z\left(W_{\nu,\mu}(z)\right) == \{\{(-\infty,\ 0),\ -i\}\} \text{ /; } \nu - \mu - \frac{1}{2} \notin \mathbb{N} \bigwedge \nu \in \mathbb{Z}$$

07.45.04.0014.01

$$\lim_{\epsilon \to 0} W_{\nu,\mu}(x + i\,\epsilon) == W_{\nu,\mu}(x) \text{ /; } x < 0$$

07.45.04.0015.01

$$\lim_{\epsilon \to 0} W_{\nu,\mu}(x - i\,\epsilon) == -\frac{(2\,i\,\pi)\,M_{\nu,\mu}(x)}{\Gamma(2\,\mu + 1)\,\Gamma\left(-\mu - \nu + \frac{1}{2}\right)} - e^{2\,i\,\pi\,\mu}\,W_{\nu,\mu}(x) \text{ /; } x < 0$$

With respect to μ

The function $W_{\nu,\mu}(z)$ does not have branch cuts with respect to μ.

07.45.04.0016.01

$$\mathcal{BC}_\mu\left(W_{\nu,\mu}(z)\right) == \{\}$$

With respect to ν

The function $W_{\nu,\mu}(z)$ does not have branch cuts with respect to ν.

07.45.04.0017.01

$$\mathcal{BC}_\nu\left(W_{\nu,\mu}(z)\right) == \{\}$$

Series representations

Generalized power series

Expansions at generic point $z == z_0$

For the function itself

07.45.06.0001.01

$$W_{\nu,\mu}(z) \propto \left(\frac{1}{z_0}\right)^{\left(-\mu-\frac{1}{2}\right)\left\lfloor\frac{\arg(z-z_0)}{2\pi}\right\rfloor} z_0^{\left(-\mu-\frac{1}{2}\right)\left\lfloor\frac{\arg(z-z_0)}{2\pi}\right\rfloor} W_{\nu,\mu}(z_0) - \frac{2\pi i e^{2i\pi\mu}}{\Gamma(2\mu+1)\Gamma\left(\frac{1}{2}-\mu-\nu\right)} \left\lfloor\frac{\arg(z-z_0)}{2\pi}\right\rfloor \left\lfloor\frac{\arg(z_0)+\pi}{2\pi}\right\rfloor \left(\frac{1}{z_0}\right)^{\left(\mu+\frac{1}{2}\right)\left\lfloor\frac{\arg(z-z_0)}{2\pi}\right\rfloor} z_0^{\left(\mu+\frac{1}{2}\right)\left\lfloor\frac{\arg(z-z_0)}{2\pi}\right\rfloor} M_{\nu,\mu}(z_0) + \left(\frac{1}{z_0}\right)^{-\left(\mu+\frac{1}{2}\right)\left\lfloor\frac{\arg(z-z_0)}{2\pi}\right\rfloor} z_0^{-\left(\mu+\frac{1}{2}\right)\left\lfloor\frac{\arg(z-z_0)}{2\pi}\right\rfloor-1} \frac{1}{2\Gamma(2\mu+2)\Gamma\left(\frac{1}{2}-\mu-\nu\right)} \left(\Gamma(2\mu+2)\Gamma\left(-\mu-\nu+\frac{1}{2}\right)\left((2\mu-z_0+1)W_{\nu,\mu}(z_0)+(2\nu-2\mu-1)\sqrt{z_0}\,W_{\nu-\frac{1}{2},\mu+\frac{1}{2}}(z_0)\right) - 2ie^{2i\pi\mu}\pi \left\lfloor\frac{\arg(z-z_0)}{2\pi}\right\rfloor \left\lfloor\frac{\arg(z_0)+\pi}{2\pi}\right\rfloor \left(\frac{1}{z_0}\right)^{(2\mu+1)\left\lfloor\frac{\arg(z-z_0)}{2\pi}\right\rfloor} z_0^{(2\mu+1)\left\lfloor\frac{\arg(z-z_0)}{2\pi}\right\rfloor} \left((2\mu+1)(2\mu-z_0+1)M_{\nu,\mu}(z_0)+(2\mu-2\nu+1)\sqrt{z_0}\,M_{\nu-\frac{1}{2},\mu+\frac{1}{2}}(z_0)\right)\right)(z-z_0)+\ldots /; (z\to z_0)$$

07.45.06.0002.01

$$W_{\nu,\mu}(z) = \sum_{k=0}^{\infty}\frac{1}{k!}\sum_{k_1=0}^{k}\sum_{k_2=0}^{k}\sum_{k_3=0}^{k}(-1)^{k_1+k_3}\delta_{k,k_1+k_2+k_3}(k_1+k_2+k_3;k_1,k_2,k_3)\,2^{-k_1} \left(\mu-k_2+\frac{3}{2}\right)_{k_2}\left(\mu-\nu+\frac{1}{2}\right)_{k_3} z_0^{-k_2-\frac{k_3}{2}} \left(\left(\frac{1}{z_0}\right)^{-\left(\mu+\frac{1}{2}\right)\left\lfloor\frac{\arg(z-z_0)}{2\pi}\right\rfloor} z_0^{-\left(\mu+\frac{1}{2}\right)\left\lfloor\frac{\arg(z-z_0)}{2\pi}\right\rfloor} W_{\nu-\frac{k_3}{2},\mu+\frac{k_3}{2}}(z_0) - \left(\frac{1}{z_0}\right)^{\left(\mu+\frac{1}{2}\right)\left\lfloor\frac{\arg(z-z_0)}{2\pi}\right\rfloor} z_0^{\left(\mu+\frac{1}{2}\right)\left\lfloor\frac{\arg(z-z_0)}{2\pi}\right\rfloor} \frac{2\pi i(-1)^{k_3}e^{2i\pi\mu}}{\Gamma\left(\frac{1}{2}-\mu-\nu\right)\Gamma(2\mu+k_3+1)} \left\lfloor\frac{\arg(z-z_0)}{2\pi}\right\rfloor \left\lfloor\frac{\arg(z_0)+\pi}{2\pi}\right\rfloor M_{\nu-\frac{k_3}{2},\mu+\frac{k_3}{2}}(z_0)\right)(z-z_0)^k$$

07.45.06.0003.01

$$W_{\nu,\mu}(z) \propto \left(\left(\frac{1}{z_0}\right)^{\left(-\mu-\frac{1}{2}\right)\left\lfloor\frac{\arg(z-z_0)}{2\pi}\right\rfloor} z_0^{\left(-\mu-\frac{1}{2}\right)\left\lfloor\frac{\arg(z-z_0)}{2\pi}\right\rfloor} W_{\nu,\mu}(z_0) - \frac{2\pi i e^{2i\pi\mu}}{\Gamma(2\mu+1)\Gamma\left(-\mu-\nu+\frac{1}{2}\right)} \left\lfloor\frac{\arg(z-z_0)}{2\pi}\right\rfloor \left\lfloor\frac{\arg(z_0)+\pi}{2\pi}\right\rfloor \left(\frac{1}{z_0}\right)^{\left(\mu+\frac{1}{2}\right)\left\lfloor\frac{\arg(z-z_0)}{2\pi}\right\rfloor} z_0^{\left(\mu+\frac{1}{2}\right)\left\lfloor\frac{\arg(z-z_0)}{2\pi}\right\rfloor} M_{\nu,\mu}(z_0)\right)(1+O(z-z_0))$$

Expansions on branch cuts

For the function itself

07.45.06.0004.01

$$W_{\nu,\mu}(z) \propto -\frac{2\pi i e^{2i\pi\mu}}{\Gamma(2\mu+1)\Gamma\left(\frac{1}{2}-\mu-\nu\right)} \left\lfloor \frac{\arg(z-x)}{2\pi} \right\rfloor e^{(2\mu+1)\pi i \left\lfloor \frac{\arg(z-x)}{2\pi} \right\rfloor} M_{\nu,\mu}(x) + e^{-(2\mu+1)\pi i \left\lfloor \frac{\arg(z-x)}{2\pi} \right\rfloor} W_{\nu,\mu}(x) + \frac{e^{-(2\mu+1)\pi i \left\lfloor \frac{\arg(z-x)}{2\pi} \right\rfloor}}{2\Gamma(2\mu+2)\Gamma\left(-\mu-\nu+\frac{1}{2}\right) x} \left(\Gamma(2\mu+2)\Gamma\left(\frac{1}{2}-\mu-\nu\right)\left((1-x+2\mu) W_{\nu,\mu}(x) + \sqrt{x}\,(2\nu-2\mu-1) W_{\nu-\frac{1}{2},\mu+\frac{1}{2}}(x)\right) - 2 i e^{2i\pi\mu}\pi \left\lfloor \frac{\arg(z-x)}{2\pi} \right\rfloor e^{2(2\mu+1)\pi i \left\lfloor \frac{\arg(z-x)}{2\pi} \right\rfloor} \left((1-x+2\mu)(2\mu+1) M_{\nu,\mu}(x) + \sqrt{x}\,(2\mu-2\nu+1) M_{\nu-\frac{1}{2},\mu+\frac{1}{2}}(x)\right)\right)(z-x) + \ldots \;/;\; (z \to x) \land x < 0$$

07.45.06.0005.01

$$W_{\nu,\mu}(z) = \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{k_1=0}^{k} \sum_{k_2=0}^{k} \sum_{k_3=0}^{k} (-1)^{k_1+k_3} \delta_{k,k_1+k_2+k_3} (k_1+k_2+k_3;\, k_1,\, k_2,\, k_3)\, 2^{-k_1} \left(\mu-k_2+\frac{3}{2}\right)_{k_2} \left(\mu-\nu+\frac{1}{2}\right)_{k_3} x^{-k_2-\frac{k_3}{2}} \left(e^{-(2\mu+1)\pi i \left\lfloor \frac{\arg(z-x)}{2\pi} \right\rfloor} W_{\nu-\frac{k_3}{2},\mu+\frac{k_3}{2}}(x) - \frac{2 i \pi (-1)^{k_3} e^{2\mu\pi i}}{\Gamma\left(\frac{1}{2}-\mu-\nu\right)\Gamma(2\mu+k_3+1)} e^{(2\mu+1)\pi i \left\lfloor \frac{\arg(z-x)}{2\pi} \right\rfloor} \left\lfloor \frac{\arg(z-x)}{2\pi} \right\rfloor M_{\nu-\frac{k_3}{2},\mu+\frac{k_3}{2}}(x)\right) (z-x)^k \;/;\; x \in \mathbb{R} \land x < 0$$

07.45.06.0006.01

$$W_{\nu,\mu}(z) \propto \left(e^{-(2\mu+1)\pi i \left\lfloor \frac{\arg(z-x)}{2\pi} \right\rfloor} W_{\nu,\mu}(x) - \frac{2\pi i e^{2i\pi\mu}}{\Gamma(2\mu+1)\Gamma\left(\frac{1}{2}-\mu-\nu\right)} \left\lfloor \frac{\arg(z-x)}{2\pi} \right\rfloor e^{(2\mu+1)\pi i \left\lfloor \frac{\arg(z-x)}{2\pi} \right\rfloor} M_{\nu,\mu}(x)\right)(1 + O(z-x)) \;/;\; x < 0$$

Expansions at $z == 0$

General case

07.45.06.0007.01

$$W_{\nu,\mu}(z) \propto \frac{\Gamma(2\mu)\, z^{\frac{1}{2}-\mu}}{\Gamma\left(\mu-\nu+\frac{1}{2}\right)} \left(1 - \frac{\nu z}{1-2\mu} + \frac{\left(4\nu^2-2\mu+1\right) z^2}{16(1-\mu)(1-2\mu)} + \ldots\right) + \frac{\Gamma(-2\mu)\, z^{\mu+\frac{1}{2}}}{\Gamma\left(-\mu-\nu+\frac{1}{2}\right)} \left(1 - \frac{\nu z}{2\mu+1} + \frac{\left(4\nu^2+2\mu+1\right) z^2}{16(\mu+1)(2\mu+1)} + \ldots\right) /; (z \to 0) \bigwedge 2\mu \notin \mathbb{Z}$$

07.45.06.0008.01

$$W_{\nu,\mu}(z) \propto \frac{\Gamma(2\mu)\, z^{\frac{1}{2}-\mu}}{\Gamma\left(\mu-\nu+\frac{1}{2}\right)} \left(1 - \frac{\nu z}{1-2\mu} + \frac{\left(4\nu^2-2\mu+1\right) z^2}{16(1-\mu)(1-2\mu)} + O\left(z^3\right)\right) + \frac{\Gamma(-2\mu)\, z^{\mu+\frac{1}{2}}}{\Gamma\left(-\mu-\nu+\frac{1}{2}\right)} \left(1 - \frac{\nu z}{2\mu+1} + \frac{\left(4\nu^2+2\mu+1\right) z^2}{16(\mu+1)(2\mu+1)} + O\left(z^3\right)\right) /; 2\mu \notin \mathbb{Z}$$

07.45.06.0009.01

$$W_{\nu,\mu}(z) = \frac{\Gamma(-2\mu)\, z^{\frac{1}{2}+\mu}}{\Gamma\left(\frac{1}{2}-\mu-\nu\right)} \sum_{k=0}^{\infty} \frac{\left(-\frac{1}{2}\right)^k}{k!}\, {}_2F_1\left(-k,\, \mu-\nu+\frac{1}{2};\, 2\mu+1;\, 2\right) z^k + \frac{\Gamma(2\mu)\, z^{\frac{1}{2}-\mu}}{\Gamma\left(\mu-\nu+\frac{1}{2}\right)} \sum_{k=0}^{\infty} \frac{\left(-\frac{1}{2}\right)^k}{k!}\, {}_2F_1\left(-k,\, -\mu-\nu+\frac{1}{2};\, 2\mu+1;\, 2\right) z^k \;/;\; 2\mu \notin \mathbb{Z}$$

07.45.06.0010.01

$$W_{\nu,\mu}(z) = \frac{\Gamma(-2\mu)\, z^{\mu+\frac{1}{2}}}{\Gamma\left(\frac{1}{2}-\mu-\nu\right)} \sum_{k=0}^{\infty} \left(\sum_{j=0}^{k} \frac{(-1)^{k-j}\, 2^{j-k} \left(\mu-\nu+\frac{1}{2}\right)_j}{(k-j)!\, (2\mu+1)_j\, j!} \right) z^k + \frac{\Gamma(2\mu)\, z^{\frac{1}{2}-\mu}}{\Gamma\left(\mu-\nu+\frac{1}{2}\right)} \sum_{k=0}^{\infty} \left(\sum_{j=0}^{k} \frac{(-1)^{k-j}\, 2^{j-k} \left(\frac{1}{2}-\mu-\nu\right)_j}{(k-j)!\, (1-2\mu)_j\, j!} \right) z^k \;/;\; 2\mu \notin \mathbb{Z}$$

07.45.06.0011.01

$$W_{\nu,\mu}(z) = \frac{e^{-\frac{z}{2}}\, \Gamma(-2\mu)\, z^{\mu+\frac{1}{2}}}{\Gamma\left(\frac{1}{2}-\mu-\nu\right)} \sum_{k=0}^{\infty} \frac{\left(\mu-\nu+\frac{1}{2}\right)_k z^k}{(2\mu+1)_k\, k!} + \frac{e^{-\frac{z}{2}}\, \Gamma(2\mu)\, z^{\frac{1}{2}-\mu}}{\Gamma\left(\mu-\nu+\frac{1}{2}\right)} \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}-\mu-\nu\right)_k z^k}{(1-2\mu)_k\, k!} \;/;\; 2\mu \notin \mathbb{Z}$$

07.45.06.0012.01

$$W_{\nu,\mu}(z) = \frac{\Gamma(-2\mu)}{\Gamma\left(\frac{1}{2}-\mu-\nu\right)} M_{\nu,\mu}(z) + \frac{\Gamma(2\mu)}{\Gamma\left(\mu-\nu+\frac{1}{2}\right)} M_{\nu,-\mu}(z) \;/;\; 2\mu \notin \mathbb{Z}$$

07.45.06.0013.01

$$W_{\nu,\mu}(z) \propto \frac{\Gamma(2\mu)\, z^{\frac{1}{2}-\mu}}{\Gamma\left(\mu-\nu+\frac{1}{2}\right)} (1+O(z)) + \frac{\Gamma(-2\mu)\, z^{\mu+\frac{1}{2}}}{\Gamma\left(-\mu-\nu+\frac{1}{2}\right)} (1+O(z)) \;/;\; 2\mu \notin \mathbb{Z}$$

Logarithmic case

07.45.06.0014.01

$$W_{\nu,0}(z) \propto -\frac{\sqrt{z}}{\Gamma\left(\frac{1}{2}-\nu\right)} \left(\left(1 - \nu z + \frac{1}{16}\left(4\nu^2+1\right) z^2 + \ldots\right) \log(z) + \psi\left(\frac{1}{2}-\nu\right) + 2\gamma - \frac{1}{2}\left(4\gamma\nu - 4\nu + \psi\left(\frac{1}{2}-\nu\right) + (2\nu-1)\,\psi\left(\frac{3}{2}-\nu\right) + 2\right) z + \frac{1}{8}\left(\psi\left(\frac{1}{2}-\nu\right) + 2(2\nu-1)\left(\psi\left(\frac{3}{2}-\nu\right) + 2\gamma - 2\right) + \frac{1}{2}(2\nu-3)(2\nu-1)\left(\psi\left(\frac{5}{2}-\nu\right) + 2\gamma - 3\right) + 2\gamma\right) z^2 + \ldots \right) /;\; (z \to 0)$$

07.45.06.0015.01

$$W_{\nu,0}(z) = -\frac{\sqrt{z}}{\Gamma\left(\frac{1}{2}-\nu\right)} \left(\log(z) \sum_{k=0}^{\infty} \sum_{j=0}^{k} \frac{(-1)^{k-j}\, 2^{j-k} \left(\frac{1}{2}-\nu\right)_j}{(j!)^2\, (k-j)!} z^k - \sum_{k=0}^{\infty} \sum_{j=0}^{k} \frac{(-1)^{k-j}\, 2^{j-k} \left(\frac{1}{2}-\nu\right)_j}{(j!)^2\, (k-j)!} \left(2\psi(j+1) - \psi\left(j-\nu+\frac{1}{2}\right)\right) z^k \right)$$

07.45.06.0016.01

$$W_{\nu,\mu}(z) = \frac{(-1)^{2\mu+1} z^{\mu+\frac{1}{2}}}{\Gamma\left(\frac{1}{2}-\mu-\nu\right)} \left(-z^{-2\mu} \sum_{k=0}^{2\mu-1} \sum_{j=0}^{k} \frac{\left(-\frac{1}{2}\right)^j (j-k+2\mu-1)!}{j!\,(k-j)!\left(\nu-\mu+\frac{1}{2}\right)_{j-k+2\mu}} z^k + \right.$$

$$\frac{\log(z)}{(2\mu)!} \sum_{k=0}^{\infty} \sum_{j=0}^{k} \frac{\left(-\frac{1}{2}\right)^{k-j} \left(\mu-\nu+\frac{1}{2}\right)_j}{j!\,(k-j)!\,(2\mu+1)_j} z^k + \sum_{k=0}^{\infty} \sum_{j=0}^{2\mu-1} \frac{(-1)^{j+k} 2^{-j-k-1} j!\, z^k}{(j+k+1)!\,(2\mu-j-1)!\left(\nu-\mu+\frac{1}{2}\right)_{j+1}} -$$

$$\left. \sum_{k=0}^{\infty} \sum_{j=0}^{k} \frac{\left(-\frac{1}{2}\right)^{k-j} \left(\mu-\nu+\frac{1}{2}\right)_j}{j!\,(k-j)!\,(j+2\mu)!} \left(\psi(j+1) + \psi(j+2\mu+1) - \psi\left(j+\mu-\nu+\frac{1}{2}\right)\right) z^k \right) /;\ 2\mu \in \mathbb{N}$$

07.45.06.0017.01

$$W_{\nu,\mu}(z) \propto \frac{(-1)^{2\mu} z^{\mu+\frac{1}{2}}}{\Gamma\left(\frac{1}{2}-\mu-\nu\right)} \left(\frac{(2\mu-1)!\, z^{-2\mu}}{\left(-\mu+\nu+\frac{1}{2}\right)_{2\mu}} (1+O(z)) - \frac{\log(z)}{(2\mu)!} (1+O(z)) + \right.$$

$$\left. \left(\frac{\psi(2\mu+1) - \psi\left(\mu-\nu+\frac{1}{2}\right) - \gamma}{(2\mu)!} - \sum_{j=0}^{2\mu-1} \frac{(-1)^j 2^{-j-1} j!}{(j+1)!\,(2\mu-j-1)!\left(\nu-\mu+\frac{1}{2}\right)_{j+1}} \right) (1+O(z)) \right) /;\ 2\mu \in \mathbb{N}^+$$

07.45.06.0018.01

$$W_{\nu,0}(z) \propto -\frac{\sqrt{z}}{\Gamma\left(\frac{1}{2}-\nu\right)} \left(\log(z)\,(1+O(z)) + \psi\left(\frac{1}{2}-\nu\right) + 2\gamma + O(z) \right) /;\ (z \to 0)$$

07.45.06.0019.01

$$W_{\nu,\mu}(z) = -\frac{z^{\mu+\frac{1}{2}}}{\Gamma\left(\mu-\nu+\frac{1}{2}\right)} \left(-(-z)^{-2\mu} \sum_{k=0}^{\infty} \sum_{j=0}^{k} \frac{\left(-\frac{1}{2}\right)^{k-j} \left(\frac{1}{2}-\mu-\nu\right)_j}{j!\,(j-2\mu)!\,(k-j)!} \left(\psi(j+1) + \psi(j-2\mu+1) - \psi\left(j-\mu-\nu+\frac{1}{2}\right)\right) z^k + \right.$$

$$\frac{(-z)^{-2\mu} \log(z)}{(-2\mu)!} \sum_{k=0}^{\infty} \sum_{j=0}^{k} \frac{\left(-\mu-\nu+\frac{1}{2}\right)_j}{j!\,(1-2\mu)_j\,(k-j)!} \left(-\frac{1}{2}\right)^{k-j} z^k -$$

$$\frac{1}{\left(\mu-\nu+\frac{1}{2}\right)_{-2\mu}} \sum_{k=0}^{-2\mu-1} \sum_{j=0}^{k} \frac{(-1)^{k-j} (j-k-2\mu-1)! \left(\mu-\nu+\frac{1}{2}\right)_{k-j}}{(k-j)!\, j!} \left(-\frac{1}{2}\right)^j z^k +$$

$$\left. \frac{z^{-2\mu}}{\left(\mu-\nu+\frac{1}{2}\right)_{-2\mu}} \sum_{k=0}^{\infty} \sum_{j=0}^{-2\mu-1} \frac{(-1)^{j+2\mu} j! \left(\mu-\nu+\frac{1}{2}\right)_{-j-2\mu-1}}{(-j-2\mu-1)!\,(j+k+1)!} \left(-\frac{1}{2}\right)^{j+k+1} z^k \right) /;\ -2\mu \in \mathbb{N}^+$$

07.45.06.0020.01

$$W_{\nu,\mu}(z) \propto -\frac{z^{\mu+\frac{1}{2}}}{\Gamma\left(\mu-\nu+\frac{1}{2}\right)} \left(\frac{(-z)^{-2\mu} \log(z)}{(-2\mu)!} (1+O(z)) - \frac{(-2\mu-1)!}{\left(\mu-\nu+\frac{1}{2}\right)_{-2\mu}} (1+O(z)) \right) /;\ -2\mu \in \mathbb{N}^+$$

Asymptotic series expansions

07.45.06.0021.01

$$W_{\nu,\mu}(z) \propto z^{\nu}\, e^{-\frac{z}{2}} \left(1 + \frac{\mu^2 - \nu^2 + \nu - \frac{1}{4}}{z} + \frac{16\,\mu^4 - 8\left(4\,\nu^2 - 8\,\nu + 5\right)\mu^2 + \left(4\,\nu^2 - 8\,\nu + 3\right)^2}{32\,z^2} + \ldots\right) /;\ (|z| \to \infty)$$

07.45.06.0022.01

$$W_{\nu,\mu}(z) \propto z^{\nu}\, e^{-\frac{z}{2}} \left(\sum_{k=0}^{n} \frac{(-1)^k \left(-\mu - \nu + \frac{1}{2}\right)_k \left(\mu - \nu + \frac{1}{2}\right)_k}{k!}\, z^{-k} + O\!\left(\frac{1}{z^{n+1}}\right)\right) /;\ (|z| \to \infty)$$

07.45.06.0023.01

$$W_{\nu,\mu}(z) \propto z^{\nu}\, e^{-\frac{z}{2}}\, {}_2F_0\!\left(-\mu - \nu + \frac{1}{2}, \mu - \nu + \frac{1}{2};\, ;\, -\frac{1}{z}\right) /;\ (|z| \to \infty)$$

07.45.06.0024.01

$$W_{\nu,\mu}(z) \propto z^{\nu}\, e^{-\frac{z}{2}} \left(1 + O\!\left(\frac{1}{z}\right)\right) /;\ (|z| \to \infty)$$

Integral representations

On the real axis

Of the direct function

07.45.07.0001.01

$$W_{\nu,\mu}(z) = \frac{z^{\mu+\frac{1}{2}}}{\Gamma\!\left(\mu - \nu + \frac{1}{2}\right)} \int_{\frac{1}{2}}^{\infty} e^{-t\,z} \left(t - \frac{1}{2}\right)^{\mu-\nu-\frac{1}{2}} \left(t + \frac{1}{2}\right)^{\mu+\nu-\frac{1}{2}} dt /;\ \mathrm{Re}(z) > 0 \bigwedge \mathrm{Re}(\mu - \nu) > -\frac{1}{2}$$

Limit representations

07.45.09.0001.01

$$W_{\nu,\mu}(z) \to e^{-\frac{z}{2}}\, z^{\nu} \left(\lim_{c\to\infty} {}_2F_1\!\left(\mu - \nu + \frac{1}{2}, -\mu - \nu + \frac{1}{2}; c; 1 - \frac{c}{z}\right)\right)$$

Differential equations

Ordinary linear differential equations and wronskians

For the direct function itself

07.45.13.0001.01

$$w''(z) - \frac{\left(z^2 - 4\,\nu\,z + 4\,\mu^2 - 1\right) w(z)}{4\,z^2} = 0 /;\ w(z) = c_1\, W_{\nu,\mu}(z) + c_2\, M_{\nu,\mu}(z)$$

07.45.13.0002.01

$$W_z\!\left(W_{\nu,\mu}(z), M_{\nu,\mu}(z)\right) = \frac{\Gamma(2\,\mu + 1)}{\Gamma\!\left(\mu - \nu + \frac{1}{2}\right)}$$

07.45.13.0003.01

$$w''(z) - \frac{\left(z^2 - 4\,\nu\,z + 4\,\mu^2 - 1\right) w(z)}{4\,z^2} = 0\;/;\; c_1\,W_{\nu,\mu}(z) + c_2\,W_{-\nu,\mu}(-z)$$

07.45.13.0004.01

$$W_z\left(W_{\nu,\mu}(z),\,W_{-\nu,\mu}(-z)\right) == \csc(2\,\pi\,\mu)\left((-z)^{\mu-\frac{1}{2}}\,z^{\frac{1}{2}-\mu}\cos(\pi\,(\mu-\nu)) - (-z)^{-\mu-\frac{1}{2}}\,z^{\mu+\frac{1}{2}}\cos(\pi\,(\mu+\nu))\right)$$

07.45.13.0005.01

$$w''(z) - \frac{g''(z)}{g'(z)}\,w'(z) + \left(-\frac{1}{4}\,g'(z)^2 + \frac{\nu\,g'(z)^2}{g(z)} + \frac{\left(1-4\,\mu^2\right) g'(z)^2}{4\,g(z)^2}\right) w(z) == 0\;/;\; w(z) = c_1\,W_{\nu,\mu}(g(z)) + c_2\,M_{\nu,\mu}(g(z))$$

07.45.13.0006.01

$$W_z\left(W_{\nu,\mu}(g(z)),\,M_{\nu,\mu}(g(z))\right) == \frac{g'(z)\,\Gamma(2\,\mu+1)}{\Gamma\left(\mu-\nu+\frac{1}{2}\right)}$$

07.45.13.0007.01

$$w''(z) - \left(\frac{2\,h'(z)}{h(z)} + \frac{g''(z)}{g'(z)}\right) w'(z) + \left(-\frac{1}{4}\,g'(z)^2 + \frac{\nu\,g'(z)^2}{g(z)} + \frac{\left(1-4\,\mu^2\right) g'(z)^2}{4\,g(z)^2} + \frac{2\,h'(z)^2}{h(z)^2} + \frac{h'(z)\,g''(z)}{h(z)\,g'(z)} - \frac{h''(z)}{h(z)}\right) w(z) == 0\;/;$$
$$w(z) = c_1\,h(z)\;W_{\nu,\mu}(g(z)) + c_2\,h(z)\;M_{\nu,\mu}(g(z))$$

07.45.13.0008.01

$$W_z\left(h(z)\,W_{\nu,\mu}(g(z)),\,h(z)\,M_{\nu,\mu}(g(z))\right) == \frac{h(z)^2\,g'(z)\,\Gamma(2\,\mu+1)}{\Gamma\left(\mu-\nu+\frac{1}{2}\right)}$$

07.45.13.0009.01

$$4\,w''(z)\,z^2 - 4\,z\,(r+2\,s-1)\,w'(z) + \left(\left(4\,a\,\nu\,z^r - a^2\,z^{2\,r} - 4\,\mu^2 + 1\right) r^2 + 4\,s\,r + 4\,s^2\right) w(z) = 0\;/;$$
$$w(z) = c_1\,z^s\,W_{\nu,\mu}\left(a\,z^r\right) + c_2\,z^s\,M_{\nu,\mu}\left(a\,z^r\right)$$

07.45.13.0010.01

$$W_z\left(z^s\,W_{\nu,\mu}(a\,z^r),\,z^s\,M_{\nu,\mu}(a\,z^r)\right) == \frac{r\,\sqrt{a}\;z^{\frac{r}{2}+2\,s-1}\,\sqrt{a\,z^r}\;\Gamma(2\,\mu+1)}{\Gamma\left(\mu-\nu+\frac{1}{2}\right)}$$

07.45.13.0011.01

$$4\,w''(z)\,z^2 - 4\,z\,(r+2\,s-1)\,w'(z) + \left(\left(4\,a\,\nu\,z^r - a^2\,z^{2\,r} - 4\,\mu^2 + 1\right) r^2 + 4\,s\,r + 4\,s^2\right) w(z) = 0\;/;$$
$$w(z) = c_1\,z^s\,W_{\nu,\mu}\left(a\,z^r\right) + c_2\,z^s\,M_{\nu,\mu}\left(a\,z^r\right)$$

07.45.13.0012.01

$$W_z\left(s^z\,W_{\nu,\mu}(a\,r^z),\,s^z\,M_{\nu,\mu}(a\,r^z)\right) == \frac{1}{2}\,s^{2\,z}\log(r)\left((2\,\mu+2\,\nu+1)\,M_{\nu+1,\mu}(a\,r^z)\,W_{\nu,\mu}(a\,r^z) + 2\,M_{\nu,\mu}(a\,r^z)\,W_{\nu+1,\mu}(a\,r^z)\right)$$

Transformations

Transformations and argument simplifications

Argument involving basic arithmetic operations

07.45.16.0001.01

$$W_{\nu,-\mu}(z) = W_{\nu,\mu}(z)$$

07.45.16.0002.01

$$W_{\nu,\mu}(-z) = \frac{\Gamma\left(\frac{1}{2}-\mu+\nu\right) z^{-\mu-\frac{1}{2}} (-z)^{\mu+\frac{1}{2}}}{\Gamma\left(\frac{1}{2}-\mu-\nu\right)} W_{-\nu,\mu}(z) + \frac{\Gamma(2\,\mu)}{\Gamma\left(\mu-\nu+\frac{1}{2}\right)} \left((-z)^{\mu-\frac{1}{2}} z^{\frac{1}{2}-\mu} \cos(\pi(\mu+\nu)) \sec(\pi(\mu-\nu)) - (-z)^{-\mu-\frac{1}{2}} z^{\mu+\frac{1}{2}}\right) M_{-\nu,-\mu}(z)$$

Products, sums, and powers of the direct function

Products of the direct function

07.45.16.0003.01

$$W_{\nu,\mu}(-z)\, W_{\nu,\mu}(z) = \frac{\left(-z^2\right)^{-\mu}}{2\left(4\mu^3-\mu\right)\Gamma\left(-\mu-\nu+\frac{1}{2}\right)^2 \Gamma\left(\mu-\nu+\frac{1}{2}\right)^2} \left(4\pi z^{\mu+2} (-z)^{\mu} \mu\,\nu \sec(\pi\mu)\, \Gamma\left(-\mu-\nu+\frac{1}{2}\right)\Gamma\left(\mu-\nu+\frac{1}{2}\right) {}_2F_3\left(1-\nu, \nu+1; \frac{3}{2}, \frac{3}{2}-\mu, \mu+\frac{3}{2}; \frac{z^2}{4}\right) - \pi\left(-z^2\right)^{\mu+\frac{1}{2}} \left(4\mu^2-1\right)\csc(\pi\mu)\,\Gamma\left(-\mu-\nu+\frac{1}{2}\right)\Gamma\left(\mu-\nu+\frac{1}{2}\right) {}_2F_3\left(\frac{1}{2}-\nu, \nu+\frac{1}{2}; \frac{1}{2}, 1-\mu, \mu+1; \frac{z^2}{4}\right) + 2\sqrt{-z^2}\,\mu\left(4\mu^2-1\right)\Gamma(2\mu)^2\,\Gamma\left(-\mu-\nu+\frac{1}{2}\right)^2 {}_2F_3\left(-\mu-\nu+\frac{1}{2}, -\mu+\nu+\frac{1}{2}; 1-2\mu, \frac{1}{2}-\mu, 1-\mu; \frac{z^2}{4}\right) + 2\left(-z^2\right)^{2\mu+\frac{1}{2}} \mu\left(4\mu^2-1\right)\Gamma(-2\mu)^2\,\Gamma\left(\mu-\nu+\frac{1}{2}\right)^2 {}_2F_3\left(\mu-\nu+\frac{1}{2}, \mu+\nu+\frac{1}{2}; \mu+\frac{1}{2}, \mu+1, 2\mu+1; \frac{z^2}{4}\right)\right)$$

Identities

Recurrence identities

Consecutive neighbors

07.45.17.0001.01

$$W_{\nu,\mu}(z) = \frac{4(-z+2\nu+2)}{4\mu^2-(2\nu+1)^2} W_{\nu+1,\mu}(z) + \frac{4}{4\mu^2-(2\nu+1)^2} W_{\nu+2,\mu}(z)$$

07.45.17.0002.01

$$W_{\nu,\mu}(z) = \frac{1}{4}\left(4\mu^2-(3-2\nu)^2\right) W_{\nu-2,\mu}(z) + (z-2\nu+2)\, W_{\nu-1,\mu}(z)$$

07.45.17.0003.01

$$W_{\nu,\mu}(z) = -\frac{4(\mu+1)\left(4\mu^2+8\mu-2z\nu+3\right)}{z(2\mu+3)(2(\mu+\nu)+1)} W_{\nu,\mu+1}(z) + \frac{(2\mu+1)(2\mu-2\nu+3)}{(2\mu+3)(2(\mu+\nu)+1)} W_{\nu,\mu+2}(z)$$

07.45.17.0004.01

$$W_{\nu,\mu}(z) = \frac{4(\mu-1)\left(4\mu^2-8\mu-2z\nu+3\right)}{z(2\mu-3)(2\mu-2\nu-1)} W_{\nu,\mu-1}(z) + \frac{(2\mu-1)(2\mu+2\nu-3)}{(2\mu-3)(2\mu-2\nu-1)} W_{\nu,\mu-2}(z)$$

Distant neighbors

07.45.17.0005.01

$$W_{\nu,\mu}(z) = C_n(\nu, \mu, z)\, W_{\nu+n,\mu}(z) + \frac{4}{(-2n+2\mu-2\nu+1)(2n+2\mu+2\nu-1)} C_{n-1}(\nu, \mu, z)\, W_{\nu+n+1,\mu}(z) \;/;$$
$$C_0(\nu, \mu, z) = 1 \bigwedge C_1(\nu, \mu, z) = \frac{4(z-2(\nu+1))}{(2\nu+1)^2-4\mu^2} \bigwedge$$
$$C_n(\nu, \mu, z) = \frac{4(z-2(\nu+1))}{(2\nu+1)^2-4\mu^2} C_{n-1}(\nu, \mu, z) - \frac{2}{\left(n-\mu+\nu-\frac{3}{2}\right)(2n+2\mu+2\nu-3)} C_{n-2}(\nu, \mu, z) \bigwedge n \in \mathbb{N}^+$$

07.45.17.0006.01

$$W_{\nu,\mu}(z) = C_n(\nu, \mu, z)\, W_{\nu-n,\mu}(z) + \frac{1}{4}(-2n+2\mu+2\nu-1)(2n+2\mu-2\nu+1)\, C_{n-1}(\nu, \mu, z)\, W_{-n+\nu-1,\mu}(z) \;/;$$
$$C_0(\nu, \mu, z) = 1 \bigwedge C_1(\nu, \mu, z) = z-2\nu+2 \bigwedge$$
$$C_n(\nu, \mu, z) = (2n+z-2\nu)\, C_{n-1}(\nu, \mu, z) - \left(n-\mu-\nu-\frac{1}{2}\right)\left(n+\mu-\nu-\frac{1}{2}\right) C_{n-2}(\nu, \mu, z) \bigwedge n \in \mathbb{N}^+$$

Functional identities

Relations between contiguous functions

07.45.17.0007.01

$$-\left(\mu^2-\nu^2+\nu-\frac{1}{4}\right) W_{\nu-1,\mu}(z) - (z-2\nu)\, W_{\nu,\mu}(z) + W_{\nu+1,\mu}(z) = 0$$

07.45.17.0008.01

$$-z(2\mu+1)(2\mu+2\nu-1)\, W_{\nu,\mu-1}(z) + 4\mu\left(-4\mu^2+2z\nu+1\right) W_{\nu,\mu}(z) + z(2\mu-1)(2\mu-2\nu+1)\, W_{\nu,\mu+1}(z) = 0$$

Relations of special kind

07.45.17.0009.01

$$W_{\nu,\mu}(z) = \frac{\Gamma(2\mu)\,\Gamma\left(-\mu+\nu+\frac{1}{2}\right) z^{\frac{1}{2}-\mu} (-z)^{-\mu-\frac{1}{2}}}{\pi} \left((-z)^{2\mu}\cos(\pi(\mu-\nu)) - z^{2\mu}\cos(\pi(\mu+\nu))\right) M_{-\nu,-\mu}(-z) +$$
$$\frac{(-z)^{-\mu-\frac{1}{2}}\, z^{\mu+\frac{1}{2}}\, \Gamma\left(-\mu+\nu+\frac{1}{2}\right)}{\Gamma\left(-\mu-\nu+\frac{1}{2}\right)} W_{-\nu,\mu}(-z)$$

Division on even and odd parts and generalization

07.45.17.0010.01

$$W_{\nu,\mu}(z) = A^-[z] + A^+[z] \;/;$$
$$A^+[z] = \frac{1}{2}\left(e^{-z} z^{\mu+\frac{1}{2}} W_{\nu,\mu}(-z)(-z)^{-\mu-\frac{1}{2}} + W_{\nu,\mu}(z)\right) \bigwedge A^-[z] = \frac{1}{2}\left(W_{\nu,\mu}(z) - e^{-z}(-z)^{-\mu-\frac{1}{2}} z^{\mu+\frac{1}{2}} W_{\nu,\mu}(-z)\right)$$

Differentiation

Low-order differentiation

With respect to ν

07.45.20.0001.01

$$W_{\nu,\mu}^{(1,0,0)}(z) = -\frac{\Gamma(-2\mu)\, e^{-\frac{z}{2}}}{\Gamma\left(\frac{1}{2}-\mu-\nu\right)} z^{\mu+\frac{1}{2}} \sum_{k=0}^{\infty} \frac{\left(\mu-\nu+\frac{1}{2}\right)_k \psi\left(k+\mu-\nu+\frac{1}{2}\right) z^k}{(2\mu+1)_k\, k!} - \frac{\Gamma(2\mu)\, e^{-\frac{z}{2}}}{\Gamma\left(\mu-\nu+\frac{1}{2}\right)} z^{\frac{1}{2}-\mu} \sum_{k=0}^{\infty} \frac{\left(-\mu-\nu+\frac{1}{2}\right)_k \psi\left(k-\mu-\nu+\frac{1}{2}\right) z^k}{(1-2\mu)_k\, k!} + \left(\psi\left(-\mu-\nu+\frac{1}{2}\right)+\psi\left(\mu-\nu+\frac{1}{2}\right)\right) W_{\nu,\mu}(z) \;/;\; 2\mu \notin \mathbb{Z}$$

07.45.20.0002.01

$$W_{\nu,\mu}^{(1,0,0)}(z) = \frac{1}{\left(4\mu^2-1\right)\Gamma\left(\frac{1}{2}-\mu-\nu\right)\Gamma\left(\mu-\nu+\frac{1}{2}\right)} \left(e^{-\frac{z}{2}} (2\mu+1)\,\Gamma(2\mu)\,\Gamma\left(-\mu-\nu+\frac{1}{2}\right) z^{\frac{3}{2}-\mu} F_{2\times 0\times 1}^{1\times 1\times 2}\left(\begin{matrix} \frac{3}{2}-\mu-\nu;\ 1;\ 1,\ \frac{1}{2}-\mu-\nu; \\ 2,\ 2-2\mu;;\ \frac{3}{2}-\mu-\nu; \end{matrix}\ z,\ z\right) - e^{-\frac{z}{2}} (2\mu-1)\,\Gamma(-2\mu)\,\Gamma\left(\mu-\nu+\frac{1}{2}\right) z^{\mu+\frac{3}{2}} F_{2\times 0\times 1}^{1\times 1\times 2}\left(\begin{matrix} \frac{3}{2}+\mu-\nu;\ 1;\ 1,\ \frac{1}{2}+\mu-\nu; \\ 2,\ 2+2\mu;;\ \frac{3}{2}+\mu-\nu; \end{matrix}\ z,\ z\right) + \left(4\mu^2-1\right)\Gamma\left(\mu-\nu+\frac{1}{2}\right) \left(\Gamma(-2\mu)\left(\psi\left(-\mu-\nu+\frac{1}{2}\right)-\psi\left(\mu-\nu+\frac{1}{2}\right)\right) M_{\nu,\mu}(z) + \Gamma\left(-\mu-\nu+\frac{1}{2}\right)\psi\left(\mu-\nu+\frac{1}{2}\right) W_{\nu,\mu}(z)\right)\right) \;/;\; 2\mu \notin \mathbb{Z}$$

With respect to μ

07.45.20.0003.01

$$W_{\nu,\mu}^{(0,1,0)}(z) = -\frac{\Gamma(2\mu)\, e^{-\frac{z}{2}}}{\Gamma\left(\mu-\nu+\frac{1}{2}\right)} z^{\frac{1}{2}-\mu} \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}-\mu-\nu\right)_k}{(1-2\mu)_k\, k!} \left(\psi\left(k-\mu-\nu+\frac{1}{2}\right) - 2\psi(k-2\mu+1)\right) z^k + \frac{e^{-\frac{z}{2}}\,\Gamma(-2\mu)}{\Gamma\left(\frac{1}{2}-\mu-\nu\right)} z^{\mu+\frac{1}{2}} \sum_{k=0}^{\infty} \frac{\left(\mu-\nu+\frac{1}{2}\right)_k}{(2\mu+1)_k\, k!} \left(\psi\left(k+\mu-\nu+\frac{1}{2}\right) - 2\psi(k+2\mu+1)\right) z^k + \frac{2\,\Gamma(-2\mu)}{\Gamma\left(\frac{1}{2}-\mu-\nu\right)} \log(z)\, M_{\nu,\mu}(z) - \left(2\pi\cot(2\pi\mu) + \log(z) - \psi\left(\frac{1}{2}-\mu-\nu\right) + \psi\left(\mu-\nu+\frac{1}{2}\right)\right) W_{\nu,\mu}(z) \;/;\; 2\mu \notin \mathbb{Z}$$

07.45.20.0004.01

$$W_{\nu,\mu}^{(0,1,0)}(z) = -\frac{e^{-\frac{z}{2}} z^{\frac{3}{2}-\mu} \Gamma(2\mu)}{(1-2\mu)^2 \Gamma\left(\mu-\nu+\frac{1}{2}\right)} \left((2\mu+2\nu-1) F_{2\times0\times1}^{1\times1\times2}\left(\begin{matrix} \frac{3}{2}-\mu-\nu; 1; 1, 1-2\mu; \\ 2, 2-2\mu;; 2-2\mu; \end{matrix} z, z \right) + (1-2\mu) F_{2\times0\times1}^{1\times1\times2}\left(\begin{matrix} \frac{3}{2}-\mu-\nu; 1; 1, \frac{1}{2}-\mu-\nu; \\ 2, 2-2\mu;; \frac{3}{2}-\mu-\nu; \end{matrix} z, z \right) \right) - \frac{e^{-\frac{z}{2}} z^{\mu+\frac{3}{2}} \Gamma(-2\mu)}{(2\mu+1)^2 \Gamma\left(\frac{1}{2}-\mu-\nu\right)} \left((2\mu-2\nu+1) F_{2\times0\times1}^{1\times1\times2}\left(\begin{matrix} \frac{3}{2}+\mu-\nu; 1; 1, 1+2\mu; \\ 2, 2+2\mu;; 2+2\mu; \end{matrix} z, z \right) - (2\mu+1) F_{2\times0\times1}^{1\times1\times2}\left(\begin{matrix} \frac{3}{2}+\mu-\nu; 1; 1, \frac{1}{2}+\mu-\nu; \\ 2, 2+2\mu;; \frac{3}{2}+\mu-\nu; \end{matrix} z, z \right) \right) + \frac{1}{\Gamma\left(\frac{1}{2}-\mu-\nu\right)} \left(\Gamma(-2\mu) \left(2\log(z) - 2\psi(2\mu+1) - 2\psi(1-2\mu) + \psi\left(\frac{1}{2}-\mu-\nu\right) + \psi\left(\mu-\nu+\frac{1}{2}\right) \right) M_{\nu,\mu}(z) - \Gamma\left(\frac{1}{2}-\mu-\nu\right) \left(2\pi\cot(2\pi\mu) + \log(z) - 2\psi(1-2\mu) + \psi\left(\mu-\nu+\frac{1}{2}\right) \right) W_{\nu,\mu}(z) \right) /; 2\mu \notin \mathbb{Z}$$

With respect to z

07.45.20.0005.01

$$\frac{\partial W_{\nu,\mu}(z)}{\partial z} = \left(\frac{1}{2} - \frac{\nu}{z}\right) W_{\nu,\mu}(z) - \frac{W_{\nu+1,\mu}(z)}{z}$$

07.45.20.0006.01

$$\frac{\partial^2 W_{\nu,\mu}(z)}{\partial z^2} = \frac{z^2 - 4\nu z + 4\mu^2 - 1}{4z^2} W_{\nu,\mu}(z)$$

07.45.20.0007.01

$$\frac{\partial^2 W_{\nu,\mu}(z)}{\partial z^2} = \frac{1}{4z^2} \left(\left(z^2 - 4\nu z + 4\nu(\nu+1)\right) W_{\nu,\mu}(z) + 4(-z+2\nu+2) W_{\nu+1,\mu}(z) + 4 W_{\nu+2,\mu}(z) \right)$$

Symbolic differentiation

With respect to ν

07.45.20.0008.01

$$W_{\nu,\mu}^{(n,0,0)}(z) = e^{-\frac{z}{2}} \Gamma(-2\mu) z^{\mu+\frac{1}{2}} \sum_{k=0}^{\infty} \frac{\partial^n \frac{\left(\mu-\nu+\frac{1}{2}\right)_k}{\Gamma\left(\frac{1}{2}-\mu-\nu\right)}}{\partial \nu^n} \frac{z^k}{(2\mu+1)_k\, k!} + e^{-\frac{z}{2}} \Gamma(2\mu) z^{\frac{1}{2}-\mu} \sum_{k=0}^{\infty} \frac{\partial^n \frac{\left(\frac{1}{2}-\mu-\nu\right)_k}{\Gamma\left(\mu-\nu+\frac{1}{2}\right)}}{\partial \nu^n} \frac{z^k}{(1-2\mu)_k\, k!} /; n \in \mathbb{N} \bigwedge 2\mu \notin \mathbb{Z}$$

With respect to μ

07.45.20.0009.01

$$W_{\nu,\mu}^{(0,n,0)}(z) = \sqrt{z}\, e^{-\frac{z}{2}} \sum_{k=0}^{\infty} \frac{\partial^n \frac{\Gamma(-2\mu) \left(\mu-\nu+\frac{1}{2}\right)_k z^{\mu}}{\Gamma\left(-\mu-\nu+\frac{1}{2}\right) (2\mu+1)_k}}{\partial \mu^n} \frac{z^k}{k!} + \sqrt{z}\, e^{-\frac{z}{2}} \sum_{k=0}^{\infty} \frac{\partial^n \frac{\Gamma(2\mu) \left(-\mu-\nu+\frac{1}{2}\right)_k z^{-\mu}}{\Gamma\left(\mu-\nu+\frac{1}{2}\right) (1-2\mu)_k}}{\partial \mu^n} \frac{z^k}{k!} /; 2\mu \notin \mathbb{Z}$$

With respect to z

07.45.20.0010.01

$$\frac{\partial^n W_{\nu,\mu}(z)}{\partial z^n} = \sum_{k_1=0}^{n}\sum_{k_2=0}^{n}\sum_{k_3=0}^{n}(-1)^{k_1+k_3}\,\delta_{n,k_1+k_2+k_3}\,(k_1+k_2+k_3;\,k_1,\,k_2,\,k_3)\,2^{-k_1}\,z^{-k_2-\frac{k_3}{2}}\left(\mu-k_2+\frac{3}{2}\right)_{k_2}\left(\mu-\nu+\frac{1}{2}\right)_{k_3} W_{\nu-\frac{k_3}{2},\mu+\frac{k_3}{2}}(z)\;/;\;n\in\mathbb{N}$$

07.45.20.0011.01

$$\frac{\partial^n \left(z^\alpha W_{\nu,\mu}(z)\right)}{\partial z^n} = \sum_{k_1=0}^{n}\sum_{k_2=0}^{n}\sum_{k_3=0}^{n}(-1)^{k_1+k_3}\,\delta_{n,k_1+k_2+k_3}\,(k_1+k_2+k_3;\,k_1,\,k_2,\,k_3)\,2^{-k_1}\,z^{\alpha-k_2-\frac{k_3}{2}}\left(\mu-\nu+\frac{1}{2}\right)_{k_3}\left(\alpha+\mu-k_2+\frac{3}{2}\right)_{k_2} W_{\nu-\frac{k_3}{2},\mu+\frac{k_3}{2}}(z)\;/;\;n\in\mathbb{N}$$

Fractional integro-differentiation

With respect to z

07.45.20.0012.01

$$\frac{\partial^\alpha W_{\nu,\mu}(z)}{\partial z^\alpha} = \frac{\Gamma(2\mu)}{\Gamma\left(\mu-\nu+\frac{1}{2}\right)}\sum_{k=0}^{\infty}\frac{\left(-\frac{1}{2}\right)^k\Gamma\left(k-\mu+\frac{3}{2}\right)}{k!\,\Gamma\left(k-\alpha-\mu+\frac{3}{2}\right)}\,{}_2F_1\left(-k,\,-\mu-\nu+\frac{1}{2};\,2\mu+1;\,2\right)z^{k-\alpha-\mu+\frac{1}{2}} + \frac{\Gamma(-2\mu)}{\Gamma\left(\frac{1}{2}-\mu-\nu\right)}\sum_{k=0}^{\infty}\frac{\left(-\frac{1}{2}\right)^k\Gamma\left(k+\mu+\frac{3}{2}\right)}{k!\,\Gamma\left(k-\alpha+\mu+\frac{3}{2}\right)}\,{}_2F_1\left(-k,\,\mu-\nu+\frac{1}{2};\,2\mu+1;\,2\right)z^{k-\alpha+\mu+\frac{1}{2}}\;/;\;2\mu\notin\mathbb{Z}$$

07.45.20.0013.01

$$\frac{\partial^\alpha W_{\nu,\mu}(z)}{\partial z^\alpha} = \frac{\Gamma(2\mu)}{\Gamma\left(\mu-\nu+\frac{1}{2}\right)}\sum_{k=0}^{\infty}\left(\sum_{j=0}^{k}\frac{(-1)^{k-j}\,2^{j-k}\left(-\mu-\nu+\frac{1}{2}\right)_j\Gamma\left(k-\mu+\frac{3}{2}\right)}{(k-j)!\,(1-2\mu)_j\,j!\,\Gamma\left(k-\alpha-\mu+\frac{3}{2}\right)}\right)z^{k-\alpha-\mu+\frac{1}{2}} + \frac{\Gamma(-2\mu)}{\Gamma\left(-\mu-\nu+\frac{1}{2}\right)}\sum_{k=0}^{\infty}\left(\sum_{j=0}^{k}\frac{(-1)^{k-j}\,2^{j-k}\left(\mu-\nu+\frac{1}{2}\right)_j\Gamma\left(k+\mu+\frac{3}{2}\right)}{(k-j)!\,(2\mu+1)_j\,j!\,\Gamma\left(k-\alpha+\mu+\frac{3}{2}\right)}\right)z^{k-\alpha+\mu+\frac{1}{2}}\;/;\;2\mu\notin\mathbb{Z}$$

07.45.20.0014.01

$$\frac{\partial^\alpha W_{\nu,0}(z)}{\partial z^\alpha} = -\frac{z^{\frac{1}{2}-\alpha}}{\Gamma\left(\frac{1}{2}-\nu\right)}\left(\sum_{k=0}^{\infty}\sum_{j=0}^{k}\frac{(-1)^{k-j}\,2^{j-k}\left(\frac{1}{2}-\nu\right)_j}{(j!)^2\,(k-j)!}\mathcal{FC}_{\log}^{(\alpha)}\left(z,\,k+\frac{1}{2}\right)z^k - \sum_{k=0}^{\infty}\sum_{j=0}^{k}\frac{(-1)^{k-j}\,2^{j-k}\left(\frac{1}{2}-\nu\right)_j\Gamma\left(k+\frac{3}{2}\right)}{(j!)^2\,(k-j)!\,\Gamma\left(k-\alpha+\frac{3}{2}\right)}\left(2\psi(j+1)-\psi\left(j-\nu+\frac{1}{2}\right)\right)z^k\right)$$

07.45.20.0015.01

$$\frac{\partial^{\alpha} W_{\nu,\mu}(z)}{\partial z^{\alpha}} = \frac{(-1)^{2\mu+1} z^{\mu+\frac{1}{2}-\alpha}}{\Gamma\left(\frac{1}{2}-\mu-\nu\right)} \left(-z^{-2\mu} \sum_{k=0}^{2\mu-1} \sum_{j=0}^{k} \frac{\left(-\frac{1}{2}\right)^{j} (j-k+2\mu-1)!}{j!\,(k-j)!\,\left(\nu-\mu+\frac{1}{2}\right)_{j-k+2\mu}} \mathcal{FC}_{\exp}^{(\alpha)}\left(z, k-\mu+\frac{1}{2}\right) z^{k} + \right.$$

$$\frac{1}{(2\mu)!} \sum_{k=0}^{\infty} \sum_{j=0}^{k} \frac{\left(-\frac{1}{2}\right)^{k-j} \left(\mu-\nu+\frac{1}{2}\right)_{j}}{j!\,(k-j)!\,(2\mu+1)_{j}} \mathcal{FC}_{\log}^{(\alpha)}\left(z, k+\mu+\frac{1}{2}\right) z^{k} -$$

$$\sum_{k=0}^{\infty} \sum_{j=0}^{k} \frac{\left(-\frac{1}{2}\right)^{k-j} \left(\mu-\nu+\frac{1}{2}\right)_{j} \Gamma\left(k+\mu+\frac{3}{2}\right)}{j!\,(k-j)!\,(j+2\mu)!\,\Gamma\left(k-\alpha+\mu+\frac{3}{2}\right)} \left(\psi(j+1)+\psi(j+2\mu+1)-\psi\left(j+\mu-\nu+\frac{1}{2}\right)\right) z^{k} +$$

$$\left. \sum_{k=0}^{\infty} \sum_{j=0}^{2\mu-1} \frac{(-1)^{j+k}\, 2^{-j-k-1}\, j!\, \Gamma\left(k+\mu+\frac{3}{2}\right) z^{k}}{(j+k+1)!\,(2\mu-j-1)!\,\left(\nu-\mu+\frac{1}{2}\right)_{j+1} \Gamma\left(k-\alpha+\mu+\frac{3}{2}\right)} \right) /;\ 2\mu \in \mathbb{N}$$

07.45.20.0016.01

$$\frac{\partial^{\alpha} W_{\nu,\mu}(z)}{\partial z^{\alpha}} = -\frac{z^{\frac{1}{2}-\alpha-\mu}}{\Gamma\left(\mu-\nu+\frac{1}{2}\right)} \left(\frac{(-1)^{2\mu}}{(-2\mu)!} \sum_{k=0}^{\infty} \sum_{j=0}^{k} \frac{\left(\frac{1}{2}-\mu-\nu\right)_{j} \left(-\frac{1}{2}\right)^{k-j}}{j!\,(k-j)!\,(1-2\mu)_{j}} \mathcal{FC}_{\log}^{(\alpha)}\left(z, k-\mu+\frac{1}{2}\right) z^{k} - \right.$$

$$\frac{z^{2\mu}}{\left(\mu-\nu+\frac{1}{2}\right)_{-2\mu}} \sum_{k=0}^{-2\mu-1} \sum_{j=0}^{k} \frac{(-1)^{k-j} (j-k-2\mu-1)!\, \left(\mu-\nu+\frac{1}{2}\right)_{k-j} \left(-\frac{1}{2}\right)^{j}}{(k-j)!\, j!} \mathcal{FC}_{\exp}^{(\alpha)}\left(z, k+\mu+\frac{1}{2}\right) z^{k} +$$

$$\frac{1}{\left(\mu-\nu+\frac{1}{2}\right)_{-2\mu}} \sum_{k=0}^{\infty} \sum_{j=0}^{-2\mu-1} \frac{(-1)^{j+2\mu} \left(-\frac{1}{2}\right)^{j+k+1} j!\, \left(\mu-\nu+\frac{1}{2}\right)_{-j-2\mu-1} \Gamma\left(k-\mu+\frac{3}{2}\right) z^{k}}{(-j-2\mu-1)!\,(j+k+1)!\,\Gamma\left(k-\alpha-\mu+\frac{3}{2}\right)} -$$

$$\left. (-1)^{2\mu} \sum_{k=0}^{\infty} \sum_{j=0}^{k} \frac{\left(-\frac{1}{2}\right)^{k-j} \left(\frac{1}{2}-\mu-\nu\right)_{j} \Gamma\left(k-\mu+\frac{3}{2}\right)}{j!\,(j-2\mu)!\,(k-j)!\,\Gamma\left(k-\alpha-\mu+\frac{3}{2}\right)} \left(\psi(j+1)+\psi(j-2\mu+1)-\psi\left(j-\mu-\nu+\frac{1}{2}\right)\right) z^{k} \right) /;\ -2\mu \in \mathbb{N}^{+}$$

Integration

Indefinite integration

Involving one direct function and elementary functions

Involving exponential function

07.45.21.0001.01

$$\int e^{-\frac{z}{2}}\, W_{\nu,\mu}(z)\, dz = G_{2,3}^{2,1}\left(z \,\middle|\, \begin{array}{c} 1,\ 2-\nu \\ \frac{3}{2}-\mu,\ \mu+\frac{3}{2},\ 0 \end{array}\right)$$

07.45.21.0002.01

$$\int e^{z/2} W_{\nu,\mu}(z)\, dz = \frac{1}{\Gamma\left(-\mu-\nu+\frac{1}{2}\right)\Gamma\left(\mu-\nu+\frac{1}{2}\right)} G_{2,3}^{2,2}\left(z \,\middle|\, \begin{array}{c} 1,\, \nu+2 \\ \frac{3}{2}-\mu,\, \mu+\frac{3}{2},\, 0 \end{array}\right)$$

Involving exponential function and a power function

07.45.21.0003.01

$$\int z^{\alpha-1} e^{\frac{1}{2}(-c)z} W_{\nu,\mu}(c\,z)\, dz = c\, z^{\alpha+1} G_{2,3}^{2,1}\left(c\,z \,\middle|\, \begin{array}{c} -\alpha,\, -\nu \\ -\mu-\frac{1}{2},\, \mu-\frac{1}{2},\, -\alpha-1 \end{array}\right)$$

07.45.21.0004.01

$$\int z^{\alpha-1} e^{\frac{c\,z}{2}} W_{\nu,\mu}(c\,z)\, dz = \frac{c\, z^{\alpha+1}}{\Gamma\left(-\mu-\nu+\frac{1}{2}\right)\Gamma\left(\mu-\nu+\frac{1}{2}\right)} G_{2,3}^{2,2}\left(c\,z \,\middle|\, \begin{array}{c} -\alpha,\, \nu \\ -\mu-\frac{1}{2},\, \mu-\frac{1}{2},\, -\alpha-1 \end{array}\right)$$

Definite integration

Involving the direct function

07.45.21.0005.01

$$\int_0^\infty t^{\alpha-1} e^{-c\,t} W_{\nu,\mu}(t)\, dt = \frac{\pi \csc(\pi(\alpha+\nu))}{\Gamma(-\alpha-\nu+1)} \left(c+\frac{1}{2}\right)^{-\alpha-\nu} {}_2F_1\left(\mu-\nu+\frac{1}{2}, -\mu-\nu+\frac{1}{2}; -\alpha-\nu+1; c+\frac{1}{2}\right) -$$

$$\frac{\pi^2 \csc(2\pi\mu)\, \Gamma(-\alpha-\nu)\, (\tan(\pi(\alpha-\mu)) - \tan(\pi(\alpha+\mu)))}{\Gamma\left(-\alpha-\mu+\frac{1}{2}\right)\Gamma\left(-\alpha+\mu+\frac{1}{2}\right)\Gamma\left(-\mu-\nu+\frac{1}{2}\right)\Gamma\left(\mu-\nu+\frac{1}{2}\right)} {}_2F_1\left(\alpha+\mu+\frac{1}{2}, \alpha-\mu+\frac{1}{2}; \alpha+\nu+1; c+\frac{1}{2}\right) /;$$

$$\operatorname{Re}(c) > -\frac{1}{2} \bigwedge \operatorname{Re}(\alpha+\mu) > -\frac{1}{2} \bigwedge \operatorname{Re}(\alpha-\mu) > -\frac{1}{2}$$

Representations through more general functions

Through hypergeometric functions

Involving ${}_1F_1$

07.45.26.0001.01

$$W_{\nu,\mu}(z) = e^{-\frac{z}{2}} z^{\mu+\frac{1}{2}} \left(\frac{z^{-2\mu}\,\Gamma(2\mu)}{\Gamma\left(\mu-\nu+\frac{1}{2}\right)} {}_1F_1\left(-\mu-\nu+\frac{1}{2}; 1-2\mu; z\right) + \frac{\Gamma(-2\mu)}{\Gamma\left(-\mu-\nu+\frac{1}{2}\right)} {}_1F_1\left(\mu-\nu+\frac{1}{2}; 2\mu+1; z\right)\right) /; 2\mu \notin \mathbb{Z}$$

Involving ${}_1\tilde{F}_1$

07.45.26.0002.01

$$W_{\nu,\mu}(z) = \pi \csc(2\pi\mu)\, z^{\mu+\frac{1}{2}} e^{-\frac{z}{2}} \left(\frac{z^{-2\mu}}{\Gamma\left(\mu-\nu+\frac{1}{2}\right)} {}_1\tilde{F}_1\left(-\mu-\nu+\frac{1}{2}; 1-2\mu; z\right) - \frac{1}{\Gamma\left(-\mu-\nu+\frac{1}{2}\right)} {}_1\tilde{F}_1\left(\mu-\nu+\frac{1}{2}; 2\mu+1; z\right)\right) /;$$

$$2\mu \notin \mathbb{Z}$$

Involving ${}_pF_q$

07.45.26.0003.01

$$W_{\nu,\mu}(-z)\,W_{\nu,\mu}(z) = \frac{\left(-z^2\right)^{-\mu}}{2\,\mu\left(4\,\mu^2-1\right)\Gamma\left(-\mu-\nu+\frac{1}{2}\right)^2\Gamma\left(\mu-\nu+\frac{1}{2}\right)^2}$$
$$\left(4\,\pi\,z^{\mu+2}\,\mu\,\nu\sec(\pi\,\mu)\,(-z)^{\mu}\,\Gamma\left(-\mu-\nu+\frac{1}{2}\right)\Gamma\left(\mu-\nu+\frac{1}{2}\right){}_2F_3\left(1-\nu,\,\nu+1;\,\frac{3}{2},\,\frac{3}{2}-\mu,\,\mu+\frac{3}{2};\,\frac{z^2}{4}\right)-\right.$$
$$\pi\left(-z^2\right)^{\mu+\frac{1}{2}}\left(4\,\mu^2-1\right)\csc(\pi\,\mu)\,\Gamma\left(-\mu-\nu+\frac{1}{2}\right)\Gamma\left(\mu-\nu+\frac{1}{2}\right){}_2F_3\left(\frac{1}{2}-\nu,\,\nu+\frac{1}{2};\,\frac{1}{2},\,1-\mu,\,\mu+1;\,\frac{z^2}{4}\right)+$$
$$2\sqrt{-z^2}\,\mu\left(4\,\mu^2-1\right)\Gamma(2\,\mu)^2\,\Gamma\left(-\mu-\nu+\frac{1}{2}\right)^2{}_2F_3\left(-\mu-\nu+\frac{1}{2},\,-\mu+\nu+\frac{1}{2};\,1-2\,\mu,\,\frac{1}{2}-\mu,\,1-\mu;\,\frac{z^2}{4}\right)+$$
$$\left.2\left(-z^2\right)^{2\,\mu+\frac{1}{2}}\mu\left(4\,\mu^2-1\right)\Gamma(-2\,\mu)^2\,\Gamma\left(\mu-\nu+\frac{1}{2}\right)^2{}_2F_3\left(\mu-\nu+\frac{1}{2},\,\mu+\nu+\frac{1}{2};\,\mu+\frac{1}{2},\,\mu+1,\,2\,\mu+1;\,\frac{z^2}{4}\right)\right)\,/;\,2\,\mu\notin\mathbb{Z}$$

Involving ${}_pF_q$

07.45.26.0004.01

$$W_{\nu,\mu}(-z)\,W_{\nu,\mu}(z) = \pi^{5/2}\left(-\frac{z^2\,\nu\sec^2(\pi\,\mu)}{4\,\Gamma\left(\mu-\nu+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}-\mu-\nu\right)}\,{}_2\tilde{F}_3\left(1-\nu,\,\nu+1;\,\frac{3}{2},\,\frac{3}{2}-\mu,\,\mu+\frac{3}{2};\,\frac{z^2}{4}\right)-\right.$$
$$\frac{\sqrt{-z^2}\,\csc^2(\pi\,\mu)}{2\,\Gamma\left(\frac{1}{2}-\mu-\nu\right)\Gamma\left(\mu-\nu+\frac{1}{2}\right)}\,{}_2\tilde{F}_3\left(\frac{1}{2}-\nu,\,\nu+\frac{1}{2};\,\frac{1}{2},\,1-\mu,\,\mu+1;\,\frac{z^2}{4}\right)+$$
$$\frac{4^{\mu}\left(-z^2\right)^{\frac{1}{2}-\mu}\csc^2(2\,\pi\,\mu)}{\Gamma\left(\mu-\nu+\frac{1}{2}\right)^2}\,{}_2\tilde{F}_3\left(\frac{1}{2}-\mu-\nu,\,-\mu+\nu+\frac{1}{2};\,1-2\,\mu,\,\frac{1}{2}-\mu,\,1-\mu;\,\frac{z^2}{4}\right)+$$
$$\left.\frac{4^{-\mu}\left(-z^2\right)^{\mu+\frac{1}{2}}\csc^2(2\,\pi\,\mu)}{\Gamma\left(-\mu-\nu+\frac{1}{2}\right)^2}\,{}_2\tilde{F}_3\left(\mu-\nu+\frac{1}{2},\,\mu+\nu+\frac{1}{2};\,\mu+\frac{1}{2},\,\mu+1,\,2\,\mu+1;\,\frac{z^2}{4}\right)\right)\,/;\,2\,\mu\notin\mathbb{Z}$$

Through Meijer *G*

Classical cases involving exp

07.45.26.0005.01

$$e^{z/2}\,W_{\nu,\mu}(z) = \frac{1}{\Gamma\left(\frac{1}{2}-\mu-\nu\right)\Gamma\left(\mu-\nu+\frac{1}{2}\right)}\,G_{1,2}^{2,1}\left(z\,\middle|\,\begin{array}{c}\nu+1\\ \mu+\frac{1}{2},\,\frac{1}{2}-\mu\end{array}\right)$$

07.45.26.0006.01

$$e^{-\frac{z}{2}}\,W_{\nu,\mu}(z) = G_{1,2}^{2,0}\left(z\,\middle|\,\begin{array}{c}1-\nu\\ \mu+\frac{1}{2},\,\frac{1}{2}-\mu\end{array}\right)$$

Classical cases involving cosh

07.45.26.0007.01

$$\cosh\left(\frac{z}{2}\right) W_{\nu,\mu}(z) = \frac{1}{2\,\Gamma\left(-\mu-\nu+\frac{1}{2}\right)\Gamma\left(\mu-\nu+\frac{1}{2}\right)} G_{1,2}^{2,1}\left(z \,\middle|\, \begin{array}{c} \nu+1 \\ \mu+\frac{1}{2}, \frac{1}{2}-\mu \end{array}\right) + \frac{1}{2} G_{1,2}^{2,0}\left(z \,\middle|\, \begin{array}{c} 1-\nu \\ \mu+\frac{1}{2}, \frac{1}{2}-\mu \end{array}\right)$$

Classical cases involving sinh

07.45.26.0008.01

$$\sinh\left(\frac{z}{2}\right) W_{\nu,\mu}(z) = \frac{1}{2\,\Gamma\left(-\mu-\nu+\frac{1}{2}\right)\Gamma\left(\mu-\nu+\frac{1}{2}\right)} G_{1,2}^{2,1}\left(z \,\middle|\, \begin{array}{c} \nu+1 \\ \mu+\frac{1}{2}, \frac{1}{2}-\mu \end{array}\right) - \frac{1}{2} G_{1,2}^{2,0}\left(z \,\middle|\, \begin{array}{c} 1-\nu \\ \mu+\frac{1}{2}, \frac{1}{2}-\mu \end{array}\right)$$

Classical cases for products of Whittaker W

07.45.26.0009.01

$$W_{\nu,\mu}(-z)\, W_{\nu,\mu}(z) = \frac{1}{\sqrt{\pi}\, \Gamma\left(\frac{1}{2}-\mu-\nu\right)\Gamma\left(\mu-\nu+\frac{1}{2}\right)} G_{2,4}^{4,1}\left(-\frac{z^2}{4} \,\middle|\, \begin{array}{c} \nu+1, 1-\nu \\ \frac{1}{2}, 1, \mu+\frac{1}{2}, \frac{1}{2}-\mu \end{array}\right)$$

07.45.26.0010.01

$$W_{\nu,-\mu}(-z)\, W_{\nu,\mu}(z) = \frac{1}{\sqrt{\pi}\, \Gamma\left(\frac{1}{2}-\mu-\nu\right)\Gamma\left(\mu-\nu+\frac{1}{2}\right)} G_{2,4}^{4,1}\left(-\frac{z^2}{4} \,\middle|\, \begin{array}{c} \nu+1, 1-\nu \\ \frac{1}{2}, 1, \mu+\frac{1}{2}, \frac{1}{2}-\mu \end{array}\right)$$

07.45.26.0011.01

$$W_{\nu,\mu}(-z)\, W_{\nu,-\mu}(z) = \frac{1}{\sqrt{\pi}\, \Gamma\left(\frac{1}{2}-\mu-\nu\right)\Gamma\left(\mu-\nu+\frac{1}{2}\right)} G_{2,4}^{4,1}\left(-\frac{z^2}{4} \,\middle|\, \begin{array}{c} \nu+1, 1-\nu \\ \frac{1}{2}, 1, \mu+\frac{1}{2}, \frac{1}{2}-\mu \end{array}\right)$$

07.45.26.0012.01

$$W_{-\nu,\mu}(z)\, W_{\nu,\mu}(z) = \frac{2^{-2\mu-1} z^{2\mu+1}}{\sqrt{\pi}} G_{2,4}^{4,0}\left(\frac{z^2}{4} \,\middle|\, \begin{array}{c} -\mu-\nu+\frac{1}{2}, -\mu+\nu+\frac{1}{2} \\ 0, \frac{1}{2}-\mu, -2\mu, -\mu \end{array}\right) /; -\frac{\pi}{2} < \arg(z) \leq \frac{\pi}{2}$$

07.45.26.0013.01

$$W_{\nu,\mu}\left(-\sqrt{z}\right) W_{\nu,\mu}\left(\sqrt{z}\right) = \frac{1}{\sqrt{\pi}\, \Gamma\left(-\mu-\nu+\frac{1}{2}\right)\Gamma\left(\mu-\nu+\frac{1}{2}\right)} G_{2,4}^{4,1}\left(-\frac{z}{4} \,\middle|\, \begin{array}{c} \nu+1, 1-\nu \\ \frac{1}{2}, 1, \mu+\frac{1}{2}, \frac{1}{2}-\mu \end{array}\right)$$

07.45.26.0014.01

$$W_{\nu,-\mu}\left(-\sqrt{z}\right) W_{\nu,\mu}\left(\sqrt{z}\right) = \frac{1}{\sqrt{\pi}\, \Gamma\left(\frac{1}{2}-\mu-\nu\right)\Gamma\left(\mu-\nu+\frac{1}{2}\right)} G_{2,4}^{4,1}\left(-\frac{z}{4} \,\middle|\, \begin{array}{c} \nu+1, 1-\nu \\ \frac{1}{2}, 1, \mu+\frac{1}{2}, \frac{1}{2}-\mu \end{array}\right)$$

07.45.26.0015.01

$$W_{\nu,\mu}\left(-\sqrt{z}\right) W_{\nu,-\mu}\left(\sqrt{z}\right) = \frac{1}{\sqrt{\pi}\, \Gamma\left(\frac{1}{2}-\mu-\nu\right)\Gamma\left(\mu-\nu+\frac{1}{2}\right)} G_{2,4}^{4,1}\left(-\frac{z}{4} \,\middle|\, \begin{array}{c} \nu+1, 1-\nu \\ \frac{1}{2}, 1, \mu+\frac{1}{2}, \frac{1}{2}-\mu \end{array}\right)$$

07.45.26.0016.01

$$W_{\nu,\mu}\left(-i\sqrt{z}\right) W_{\nu,\mu}\left(i\sqrt{z}\right) = \frac{1}{\sqrt{\pi}\, \Gamma\left(-\mu-\nu+\frac{1}{2}\right)\Gamma\left(\mu-\nu+\frac{1}{2}\right)} G_{2,4}^{4,1}\left(\frac{z}{4} \,\middle|\, \begin{array}{c} \nu+1, 1-\nu \\ \frac{1}{2}, 1, \mu+\frac{1}{2}, \frac{1}{2}-\mu \end{array}\right)$$

Classical cases involving exp and ${}_1F_1$

07.45.26.0017.01

$$e^{z/2}\,{}_1F_1\left(\mu-\nu+\frac{1}{2};\,2\mu+1;\,-z\right)W_{\nu,\mu}(z)=\frac{2^{-2\mu-1}\,\Gamma(2\mu+1)\,z^{\mu+\frac{1}{2}}}{\sqrt{\pi}\,\Gamma\left(\mu-\nu+\frac{1}{2}\right)}\,G_{2,4}^{3,1}\left(\frac{z^2}{4}\,\middle|\,\begin{array}{c}-\mu+\nu+\frac{1}{2},\,-\mu-\nu+\frac{1}{2}\\0,\,\frac{1}{2}-\mu,\,-\mu,\,-2\mu\end{array}\right)/;\,-\frac{\pi}{2}<\arg(z)\leq\frac{\pi}{2}$$

07.45.26.0018.01

$$e^{-\frac{z}{2}}\,{}_1F_1\left(\mu+\nu+\frac{1}{2};\,2\mu+1;\,z\right)W_{\nu,\mu}(z)=\frac{2^{-2\mu-1}\,\Gamma(2\mu+1)\,z^{\mu+\frac{1}{2}}}{\sqrt{\pi}\,\Gamma\left(\mu-\nu+\frac{1}{2}\right)}\,G_{2,4}^{3,1}\left(\frac{z^2}{4}\,\middle|\,\begin{array}{c}-\mu+\nu+\frac{1}{2},\,-\mu-\nu+\frac{1}{2}\\0,\,\frac{1}{2}-\mu,\,-\mu,\,-2\mu\end{array}\right)/;\,-\frac{\pi}{2}<\arg(z)\leq\frac{\pi}{2}$$

07.45.26.0019.01

$$e^{\frac{\sqrt{z}}{2}}\,{}_1F_1\left(\mu-\nu+\frac{1}{2};\,2\mu+1;\,-\sqrt{z}\right)W_{\nu,\mu}\left(\sqrt{z}\right)=\frac{2^{-\mu-\frac{1}{2}}\,\Gamma(2\mu+1)}{\sqrt{\pi}\,\Gamma\left(\mu-\nu+\frac{1}{2}\right)}\,G_{2,4}^{3,1}\left(\frac{z}{4}\,\middle|\,\begin{array}{c}\frac{1}{4}(-2\mu+4\nu+3),\,\frac{1}{4}(-2\mu-4\nu+3)\\\frac{1}{4}(1-2\mu),\,\frac{1}{4}(3-2\mu),\,\frac{1}{4}(2\mu+1),\,\frac{1}{4}(1-6\mu)\end{array}\right)$$

07.45.26.0020.01

$$e^{-\frac{\sqrt{z}}{2}}\,{}_1F_1\left(\mu+\nu+\frac{1}{2};\,2\mu+1;\,\sqrt{z}\right)W_{\nu,\mu}\left(\sqrt{z}\right)=\frac{2^{-\mu-\frac{1}{2}}\,\Gamma(2\mu+1)}{\sqrt{\pi}\,\Gamma\left(\mu-\nu+\frac{1}{2}\right)}\,G_{2,4}^{3,1}\left(\frac{z}{4}\,\middle|\,\begin{array}{c}\frac{1}{4}(-2\mu+4\nu+3),\,\frac{1}{4}(-2\mu-4\nu+3)\\\frac{1}{4}(1-2\mu),\,\frac{1}{4}(3-2\mu),\,\frac{1}{4}(2\mu+1),\,\frac{1}{4}(1-6\mu)\end{array}\right)$$

Classical cases involving exp and ${}_1\tilde{F}_1$

07.45.26.0021.01

$$e^{z/2}\,{}_1\tilde{F}_1\left(\mu-\nu+\frac{1}{2};\,2\mu+1;\,-z\right)W_{\nu,\mu}(z)=\frac{2^{-2\mu-1}\,z^{\mu+\frac{1}{2}}}{\sqrt{\pi}\,\Gamma\left(\mu-\nu+\frac{1}{2}\right)}\,G_{2,4}^{3,1}\left(\frac{z^2}{4}\,\middle|\,\begin{array}{c}-\mu+\nu+\frac{1}{2},\,-\mu-\nu+\frac{1}{2}\\0,\,\frac{1}{2}-\mu,\,-\mu,\,-2\mu\end{array}\right)/;\,-\frac{\pi}{2}<\arg(z)\leq\frac{\pi}{2}$$

07.45.26.0022.01

$$e^{-\frac{z}{2}}\,{}_1\tilde{F}_1\left(\mu+\nu+\frac{1}{2};\,2\mu+1;\,z\right)W_{\nu,\mu}(z)=\frac{2^{-2\mu-1}\,z^{\mu+\frac{1}{2}}}{\sqrt{\pi}\,\Gamma\left(\mu-\nu+\frac{1}{2}\right)}\,G_{2,4}^{3,1}\left(\frac{z^2}{4}\,\middle|\,\begin{array}{c}-\mu+\nu+\frac{1}{2},\,-\mu-\nu+\frac{1}{2}\\0,\,\frac{1}{2}-\mu,\,-\mu,\,-2\mu\end{array}\right)/;\,-\frac{\pi}{2}<\arg(z)\leq\frac{\pi}{2}$$

07.45.26.0023.01

$$e^{\frac{\sqrt{z}}{2}}\,{}_1\tilde{F}_1\left(\mu-\nu+\frac{1}{2};\,2\mu+1;\,-\sqrt{z}\right)W_{\nu,\mu}\left(\sqrt{z}\right)=\frac{2^{-\mu-\frac{1}{2}}}{\sqrt{\pi}\,\Gamma\left(\mu-\nu+\frac{1}{2}\right)}\,G_{2,4}^{3,1}\left(\frac{z}{4}\,\middle|\,\begin{array}{c}\frac{1}{4}(-2\mu+4\nu+3),\,\frac{1}{4}(-2\mu-4\nu+3)\\\frac{1}{4}(1-2\mu),\,\frac{1}{4}(3-2\mu),\,\frac{1}{4}(2\mu+1),\,\frac{1}{4}(1-6\mu)\end{array}\right)$$

07.45.26.0024.01

$$e^{-\frac{\sqrt{z}}{2}}\,{}_1\tilde{F}_1\left(\mu+\nu+\frac{1}{2};\,2\mu+1;\,\sqrt{z}\right)W_{\nu,\mu}\left(\sqrt{z}\right)=\frac{2^{-\mu-\frac{1}{2}}}{\sqrt{\pi}\,\Gamma\left(\mu-\nu+\frac{1}{2}\right)}\,G_{2,4}^{3,1}\left(\frac{z}{4}\,\middle|\,\begin{array}{c}\frac{1}{4}(-2\mu+4\nu+3),\,\frac{1}{4}(-2\mu-4\nu+3)\\\frac{1}{4}(1-2\mu),\,\frac{1}{4}(3-2\mu),\,\frac{1}{4}(2\mu+1),\,\frac{1}{4}(1-6\mu)\end{array}\right)$$

Classical cases involving exp and hypergeometric *U*

07.45.26.0025.01

$$e^{z/2}\,U\left(\mu-\nu+\frac{1}{2},\,2\mu+1,\,-z\right)W_{\nu,\mu}(z)=\frac{2^{-2\mu-1}\,z^{\mu+\frac{1}{2}}}{\sqrt{\pi}\,\Gamma\left(-\mu-\nu+\frac{1}{2}\right)\Gamma\left(\mu-\nu+\frac{1}{2}\right)}\,G_{2,4}^{4,1}\left(-\frac{z^2}{4}\,\middle|\,\begin{array}{c}-\mu+\nu+\frac{1}{2},\,-\mu-\nu+\frac{1}{2}\\0,\,\frac{1}{2}-\mu,\,-2\mu,\,-\mu\end{array}\right)$$

07.45.26.0026.01

$$e^{z/2}\,U\left(\frac{1}{2}-\mu-\nu,\,1-2\mu,\,-z\right)W_{\nu,\mu}(z)=\frac{2^{2\mu-1}\,z^{\frac{1}{2}-\mu}}{\sqrt{\pi}\,\Gamma\left(\frac{1}{2}-\mu-\nu\right)\Gamma\left(\mu-\nu+\frac{1}{2}\right)}\,G_{2,4}^{4,1}\left(-\frac{z^2}{4}\,\middle|\,\begin{array}{c}\mu+\nu+\frac{1}{2},\,\mu-\nu+\frac{1}{2}\\0,\,2\mu,\,\mu+\frac{1}{2},\,\mu\end{array}\right)$$

Classical cases involving exp and Laguerre L

07.45.26.0027.01

$$e^{z/2} L_{\nu-\frac{1}{2}}(-z)\, W_{\nu,0}(z) = \frac{\cos(\pi\nu)\,\Gamma\left(\nu+\frac{1}{2}\right)\sqrt{z}}{2\,\pi^{3/2}}\, G_{2,4}^{3,1}\left(\frac{z^2}{4}\,\middle|\,\begin{matrix}\nu+\frac{1}{2},\ \frac{1}{2}-\nu\\ 0,\ 0,\ \frac{1}{2},\ 0\end{matrix}\right)\ /;\ -\frac{\pi}{2}<\arg(z)\le\frac{\pi}{2}$$

07.45.26.0028.01

$$e^{z/2} L_{-\mu+\nu-\frac{1}{2}}^{2\mu}(-z)\, W_{\nu,\mu}(z) = \frac{2^{-2\mu-1}\cos(\pi(\mu-\nu))\,\Gamma\left(\mu+\nu+\frac{1}{2}\right) z^{\mu+\frac{1}{2}}}{\pi^{3/2}}\, G_{2,4}^{3,1}\left(\frac{z^2}{4}\,\middle|\,\begin{matrix}-\mu+\nu+\frac{1}{2},\ -\mu-\nu+\frac{1}{2}\\ 0,\ \frac{1}{2}-\mu,\ -\mu,\ -2\mu\end{matrix}\right)\ /;\ -\frac{\pi}{2}<\arg(z)\le\frac{\pi}{2}$$

07.45.26.0029.01

$$e^{-\frac{z}{2}} L_{-\nu-\frac{1}{2}}(z)\, W_{\nu,0}(z) = \frac{\sqrt{z}}{2\sqrt{\pi}\,\Gamma\left(\frac{1}{2}-\nu\right)}\, G_{2,4}^{3,1}\left(\frac{z^2}{4}\,\middle|\,\begin{matrix}\nu+\frac{1}{2},\ \frac{1}{2}-\nu\\ 0,\ 0,\ \frac{1}{2},\ 0\end{matrix}\right)\ /;\ -\frac{\pi}{2}<\arg(z)\le\frac{\pi}{2}$$

07.45.26.0030.01

$$e^{-\frac{z}{2}} L_{-\mu-\nu-\frac{1}{2}}^{2\mu}(z)\, W_{\nu,\mu}(z) = \frac{2^{-2\mu-1} z^{\mu+\frac{1}{2}}}{\sqrt{\pi}\,\Gamma\left(\frac{1}{2}-\mu-\nu\right)}\, G_{2,4}^{3,1}\left(\frac{z^2}{4}\,\middle|\,\begin{matrix}-\mu+\nu+\frac{1}{2},\ -\mu-\nu+\frac{1}{2}\\ 0,\ \frac{1}{2}-\mu,\ -\mu,\ -2\mu\end{matrix}\right)\ /;\ -\frac{\pi}{2}<\arg(z)\le\frac{\pi}{2}$$

Classical cases involving Whittaker M

07.45.26.0031.01

$$M_{-\nu,\mu}(z)\, W_{\nu,\mu}(z) = \frac{\Gamma(2\mu+1)}{\sqrt{\pi}\,\Gamma\left(\mu-\nu+\frac{1}{2}\right)}\, G_{2,4}^{3,1}\left(\frac{z^2}{4}\,\middle|\,\begin{matrix}\nu+1,\ 1-\nu\\ \frac{1}{2},\ 1,\ \mu+\frac{1}{2},\ \frac{1}{2}-\mu\end{matrix}\right)\ /;\ -\frac{\pi}{2}<\arg(z)\le\frac{\pi}{2}$$

07.45.26.0032.01

$$M_{\nu,\mu}(-z)\, W_{\nu,\mu}(z) = \frac{2^{-2\mu-1}\,\Gamma(2\mu+1)\left(-z^2\right)^{\mu+\frac{1}{2}}}{\sqrt{\pi}\,\Gamma\left(\mu-\nu+\frac{1}{2}\right)}\, G_{2,4}^{3,1}\left(\frac{z^2}{4}\,\middle|\,\begin{matrix}-\mu+\nu+\frac{1}{2},\ -\mu-\nu+\frac{1}{2}\\ 0,\ \frac{1}{2}-\mu,\ -\mu,\ -2\mu\end{matrix}\right)\ /;\ -\frac{\pi}{2}<\arg(z)\le\frac{\pi}{2}$$

07.45.26.0033.01

$$M_{-\nu,\mu}\left(\sqrt{z}\right) W_{\nu,\mu}\left(\sqrt{z}\right) = \frac{\Gamma(2\mu+1)}{\sqrt{\pi}\,\Gamma\left(\mu-\nu+\frac{1}{2}\right)}\, G_{2,4}^{3,1}\left(\frac{z}{4}\,\middle|\,\begin{matrix}\nu+1,\ 1-\nu\\ \frac{1}{2},\ 1,\ \mu+\frac{1}{2},\ \frac{1}{2}-\mu\end{matrix}\right)$$

07.45.26.0034.01

$$M_{\nu,\mu}\left(-\sqrt{z}\right) W_{\nu,\mu}\left(\sqrt{z}\right) = \frac{2^{-\mu-\frac{1}{2}}\,\Gamma(2\mu+1)\left(-\sqrt{z}\right)^{\mu+\frac{1}{2}}}{\sqrt{\pi}\,\Gamma\left(\mu-\nu+\frac{1}{2}\right)}\, G_{2,4}^{3,1}\left(\frac{z}{4}\,\middle|\,\begin{matrix}-\frac{\mu}{2}+\nu+\frac{3}{4},\ \frac{1}{4}(-2\mu-4\nu+3)\\ \frac{1}{4}(1-2\mu),\ \frac{1}{4}(3-2\mu),\ \frac{1}{4}(2\mu+1),\ \frac{1}{4}(1-6\mu)\end{matrix}\right)$$

Generalized cases for products of Whittaker W

07.45.26.0035.01

$$W_{\nu,\mu}(-z)\, W_{\nu,\mu}(z) = \frac{1}{\sqrt{\pi}\,\Gamma\left(-\mu-\nu+\frac{1}{2}\right)\Gamma\left(\mu-\nu+\frac{1}{2}\right)}\, G_{2,4}^{4,1}\left(\frac{\sqrt{-z^2}}{2},\ \frac{1}{2}\,\middle|\,\begin{matrix}\nu+1,\ 1-\nu\\ \frac{1}{2},\ 1,\ \mu+\frac{1}{2},\ \frac{1}{2}-\mu\end{matrix}\right)$$

07.45.26.0036.01

$$W_{\nu,-\mu}(-z)\, W_{\nu,\mu}(z) = \frac{1}{\sqrt{\pi}\,\Gamma\left(-\mu-\nu+\frac{1}{2}\right)\Gamma\left(\mu-\nu+\frac{1}{2}\right)}\, G_{2,4}^{4,1}\left(\frac{\sqrt{-z^2}}{2},\ \frac{1}{2}\,\middle|\,\begin{matrix}\nu+1,\ 1-\nu\\ \frac{1}{2},\ 1,\ \mu+\frac{1}{2},\ \frac{1}{2}-\mu\end{matrix}\right)$$

07.45.26.0037.01

$$W_{\nu,\mu}(-z)\,W_{\nu,-\mu}(z)=\frac{1}{\sqrt{\pi}\,\Gamma\left(\frac{1}{2}-\mu-\nu\right)\Gamma\left(\mu-\nu+\frac{1}{2}\right)}\,G_{2,4}^{4,1}\left(\frac{\sqrt{-z^2}}{2},\frac{1}{2}\,\middle|\,\begin{array}{c}\nu+1,1-\nu\\ \frac{1}{2},1,\mu+\frac{1}{2},\frac{1}{2}-\mu\end{array}\right)$$

07.45.26.0038.01

$$W_{-\nu,\mu}(z)\,W_{\nu,\mu}(z)=\frac{1}{\sqrt{\pi}}\,G_{2,4}^{4,0}\left(\frac{z}{2},\frac{1}{2}\,\middle|\,\begin{array}{c}1-\nu,\nu+1\\ \frac{1}{2},1,\mu+\frac{1}{2},\frac{1}{2}-\mu\end{array}\right)$$

Generalized cases involving exp and ${}_1F_1$

07.45.26.0039.01

$$e^{z/2}\,{}_1F_1\left(\mu-\nu+\frac{1}{2};2\mu+1;-z\right)W_{\nu,\mu}(z)=\frac{2^{-\mu-\frac{1}{2}}\,\Gamma(2\mu+1)}{\sqrt{\pi}\,\Gamma\left(\mu-\nu+\frac{1}{2}\right)}\,G_{2,4}^{3,1}\left(\frac{z}{2},\frac{1}{2}\,\middle|\,\begin{array}{c}\frac{1}{4}(-2\mu+4\nu+3),\frac{1}{4}(-2\mu-4\nu+3)\\ \frac{1}{4}(1-2\mu),\frac{1}{4}(3-2\mu),\frac{1}{4}(2\mu+1),\frac{1}{4}(1-6\mu)\end{array}\right)$$

07.45.26.0040.01

$$e^{-\frac{z}{2}}\,{}_1F_1\left(\mu+\nu+\frac{1}{2};2\mu+1;z\right)W_{\nu,\mu}(z)==\frac{2^{-\mu-\frac{1}{2}}\,\Gamma(2\mu+1)}{\sqrt{\pi}\,\Gamma\left(\mu-\nu+\frac{1}{2}\right)}\,G_{2,4}^{3,1}\left(\frac{z}{2},\frac{1}{2}\,\middle|\,\begin{array}{c}\frac{1}{4}(-2\mu+4\nu+3),\frac{1}{4}(-2\mu-4\nu+3)\\ \frac{1}{4}(1-2\mu),\frac{1}{4}(3-2\mu),\frac{1}{4}(2\mu+1),\frac{1}{4}(1-6\mu)\end{array}\right)$$

Generalized cases involving exp and ${}_1\tilde{F}_1$

07.45.26.0041.01

$$e^{z/2}\,{}_1\tilde{F}_1\left(\mu-\nu+\frac{1}{2};2\mu+1;-z\right)W_{\nu,\mu}(z)=\frac{2^{-\mu-\frac{1}{2}}}{\sqrt{\pi}\,\Gamma\left(\mu-\nu+\frac{1}{2}\right)}\,G_{2,4}^{3,1}\left(\frac{z}{2},\frac{1}{2}\,\middle|\,\begin{array}{c}\frac{1}{4}(-2\mu+4\nu+3),\frac{1}{4}(-2\mu-4\nu+3)\\ \frac{1}{4}(1-2\mu),\frac{1}{4}(3-2\mu),\frac{1}{4}(2\mu+1),\frac{1}{4}(1-6\mu)\end{array}\right)$$

07.45.26.0042.01

$$e^{-\frac{z}{2}}\,{}_1\tilde{F}_1\left(\mu+\nu+\frac{1}{2};2\mu+1;z\right)W_{\nu,\mu}(z)=\frac{2^{-\mu-\frac{1}{2}}}{\sqrt{\pi}\,\Gamma\left(\mu-\nu+\frac{1}{2}\right)}\,G_{2,4}^{3,1}\left(\frac{z}{2},\frac{1}{2}\,\middle|\,\begin{array}{c}\frac{1}{4}(-2\mu+4\nu+3),\frac{1}{4}(-2\mu-4\nu+3)\\ \frac{1}{4}(1-2\mu),\frac{1}{4}(3-2\mu),\frac{1}{4}(2\mu+1),\frac{1}{4}(1-6\mu)\end{array}\right)$$

Generalized cases involving exp and hypergeometric U

07.45.26.0043.01

$$U(a,b,-z)\,U(a,b,z)=\frac{2^{-b}}{\sqrt{\pi}\,\Gamma(a)\,\Gamma(a-b+1)}\,G_{2,4}^{4,1}\left(\frac{\sqrt{-z^2}}{2},\frac{1}{2}\,\middle|\,\begin{array}{c}1-a,a-b+1\\ 0,\frac{1-b}{2},1-\frac{b}{2},1-b\end{array}\right)$$

07.45.26.0044.01

$$e^{z/2}\,U\left(\frac{1}{2}-\mu-\nu,1-2\mu,-z\right)W_{\nu,\mu}(z)=\frac{2^{2\mu-1}}{\sqrt{\pi}\,z^{\mu-\frac{1}{2}}\,\Gamma\left(\frac{1}{2}-\mu-\nu\right)\Gamma\left(\mu-\nu+\frac{1}{2}\right)}\,G_{2,4}^{4,1}\left(\frac{\sqrt{-z^2}}{2},\frac{1}{2}\,\middle|\,\begin{array}{c}\mu+\nu+\frac{1}{2},\mu-\nu+\frac{1}{2}\\ 0,\mu+\frac{1}{2},2\mu,+\mu\end{array}\right)$$

Generalized cases involving exp and Laguerre L

07.45.26.0045.01

$$e^{z/2}\,L_{\nu-\frac{1}{2}}(-z)\,W_{\nu,0}(z)=\frac{\cos(\pi\nu)\,\Gamma\left(\nu+\frac{1}{2}\right)}{\sqrt{2}\,\pi^{3/2}}\,G_{2,4}^{3,1}\left(\frac{z}{2},\frac{1}{2}\,\middle|\,\begin{array}{c}\nu+\frac{3}{4},\frac{3}{4}-\nu\\ \frac{1}{4},\frac{1}{4},\frac{3}{4},\frac{1}{4}\end{array}\right)$$

07.45.26.0046.01

$$e^{z/2} L_{-\mu+\nu-\frac{1}{2}}^{2\mu}(-z)\, W_{\nu,\mu}(z) = \frac{2^{-\mu-\frac{1}{2}} \cos(\pi(\mu-\nu))\, \Gamma\left(\mu+\nu+\frac{1}{2}\right)}{\pi^{3/2}} G_{2,4}^{3,1}\left(\frac{z}{2}, \frac{1}{2} \middle| \begin{array}{c} \frac{1}{4}(-2\mu+4\nu+3), \frac{1}{4}(-2\mu-4\nu+3) \\ \frac{1}{4}(1-2\mu), \frac{1}{4}(3-2\mu), \frac{1}{4}(2\mu+1), \frac{1}{4}(1-6\mu) \end{array}\right)$$

07.45.26.0047.01

$$e^{-\frac{z}{2}} L_{-\nu-\frac{1}{2}}(z)\, W_{\nu,0}(z) = \frac{1}{\sqrt{2\pi}\, \Gamma\left(\frac{1}{2}-\nu\right)} G_{2,4}^{3,1}\left(\frac{z}{2}, \frac{1}{2} \middle| \begin{array}{c} \nu+\frac{3}{4}, \frac{3}{4}-\nu \\ \frac{1}{4}, \frac{1}{4}, \frac{3}{4}, \frac{1}{4} \end{array}\right)$$

07.45.26.0048.01

$$e^{-\frac{z}{2}} L_{-\mu-\nu-\frac{1}{2}}^{2\mu}(z)\, W_{\nu,\mu}(z) = \frac{2^{-\mu-\frac{1}{2}}}{\sqrt{\pi}\, \Gamma\left(-\mu-\nu+\frac{1}{2}\right)} G_{2,4}^{3,1}\left(\frac{z}{2}, \frac{1}{2} \middle| \begin{array}{c} \frac{1}{4}(-2\mu+4\nu+3), \frac{1}{4}(-2\mu-4\nu+3) \\ \frac{1}{4}(1-2\mu), \frac{1}{4}(3-2\mu), \frac{1}{4}(2\mu+1), \frac{1}{4}(1-6\mu) \end{array}\right)$$

Generalized cases involving Whittaker *M*

07.45.26.0049.01

$$M_{\nu,\mu}(-z)\, W_{\nu,\mu}(z) = \frac{2^{-\mu-\frac{1}{2}} (-z)^{\mu+\frac{1}{2}}\, \Gamma(2\mu+1)}{\sqrt{\pi}\, \Gamma\left(\mu-\nu+\frac{1}{2}\right)} G_{2,4}^{3,1}\left(\frac{z}{2}, \frac{1}{2} \middle| \begin{array}{c} -\frac{\mu}{2}+\nu+\frac{3}{4}, \frac{1}{4}(-2\mu-4\nu+3) \\ \frac{1}{4}(1-2\mu), \frac{1}{4}(3-2\mu), \frac{1}{4}(2\mu+1), \frac{1}{4}(1-6\mu) \end{array}\right)$$

07.45.26.0050.01

$$M_{\nu,-\mu}(-z)\, W_{\nu,\mu}(z) = \frac{2^{\mu-\frac{1}{2}} (-z)^{\frac{1}{2}-\mu}\, \Gamma(1-2\mu)}{\sqrt{\pi}\, \Gamma\left(-\mu-\nu+\frac{1}{2}\right)} G_{2,4}^{3,1}\left(\frac{z}{2}, \frac{1}{2} \middle| \begin{array}{c} \frac{\mu}{2}+\nu+\frac{3}{4}, \frac{1}{4}(2\mu-4\nu+3) \\ \frac{1}{4}(2\mu+1), \frac{1}{4}(2\mu+3), \frac{1}{4}(1-2\mu), \frac{1}{4}(6\mu+1) \end{array}\right)$$

07.45.26.0051.01

$$M_{-\nu,\mu}(z)\, W_{\nu,\mu}(z) = \frac{\Gamma(2\mu+1)}{\sqrt{\pi}\, \Gamma\left(\mu-\nu+\frac{1}{2}\right)} G_{2,4}^{3,1}\left(\frac{z}{2}, \frac{1}{2} \middle| \begin{array}{c} \nu+1, 1-\nu \\ \frac{1}{2}, 1, \mu+\frac{1}{2}, \frac{1}{2}-\mu \end{array}\right)$$

07.45.26.0052.01

$$M_{-\nu,\mu}(z)\, W_{\nu,-\mu}(z) = \frac{\Gamma(2\mu+1)}{\sqrt{\pi}\, \Gamma\left(\mu-\nu+\frac{1}{2}\right)} G_{2,4}^{3,1}\left(\frac{z}{2}, \frac{1}{2} \middle| \begin{array}{c} \nu+1, 1-\nu \\ \frac{1}{2}, 1, \mu+\frac{1}{2}, \frac{1}{2}-\mu \end{array}\right)$$

07.45.26.0053.01

$$M_{-\nu,-\mu}(z)\, W_{\nu,\mu}(z) = \frac{\Gamma(1-2\mu)}{\sqrt{\pi}\, \Gamma\left(-\mu-\nu+\frac{1}{2}\right)} G_{2,4}^{3,1}\left(\frac{z}{2}, \frac{1}{2} \middle| \begin{array}{c} \nu+1, 1-\nu \\ \frac{1}{2}, 1, \frac{1}{2}-\mu, \mu+\frac{1}{2} \end{array}\right)$$

Representations through equivalent functions

With related functions

07.45.27.0001.01

$$W_{\nu,\mu}(z) = e^{-\frac{z}{2}} z^{\mu+\frac{1}{2}} \left(\frac{z^{-2\mu}\, \Gamma(2\mu)}{\Gamma\left(\mu-\nu+\frac{1}{2}\right)} {}_1F_1\left(-\mu-\nu+\frac{1}{2}; 1-2\mu; z\right) + \frac{\Gamma(-2\mu)}{\Gamma\left(-\mu-\nu+\frac{1}{2}\right)} {}_1F_1\left(\mu-\nu+\frac{1}{2}; 2\mu+1; z\right) \right) /; \neg\, 2\mu \in \mathbb{Z}$$

07.45.27.0002.01

$$W_{\nu,\mu}(z) = \pi \csc(2\pi\mu)\, z^{\mu+\frac{1}{2}}\, e^{-\frac{z}{2}} \left(\frac{z^{-2\mu}}{\Gamma\left(\mu-\nu+\frac{1}{2}\right)}\, {}_1\tilde{F}_1\left(-\mu-\nu+\frac{1}{2}; 1-2\mu; z\right) - \frac{1}{\Gamma\left(-\mu-\nu+\frac{1}{2}\right)}\, {}_1\tilde{F}_1\left(\mu-\nu+\frac{1}{2}; 2\mu+1; z\right)\right) /; \neg\, 2\mu \in \mathbb{Z}$$

07.45.27.0003.01

$$W_{\nu,\mu}(z) = z^{\mu+\frac{1}{2}}\, e^{-\frac{z}{2}}\, U\left(\mu-\nu+\frac{1}{2}, 2\mu+1, z\right)$$

07.45.27.0004.01

$$W_{\nu,\mu}(z) = e^{-\frac{z}{2}}\, z^{\frac{1}{2}-\mu} \csc(2\pi\mu) \left(\cos(\pi(\mu-\nu))\, \Gamma\left(\mu+\nu+\frac{1}{2}\right) L_{\mu+\nu-\frac{1}{2}}^{-2\mu}(z) - z^{2\mu} \cos(\pi(\mu+\nu))\, \Gamma\left(-\mu+\nu+\frac{1}{2}\right) L_{-\mu+\nu-\frac{1}{2}}^{2\mu}(z) \right) /; \neg\, 2\mu \in \mathbb{Z}$$

07.45.27.0005.01

$$W_{\nu,\mu}(z) = \frac{\Gamma(2\mu)}{\Gamma\left(\mu-\nu+\frac{1}{2}\right)} M_{\nu,-\mu}(z) + \frac{\Gamma(-2\mu)}{\Gamma\left(-\mu-\nu+\frac{1}{2}\right)} M_{\nu,\mu}(z) /; \neg\, 2\mu \in \mathbb{Z}$$

Theorems

History

– E. T. Whittaker (1904);

– C. S. Meijer (1936)

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