

# SpheroidalS1

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## Notations

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### Traditional name

Radial spheroidal function of the first kind

### Traditional notation

$$S_{\nu,\mu}^{(1)}(\gamma, z)$$

### Mathematica StandardForm notation

SpheroidalS1[ $\nu, \mu, \gamma, z$ ]

## Primary definition

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11.10.02.0001.01

$$S_{\nu,\mu}^{(1)}(\gamma, z)$$

$S_{\nu,\mu}^{(1)}(\gamma, z)$  is the radial spheroidal function of the first kind with variable  $z$  and parameters  $\nu, \mu, \gamma$ . It is defined as the normalizable solution  $w(z) = S_{\nu,\mu}^{(1)}(\gamma, z)$  of the wave differential equation  $(1 - z^2)w''(z) - 2zw'(z) + (\lambda + \gamma^2(1 - z^2) - \mu^2/(1 - z^2))w(z) = 0$  with parameter  $\lambda$  equal to spheroidal eigenvalue  $\lambda = \lambda_{\nu,\mu}(\gamma)$ . The parameter  $\nu$  enumerates the spheroidal eigenvalues in such a manner that in the limit ( $\gamma \rightarrow 0$ ), the eigenvalues are  $\lambda_{\nu,\mu}(0) = \nu(\nu + 1)$  and  $\lim_{z \rightarrow 0} S_{\nu,\mu}^{(1)}(\gamma, z/\gamma) = j_\nu(z)$ , where  $j_\nu(z)$  is the spherical Bessel function of the first kind. The radial spheroidal functions are normalized according to the Meixner-Schäfke normalization scheme, meaning  $\lim_{z \rightarrow \infty} S_{\nu,\mu}^{(1)}(\gamma, z/\gamma) / j_\nu(z) = 1$ .  $S_{\nu,\mu}^{(1)}(\gamma, z)$  is an analytical function in the variables  $\nu, \mu, \gamma$  and  $z$ .

## Specific values

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### General characteristics

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#### Domain and analyticity

 $S_{\nu,\mu}^{(1)}(\gamma, z)$  is an analytical function of  $\nu, \mu, \gamma, z$  which is defined in  $\mathbb{C}^4$ .

11.10.04.0001.01

$$(\nu * \mu * \gamma * z) \rightarrow S_{\nu,\mu}^{(1)}(\gamma, z) :: (\mathbb{C} \otimes \mathbb{C} \otimes \mathbb{C} \otimes \mathbb{C}) \rightarrow \mathbb{C}$$

#### Symmetries and periodicities

**Parity**

$S_{\nu,\mu}^{(1)}(\gamma, z)$  is an even function with respect to  $\mu$ .

11.10.04.0002.01

$$S_{\nu,-\mu}^{(1)}(\gamma, z) = S_{\nu,\mu}^{(1)}(\gamma, z)$$

**Mirror symmetry**

11.10.04.0003.01

$$S_{\nu,\mu}^{(1)}(\bar{\gamma}, \bar{z}) = \overline{S_{\nu,\mu}^{(1)}(\gamma, z)}$$

**Periodicity**

No periodicity

**Series representations**

**Generalized power series**

**Expansions at generic point  $z = z_0$**

11.10.06.0001.01

$$S_{\nu,\mu}^{(1)}(\gamma, z) \propto S_{\nu,\mu}^{(1)}(\gamma, z_0) + S_{\nu,\mu}^{(1)\prime}(\gamma, z_0)(z - z_0) + \frac{1}{2(1 - z_0^2)} \left( 2 S_{\nu,\mu}^{(1)\prime}(\gamma, z_0) z_0 + S_{\nu,\mu}^{(1)}(\gamma, z_0) \left( (z_0^2 - 1) \gamma^2 - \lambda_{\nu,\mu}(\gamma) + \frac{\mu^2}{1 - z_0^2} \right) \right) (z - z_0)^2 -$$

$$\frac{1}{6(z_0^2 - 1)^3} \left( S_{\nu,\mu}^{(1)\prime}(\gamma, z_0) (z_0^2 - 1) (\gamma^2 z_0^4 - 2(\gamma^2 + 3) z_0^2 + \gamma^2 - \mu^2 - \lambda_{\nu,\mu}(\gamma) (z_0^2 - 1) - 2) - \right.$$

$$\left. 2 S_{\nu,\mu}^{(1)}(\gamma, z_0) z_0 (-3 \mu^2 + \gamma^2 (z_0^2 - 1)^2 - 2 \lambda_{\nu,\mu}(\gamma) (z_0^2 - 1)) \right) (z - z_0)^3 +$$

$$\frac{1}{24(z_0^2 - 1)^4} \left( 4 S_{\nu,\mu}^{(1)\prime}(\gamma, z_0) z_0 (z_0^2 - 1) (\gamma^2 z_0^4 - 2(\gamma^2 + 3) z_0^2 + \gamma^2 - 3 \mu^2 - 2 \lambda_{\nu,\mu}(\gamma) (z_0^2 - 1) - 6) + \right.$$

$$S_{\nu,\mu}^{(1)}(\gamma, z_0) \left( \gamma^4 (z_0^2 - 1)^4 + \lambda_{\nu,\mu}(\gamma)^2 (z_0^2 - 1)^2 - 2 \gamma^2 (\mu^2 + 4 z_0^2 + 2) (z_0^2 - 1)^2 - \right.$$

$$\left. \left. 2 \lambda_{\nu,\mu}(\gamma) (\gamma^2 z_0^4 - (2 \gamma^2 + 9) z_0^2 + \gamma^2 - \mu^2 - 3) (z_0^2 - 1) + \mu^2 (\mu^2 + 36 z_0^2 + 8) \right) \right) (z - z_0)^4 + \dots /; (z \rightarrow z_0)$$

11.10.06.0002.01

$$S_{\nu,\mu}^{(1)}(\gamma, z) \propto S_{\nu,\mu}^{(1)}(\gamma, z_0) (1 + O(z - z_0))$$

**Expansions at  $z = 0$**

11.10.06.0003.01

$$S_{\nu,\mu}^{(1)}(\gamma, z) \propto S_{\nu,\mu}^{(1)}(\gamma, 0) + S_{\nu,\mu}^{(1)\prime}(\gamma, 0) z - \frac{1}{2} (\gamma^2 - \mu^2 + \lambda_{\nu,\mu}(\gamma)) S_{\nu,\mu}^{(1)}(\gamma, 0) z^2 +$$

$$\frac{1}{6} \left( 2 S_{\nu,\mu}^{(1)\prime}(\gamma, 0) - (\gamma^2 - \mu^2 + \lambda_{\nu,\mu}(\gamma)) S_{\nu,\mu}^{(1)\prime}(\gamma, 0) \right) z^3 + \dots /; (z \rightarrow 0)$$

11.10.06.0004.01

$$S_{\nu,\mu}^{(1)}(\gamma, z) \propto S_{\nu,\mu}^{(1)}(\gamma, 0) (1 + O(z))$$

## Differential equations

### Ordinary linear differential equations and wronskians

#### For the direct function itself

11.10.13.0001.01

$$(1 - z^2) w''(z) - 2z w'(z) + \left( (1 - z^2) \gamma^2 + \lambda_{\nu, \mu}(\gamma) - \frac{\mu^2}{1 - z^2} \right) w(z) = 0 /; w(z) = c_1 S_{\nu, \mu}^{(1)}(\gamma, z) + c_2 S_{\nu, \mu}^{(2)}(\gamma, z)$$

11.10.13.0002.01

$$W_z(S_{\nu, \mu}^{(1)}(\gamma, z), S_{\nu, \mu}^{(2)}(\gamma, z)) = \frac{1}{1 - z^2} (S_{\nu, \mu}^{(1)}(\gamma, 0) S_{\nu, \mu}^{(2)'}(\gamma, 0) - S_{\nu, \mu}^{(1)'}(\gamma, 0) S_{\nu, \mu}^{(2)}(\gamma, 0))$$

11.10.13.0003.01

$$(1 - g(z)^2) w''(z) + \left( (g(z)^2 - 1) \frac{g''(z)}{g'(z)} - 2g(z)g'(z) \right) w'(z) + \left( \frac{(\gamma g(z)^2 + \mu - \gamma)(-\gamma g(z)^2 + \mu + \gamma)g'(z)^2}{g(z)^2 - 1} + \lambda_{\nu, \mu}(\gamma)g'(z)^2 \right) w(z) = 0 /;$$

$$w(z) = c_1 S_{\nu, \mu}^{(1)}(\gamma, g(z)) + c_2 S_{\nu, \mu}^{(2)}(\gamma, g(z))$$

11.10.13.0004.01

$$W_z(S_{\nu, \mu}^{(1)}(\gamma, g(z)), S_{\nu, \mu}^{(2)}(\gamma, g(z))) = \frac{g'(z)}{1 - g[z]^2} (S_{\nu, \mu}^{(1)}(\gamma, 0) S_{\nu, \mu}^{(2)'}(\gamma, 0) - S_{\nu, \mu}^{(1)'}(\gamma, 0) S_{\nu, \mu}^{(2)}(\gamma, 0))$$

11.10.13.0005.01

$$(1 - g(z)^2) w''(z) + \left( (g(z)^2 - 1) \left( \frac{2h'(z)}{h(z)} + \frac{g''(z)}{g'(z)} \right) - 2g(z)g'(z) \right) w'(z) +$$

$$\left( \frac{(\gamma g(z)^2 + \mu - \gamma)(-\gamma g(z)^2 + \mu + \gamma)g'(z)^2}{g(z)^2 - 1} + \lambda_{\nu, \mu}(\gamma)g'(z)^2 + \frac{2g(z)h'(z)g'(z)}{h(z)} + \frac{(g(z)^2 - 1)(h(z)h''(z) - 2h'(z)^2)}{h(z)^2} - \frac{(g(z)^2 - 1)h'(z)g''(z)}{h(z)g'(z)} \right) w(z) = 0 /; w(z) = c_1 h(z) S_{\nu, \mu}^{(1)}(\gamma, g(z)) + c_2 h(z) S_{\nu, \mu}^{(2)}(\gamma, g(z))$$

11.10.13.0006.01

$$W_z(h(z) S_{\nu, \mu}^{(1)}(\gamma, g(z)), h(z) S_{\nu, \mu}^{(2)}(\gamma, g(z))) = \frac{h(z)^2 g'(z)}{1 - g[z]^2} (S_{\nu, \mu}^{(1)}(\gamma, 0) S_{\nu, \mu}^{(2)'}(\gamma, 0) - S_{\nu, \mu}^{(1)'}(\gamma, 0) S_{\nu, \mu}^{(2)}(\gamma, 0))$$

11.10.13.0007.01

$$(1 - a^2 z^{2r}) w''(z) - \frac{a^2 (r - 2s + 1) z^{2r} + r + 2s - 1}{z} w'(z) +$$

$$\left( a^2 r^2 \lambda_{\nu, \mu}(\gamma) z^{2r-2} + \frac{-a^2 r^2 \left( (a^2 z^{2r} - 1)^2 \gamma^2 - \mu^2 \right) z^{2r} - s^2 (a^2 z^{2r} - 1)^2 + r s (a^4 z^{4r} - 1)}{z^2 (a^2 z^{2r} - 1)} \right) w(z) =$$

$$0 /; w(z) = c_1 z^s S_{\nu, \mu}^{(1)}(\gamma, a z^r) + c_2 z^s S_{\nu, \mu}^{(2)}(\gamma, a z^r)$$

11.10.13.0008.01

$$W_z(z^s S_{\nu, \mu}^{(1)}(\gamma, a z^r), z^s S_{\nu, \mu}^{(2)}(\gamma, a z^r)) = \frac{a r z^{r+2s-1}}{1 - a^2 z^{2r}} (S_{\nu, \mu}^{(1)}(\gamma, 0) S_{\nu, \mu}^{(2)'}(\gamma, 0) - S_{\nu, \mu}^{(1)'}(\gamma, 0) S_{\nu, \mu}^{(2)}(\gamma, 0))$$

11.10.13.0009.01

$$(1 - a^2 r^{2z}) w''(z) + (-a^2 \log(r) r^{2z} + 2 a^2 \log(s) r^{2z} - \log(r) - 2 \log(s)) w'(z) + \left( a^2 \log^2(r) \lambda_{v,\mu}(\gamma) r^{2z} + \frac{1}{a^2 r^{2z} - 1} \left( -a^2 \left( (a^2 r^{2z} - 1)^2 \gamma^2 - \mu^2 \right) \log^2(r) r^{2z} - (a^2 r^{2z} - 1)^2 \log^2(s) + (a^4 r^{4z} - 1) \log(r) \log(s) \right) \right) w(z) = 0 /; w(z) = c_1 s^z S_{v,\mu}^{(1)}(\gamma, a r^z) + c_2 s^z S_{v,\mu}^{(2)}(\gamma, a r^z)$$

11.10.13.0010.01

$$W_z(s^z S_{v,\mu}^{(1)}(\gamma, a r^z), s^z S_{v,\mu}^{(2)}(\gamma, a r^z)) = \frac{a r^z s^{2z} \log(r)}{1 - a^2 r^{2z}} (S_{v,\mu}^{(1)}(\gamma, 0) S_{v,\mu}^{(2)'}(\gamma, 0) - S_{v,\mu}^{(1)'}(\gamma, 0) S_{v,\mu}^{(2)}(\gamma, 0))$$

## Differentiation

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### Low-order differentiation

With respect to  $z$

11.10.20.0001.01

$$\text{SpheroidalSI}^{(0,0,0,1)}(v, \mu, \gamma, z) = S_{v,\mu}^{(1)'}(\gamma, z)$$

11.10.20.0002.01

$$\text{SpheroidalSI}^{(0,0,0,2)}(v, \mu, \gamma, z) = \frac{1}{1 - z^2} \left( 2 z S_{v,\mu}^{(1)'}(\gamma, z) - \left( (1 - z^2) \gamma^2 + \lambda_{v,\mu}(\gamma) - \frac{\mu^2}{1 - z^2} \right) S_{v,\mu}^{(1)}(\gamma, z) \right)$$

## Integration

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### Operations

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#### Limit operation

11.10.25.0001.01

$$\lim_{z \rightarrow \infty} \frac{S_{v,\mu}^{(1)}\left(\gamma, \frac{z}{\gamma}\right)}{j_v(z)} = 1$$

## Representations through equivalent functions

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### Theorems

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### History

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