

ProductLog

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Notations

Traditional name

Lambert function

Traditional notation

$W(z)$

Mathematica StandardForm notation

ProductLog[z]

LambertW[z]

Primary definition

01.31.02.0001.01

$$W(z) = W_0(z)$$

Specific values

Values at fixed points

01.31.03.0001.01

$$W(0) = 0$$

01.31.03.0007.01

$$W\left(-\frac{\pi}{2}\right) = \frac{i\pi}{2}$$

01.31.03.0008.01

$$W\left(-\frac{1}{e}\right) = -1$$

Values at infinities

01.31.03.0002.01

$$W(\infty) = \infty$$

01.31.03.0003.01

$$W(-\infty) = \infty$$

01.31.03.0004.01

$$W(i\infty) = \infty$$

01.31.03.0005.01

$$W(-i\infty) = \infty$$

01.31.03.0006.01

$$W(\tilde{\infty}) = \infty$$

General characteristics

Domain and analyticity

$W(z)$ is an analytical function of z which is defined over the whole complex z -plane.

01.31.04.0001.01

$$z \rightarrow W(z) :: \mathbb{C} \rightarrow \mathbb{C}$$

Symmetries and periodicities

Mirror symmetry

01.31.04.0002.01

$$W(\bar{z}) = \overline{W(z)} /; z \notin \left(-\infty, -\frac{1}{e}\right)$$

Periodicity

No periodicity

Poles and essential singularities

The function $W(z)$ does not have poles and essential singularities.

01.31.04.0003.01

$$\text{Sing}_z(W(z)) = \{\}$$

Branch points

The function $W(z)$ has two branch points: $z = -\frac{1}{e}$, $z = \tilde{\infty}$.

01.31.04.0004.01

$$\mathcal{BP}_z(W(z)) = \left\{-\frac{1}{e}, \tilde{\infty}\right\}$$

01.31.04.0005.01

$$\mathcal{R}_z(W(z), 0) = \log$$

01.31.04.0006.01

$$\mathcal{R}_z\left(W(z), -\frac{1}{e}\right) = 2$$

01.31.04.0007.01

$$\mathcal{R}_z(W(z), \tilde{\infty}) = \log$$

Branch cuts

The function $W(z)$ is a single-valued function on the z -plane cut along the interval $(-\infty, -\frac{1}{e})$, where it is continuous from above.

01.31.04.0008.01

$$\mathcal{BC}_z(W(z)) = \left\{ \left(-\infty, -\frac{1}{e} \right), -i \right\}$$

01.31.04.0009.01

$$\lim_{\epsilon \rightarrow +0} W(x + i\epsilon) = W(x) /; x < -\frac{1}{e}$$

01.31.04.0010.01

$$\lim_{\epsilon \rightarrow +0} W(x - i\epsilon) = W_{-1}(x) /; x < -\frac{1}{e}$$

Series representations

Generalized power series

Expansions at $z = 0$

01.31.06.0001.02

$$W(z) \propto z - z^2 + \frac{3z^3}{2} - \dots /; (z \rightarrow 0)$$

01.31.06.0008.01

$$W(z) \propto z - z^2 + \frac{3z^3}{2} - O(z^4)$$

01.31.06.0002.01

$$W(z) = \sum_{k=1}^{\infty} \frac{(-k)^{k-1} z^k}{k!} /; |z| < \frac{1}{e}$$

01.31.06.0003.02

$$W(z) \propto z + O(z^2)$$

Expansions at $z = -\frac{1}{e}$

01.31.06.0004.02

$$W(z) \propto -1 + \sqrt{2e} \sqrt{z + \frac{1}{e}} - \frac{2e}{3} \left(z + \frac{1}{e} \right) + \frac{11e^{3/2}}{18\sqrt{2}} \left(z + \frac{1}{e} \right)^{3/2} + \dots /; \left(z \rightarrow -\frac{1}{e} \right)$$

01.31.06.0009.01

$$W(z) \propto -1 + \sqrt{2e} \sqrt{z + \frac{1}{e}} - \frac{2e}{3} \left(z + \frac{1}{e} \right) + \frac{11e^{3/2}}{18\sqrt{2}} \left(z + \frac{1}{e} \right)^{3/2} + O\left(\left(z + \frac{1}{e} \right)^2 \right)$$

01.31.06.0005.01

$$W(z) = \sum_{k=0}^{\infty} c_k p^k /; p = \sqrt{2} \sqrt{e z + 1} \wedge |e z + 1| < 1 \wedge c_0 = -1 \wedge c_1 = 1 \wedge$$

$$a_0 = 2 \wedge a_1 = -1 \wedge a_k = \sum_{j=2}^{k-1} c_j c_{1-j+k} \wedge c_k = -\frac{a_k}{2} + \frac{k-1}{k+1} \left(\frac{a_{k-2}}{4} + \frac{c_{k-2}}{2} \right) - \frac{c_{k-1}}{k+1}$$

01.31.06.0006.02

$$W(z) \propto -1 + O\left(\sqrt{z + \frac{1}{e}}\right)$$

Expansions at $z = \infty$

01.31.06.0007.02

$$W(z) = \log(z) - \log(\log(z)) - \sum_{k=0}^{\infty} \frac{(-1)^k}{\log^k(z)} \sum_{j=1}^k \frac{S_k^{(k-j+1)}}{j!} \log^j(\log(z)) /; (|z| \rightarrow \infty)$$

Integral representations

On the real axis

Of the direct function

01.31.07.0001.01

$$W(z) = 1 + (\log(z) - 1) \exp\left(\frac{i}{2\pi} \int_0^{\infty} \frac{1}{t+1} \log\left(\frac{\log(z) + t - \log(t) - i\pi}{\log(z) + t - \log(t) + i\pi}\right) dt\right) /; z \notin \left(-\frac{1}{e}, 0\right)$$

Dispersion relation

01.31.07.0002.01

$$W(z) = \int_{-\infty}^{-\frac{1}{e}} -\frac{1}{\pi} \operatorname{Im}\left(\frac{\partial W(x)}{\partial x}\right) \log\left(1 - \frac{z}{x}\right) dx /; z \notin \left(-\infty, -\frac{1}{e}\right)$$

Limit representations

01.31.09.0001.01

$$W(x) = \lim_{n \rightarrow \infty} \underbrace{x \exp(-x \exp(-\dots x \exp(-x)))}_{n\text{-times}} /; 0 < x < e$$

01.31.09.0002.01

$$W(-\log(x)) = -\log(x) \lim_{n \rightarrow \infty} \underbrace{x^x}_{n\text{-times}} /; e^{-e} < x < e^{1/e}$$

Differential equations

Ordinary nonlinear differential equations

01.31.13.0001.01

$$w'(z) z (w(z) + 1) - w(z) = 0 /; w(z) = W(c_1 z)$$

Transformations

Transformations and argument simplifications

01.31.16.0001.01

$$W(x \log(x)) = \log(x) /; x > \frac{1}{e}$$

01.31.16.0002.01

$$W\left(-\frac{\log(x)}{x}\right) = -\log(x) /; 0 \leq x \leq e$$

01.31.16.0003.01

$$W\left(-\frac{\log(x+1)}{x(x+1)^{1/x}}\right) = -\frac{x+1}{x} \log(x+1) /; -1 < x < 0$$

Products, sums, and powers of the direct function

Sums of the direct function

01.31.16.0004.01

$$W(x) + W(y) = W\left(xy \left(\frac{1}{W(x)} + \frac{1}{W(y)}\right)\right) /; x > 0 \vee y > 0$$

01.31.16.0005.01

$$w^w = z /; w = \frac{\log(z)}{W(\log(z))}$$

Related transformations

01.31.16.0006.01

$$e^{\alpha W(z)} = z^{\alpha} W(z)^{-\alpha}$$

Identities

Functional identities

01.31.17.0001.01

$$e^{n w(z)} = z^n w(z)^{-n} /; w(z) = W(z) \wedge n \in \mathbb{Z}$$

Complex characteristics

Real part

01.31.19.0001.01

$$\operatorname{Re}(W(x + i y)) = \sum_{k=1}^{\infty} \frac{(-k)^{k-1}}{k!} (x^2 + y^2)^{k/2} \cos(k \tan^{-1}(x, y)) /; x^2 + y^2 < \frac{1}{e^2}$$

Imaginary part

01.31.19.0002.01

$$\operatorname{Im}(W(x + i y)) = \sum_{k=1}^{\infty} \frac{(-k)^{k-1}}{k!} (x^2 + y^2)^{k/2} \sin(k \tan^{-1}(x, y)) /; x^2 + y^2 < \frac{1}{e^2}$$

Absolute value

01.31.19.0003.01

$$|W(x + i y)| = W\left(\sqrt{x^2 + y^2}\right)$$

Argument

01.31.19.0004.01

$$\arg(W(x + i y)) = \sum_{k=1}^{\infty} \frac{(-k)^{k-1}}{k!} \tan^{-1}(\cos(k \tan^{-1}(x, y)), \sin(k \tan^{-1}(x, y))) /; x^2 + y^2 < \frac{1}{e^2}$$

Conjugate value

01.31.19.0005.01

$$\overline{W(x + i y)} = \sum_{k=1}^{\infty} \frac{(-k)^{k-1}}{k!} (\cos(k \tan^{-1}(x, y)) - i \sin(k \tan^{-1}(x, y))) (x^2 + y^2)^{k/2} /; x^2 + y^2 < \frac{1}{e^2}$$

Differentiation

Low-order differentiation

01.31.20.0001.01

$$\frac{\partial W(z)}{\partial z} = \frac{W(z)}{z(W(z) + 1)}$$

01.31.20.0002.01

$$\frac{\partial^2 W(z)}{\partial z^2} = -\frac{W(z)^2 (W(z) + 2)}{z^2 (W(z) + 1)^3}$$

Symbolic differentiation

01.31.20.0003.01

$$\frac{\partial^n W(z)}{\partial z^n} = \sum_{k=0}^{\infty} \frac{(-k - n)^{k+n-1} (k + 1)_n z^k}{(k + n)!} /; |z| < \frac{1}{e} \wedge n \in \mathbb{N}^+$$

01.31.20.0006.01

$$\frac{\partial^n W(z)}{\partial z^n} = \operatorname{boole}(n = 0, W(z)) + \frac{n! W(z)^n}{z^n (W(z) + 1)^{2n-1}} \sum_{q=0}^n \sum_{m=0}^q \sum_{k=0}^{q-m} \sum_{j=0}^{q-k-m} \frac{(-1)^{q-j+n-1} (k + n)^{j+k+n-1} \binom{k+n}{n} (-2n)_{q-j-k-m}}{j! (k + n)! (q - j - k - m)!} W(z)^q /;$$

$n \in \mathbb{N}^+$

Eric Weisstein and Oleg Marichev

01.31.20.0004.01

$$\frac{\partial^n W(z)}{\partial z^n} = \frac{e^{-n W(z)} \text{Pol}(n, W(z))}{(W(z) + 1)^{2n-1}} ; \text{Pol}(n + 1, w) = (w + 1) \text{Pol}^{(0,1)}(n, w) - (wn + 3n - 1) \text{Pol}(n, w) \wedge \text{Pol}(1, w) = 1 \wedge n \in \mathbb{N}^+$$

Fractional integro-differentiation

01.31.20.0005.01

$$\frac{\partial^\alpha W(z)}{\partial z^\alpha} = \sum_{k=1}^{\infty} \frac{(-k)^{k-1} z^{k-\alpha}}{\Gamma(k - \alpha + 1)} ; |z| < \frac{1}{e}$$

Integration

Indefinite integration

Involving only one direct function

01.31.21.0001.01

$$\int W(z) dz = \frac{z(W(z)^2 - W(z) + 1)}{W(z)}$$

Involving one direct function and elementary functions

Involving power function

01.31.21.0002.01

$$\int z^{\alpha-1} W(z) dz = \frac{1}{\alpha^2} (e^{-(\alpha-1)W(z)-W(z)} z^\alpha (\alpha \Gamma(\alpha + 1, -\alpha W(z)) - \Gamma(\alpha + 2, -\alpha W(z))) (-\alpha W(z))^{-\alpha})$$

Definite integration

For the direct function itself

01.31.21.0003.01

$$\int_0^1 W(t) dt = \frac{(2W(1) - 1)(2W(1)^2 + 1)}{8W(1)^2} + \frac{1}{8}$$

01.31.21.0004.01

$$\int_0^1 -\frac{\log(e^{W[x \log(x)]})}{x} dx = \frac{\pi^2}{6}$$

Operations

Limit operation

01.31.25.0001.01

$$\lim_{n \rightarrow \infty} \left(\frac{m_n(n)}{n} - W(n) \right) = 0 ; S_n^{(m_n(n))} \geq S_n^{(m)} \wedge 0 \leq m \leq n$$

Representations through more general functions

Through other functions

01.31.26.0001.01

$$W(z) = W_0(z)$$

Representations through equivalent functions

With inverse function

01.31.27.0001.01

$$e^{W(z)} W(z) = z$$

With related functions

01.31.27.0002.01

$$e^{\alpha W(z)} = z^{\alpha} W(z)^{-\alpha}$$

Theorems

Periodic solutions of the two species Lotka-Volterra model

A closed form solution for periodic solutions of the two species Lotka-Volterra model

$$x'(t) = \alpha x(t)(1 - y(t)), \quad y'(t) = \beta y(t)(1 - x(t)) \text{ is given by } y(t) = -W\left(-c_1 x(t)^{\frac{\beta}{\alpha}} e^{-\frac{\beta}{\alpha} x(t)}\right).$$

Solution of the bootstrap equation

A solution of the bootstrap equation $z = 2\Omega(z) - e^{\Omega(z)} + 1$, appearing for instance in the renormalization group

$$\text{equations, is given as } \Omega(z) = \left(z - 2W\left(-e^{\frac{z-1}{2}}/2\right) - 1\right)/2.$$

The number of rooted trees

The number of rooted trees on n labeled points is $[z^n] W(z)$.

History

–J. H. Lambert (1758)

–L. Euler (1764, 1779)

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