RiemannSiegelTheta

Notations

Traditional name

Riemann–Siegel function ϑ

Traditional notation

$\vartheta(z)$

Mathematica StandardForm notation

`RiemannSiegelTheta[z]`

Primary definition

10.03.02.0001.01

$$\vartheta(z) = -\frac{\log(\pi)}{2} z - \frac{i}{2}\left(\log\Gamma\left(\frac{1}{4}+\frac{i z}{2}\right) - \log\Gamma\left(\frac{1}{4}-\frac{i z}{2}\right)\right)$$

Specific values

Specialized values

10.03.03.0001.01

$$\vartheta(x) = \operatorname{Im}\left(\log\Gamma\left(\frac{i x}{2}+\frac{1}{4}\right)\right) - \frac{1}{2}\log(\pi)\, x \;/;\; x \in \mathbb{R}$$

10.03.03.0002.01

$$\vartheta\left(\frac{i}{2} + 2 i n\right) = -i \infty \;/;\; n \in \mathbb{N}$$

10.03.03.0003.01

$$\vartheta\left(-\frac{i}{2} - 2 i n\right) = i \infty \;/;\; n \in \mathbb{N}$$

10.03.03.0004.01

$$\vartheta\left(\frac{3 i}{2} + 2 i n\right) = \frac{1}{2}\left(-(n+1)\left(i \log\left(2\pi^2\right)+\pi\right) + i \log(n!) + i \sum_{k=1}^{n+1} \log(2k-1)\right) \;/;\; n \in \mathbb{N}$$

10.03.03.0005.01

$$\vartheta\left(-\frac{3 i}{2} - 2 i n\right) = \frac{1}{2}\left((n+1)\left(i \log\left(2\pi^2\right)+\pi\right) - i \log(n!) - i \sum_{k=1}^{n+1} \log(2k-1)\right) \;/;\; n \in \mathbb{N}$$

10.03.03.0016.01

$$\vartheta\left(\frac{i\,(q-4\,p)}{2\,q}-2\,i\,n\right)=\frac{i}{2\,q}\left(-q\log(2)+2\,p\log(2\,\pi)+n\,q\,(-i\,\pi+\log(2)+2\log(\pi\,q))-q\log\left(\cos\left(\frac{p\,\pi}{q}\right)\Gamma\left(\frac{2\,p}{q}\right)\right)-q\sum_{k=1}^{n}\log((2\,p+2\,k\,q-q)\,(p+k\,q-q))\right)/;\,n\in\mathbb{N}\land p\in\mathbb{N}^+\land q\in\mathbb{N}^+\land p<q$$

10.03.03.0017.01

$$\vartheta\left(2\,i\,n+\frac{i\,(q-4\,p)}{2\,q}\right)=\frac{i}{2\,q}\left(-q\log(2)+2\,p\log(2\,\pi)+n\,q\,(i\,\pi-\log(2)-2\log(\pi\,q))-q\log\left(\cos\left(\frac{p\,\pi}{q}\right)\Gamma\left(\frac{2\,p}{q}\right)\right)+q\sum_{k=1}^{n}\log(q\,k-p)+q\sum_{k=1}^{n}\log(-2\,p+2\,k\,q-q)\right)/;\,n\in\mathbb{N}\land p\in\mathbb{N}^+\land q\in\mathbb{N}^+\land p<q$$

10.03.03.0018.01

$$\vartheta\left(\frac{i\,(4\,p-q)}{2\,q}-2\,i\,n\right)=\frac{1}{2\,q}\left(i\left(q\log(2)-2\,p\log(2\,\pi)+n\,q\,(-i\,\pi+\log(2)+2\log(\pi\,q))+q\log\left(\cos\left(\frac{p\,\pi}{q}\right)\Gamma\left(\frac{2\,p}{q}\right)\right)-q\sum_{k=1}^{n}\log(q\,k-p)-q\sum_{k=1}^{n}\log(-2\,p+2\,k\,q-q)\right)\right)/;\,n\in\mathbb{N}\land p\in\mathbb{N}^+\land q\in\mathbb{N}^+\land p<q$$

10.03.03.0019.01

$$\vartheta\left(2\,i\,n+\frac{i\,(4\,p-q)}{2\,q}\right)=\frac{i}{2\,q}\left(q\log(2)-2\,p\log(2\,\pi)+n\,q\,(i\,\pi-\log(2)-2\log(\pi\,q))+q\log\left(\cos\left(\frac{p\,\pi}{q}\right)\Gamma\left(\frac{2\,p}{q}\right)\right)+q\sum_{k=1}^{n}\log((2\,p+2\,k\,q-q)\,(p+k\,q-q))\right)/;\,n\in\mathbb{N}\land p\in\mathbb{N}^+\land q\in\mathbb{N}^+\land p<q$$

Values at fixed points

10.03.03.0006.01

$$\vartheta(0)==0$$

10.03.03.0007.01

$$\vartheta\left(\frac{i}{2}\right)==-i\,\infty$$

10.03.03.0008.01

$$\vartheta\left(\frac{3\,i}{2}\right)==-\frac{1}{2}\left(i\log\left(2\,\pi^2\right)+\pi\right)$$

10.03.03.0009.01

$$\vartheta\left(-\frac{i}{2}\right)==i\,\infty$$

10.03.03.0010.01

$$\vartheta\left(-\frac{3\,i}{2}\right)==\frac{1}{2}\left(i\log\left(2\,\pi^2\right)+\pi\right)$$

Values at infinities

10.03.03.0011.01

$\vartheta(\infty) == \infty$

10.03.03.0012.01

$\vartheta(-\infty) == -\infty$

10.03.03.0013.01

$\vartheta(i\,\infty) == \tilde{\infty}$

10.03.03.0014.01

$\vartheta(-i\,\infty) == \tilde{\infty}$

10.03.03.0015.01

$\vartheta(\tilde{\infty}) == \tilde{\infty}$

General characteristics

Domain and analyticity

$\vartheta(z)$ is an analytical function of z which is defined over the whole complex z-plane.

10.03.04.0001.01

$z \longrightarrow \vartheta(z) :: \mathbb{C} \longrightarrow \mathbb{C}$

Symmetries and periodicities

Parity

$\vartheta(z)$ is an odd function.

10.03.04.0002.01

$\vartheta(-z) == -\vartheta(z)$

Mirror symmetry

10.03.04.0003.01

$$\vartheta(\bar{z}) == \overline{\vartheta(z)} \;/;\; i\,z \notin \left(-\infty, -\frac{1}{2}\right) \bigwedge i\,z \notin \left(\frac{1}{2}, \infty\right)$$

Periodicity

No periodicity

Poles and essential singularities

The function $\vartheta(z)$ does not have poles and essential singularities.

10.03.04.0004.01

$\mathcal{S}ing_z(\vartheta(z)) == \{\}$

Branch points

The function $\vartheta(z)$ has infinitely many branch points: $z == \pm i\left(\frac{1}{2} + 2\,k\right) /; k \in \mathbb{N}$ and $z == \tilde{\infty}$. All these are logarithmic branch points.

10.03.04.0005.01

$$\mathcal{BP}_z(\vartheta(z)) == \left\{\left\{\frac{i}{2}+2\,k\,i\;/;\;k\in\mathbb{N}\right\},\;\left\{-\frac{i}{2}-2\,k\,i\;/;\;k\in\mathbb{N}\right\},\;\tilde{\infty}\right\}$$

10.03.04.0006.01

$$\mathcal{R}_z\left(\vartheta(z),\;\frac{i}{2}+2\,k\,i\right) == \log\;/;\;k\in\mathbb{N}$$

10.03.04.0007.01

$$\mathcal{R}_z\left(\vartheta(z),\;-\frac{i}{2}-2\,k\,i\right) == \log\;/;\;k\in\mathbb{N}$$

10.03.04.0008.01

$$\mathcal{R}_z(\vartheta(z),\;\tilde{\infty}) == \log$$

Branch cuts

The function $\vartheta(z)$ is a single-valued function on the z-plane cut along the intervals $\left\{-i\,\infty,\,-\frac{i}{2}\right\}$ and $\left\{\frac{i}{2},\,i\,\infty\right\}$. At $i\,z\in\left\{-i\,\infty,\,-\frac{i}{2}\right\}\bigvee i\,z\in\left\{\frac{i}{2},\,i\,\infty\right\}$ potentially multiple branch cuts are situated over each other (at $i\,z$ there are $\left\lfloor\frac{i\,z}{2}+\frac{1}{4}\right\rfloor$, respectively $\left\lfloor\frac{1}{4}-\frac{i\,z}{2}\right\rfloor$ branch cuts overlapping).

The function $\vartheta(z)$ is continuous from the left on the interval $\left\{-i\,\infty,\,-\frac{i}{2}\right\}$ and from the right on the interval $\left\{\frac{i}{2},\,i\,\infty\right\}$.

10.03.04.0009.01

$$\mathcal{BC}_z(\vartheta(z)) == \left\{\left\{\left\{-i\,\infty,\,-\frac{i}{2}\right\},\,1\right\},\;\left\{\left\{\frac{i}{2},\,i\,\infty\right\},\,-1\right\}\right\}$$

10.03.04.0010.01

$$\lim_{\epsilon\to+0}\vartheta(x-\epsilon) == \vartheta(x)\;/;\;i\,x>\frac{1}{2}$$

10.03.04.0011.01

$$\lim_{\epsilon\to+0}\vartheta(x+\epsilon) == \pi\left\lfloor\frac{1}{4}-\frac{i\,x}{2}\right\rfloor+\vartheta(x)\;/;\;i\,x>\frac{1}{2}$$

10.03.04.0012.01

$$\lim_{\epsilon\to+0}\vartheta(x-\epsilon) == \vartheta(x)\;/;\;i\,x<-\frac{1}{2}$$

10.03.04.0013.01

$$\lim_{\epsilon\to+0}\vartheta(x-\epsilon) == \vartheta(x)-\pi\left\lfloor\frac{i\,x}{2}+\frac{1}{4}\right\rfloor\;/;\;i\,x<-\frac{1}{2}$$

Series representations

Generalized power series

Expansions on branch cuts

In the lower half-plane

10.03.06.0021.01

$$\vartheta(z) \propto \vartheta(z_0) - \pi \left\lfloor \frac{\arg(-i(z-z_0))}{2\pi} \right\rfloor \left\lfloor \frac{1-2 i z_0}{4} \right\rfloor - \frac{1}{4}\left(2\log(\pi) - \psi\left(\frac{i z_0}{2} + \frac{1}{4}\right) - \psi\left(\frac{1}{4} - \frac{i z_0}{2}\right)\right)(z-z_0) + \frac{i}{16}\left(\psi^{(1)}\left(\frac{i z_0}{2} + \frac{1}{4}\right) - \psi^{(1)}\left(\frac{1}{4} - \frac{i z_0}{2}\right)\right)(z-z_0)^2 + \ldots /; (z \to z_0) \bigwedge i z_0 \in \mathbb{R} \bigwedge i z_0 > \frac{1}{2} \bigwedge \frac{i z_0}{2} \notin \mathbb{Z}$$

10.03.06.0022.01

$$\vartheta(z) \propto \vartheta(z_0) - \pi \left\lfloor \frac{\arg(-i(z-z_0))}{2\pi} \right\rfloor \left\lfloor \frac{1-2 i z_0}{4} \right\rfloor - \frac{1}{4}\left(2\log(\pi) - \psi\left(\frac{i z_0}{2} + \frac{1}{4}\right) - \psi\left(\frac{1}{4} - \frac{i z_0}{2}\right)\right)(z-z_0) + \frac{i}{16}\left(\psi^{(1)}\left(\frac{i z_0}{2} + \frac{1}{4}\right) - \psi^{(1)}\left(\frac{1}{4} - \frac{i z_0}{2}\right)\right)(z-z_0)^2 + O\left((z-z_0)^3\right) /; i z_0 \in \mathbb{R} \bigwedge i z_0 > \frac{1}{2} \bigwedge \frac{i z_0}{2} \notin \mathbb{Z}$$

10.03.06.0023.01

$$\vartheta(z) = \vartheta(z_0) - \pi \left\lfloor \frac{1-2 i z_0}{4} \right\rfloor \left\lfloor \frac{\arg(-i(z-z_0))}{2\pi} \right\rfloor - \frac{i}{2}\sum_{k=1}^{\infty} \frac{(-i)^k 2^{-k}}{k!}\left(2\delta_{k-1}\log(\pi) - \psi^{(k-1)}\left(\frac{1}{4} - \frac{i z_0}{2}\right) + (-1)^k \psi^{(k-1)}\left(\frac{1}{4} + \frac{i z_0}{2}\right)\right)(z-z_0)^k /; i z_0 \in \mathbb{R} \bigwedge i z_0 > \frac{1}{2} \bigwedge \frac{i z_0}{2} \notin \mathbb{Z}$$

10.03.06.0024.01

$$\vartheta(z) \propto \vartheta(z_0) - \pi \left\lfloor \frac{1}{4}(1-2 i z_0) \right\rfloor \left\lfloor \frac{\arg(-i(z-z_0))}{2\pi} \right\rfloor + O(z-z_0) /; i z_0 \in \mathbb{R} \bigwedge i z_0 > \frac{1}{2} \bigwedge \frac{i z_0}{2} \notin \mathbb{Z}$$

In the upper half-plane

10.03.06.0025.01

$$\vartheta(z) \propto \vartheta(z_0) + \pi \left\lfloor \frac{\arg(i(z-z_0))}{2\pi} \right\rfloor \left\lfloor \frac{1+2 i z_0}{4} \right\rfloor - \frac{1}{4}\left(2\log(\pi) - \psi\left(\frac{i z_0}{2} + \frac{1}{4}\right) - \psi\left(\frac{1}{4} - \frac{i z_0}{2}\right)\right)(z-z_0) - \frac{i}{16}\left(\psi^{(1)}\left(\frac{1}{4} - \frac{i z_0}{2}\right) - \psi^{(1)}\left(\frac{i z_0}{2} + \frac{1}{4}\right)\right)(z-z_0)^2 + \ldots /; (z \to z_0) \bigwedge i z_0 \in \mathbb{R} \bigwedge i z_0 < -\frac{1}{2} \bigwedge \frac{i z_0}{2} \notin \mathbb{Z}$$

10.03.06.0026.01

$$\vartheta(z) \propto \vartheta(z_0) + \pi \left\lfloor \frac{\arg(i(z-z_0))}{2\pi} \right\rfloor \left\lfloor \frac{1+2 i z_0}{4} \right\rfloor - \frac{1}{4}\left(2\log(\pi) - \psi\left(\frac{i z_0}{2} + \frac{1}{4}\right) - \psi\left(\frac{1}{4} - \frac{i z_0}{2}\right)\right)(z-z_0) - \frac{i}{16}\left(\psi^{(1)}\left(\frac{1}{4} - \frac{i z_0}{2}\right) - \psi^{(1)}\left(\frac{i z_0}{2} + \frac{1}{4}\right)\right)(z-z_0)^2 + O\left((z-z_0)^3\right) /; i z_0 \in \mathbb{R} \bigwedge i z_0 < -\frac{1}{2} \bigwedge \frac{i z_0}{2} \notin \mathbb{Z}$$

10.03.06.0027.01

$$\vartheta(z) = \vartheta(z_0) + \pi \left\lfloor \frac{2 i z_0 + 1}{4} \right\rfloor \left\lfloor \frac{\arg(i(z-z_0))}{2\pi} \right\rfloor + \frac{i}{2}\sum_{k=1}^{\infty} \frac{i^k 2^{-k}}{k!}\left(2\delta_{k-1}\log(\pi) - \psi^{(k-1)}\left(\frac{1}{4} + \frac{i z_0}{2}\right) + (-1)^k \psi^{(k-1)}\left(\frac{1}{4} - \frac{i z_0}{2}\right)\right)(z-z_0)^k /; i z_0 \in \mathbb{R} \bigwedge i z_0 < -\frac{1}{2} \bigwedge \frac{i z_0}{2} \notin \mathbb{Z}$$

10.03.06.0028.01

$$\vartheta(z) \propto \vartheta(z_0) + \pi \left\lfloor \frac{1}{4}(1+2 i z_0) \right\rfloor \left\lfloor \frac{\arg(i(z-z_0))}{2\pi} \right\rfloor + O(z-z_0) /; i z_0 \in \mathbb{R} \bigwedge i z_0 < -\frac{1}{2} \bigwedge \frac{i z_0}{2} \notin \mathbb{Z}$$

Expansions at $z == 0$

10.03.06.0029.01

$$\vartheta(z) \propto \left(\frac{1}{2}\psi\left(\frac{1}{4}\right) - \frac{\log(\pi)}{2}\right) z - \frac{1}{48}\psi^{(2)}\left(\frac{1}{4}\right) z^3 + \frac{1}{3840}\psi^{(4)}\left(\frac{1}{4}\right) z^5 + \ldots \,/;\, (z \to 0)$$

10.03.06.0030.01

$$\vartheta(z) \propto -\frac{2\log(8\pi) + \pi + 2\gamma}{4} z + \frac{28\zeta(3) + \pi^3}{24} z^3 - \frac{1488\zeta(5) + 5\pi^5}{480} z^5 + O\left(z^7\right)$$

10.03.06.0031.01

$$\vartheta(z) = z \sum_{k=0}^{\infty} \left(\frac{(-1)^k}{(2k+1)!\, 2^{2k+1}} \psi^{(2k)}\left(\frac{1}{4}\right) - \frac{\log(\pi)\,\delta_k}{2} \right) z^{2k} \,/;\, \left| z - \frac{i}{2} \right| < 2$$

10.03.06.0032.01

$$\vartheta(z) \propto -\frac{2\log(8\pi) + \pi + 2\gamma}{4} z \left(1 + O\left(z^2\right)\right) /;\, (z \to 0)$$

Expansions at $z == \frac{i}{2}$

10.03.06.0002.02

$$\vartheta(z) \propto \frac{i}{2}\log\left(\frac{i}{2}\left(z - \frac{i}{2}\right)\right) - \frac{\log(2\pi) + \gamma}{2}\left(z - \frac{i}{2}\right) - \frac{i\pi^2}{48}\left(z - \frac{i}{2}\right)^2 + \frac{\zeta(3)}{6}\left(z - \frac{i}{2}\right)^3 + \frac{7i\pi^4}{5760}\left(z - \frac{i}{2}\right)^4 + \ldots \,/;\, \left(z \to \frac{i}{2}\right)$$

10.03.06.0033.01

$$\vartheta(z) \propto \frac{i}{2}\log\left(\frac{i}{2}\left(z - \frac{i}{2}\right)\right) - \frac{\log(2\pi) + \gamma}{2}\left(z - \frac{i}{2}\right) - \frac{i\pi^2}{48}\left(z - \frac{i}{2}\right)^2 + \frac{\zeta(3)}{6}\left(z - \frac{i}{2}\right)^3 + \frac{7i\pi^4}{5760}\left(z - \frac{i}{2}\right)^4 + O\left(\left(z - \frac{i}{2}\right)^5\right)$$

10.03.06.0001.01

$$\vartheta(z) == \frac{1}{2}\left(i\log\left(\frac{i}{2}\left(z - \frac{i}{2}\right)\right) - (\log(2\pi) + \gamma)\left(z - \frac{i}{2}\right)\right) - \frac{i}{8}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{1}{j+2}\left(\left(\frac{i}{2}\right)^j\left(k + \frac{1}{2}\right)^{-j-2} - \left(-\frac{i}{2}\right)^j (k+1)^{-j-2}\right)\left(z - \frac{i}{2}\right)^{2+j} /;$$

$$\left| z - \frac{i}{2} \right| < 2$$

10.03.06.0003.01

$$\vartheta(z) == \frac{i}{2}\log\left(\frac{i}{2}\left(z - \frac{i}{2}\right)\right) - \frac{\log(2\pi) + \gamma}{2}\left(z - \frac{i}{2}\right) + \frac{i}{8}\sum_{j=0}^{\infty}\frac{\zeta(j+2)}{j+2}\left(\left(\frac{i}{2}\right)^j + \left(-\frac{i}{2}\right)^j - 4\,i^j\right)\left(z - \frac{i}{2}\right)^{j+2} /;\, \left| z - \frac{i}{2} \right| < 2$$

10.03.06.0004.01

$$\vartheta(z) \propto \frac{i}{2}\log\left(\frac{i}{2}\left(z - \frac{i}{2}\right)\right) - \frac{\log(2\pi) + \gamma}{2}\left(z - \frac{i}{2}\right)\left(1 + O\left(z - \frac{i}{2}\right)\right) /;\, \left(z \to \frac{i}{2}\right)$$

Expansions at $z == -\frac{i}{2}$

10.03.06.0006.02

$$\vartheta(z) \propto -\frac{i}{2}\log\left(-\frac{i}{2}\left(z + \frac{i}{2}\right)\right) - \frac{\log(2\pi) + \gamma}{2}\left(z + \frac{i}{2}\right) + \frac{i\pi^2}{48}\left(z + \frac{i}{2}\right)^2 + \frac{\zeta(3)}{6}\left(z + \frac{i}{2}\right)^3 - \frac{7i\pi^4}{5760}\left(z + \frac{i}{2}\right)^4 + \ldots \,/;\, \left(z \to -\frac{i}{2}\right)$$

10.03.06.0034.01

$$\vartheta(z) \propto -\frac{i}{2}\log\left(-\frac{i}{2}\left(z + \frac{i}{2}\right)\right) - \frac{\log(2\pi) + \gamma}{2}\left(z + \frac{i}{2}\right) + \frac{i\pi^2}{48}\left(z + \frac{i}{2}\right)^2 + \frac{\zeta(3)}{6}\left(z + \frac{i}{2}\right)^3 - \frac{7i\pi^4}{5760}\left(z + \frac{i}{2}\right)^4 + O\left(\left(z + \frac{i}{2}\right)^5\right)$$

10.03.06.0005.01

$$\vartheta(z) == -\frac{1}{2}\left(i \log\left(-\frac{i}{2}\left(z+\frac{i}{2}\right)\right) + (\log(2\pi)+\gamma)\left(z+\frac{i}{2}\right)\right) + \frac{i}{8}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{1}{j+2}\left(\left(-\frac{i}{2}\right)^{j}\left(k+\frac{1}{2}\right)^{-j-2} - \left(\frac{i}{2}\right)^{j}(k+1)^{-j-2}\right)\left(z+\frac{i}{2}\right)^{2+j} /; \left|z+\frac{i}{2}\right| < 2$$

10.03.06.0007.01

$$\vartheta(z) == -\frac{i}{2}\log\left(-\frac{i}{2}\left(z+\frac{i}{2}\right)\right) - \frac{\log(2\pi)+\gamma}{2}\left(z+\frac{i}{2}\right) - \frac{i}{8}\sum_{j=0}^{\infty}\frac{\zeta(j+2)}{j+2}\left(\left(-\frac{i}{2}\right)^{j} + \left(\frac{i}{2}\right)^{j} - 4(-i)^{j}\right)\left(z+\frac{i}{2}\right)^{j+2} /; \left|z+\frac{i}{2}\right| < 2$$

10.03.06.0008.01

$$\vartheta(z) \propto -\frac{i}{2}\log\left(-\frac{i}{2}\left(z+\frac{i}{2}\right)\right) - \frac{\log(2\pi)+\gamma}{2}\left(z+\frac{i}{2}\right)\left(1+O\left(z+\frac{i}{2}\right)\right) /; \left(z \to -\frac{i}{2}\right)$$

Expansions at $z == z_0$ /; $z_0 \neq \pm\frac{i}{2} \pm 2\,i\,k$

10.03.06.0010.02

$$\vartheta(z) \propto \vartheta(z_0) + \frac{1}{4}\left(-2\log(\pi) + \psi\left(\frac{i z_0}{2}+\frac{1}{4}\right) + \psi\left(\frac{1}{4}-\frac{i z_0}{2}\right)\right)(z-z_0) + \frac{i}{16}\left(\zeta\left(2, \frac{i z_0}{2}+\frac{1}{4}\right) - \zeta\left(2, \frac{1}{4}-\frac{i z_0}{2}\right)\right)(z-z_0)^2 + \frac{1}{48}\left(\zeta\left(3, \frac{i z_0}{2}+\frac{1}{4}\right) + \zeta\left(3, \frac{1}{4}-\frac{i z_0}{2}\right)\right)(z-z_0)^3 + \ldots /; (z \to z_0) \bigwedge z_0^2 \neq -\left(2k+\frac{1}{2}\right)^2 \bigwedge k \in \mathbb{Z}$$

10.03.06.0035.01

$$\vartheta(z) \propto \vartheta(z_0) + \frac{1}{4}\left(-2\log(\pi) + \psi\left(\frac{i z_0}{2}+\frac{1}{4}\right) + \psi\left(\frac{1}{4}-\frac{i z_0}{2}\right)\right)(z-z_0) + \frac{i}{16}\left(\zeta\left(2, \frac{i z_0}{2}+\frac{1}{4}\right) - \zeta\left(2, \frac{1}{4}-\frac{i z_0}{2}\right)\right)(z-z_0)^2 + \frac{1}{48}\left(\zeta\left(3, \frac{i z_0}{2}+\frac{1}{4}\right) + \zeta\left(3, \frac{1}{4}-\frac{i z_0}{2}\right)\right)(z-z_0)^3 + O\left((z-z_0)^4\right) /; (z \to z_0) \bigwedge z_0^2 \neq -\left(2k+\frac{1}{2}\right)^2 \bigwedge k \in \mathbb{Z}$$

10.03.06.0009.01

$$\vartheta(z) == \vartheta(z_0) + \frac{1}{4}\left(-2\log(\pi) + \psi\left(\frac{1}{4}+\frac{i z_0}{2}\right) + \psi\left(\frac{1}{4}-\frac{i z_0}{2}\right)\right)(z-z_0) + \frac{i}{8}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{2^{-j}(-i)^{j}}{j+2}\left(\left(\frac{i z_0}{2}+k+\frac{1}{4}\right)^{-j-2} - \left(\frac{i z_0}{2}-k-\frac{1}{4}\right)^{-j-2}\right)(z-z_0)^{j+2} /; z_0^2 \neq -\left(2k+\frac{1}{2}\right)^2 \bigwedge k \in \mathbb{Z}$$

10.03.06.0011.01

$$\vartheta(z) == \vartheta(z_0) + \frac{1}{4}\left(-2\log(\pi) + \psi\left(\frac{1}{4}+\frac{i z_0}{2}\right) + \psi\left(\frac{1}{4}-\frac{i z_0}{2}\right)\right)(z-z_0) + \frac{i}{8}\sum_{j=0}^{\infty}\frac{i^{j}\,2^{-j}}{j+2}\left((-1)^{j}\zeta\left(j+2, \frac{1}{4}+\frac{i z_0}{2}\right) - \zeta\left(j+2, \frac{1}{4}-\frac{i z_0}{2}\right)\right)(z-z_0)^{j+2} /; z_0^2 \neq -\left(2k+\frac{1}{2}\right)^2 \bigwedge k \in \mathbb{Z}$$

10.03.06.0012.01

$$\vartheta(z) \propto \vartheta(z_0) + \frac{1}{4}\left(-2\log(\pi) + \psi\left(\frac{i z_0}{2}+\frac{1}{4}\right) + \psi\left(\frac{1}{4}-\frac{i z_0}{2}\right)\right)(z-z_0)(1+O(z-z_0)) /; (z \to z_0) \bigwedge z_0^2 \neq -\left(2k+\frac{1}{2}\right)^2 \bigwedge k \in \mathbb{Z}$$

Expansions at $z == \frac{i}{2} + 2\,i\,k$

10.03.06.0013.01

$$\vartheta(z) = \frac{i}{4}\left(2\log\Gamma\left(k+\frac{1}{2}\right)-(4k+1)\log(\pi)\right)+\frac{1}{4}\left(\psi\left(k+\frac{1}{2}\right)-2\log(\pi)-\gamma\right)\left(z-\frac{i}{2}-2ik\right)+ \frac{i}{2}\left(\log\left(\frac{i}{2}\left(z-\frac{i}{2}-2ik\right)\right)-\frac{1}{4}\sum_{j=0}^{\infty}\frac{2^{-j}i^j}{j+2}\left(\zeta\left(j+2,k+\frac{1}{2}\right)-(-1)^j\zeta(j+2)\right)\left(z-\frac{i}{2}-2ik\right)^{j+2}+ \sum_{j=0}^{k-1}\log\left(j-k+\frac{i}{2}\left(z-\frac{i}{2}-2ik\right)\right)\right)/;\left(z\to\frac{i}{2}+2ik\right)\bigwedge k\in\mathbb{N}$$

10.03.06.0014.01

$$\vartheta(z) \propto \frac{i}{2}\log\left(\frac{i}{2}\left(z-\frac{i}{2}-2ik\right)\right)+\frac{i}{4}\left(2\log\Gamma\left(k+\frac{1}{2}\right)-(4k+1)\log(\pi)\right)+ \frac{i}{2}\sum_{j=0}^{k-1}\log\left(\frac{i}{2}\left(z-\frac{i}{2}-2ik\right)-k+j\right)\left(1+O\left(z-\frac{i}{2}-2ik\right)\right)/;\left(z\to\frac{i}{2}+2ik\right)\bigwedge k\in\mathbb{N}$$

Expansions at $z == -\frac{i}{2}-2ik$

10.03.06.0015.01

$$\vartheta(z) = \frac{i}{4}\left((4k+1)\log(\pi)-2\log\Gamma\left(k+\frac{1}{2}\right)\right)+\frac{1}{4}\left(\psi\left(k+\frac{1}{2}\right)-2\log(\pi)-\gamma\right)\left(z+\frac{i}{2}+2ik\right)- \frac{i}{2}\left(\log\left(-\frac{i}{2}\left(z+\frac{i}{2}+2ik\right)\right)-\frac{1}{4}\sum_{j=0}^{\infty}\frac{2^{-j}i^j}{j+2}\left((-1)^j\zeta\left(j+2,k+\frac{1}{2}\right)-\zeta(j+2)\right)\left(z+\frac{i}{2}+2ik\right)^{j+2}+ \sum_{j=0}^{k-1}\log\left(j-k-\frac{i}{2}\left(z+\frac{i}{2}+2ik\right)\right)\right)/;\left(z\to-\frac{i}{2}-2ik\right)\bigwedge k\in\mathbb{N}$$

10.03.06.0016.01

$$\vartheta(z) \propto -\frac{i}{2}\log\left(-\frac{i}{2}\left(z+\frac{i}{2}+2ik\right)\right)+\frac{i}{4}\left((4k+1)\log(\pi)-2\log\Gamma\left(k+\frac{1}{2}\right)\right)- \frac{i}{2}\sum_{j=0}^{k-1}\log\left(-\frac{i}{2}\left(z+\frac{i}{2}+2ik\right)-k+j\right)\left(1+O\left(z+\frac{i}{2}+2ik\right)\right)/;\left(z\to-\frac{i}{2}-2ik\right)\bigwedge k\in\mathbb{N}$$

Asymptotic series expansions

10.03.06.0017.01

$$\vartheta(z) \propto \frac{z}{4}\log\left(\frac{z^2}{4}\right)-\frac{\pi\sqrt{z^2}}{8z}-\frac{z}{2}(\log(\pi)+1)+\sum_{k=0}^{\infty}(-1)^k 2^{-2k}\left(\frac{4k+3}{16(k+1)(2k+1)}-2\sum_{j=0}^{k-1}\frac{2^{4j}(2k)!\,B_{2j+2}}{(2j+2)!\,(2k-2j)!}\right)z^{-2k-1}/; \left|\arg\left(z^2\right)\right|<\pi\bigwedge(|z|\to\infty)$$

10.03.06.0018.01

$$\vartheta(z) \propto \frac{z}{4}\log\left(\frac{z^2}{4}\right)-\frac{\pi\sqrt{z^2}}{8z}-\frac{z}{2}(\log(\pi)+1)+\frac{3}{16z}\left(1+O\left(\frac{1}{z^2}\right)\right)/;\left|\arg\left(z^2\right)\right|<\pi\bigwedge(|z|\to\infty)$$

Other series representations

10.03.06.0019.01

$$\vartheta(z) = -\frac{z}{2}\log(\pi) - \frac{\gamma z}{2} + \frac{i}{2}\log(1+2\,i\,z) - \frac{i}{2}\log(1-2\,i\,z) - \frac{i}{4}\log\left(\frac{\pi\left(\frac{i\,z}{2}+\frac{1}{4}\right)}{\sin\left(\pi\left(\frac{i\,z}{2}+\frac{1}{4}\right)\right)}\right) + \frac{i}{4}\log\left(\frac{\pi\left(\frac{1}{4}-\frac{i\,z}{2}\right)}{\sin\left(\pi\left(\frac{1}{4}-\frac{i\,z}{2}\right)\right)}\right) - \frac{1}{2}\sum_{k=1}^{\infty}\sum_{j=0}^{k}\frac{(-1)^{j+k}\,2^{-2j-2k}\,(2k)!\,\zeta(2k+1)\,z^{1-2j+2k}}{(2j)!\,(1-2j+2k)!} \;/;\; |z|<2$$

10.03.06.0020.01

$$\vartheta(z) = \frac{1}{8}\left(-4\,z\,(\log(4\,\pi)+1) + (2\,z+i)\log(1+2\,i\,z) + (2\,z-i)\log(1-2\,i\,z)\right) - \frac{i}{4}\sum_{k=2}^{\infty}\frac{k-1}{k\,(k+1)}\left(\zeta\left(k,\frac{i\,z}{2}+\frac{5}{4}\right) - \zeta\left(k,\frac{5}{4}-\frac{i\,z}{2}\right)\right)$$

Integral representations

On the real axis

Of the direct function

10.03.07.0001.01

$$\vartheta(z) = -\frac{i}{2}\int_{\frac{1}{4}-\frac{i\,z}{2}}^{\frac{1}{4}+\frac{i\,z}{2}}\psi(t)\,dt - \frac{z}{2}\log(\pi)$$

10.03.07.0002.01

$$\vartheta(z) = \frac{1}{2}\left(\int_0^{\infty}\left(\frac{e^{-t}\,z}{t} + \frac{2\,e^{t/4}}{t\,(1-e^t)}\sin\left(\frac{t\,z}{2}\right)\right)dt + i\log\left(\sin\left(\pi\left(\frac{i\,z}{2}+\frac{1}{4}\right)\right)\right) - i\log\left(\sin\left(\pi\left(\frac{1}{4}-\frac{i\,z}{2}\right)\right)\right)\right) - \frac{1}{2}\log(\pi)\,z \;/;\; |\mathrm{Im}(z)| < \frac{3}{2}$$

10.03.07.0003.01

$$\vartheta(z) = \int_0^{\infty}\left(\frac{e^{-t}\,z}{2\,t} - \frac{e^{\frac{3t}{4}}}{(e^t-1)\,t}\sin\left(\frac{t\,z}{2}\right)\right)dt - \frac{1}{2}\log(\pi)\,z \;/;\; \mathrm{Im}(z) = 0$$

10.03.07.0004.01

$$\vartheta(z) = \frac{1}{8}\left((2\,z+i)\log(2\,i\,z+1) + (2\,z-i)\log(1-2\,i\,z) - 4\,z\,(\log(4\,\pi)+1)\right) - \int_0^{\infty}\frac{e^{-\frac{t}{4}}}{t}\sin\left(\frac{t\,z}{2}\right)\left(\frac{1}{2}-\frac{1}{t}+\frac{1}{e^t-1}\right)dt \;/;\; \mathrm{Im}(z) = 0$$

10.03.07.0005.01

$$\vartheta(z) = \frac{1}{2}\int_0^{\infty}\left(\frac{e^{-t}\,z}{t} + \frac{2\,e^{\frac{3t}{4}}}{t\,(1-e^t)}\sin\left(\frac{t\,z}{2}\right)\right)dt - \frac{1}{2}\log(\pi)\,z \;/;\; \mathrm{Im}(z) = 0$$

10.03.07.0006.01

$$\vartheta(z) = \frac{1}{8}\left(-2\,(\log(16)+2\log(\pi)+2)\,z + (i+2\,z)\log(2\,i\,z+1) + (-i+2\,z)\log(1-2\,i\,z)\right) - \int_0^{\infty}\frac{e^{-\frac{t}{4}}}{t}\sin\left(\frac{t\,z}{2}\right)\left(\frac{1}{2}-\frac{1}{t}+\frac{1}{e^t-1}\right)dt \;/;\; \mathrm{Im}(z) = 0$$

10.03.07.0007.01

$$\vartheta(z) == \frac{1}{2}\int_0^{\infty} \frac{e^{-t}}{t}\left(z - e^t \cosh\left(\frac{t}{4}\right)\operatorname{csch}\left(\frac{t}{2}\right)\sin\left(\frac{t\,z}{2}\right)\right)dt + \frac{1}{4}\left(-2\,z\log(\pi) + i\log\left(\sin\left(\frac{\pi}{4}(2\,i\,z+1)\right)\right) - i\log\left(\sin\left(\frac{\pi}{4}(1-2\,i\,z)\right)\right)\right) /; |\operatorname{Im}(z)| < \frac{1}{2}$$

10.03.07.0008.01

$$\vartheta(z) == -\frac{i}{2}\int_0^{\infty}\frac{1}{t}\left(i\,e^{-t}\,z - \frac{(t+1)^{-\frac{i\,z}{2}-\frac{1}{4}}\left((t+1)^{i\,z}-1\right)}{\log(t+1)}\right)dt - \frac{\log(\pi)}{2}\,z /; \operatorname{Im}(z) == 0$$

Differential equations

Ordinary linear differential equations and wronskians

For the direct function itself

10.03.13.0001.01

$$\frac{\partial w(z)}{\partial z} = \frac{1}{4}\left(\psi\left(\frac{i\,z}{2}+\frac{1}{4}\right)+\psi\left(\frac{1}{4}-\frac{i\,z}{2}\right)\right) - \frac{\log(\pi)}{2} /; w(z) = \vartheta(z)$$

Transformations

Transformations and argument simplifications

Argument involving basic arithmetic operations

10.03.16.0001.01

$$\vartheta(-z) = -\vartheta(z)$$

10.03.16.0002.01

$$\vartheta(z+2\,i) == -i\log(4\,\pi) + \frac{1}{2}\,i\,(\log(2\,i\,z-3)+\log(1-2\,i\,z)) + \vartheta(z)$$

10.03.16.0003.01

$$\vartheta(z-2\,i) == i\log(4\,\pi) - \frac{i}{2}(\log(2\,i\,z+1)+\log(-2\,i\,z-3)) + \vartheta(z)$$

10.03.16.0004.01

$$\vartheta(z+2\,i\,n) == \vartheta(z) - i\,n\log(4\,\pi) + \frac{i}{2}\sum_{k=1}^{n}(\log(-4\,k+2\,i\,z+1)+\log(4\,k-2\,i\,z-3)) /; n \in \mathbb{N}$$

10.03.16.0005.01

$$\vartheta(z-2\,i\,n) == \vartheta(z) + i\,n\log(4\,\pi) - \frac{i}{2}\sum_{k=1}^{n}(\log(4\,k+2\,i\,z-3)+\log(-4\,k-2\,i\,z+1)) /; n \in \mathbb{N}$$

Identities

Recurrence identities

Consecutive neighbors

10.03.17.0001.01

$$\vartheta(z) = \vartheta(z + 2\,i) + i\log(4\,\pi) - \frac{1}{2}\,i\,(\log(2\,i\,z - 3) + \log(1 - 2\,i\,z))$$

10.03.17.0003.01

$$\vartheta(z) = \vartheta(z - 2\,i) - i\log(4\,\pi) + \frac{1}{2}\,i\,(\log(2\,i\,z + 1) + \log(-2\,i\,z - 3))$$

Distant neighbors

10.03.17.0002.01

$$\vartheta(z) = \vartheta(z + 2\,i\,n) + i\,n\log(4\,\pi) - \frac{1}{2}\,i\sum_{k=1}^{n}(\log(-4\,k + 2\,i\,z + 1) + \log(4\,k - 2\,i\,z - 3))\;/;\; n \in \mathbb{N}$$

10.03.17.0004.01

$$\vartheta(z) = \vartheta(z - 2\,i\,n) - i\,n\log(4\,\pi) + \frac{1}{2}\,i\sum_{k=1}^{n}(\log(4\,k + 2\,i\,z - 3) + \log(-4\,k - 2\,i\,z + 1))\;/;\; n \in \mathbb{N}$$

Complex characteristics

Real part

10.03.19.0001.01

$$\operatorname{Re}(\vartheta(x + i\,y)) = \frac{1}{2}\left(\tan^{-1}(2\,y + 1,\, -2\,x) - \tan^{-1}(1 - 2\,y,\, 2\,x) - x\,(\log(\pi) + \gamma) + \sum_{k=1}^{\infty}\left(\frac{x}{k} + \tan^{-1}\left(\frac{4\,k + 2\,y + 1}{k},\, -\frac{2\,x}{k}\right) - \tan^{-1}\left(\frac{4\,k - 2\,y + 1}{k},\, \frac{2\,x}{k}\right)\right)\right)$$

Imaginary part

10.03.19.0002.01

$$\operatorname{Im}(\vartheta(x + i\,y)) = \frac{1}{4}\left(-2\,y\,(\log(\pi) + \gamma) + \log\left(\frac{4\,x^2 + (1 - 2\,y)^2}{4\,x^2 + (2\,y + 1)^2}\right) + \sum_{k=1}^{\infty}\left(\frac{2\,y}{k} + \log\left(\frac{16\,k^2 + (8 - 16\,y)\,k + 4\,x^2 + (1 - 2\,y)^2}{16\,k^2 + 8\,(2\,y + 1)\,k + 4\,x^2 + (2\,y + 1)^2}\right)\right)\right)$$

Differentiation

Low-order differentiation

10.03.20.0001.01

$$\frac{\partial \vartheta(z)}{\partial z} = \frac{1}{4}\left(\psi\left(\frac{1}{4} + \frac{i\,z}{2}\right) + \psi\left(\frac{1}{4} - \frac{i\,z}{2}\right) - 2\log(\pi)\right)$$

10.03.20.0002.01

$$\frac{\partial^2 \vartheta(z)}{\partial z^2} = \frac{i}{8}\left(\psi^{(1)}\left(\frac{i\,z}{2} + \frac{1}{4}\right) - \psi^{(1)}\left(\frac{1}{4} - \frac{i\,z}{2}\right)\right)$$

Symbolic differentiation

10.03.20.0003.02

$$\frac{\partial^n \vartheta(z)}{\partial z^n} = \frac{\log(\pi)}{2}\,(\delta_{n-1} + z\,\delta_n) - \frac{i^{n+1}}{2^{n+1}}\left(\psi^{(n-1)}\left(\frac{i\,z}{2} + \frac{1}{4}\right) - (-1)^n\,\psi^{(n-1)}\left(\frac{1}{4} - \frac{i\,z}{2}\right)\right)\;/;\; n \in \mathbb{N}$$

10.03.20.0005.01

$$\frac{\partial^n \vartheta(z)}{\partial z^n} = -\frac{\log(\pi)}{2} (2-n)_n z^{1-n} + \frac{i^{n+1} (n-1)!}{2^{n+1}} \left(\zeta\left(n, \frac{1}{4} - \frac{i z}{2}\right) - (-1)^n \zeta\left(n, \frac{i z}{2} + \frac{1}{4}\right)\right) /; n \in \mathbb{Z} \land n \geq 2$$

Fractional integro-differentiation

10.03.20.0006.01

$$\frac{\partial^\alpha \vartheta(z)}{\partial z^\alpha} = \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(2k-\alpha+2) 2^{2k+1}} \psi^{(2k)}\left(\frac{1}{4}\right) z^{2k-\alpha+1} - \frac{\log(\pi) z^{1-\alpha}}{2 \Gamma(2-\alpha)}$$

10.03.20.0004.01

$$\frac{\partial^\alpha \vartheta(z)}{\partial z^\alpha} = z^{1-\alpha} \sum_{k=1}^{\infty} \left(\frac{1}{2k \Gamma(2-\alpha)} - \frac{1}{4k+1} \left({}_2\tilde{F}_1\left(1, 1; 2-\alpha; \frac{2 i z}{4k+1}\right) + {}_2\tilde{F}_1\left(1, 1; 2-\alpha; -\frac{2 i z}{4k+1}\right)\right)\right) - \left({}_2\tilde{F}_1(1, 1; 2-\alpha; -2 i z) + {}_2\tilde{F}_1(1, 1; 2-\alpha; 2 i z)\right) z^{1-\alpha} - \frac{(\log(\pi)+\gamma) z^{1-\alpha}}{2 \Gamma(2-\alpha)}$$

Integration

Indefinite integration

Involving only one direct function

10.03.21.0002.01

$$\int \vartheta(z) \, dz = -\frac{1}{4} \log(\pi) z^2 - \psi^{(-2)}\left(\frac{i z}{2} + \frac{1}{4}\right) - \psi^{(-2)}\left(\frac{1}{4} - \frac{i z}{2}\right)$$

10.03.21.0001.01

$$\int \vartheta(z) \, dz = -\frac{1}{4} (\log(\pi)+\gamma) z^2 + \frac{1}{4} ((1+2 i z) \log(1+2 i z) + (1-2 i z) \log(1-2 i z)) + \sum_{k=1}^{\infty} \left(\frac{z^2}{4k} + \frac{1}{4} \left((4k+2 i z+1) \log\left(1 + \frac{2 i z}{4k+1}\right) + (4k-2 i z+1) \log\left(1 - \frac{2 i z}{4k+1}\right)\right)\right)$$

Involving one direct function and elementary functions

Involving power function

10.03.21.0003.01

$$\int z^n \vartheta(z) \, dz = -n! \, z^n \sum_{k=0}^{n} \frac{2^k (i z)^{-k}}{(n-k)!} \left((-1)^k \psi^{(-k-2)}\left(\frac{i z}{2} + \frac{1}{4}\right) + \psi^{(-k-2)}\left(\frac{1}{4} - \frac{i z}{2}\right)\right) - \frac{\log(\pi) z^{n+2}}{2(n+2)} /; n \in \mathbb{N}$$

10.03.21.0004.01

$$\int z^n \vartheta(a+b z) \, dz = -\frac{1}{2} \left(\frac{a}{n+1} + \frac{b z}{n+2}\right) \log(\pi) z^{n+1} - \frac{n! \, z^n}{b} \sum_{k=0}^{n} \frac{1}{(n-k)!} \left((-1)^k \psi^{(-k-2)}\left(\frac{1}{4} + \frac{i a}{2} + \frac{1}{2} (i b) z\right) + \psi^{(-k-2)}\left(\frac{1}{4} - \frac{i a}{2} - \frac{1}{2} (i b) z\right)\right) \left(\frac{i b z}{2}\right)^{-k} /; n \in \mathbb{N}$$

Definite integration

For the direct function itself

10.03.21.0005.01

$$\int_z^{z+1} \vartheta(t)\, dt = -\frac{1}{2} z \log(\pi) - \frac{\log(\pi)}{4} + \psi^{(-2)}\left(\frac{i z}{2} + \frac{1}{4}\right) + \psi^{(-2)}\left(\frac{1}{4} - \frac{i z}{2}\right) - \psi^{(-2)}\left(\frac{1}{4} i ((2 - i) + 2 z)\right) - \psi^{(-2)}\left(-\frac{1}{4} i ((2 + i) + 2 z)\right) /; i z \notin \left(-\infty, -\frac{1}{2}\right) \bigwedge i z \notin \left(\frac{1}{2}, \infty\right)$$

Representations through more general functions

Through other functions

10.03.26.0001.01

$$\vartheta(z) = -\frac{\log(\pi)}{2} z - \frac{1}{2} i \left(\zeta^{(1,0)}\left(0, \frac{1}{4} + \frac{i z}{2}\right) - \zeta^{(1,0)}\left(0, \frac{1}{4} + \frac{i z}{2}\right)\right) /; |\operatorname{Im}(z)| < \frac{1}{2}$$

Representations through equivalent functions

With related functions

10.03.27.0001.01

$$\vartheta(z) = -\frac{z}{2} \log(\pi) - \frac{i}{2} \left(\log\left(\Gamma\left(\frac{1}{4} + \frac{i z}{2}\right)\right) - \log\left(\Gamma\left(\frac{1}{4} - \frac{i z}{2}\right)\right)\right) /; |\operatorname{Re}(z)| \leq 7 \bigwedge |\operatorname{Im}(z)| < \frac{1}{2}$$

10.03.27.0002.01

$$\vartheta(z) = -i \log\left(\frac{Z(z)}{\zeta\left(i z + \frac{1}{2}\right)}\right)$$

10.03.27.0003.01

$$\vartheta(z) = \frac{1}{4} \left(z \log(4) + i \left(23 \log(2 \pi) - 2 \log\Gamma\left(\frac{47}{4} - \frac{i z}{2}\right) + 2 \log\Gamma\left(\frac{1}{4} - \frac{i z}{2}\right)\right) + 4\, \text{RamanujanTauTheta}\left(\frac{1}{4} (23 i + 2 z)\right)\right)$$

History

– B. Riemann (1859)

– C. L. Siegel (1932)

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