Root

Notations

Traditional name

The kth root of the polynomial equation $\sum_{j=0}^{n} a_j z^j == 0$

Traditional notation

$$\left(z;\ \sum_{j=0}^{n} a_j z^j\right)_k^{-1}$$

Mathematica StandardForm notation

$$\texttt{Root}\left[\texttt{Function}\left[z,\ \sum_{j=0}^{n} a_j z^j\right],\ k\right]$$

Primary definition

$\left(z;\ \sum_{j=0}^{n} a_j z^j\right)_k^{-1}$ is the kth root of the polynomial equation $\sum_{j=0}^{n} a_j z^j == 0$.

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$$\left(\sum_{j=0}^{n} a_j z^j \text{ /; } z == \left(z;\ \sum_{j=0}^{n} a_j z^j\right)_k^{-1}\right) == 0 \text{ /; } k \in \mathbb{Z} \bigwedge 1 \leq k \leq n$$

Specific values

Specialized values

01.33.03.0001.01

$$(z;\ a_0 + a_1 z)_1^{-1} == -\frac{a_0}{a_1}$$

01.33.03.0002.01

$$\left(z;\ a_0 + a_1 z + a_2 z^2\right)_k^{-1} == -\frac{a_1}{2 a_2} + \frac{(-1)^k}{2} \sqrt{\frac{a_1^2 - 4 a_2 a_0}{a_2^2}} \text{ /; } 1 \leq k \leq 2$$

01.33.03.0003.01

$$\left(z;\ z^2 - a\right)_2^{-1} == \sqrt{a}$$

01.33.03.0004.01

$$\left(z;\, a_0 + a_1 z + a_2 z^2 + a_3 z^3\right)_k^{-1} = -\frac{a_2}{3 a_3} + \frac{e^{\frac{2 i \pi (k-1)}{3}} p}{3\, 2^{1/3} a_3} - \frac{2^{1/3} e^{-\frac{2 i \pi (k-1)}{3}} r}{3 p a_3} /;$$

$$p = \sqrt[3]{q + \sqrt{4 r^3 + q^2}} \bigwedge q = 9 a_1 a_3 a_2 - 2 a_2^3 - 27 a_0 a_3^2 \bigwedge r = 3 a_1 a_3 - a_2^2 \bigwedge 1 \leq k \leq 3$$

01.33.03.0005.01

$$\left(z;\, a_0 + a_1 z + a_2 z^2 + a_3 z^3 + a_4 z^4\right)_k^{-1} = -\frac{a_3}{4 a_4} + \left(2 \left\lfloor \frac{k-1}{2} \right\rfloor - 1\right) \epsilon_1 + (-1)^k \epsilon_2 (1 - \theta(k-3)) - (-1)^k (\theta(2-k) - 1) \epsilon_3 /;$$

$$\epsilon_1 = \frac{1}{2} \sqrt{\frac{\sqrt[3]{2} \left(a_2^2 - 3 a_1 a_3 + 12 a_0 a_4\right)}{3 s a_4} + \frac{3 a_3^2 + 2\, 2^{2/3} s a_4 - 8 a_2 a_4}{12 a_4^2}} \bigwedge$$

$$\epsilon_2 = \frac{\sqrt{u+v}}{2} \bigwedge \epsilon_3 = \frac{\sqrt{u-v}}{2} \bigwedge u = 8 \epsilon_1^2 - \frac{\sqrt[3]{2} s^2 + 2 a_2^2 - 6 a_1 a_3 + 24 a_0 a_4}{2^{2/3} s a_4} \bigwedge$$

$$v = \frac{a_3^3 - 4 a_2 a_4 a_3 + 8 a_1 a_4^2}{8 a_4^3 \epsilon_1} \bigwedge s = \sqrt[3]{2 a_2^3 - 9 (a_1 a_3 + 8 a_0 a_4) a_2 + t + 27 \left(a_4 a_1^2 + a_0 a_3^2\right)} \bigwedge$$

$$t = \sqrt{\left(\left(2 a_2^3 - 9 (a_1 a_3 + 8 a_0 a_4) a_2 + 27 \left(a_4 a_1^2 + a_0 a_3^2\right)\right)^2 - 4 \left(a_2^2 - 3 a_1 a_3 + 12 a_0 a_4\right)^3\right)} \bigwedge 1 \leq k \leq 4$$

01.33.03.0006.01

$$(z;\, z^n - a)_n^{-1} = a^{1/n} /; n = 2 \bigvee n - 3 \in \mathbb{N} \bigwedge a \notin (0, \infty)$$

Values at fixed points

01.33.03.0007.01

$$\left(z;\, 1 + z^2\right)_2^{-1} = i$$

01.33.03.0008.01

$$\left(z;\, -1 - z + z^2\right)_2^{-1} = \varphi$$

General characteristics

Domain and analyticity

$\left(z;\, \sum_{j=0}^{n} a_j z^j\right)_k^{-1}$ is an analytic function of the a_j $(0 \leq j \leq n)$ in $\mathbb{C}^{n+1}$.

01.33.04.0001.01

$$(a_0 * a_1 * \ldots * a_n) \longrightarrow \left(z;\, \sum_{j=0}^{n} a_j z^j\right)_k^{-1} :: \mathbb{C}^{n+1} \longrightarrow \mathbb{C}$$

Symmetries and periodicities

Symmetry

No symmetry

Periodicity

No periodicity

Poles and essential singularities

In most cases there are no poles, but poles up to order $n-1$can be present.

Branch points

For generic values a_j, it has $n-1$ branchpoints of order 2. At most $\left(z; \sum_{j=0}^{n} a_j z^j\right)_k^{-1}$ can have one branch point of order n.

Branch cuts

The location of the branch cuts is complicated. Generically the branch cuts run asymptotically radially outwards and do not connect branch points.

Limit representations

01.33.09.0001.01

$$\left(z; \sum_{j=0}^{n} a_j z^j\right)_k^{-1} = \lim_{m\to\infty} w_k^m \;/;\; w_k^{m+1} = w_k^m - \sum_{j=0}^{n} \frac{a_j \left(w_k^m\right)^j}{\prod_{i=1}^{m} \prod_{j=1}^{m} \text{If}\left[i \neq j,\, w_i^m - w_j^m,\, 1\right]} \bigwedge w_k^0 \in \mathbb{C} \bigwedge w_k^0 \neq w_l^0$$

Differentiation

Low-order differentiation

With respect to a_k

01.33.20.0001.01

$$\frac{\partial}{\partial a_k} \left(z; \sum_{j=0}^{n} a_j z^j\right)_k^{-1} = -\frac{R^k}{k\, a_k R^{k-1} + \sum_{j=1}^{n} \left(1 - \delta_{k,j}\right) j\, a_j R^{j-1}} \;/;\; R = \left(z; \sum_{j=0}^{n} a_j z^j\right)_k^{-1}$$

Representations through equivalent functions

With related functions

01.33.27.0001.01

$$\left(x; x^2 - z\right)_2^{-1} = \sqrt{z}$$

01.33.27.0002.01

$$(z; z^n - a)_n^{-1} = a^{1/n} \;/;\; n = 2 \bigvee n - 3 \in \mathbb{N} \bigwedge a \notin (0, \infty)$$

01.33.27.0003.01

$$\sum_{j=0}^{n} a_j z^j == \prod_{k=1}^{n} (z - \alpha_k) /;$$

$$a_0 == (-1)^n \prod_{k=1}^{n} \alpha_k \bigwedge a_1 == (-1)^{n-1} \sum_{k=1}^{n} \prod_{j=1}^{n} \mathrm{If}[j \neq k, \alpha_j, 1] \bigwedge \ldots \bigwedge a_{n-2} == \sum_{k=1}^{n} \sum_{j=1}^{k-1} \mathrm{If}[j \neq k, \alpha_j \alpha_k, 0] \bigwedge a_{n-1} == -\sum_{k=1}^{n} \alpha_k$$

Inequalities

01.33.29.0001.01

$$\left| \left(z; \sum_{j=0}^{n} a_j z^j\right)_k^{-1} \right| \leq 1 + \max\left(\left|\frac{a_0}{a_n}\right|, \left|\frac{a_1}{a_n}\right|, \ldots, \left|\frac{a_{n-1}}{a_n}\right| \right)$$

Theorems

Salem number

The smallest known Salem number (a real algebraic integer greater than 1 whose conjugates have absolute value $\leq$ 1, at least one conjugate having absolute value ==1) is given by $\left(z; 1 + z - z^3 - z^4 - z^5 - z^6 - z^7 + z^9 + z^{10}\right)_2^{-1} \propto 1.17628\ldots$.

Pisot numbers

Raising simple algebraic numbers to a high power can yield numbers very near to integers. Example: $\left(\left(z; -1 - z + z^3\right)_1^{-1}\right)^{10\,000}$ is within 10^{-611} of an integer.

Unequality for rational numbers

For any algebraic α of degree greater than 1 there exists a $c(\alpha)$ such that $\left| \frac{p}{q} - \alpha \right| > \frac{c(\alpha)}{q^2}$ for all rational numbers $\frac{p}{q}$.

The Lagrange points of the restricted three-body problem

The Lagrange points L_1, L_2, and $L_3$of the restricted three-body problem with potential $V(x, y) == -\frac{1}{2}\left(x^2 + y^2\right) - \frac{1-\mu}{\sqrt{y^2 + (x - x_1)^2}} - \frac{\mu}{\sqrt{y^2 + (x - x_2)^2}}$ are given by $\{\tilde{x}_i, 0\}$, where the $\tilde{x}_i$ are the real solutions of the quintic polynomial equation

$$x^5 - 2\,(x_1 + x_2)\,x^4 + \left(x_1^2 + 4\,x_2\,x_1 + x_2^2\right) x^3 - (2\,x_1\,x_2\,(x_1 + x_2) + 1)\,x^2 + \left(x_1^2\,x_2^2 + 2\,\mu\,x_1 - 2\,(\mu - 1)\,x_2\right) x + (\mu - 1)\,x_2^2 - \mu\,x_1^2 == 0.$$

Gauss-Lucas theorem

$$\max\left(\left|\left(z;\sum_{j=0}^{n}a_j\,z^j\right)_1^{-1}\right|,\ \left|\left(z;\sum_{j=0}^{n}a_j\,z^j\right)_2^{-1}\right|,\ \ldots,\ \left|\left(z;\sum_{j=0}^{n}a_j\,z^j\right)_n^{-1}\right|\right)\geq$$

$$\max\left(\left|\left(z;\frac{\partial}{\partial z}\sum_{j=0}^{n}a_j\,z^j\right)_1^{-1}\right|,\ \left|\left(z;\frac{\partial}{\partial z}\sum_{j=0}^{n}a_j\,z^j\right)_2^{-1}\right|,\ \ldots,\ \left|\left(z;\frac{\partial}{\partial z}\sum_{j=0}^{n-1}a_j\,z^j\right)_{n-1}^{-1}\right|\right).$$

The hard hexagon entropy constant

The hard hexagon entropy constant is an algebraic number of degree 24.

History

– J. L. Lagrange (1767-1772)
– P. Ruffini (1799)
– C. F. Gauss (1801)
– N. H. Abel (1826)
– J. C. F. Sturm (1829)
– E. Galois (1832)
– G. Eisenstein (1844)
– J. Cockle, J. K. Thomae (1869)
– C. Jordan (1870)
– F. von Lindermann (1884,1892)
– L. Kronecker (1890,1891)
– R. H. Mellin (1915)
– R. Birkeland (1905-1925)

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