Power

Notations

Traditional name

Power function

Traditional notation

z^a

Mathematica StandardForm notation

Power[z, a]

Primary definition

01.02.02.0001.01

$$z^0 = 1 \;/;\; z \neq 0$$

01.02.02.0002.01

$$z^k = z \times z \times \ldots \times z = z\, z^{k-1} \;/;\; k \in \mathbb{N}^+$$

01.02.02.0003.01

$$z^a = \sum_{k=0}^{\infty} \frac{\log^k(z)\, a^k}{k!}$$

For complex numbers z and a, the function z^a gives the principal value of $e^{a \log(z)}$.

Specific values

Specialized values

For fixed z

01.02.03.0001.01

$$z^0 = 1 \;/;\; z \neq 0$$

01.02.03.0002.01

$$z^{1/2} = \sqrt{z}$$

01.02.03.0003.01

$$z^1 = z$$

For fixed a

01.02.03.0004.01

$0^a == 0$ /; $\mathrm{Re}(a) > 0$

01.02.03.0005.01

$0^a == \tilde{\infty}$ /; $\mathrm{Re}(a) < 0$

01.02.03.0006.01

$0^a ==$ ¿ /; $\mathrm{Re}(a) == 0$

01.02.03.0007.01

$1^a == 1$

01.02.03.0008.01

$(-1)^a == e^{i a \pi}$

01.02.03.0009.01

$i^a == e^{\frac{i a \pi}{2}}$

01.02.03.0010.01

$(-i)^a == e^{-\frac{1}{2} i a \pi}$

01.02.03.0011.01

$z^a == e^a == \exp(a)$ /; $z == e$

Values at fixed points

01.02.03.0012.01

$0^0 ==$ ¿

Values at infinities

01.02.03.0013.01

$z^\infty == 0$ /; $|z| < 1$

01.02.03.0014.01

$z^\infty == \tilde{\infty}$ /; $|z| > 1$

01.02.03.0015.01

$x^\infty == \infty$ /; $x > 1$

01.02.03.0016.01

$1^\infty ==$ ¿

01.02.03.0017.01

$(-1)^\infty ==$ ¿

01.02.03.0018.01

$i^\infty ==$ ¿

01.02.03.0019.01

$(-i)^\infty ==$ ¿

01.02.03.0020.01

$z^{-\infty} == 0$ /; $|z| > 1$

01.02.03.0021.01

$z^{-\infty} == \tilde{\infty}$ /; $|z| < 1$

01.02.03.0022.01

$x^{-\infty} == \infty /; x < 1$

01.02.03.0023.01

$1^{-\infty} == ¿$

01.02.03.0024.01

$(-1)^{-\infty} == ¿$

01.02.03.0025.01

$i^{-\infty} == ¿$

01.02.03.0026.01

$(-i)^{-\infty} == ¿$

01.02.03.0027.01

$\infty^{a} == ¿$

01.02.03.0028.01

$\infty^{a} == 0 /; \mathrm{Re}(a) < 0$

01.02.03.0029.01

$\infty^{a} == \infty /; a > 0$

01.02.03.0030.01

$\infty^{a} == \tilde{\infty} /; \mathrm{Re}(a) > 0 \land \mathrm{Im}(a) \neq 0$

01.02.03.0031.01

$\infty^{0} == ¿$

01.02.03.0032.01

$$\frac{1}{\infty} == 0$$

01.02.03.0033.01

$\infty^{i} == ¿$

01.02.03.0034.01

$\infty^{-\infty} == 0$

01.02.03.0035.01

$(-\infty)^{-\infty} == 0$

01.02.03.0036.01

$(i\,\infty)^{-\infty} == 0$

01.02.03.0037.01

$(-i\,\infty)^{-\infty} == 0$

01.02.03.0038.01

$\tilde{\infty}^{-\infty} == 0$

General characteristics

Domain and analyticity

z^a is an analytical function of z and a which is defined over $\mathbb{C}^2$. For fixed z, it is an entire function of a. For positive integer a, z^a degenerates to a polynomial in z.

01.02.04.0001.01

$(z * a) \longrightarrow z^a :: (\mathbb{C} \otimes \mathbb{C}) \longrightarrow \mathbb{C}$

Symmetries and periodicities

Parity

01.02.04.0002.01

$(-z)^n == (-1)^n z^n /; n \in \mathbb{Z}$

Mirror symmetry

01.02.04.0003.02

$\bar{z}^{\bar{a}} == \overline{z^a} /; z \notin (-\infty, 0)$

Periodicity

z^a is a periodic function with respect to a with period $\frac{2 i \pi}{\log(z)}$.

01.02.04.0004.01

$z^a == z^{a+\frac{2 i \pi}{\log(z)}}$

Homogeneity

01.02.04.0005.01

$(\lambda z)^a == \lambda^a z^a /; \lambda > 0 \bigvee \lambda + z \geq 0 \bigvee a \in \mathbb{Z}$

01.02.04.0006.01

$z^{\lambda a} == (z^a)^\lambda /; -\pi < \operatorname{Im}(a \log(z)) \leq \pi \bigvee -1 < a \leq 1 \bigvee \lambda \in \mathbb{Z}$

Scale symmetry

01.02.04.0007.01

$(z^a)^b == z^{a b} /; -\pi < \operatorname{Im}(a \log(z)) \leq \pi \bigvee -1 < a \leq 1 \bigvee b \in \mathbb{Z}$

Poles and essential singularities

With respect to *a*

For fixed z, the function z^a has only one singular point at $a = \tilde{\infty}$. It is an essential singular point.

01.02.04.0008.01

$\mathcal{S}ing_a(z^a) == \{\{\tilde{\infty}, \infty\}\}$

With respect to *z*

For fixed a /; $a \notin \mathbb{Z}$, the function z^a does not have poles and essential singularities.

01.02.04.0009.01

$\mathcal{S}ing_z(z^a) == \{\} /; a \notin \mathbb{Z}$

For positive integer a, the function z^a has a pole of order a at $z = \tilde{\infty}$.

01.02.04.0010.01

$\mathcal{S}ing_z(z^a) == \{\{\tilde{\infty}, a\}\} /; a \in \mathbb{N}^+$

For negative integers a, the function z^ahas a pole of order $-a$ at the point $z == 0$ with residue δ_{-a-1}(1if $a == -1$and 0 otherwise).

01.02.04.0011.01

$$\mathcal{S}ing_z(z^a) == \{\{0, -a\}\} /; -a \in \mathbb{N}^+$$

01.02.04.0012.01

$$\text{res}_z(z^a)(0) == \delta_{-a-1} /; a \in \mathbb{Z}$$

Branch points

With respect to *a*

For fixed z, the function z^a does not have branch points.

01.02.04.0013.01

$$\mathcal{BP}_a(z^a) == \{\}$$

With respect to *z*

For fixed noninteger a, the function z^a has two singular branch points: $z == 0$, $z == \tilde{\infty}$.

For integer a, the function z^a does not have branch points.

01.02.04.0014.01

$$\mathcal{BP}_z(z^a) == \{0, \tilde{\infty}\} /; a \notin \mathbb{Z}$$

01.02.04.0015.01

$$\mathcal{BP}_z(z^a) == \{\} /; a \in \mathbb{Z}$$

01.02.04.0016.01

$$\mathcal{R}_z(z^a, 0) == \log /; a \notin \mathbb{Z} \bigwedge a \notin \mathbb{Q}$$

01.02.04.0017.01

$$\mathcal{R}_z(z^a, 0) == q /; a == \frac{p}{q} \bigwedge p \in \mathbb{Z} \bigwedge q - 1 \in \mathbb{N}^+ \bigwedge \gcd(p, q) == 1$$

01.02.04.0018.01

$$\mathcal{R}_z(z^a, \tilde{\infty}) == \log /; a \notin \mathbb{Z} \bigwedge a \notin \mathbb{Q}$$

01.02.04.0019.01

$$\mathcal{R}_z(z^a, \tilde{\infty}) == q /; a == \frac{p}{q} \bigwedge p \in \mathbb{Z} \bigwedge q - 1 \in \mathbb{N}^+ \bigwedge \gcd(p, q) == 1$$

Branch cuts

With respect to *a*

For fixed z, the function z^adoes not have branch cuts.

01.02.04.0020.01

$$\mathcal{BC}_a(z^a) == \{\}$$

With respect to *z*

For fixed noninteger a, the function z^a is a single-valued function on the z-plane cut along the interval $(-\infty, 0)$, where it is continuous from above.

For integer a, the function z^a does not have branch cuts.

01.02.04.0021.01

$$\mathcal{BC}_z(z^a) == \{\{(-\infty, 0), -i\}\}$$

01.02.04.0022.01

$$\mathcal{BC}_z(z^a) == \{\} /; a \in \mathbb{Z}$$

01.02.04.0023.01

$$\lim_{\epsilon \to +0} (x + i\,\epsilon)^a == x^a /; x < 0$$

01.02.04.0026.01

$$\lim_{\epsilon \to +0} (x + i\,\epsilon)^a = e^{i\,\pi\,a}\,|x|^a /; x < 0$$

01.02.04.0027.01

$$\lim_{\epsilon \to +0} (x + i\,\epsilon)^a = e^{i\,\pi\,a}\,(-x)^a /; x < 0$$

01.02.04.0028.01

$$\lim_{\epsilon \to +0} (x - i\,\epsilon)^a = e^{-i\,\pi\,a}\,|x|^a /; x < 0$$

01.02.04.0024.01

$$\lim_{\epsilon \to +0} (x - i\,\epsilon)^a == e^{-i\,\pi\,a}\,(-x)^a /; x < 0$$

01.02.04.0025.01

$$\lim_{\epsilon \to +0} (x - i\,\epsilon)^a == e^{-2\,i\,\pi\,a}\,x^a /; x < 0$$

Series representations

Generalized power series

Expansions at generic point $a == a_0$

For the function itself

01.02.06.0022.01

$$z^a \propto z^{a_0} \left(1 + \log(z)\,(a - a_0) + \frac{1}{2} \log^2(z)\,(a - a_0)^2 + \ldots\right) /; (a \to a_0)$$

01.02.06.0023.01

$$z^a \propto z^{a_0} \left(1 + \log(z)\,(a - a_0) + \frac{1}{2} \log^2(z)\,(a - a_0)^2 + O\left((a - a_0)^3\right)\right)$$

01.02.06.0024.01

$$z^a = z^{a_0} \sum_{k=0}^{\infty} \frac{\log^k(z)}{k!}\,(a - a_0)^k$$

01.02.06.0025.01

$$z^a = z^{a_0}\,{}_0F_0(;;(a - a_0)\log(z))$$

01.02.06.0026.01

$$z^a \propto z^{a_0}\,(1 + O(a - a_0))$$

Expansions at $a == 0$

For the function itself

01.02.06.0001.02

$$z^a \propto 1 + a\log(z) + \frac{1}{6}\log^3(z)\,a^3 + \frac{1}{2}\log^2(z)\,a^2 + \ldots \;/;\; (a \to 0)$$

01.02.06.0027.01

$$z^a \propto 1 + a\log(z) + \frac{1}{6}\log^3(z)\,a^3 + \frac{1}{2}\log^2(z)\,a^2 + O\left(a^4\right)$$

01.02.06.0002.01

$$z^a == \sum_{k=0}^{\infty} \frac{\log^k(z)\,a^k}{k!}$$

01.02.06.0003.01

$$z^a == {}_0F_0(;\,;\,a\log(z))$$

01.02.06.0004.02

$$z^a \propto 1 + O(a)$$

01.02.06.0028.01

$$z^a == F_\infty(a, z) \;/;\; \left(\left(F_n(a, z) == \sum_{k=0}^{n} \frac{\log^k(z)\,a^k}{k!} == z^a\, Q(n+1, a\log(z))\right) \bigwedge n \in \mathbb{N}\right)$$

Summed form of the truncated series expansion.

Expansions at generic point $z == z_0$

For the function itself

01.02.06.0029.01

$$z^a \propto \left(\frac{1}{z_0}\right)^{a\left\lfloor \frac{\arg(z-z_0)}{2\pi} \right\rfloor} z_0^{a\left(\left\lfloor \frac{\arg(z-z_0)}{2\pi} \right\rfloor + 1\right)} \left(1 + \frac{a\,(z-z_0)}{z_0} - \frac{((1-a)\,a)\,(z-z_0)^2}{2\,z_0^2} + \ldots\right) /;\; (z \to z_0)$$

01.02.06.0030.01

$$z^a \propto \left(\frac{1}{z_0}\right)^{a\left\lfloor \frac{\arg(z-z_0)}{2\pi} \right\rfloor} z_0^{a\left(\left\lfloor \frac{\arg(z-z_0)}{2\pi} \right\rfloor + 1\right)} \left(1 + \frac{a}{z_0}(z-z_0) - \frac{(1-a)\,a}{2\,z_0^2}(z-z_0)^2 + O\left((z-z_0)^3\right)\right)$$

01.02.06.0031.01

$$z^a = \left(\frac{1}{z_0}\right)^{a\left\lfloor \frac{\arg(z-z_0)}{2\pi} \right\rfloor} z_0^{a\left(\left\lfloor \frac{\arg(z-z_0)}{2\pi} \right\rfloor + 1\right)} \sum_{k=0}^{\infty} \frac{(-1)^k\,(-a)_k\,z_0^{-k}}{k!}(z-z_0)^k$$

01.02.06.0032.01

$$z^a == \exp\left(2\pi i a \left\lfloor \frac{\pi - \arg\left(\frac{z}{z_0}\right) - \arg(z_0)}{2\pi} \right\rfloor\right) \sum_{k=0}^{\infty} \frac{(-1)^k\,(-a)_k\,z_0^{-k}}{k!}(z-z_0)^k$$

01.02.06.0033.01

$$z^a = \left(\frac{1}{z_0}\right)^{a \left\lfloor \frac{\arg(z-z_0)}{2\pi} \right\rfloor} z_0^{a\left(\left\lfloor \frac{\arg(z-z_0)}{2\pi} \right\rfloor+1\right)} {}_1F_0\left(-a;;-\frac{z-z_0}{z_0}\right)$$

01.02.06.0034.01

$$z^a \propto \left(\frac{1}{z_0}\right)^{a \left\lfloor \frac{\arg(z-z_0)}{2\pi} \right\rfloor} z_0^{a\left(\left\lfloor \frac{\arg(z-z_0)}{2\pi} \right\rfloor+1\right)} (1+O(z-z_0))$$

Expansions of $f(z)^a$ at $z == z_0$

01.02.06.0035.01

$$f(z)^a \propto f(z_0)^a \, e^{2 i a \pi \left\lfloor \frac{1}{2} - \frac{\arg(f(z_0))}{2\pi} - \frac{1}{2\pi} \arg\left(\frac{f(z)}{f(z_0)}\right) \right\rfloor} \left(1 + \frac{a f'(z_0)}{f(z_0)} (z-z_0) + \frac{a}{2 f(z_0)^2} \left((a-1) f'(z_0)^2 + f(z_0) f''(z_0)\right) (z-z_0)^2 + \ldots\right) /;$$

$$(z \to z_0)$$

01.02.06.0036.01

$$f(z)^a = a f(z_0)^a \, e^{2 i a \pi \left\lfloor \frac{1}{2} - \frac{\arg(f(z_0))}{2\pi} - \frac{1}{2\pi} \arg\left(\frac{f(z)}{f(z_0)}\right) \right\rfloor} \sum_{k=0}^{\infty} \binom{k-a}{k} \sum_{j=0}^{k} \frac{(-1)^j}{a-j} \binom{k}{j} p_{j,k} (z-z_0)^k /;$$

$$p_{j,0} = 1 \bigwedge p_{j,k} = \frac{1}{f(z_0)\, k} \sum_{m=1}^{k} \frac{j m + m - k}{m!} f^{(m)}(z_0)\, p_{j,k-m} \bigwedge k \in \mathbb{N}^+ \bigwedge f(z_0) \neq 0$$

01.02.06.0037.01

$$f(z)^n = f(z_0)^n \sum_{k=0}^{\infty} p_{n,k} (z-z_0)^k /; p_{j,0} = 1 \bigwedge p_{j,k} = \frac{1}{f(z_0)\, k} \sum_{m=1}^{k} \frac{j m + m - k}{m!} f^{(m)}(z_0)\, p_{j,k-m} \bigwedge k \in \mathbb{N}^+ \bigwedge n \in \mathbb{N}^+ \bigwedge f(z_0) \neq 0$$

01.02.06.0038.01

$$f(z)^n == f(z_0)^n \sum_{k=0}^{m n} p_{n,k} (z-z_0)^k /; f(z) == \sum_{k=0}^{m} c_k z^k \bigwedge p_{j,0} == 1 \bigwedge$$

$$p_{j,k} == \frac{1}{f(z_0)\, k} \sum_{m=1}^{k} \frac{j m + m - k}{m!} f^{(m)}(z_0)\, p_{j,k-m} \bigwedge k \in \mathbb{N}^+ \bigwedge m \in \mathbb{N}^+ \bigwedge n \in \mathbb{N}^+ \bigwedge c_j == 0 /; j > m \land f(z_0) \neq 0$$

01.02.06.0039.01

$$f(z)^a \propto e^{2 i a \pi \left\lfloor \frac{1}{2} - \frac{\arg(f(z_0))}{2\pi} - \frac{1}{2\pi} \arg\left(\frac{f(z)}{f(z_0)}\right) \right\rfloor} f(z_0)^a (1+O(z-z_0))$$

Expansions on branch cuts

For the function itself

01.02.06.0040.01

$$z^a \propto x^a \, e^{2 a \pi i \left\lfloor \frac{\arg(z-x)}{2\pi} \right\rfloor} \left(1 + \frac{a}{z_0}(z-x) - \frac{(1-a)\, a}{2 z_0^2} (z-x)^2 + \ldots\right) /; (z \to x) \land x \in \mathbb{R} \land x < 0$$

01.02.06.0041.01

$$z^a \propto x^a e^{2 a \pi i \left\lfloor \frac{\arg(z-x)}{2\pi} \right\rfloor} \left(1 + \frac{a}{z_0}(z-x) - \frac{(1-a)a}{2 z_0^2}(z-x)^2 + O\left((z-x)^3\right)\right) /; x \in \mathbb{R} \land x < 0$$

01.02.06.0042.01

$$z^a = x^a e^{2 a \pi i \left\lfloor \frac{\arg(z-x)}{2\pi} \right\rfloor} \sum_{k=0}^{\infty} \frac{(-1)^k (-a)_k x^{-k}}{k!} (z-x)^k /; x \in \mathbb{R} \land x < 0$$

01.02.06.0043.01

$$z^a = x^a e^{2 a \pi i \left\lfloor \frac{\arg(z-x)}{2\pi} \right\rfloor} {}_1F_0\left(-a; ; -\frac{z-x}{x}\right) /; x \in \mathbb{R} \land x < 0$$

01.02.06.0044.01

$$z^a = e^{2 i a \pi \left\lfloor \frac{\arg(z-x)}{2\pi} \right\rfloor} x^a \left(\frac{z}{x}\right)^a /; \left|\frac{z}{x} - 1\right| < 1 \bigwedge x \in \mathbb{R} \bigwedge x < 0$$

01.02.06.0045.01

$$z^a \propto x^a e^{2 a \pi i \left\lfloor \frac{\arg(z-x)}{2\pi} \right\rfloor} (1 + O(z-x)) /; x \in \mathbb{R} \land x < 0$$

Expansions at $z == 1$

For the function itself

General case

01.02.06.0011.02

$$z^a \propto 1 + a(z-1) + \frac{a(a-1)}{2!}(z-1)^2 + \frac{a(a-1)(a-2)}{3!}(z-1)^3 + \ldots /; (z \to 1)$$

01.02.06.0046.01

$$z^a \propto 1 + a(z-1) + \frac{a(a-1)}{2!}(z-1)^2 + \frac{a(a-1)(a-2)}{3!}(z-1)^3 + O\left((z-1)^4\right)$$

01.02.06.0012.01

$$z^a == \sum_{k=0}^{\infty} \binom{a}{k} (z-1)^k /; |z-1| < 1$$

01.02.06.0013.01

$$z^a == \sum_{k=0}^{\infty} \frac{(-1)^k (-a)_k}{k!} (z-1)^k /; |z-1| < 1$$

01.02.06.0014.01

$$z^a == {}_1F_0(-a; ; 1-z)$$

01.02.06.0015.02

$$z^a \propto 1 + O(z-1)$$

01.02.06.0047.01

$$z^a == F_\infty(z, a) /; \left(\left(F_n(z, a) == \sum_{k=0}^{n} \frac{(-1)^k (-a)_k (z-1)^k}{k!} == z^a + (-1)^n (z-1)^{n+1} (-a)_{n+1} \, {}_2\tilde{F}_1(1, -a+n+1; n+2; 1-z)\right) \bigwedge n \in \mathbb{N}\right)$$

Summed form of the truncated series expansion.

Special cases

01.02.06.0048.01

$$\frac{1}{z} \propto 1 + (z-1)^2 - (z-1)^3 + \ldots \;/;\; (z \to 1)$$

01.02.06.0049.01

$$\frac{1}{z} \propto 1 + (z-1)^2 - (z-1)^3 + O\left((z-1)^4\right)$$

01.02.06.0016.01

$$\frac{1}{z} = \sum_{k=0}^{\infty} (-1)^k (z-1)^k \;/;\; |z-1| < 1$$

01.02.06.0050.01

$$\frac{1}{z} = {}_1F_0(1;\;;1-z)$$

01.02.06.0051.01

$$\frac{1}{z} \propto 1 + O(z-1)$$

01.02.06.0052.01

$$\frac{1}{z} = F_\infty(z) \;/;\; \left(\left(F_n(z) = \sum_{k=0}^{n} (-1)^k (z-1)^k = \frac{(-1)^n (z-1)^{n+1} + 1}{z}\right) \bigwedge n \in \mathbb{N}\right)$$

Summed form of the truncated series expansion.

Expansions of $(1+z)^a$ at $z = 0$

For the function itself

General case

01.02.06.0005.02

$$(1+z)^a \propto 1 + a\,z + \frac{a\,(a-1)}{2!} z^2 + \frac{a\,(a-1)\,(a-2)}{3!} z^3 + \ldots \;/;\; (z \to 0)$$

01.02.06.0053.01

$$(1+z)^a \propto 1 + a\,z + \frac{a\,(a-1)}{2!} z^2 + \frac{a\,(a-1)\,(a-2)}{3!} z^3 + O\left(z^4\right)$$

01.02.06.0006.01

$$(1+z)^a = \sum_{k=0}^{\infty} \binom{a}{k} z^k \;/;\; |z| < 1$$

01.02.06.0007.01

$$(1+z)^a = \sum_{k=0}^{\infty} \frac{(-1)^k (-a)_k}{k!} z^k \;/;\; |z| < 1$$

01.02.06.0008.01

$$(1+z)^a = {}_1F_0(-a;\;;-z)$$

01.02.06.0009.02

$$(1+z)^a \propto 1 + O(z)$$

01.02.06.0054.01

$$(1+z)^a = F_\infty(z, a) /; \left(\left(F_n(z, a) = \sum_{k=0}^{n} \frac{(-1)^k (-a)_k z^k}{k!} = (z+1)^a + (-1)^n z^{n+1} (-a)_{n+1}\, {}_2\tilde{F}_1(1, -a+n+1; n+2; -z)\right) \bigwedge n \in \mathbb{N}\right)$$

Summed form of the truncated series expansion.

Special cases

01.02.06.0055.01

$$\frac{1}{1+z} \propto 1 - z + z^2 - z^3 + \ldots /; (z \to 0)$$

01.02.06.0056.01

$$\frac{1}{1+z} \propto 1 - z + z^2 - z^3 + O(z^4)$$

01.02.06.0010.01

$$\frac{1}{1+z} = \sum_{k=0}^{\infty} (-1)^k z^k /; |z| < 1$$

01.02.06.0057.01

$$\frac{1}{1+z} = {}_1F_0(1; ; -z)$$

01.02.06.0058.01

$$\frac{1}{1+z} \propto 1 + O(z)$$

01.02.06.0059.01

$$\frac{1}{1+z} = F_\infty(z) /; \left(\left(F_n(z) = \sum_{k=0}^{n} (-1)^k z^k = \frac{(-1)^n z^{n+1} + 1}{z+1}\right) \bigwedge n \in \mathbb{N}\right)$$

Summed form of the truncated series expansion.

Expansions of $\left(1 + \sum_{k=1}^{\infty} c_k z^k\right)^a$ at $z = 0$

01.02.06.0060.01

$$\left(1 + \sum_{k=1}^{\infty} c_k z^k\right)^a \propto 1 + a c_1 z + \frac{1}{2} a \left((a-1) c_1^2 + 2 c_2\right) z^2 + \frac{1}{6} a \left((a-2)(a-1) c_1^3 + 6 (a-1) c_2 c_1 + 6 c_3\right) z^3 + \ldots /; (z \to 0)$$

01.02.06.0061.01

$$\left(1 + \sum_{k=1}^{\infty} c_k z^k\right)^a \propto 1 + a c_1 z + \frac{1}{2} a \left((a-1) c_1^2 + 2 c_2\right) z^2 + \frac{1}{6} a \left((a-2)(a-1) c_1^3 + 6 (a-1) c_2 c_1 + 6 c_3\right) z^3 + O(z^4) /; (z \to 0)$$

01.02.06.0062.01

$$\left(1 + \sum_{k=1}^{\infty} c_k z^k\right)^a = a \sum_{k=0}^{\infty} \binom{k-a}{k} \sum_{j=0}^{k} \frac{(-1)^j}{a-j} \binom{k}{j} p_{j,k} z^k /; p_{j,0} = 1 \bigwedge p_{j,k} = \frac{1}{k} \sum_{m=1}^{k} (j m + m - k) c_m p_{j,k-m} \bigwedge k \in \mathbb{N}^+ \bigwedge a \notin \mathbb{N}^+$$

01.02.06.0063.01

$$\left(1+\sum_{k=1}^{\infty} c_k\, z^k\right)^n = \sum_{k=0}^{\infty} p(n,\,k)\, z^k \;/;\; p_{j,0}=1 \bigwedge p_{j,k} = \frac{1}{k}\sum_{m=1}^{k} (j\,m+m-k)\, c_m\, p_{j,k-m} \bigwedge k\in\mathbb{N}^+ \bigwedge n\in\mathbb{N}^+$$

01.02.06.0064.01

$$\left(1+\sum_{k=1}^{m} c_k\, z^k\right)^n = \sum_{k=0}^{m\,n} p_{n,k}\, z^k \;/;$$

$$p_{j,0}=1 \bigwedge p_{j,k} = \frac{1}{k}\sum_{m=1}^{k} (j\,m+m-k)\, c_m\, p_{j,k-m} \bigwedge k\in\mathbb{N}^+ \bigwedge m\in\mathbb{N}^+ \bigwedge n\in\mathbb{N}^+ \bigwedge c_j = 0 \;/;\; j>m$$

01.02.06.0065.01

$$\left(1+\sum_{k=1}^{\infty} c_k\, z^k\right)^a \propto 1+O(z)$$

Expansions of $(1+z)^a$ at $z == \infty$

For the function itself

General case

01.02.06.0066.01

$$(1+z)^a \propto z^a\left(1+\frac{a}{z}-\frac{(1-a)\,a}{2\,z^2}+\ldots\right) /;\; (|z|\to\infty)$$

01.02.06.0067.01

$$(1+z)^a \propto z^a\left(1+\frac{a}{z}-\frac{(1-a)\,a}{2\,z^2}+O\!\left(\frac{1}{z^3}\right)\right)$$

01.02.06.0068.01

$$(1+z)^a == z^a \sum_{k=0}^{\infty} \binom{a}{k} z^{-k} \;/;\; |z|>1$$

01.02.06.0069.01

$$(1+z)^a == z^a \sum_{k=0}^{\infty} \frac{(-1)^k\,(-a)_k\, z^{-k}}{k!} \;/;\; |z|>1$$

01.02.06.0070.01

$$(1+z)^a == e^{2\,\pi\,i\,a\left\lfloor \frac{\pi-\arg(z)-\arg\left(1+\frac{1}{z}\right)}{2\,\pi}\right\rfloor} z^a\, {}_1F_0\!\left(-a;\,;\,-\frac{1}{z}\right)$$

01.02.06.0071.01

$$(1+z)^a == z^a\, {}_1F_0\!\left(-a;\,;\,-\frac{1}{z}\right) /;\; |z|>1$$

01.02.06.0072.01

$$(1+z)^a \propto z^a\left(1+O\!\left(\frac{1}{z}\right)\right)$$

01.02.06.0073.01

$$(z+1)^a = F_\infty(z, a) /; \left(\left(F_n(z, a) = z^a \sum_{k=0}^{n} \frac{\left((-1)^k (-a)_k\right) z^{-k}}{k!} = \left(1+\frac{1}{z}\right)^a z^a + \frac{(-1)^n \Gamma(-a+n+1)}{\Gamma(-a)} z^{a-n-1} {}_2\tilde{F}_1\left(1, -a+n+1; n+2; -\frac{1}{z}\right)\right) \bigwedge n \in \mathbb{N}\right)$$

Summed form of the truncated series expansion.

Special cases

01.02.06.0074.01

$$\frac{1}{1+z} \propto \frac{1}{z}\left(1-\frac{1}{z}+\frac{1}{z^2}-\ldots\right) /; (z \to 0)$$

01.02.06.0075.01

$$\frac{1}{1+z} \propto \frac{1}{z}\left(1-\frac{1}{z}+\frac{1}{z^2}-O\left(\frac{1}{z^3}\right)\right)$$

01.02.06.0076.01

$$\frac{1}{1+z} = \frac{1}{z}\sum_{k=0}^{\infty}(-1)^k z^{-k} /; |z| > 1$$

01.02.06.0077.01

$$\frac{1}{1+z} = \frac{1}{z}\, {}_1F_0\left(1;; -\frac{1}{z}\right)$$

01.02.06.0078.01

$$\frac{1}{1+z} \propto \frac{1}{z}\left(1+O\left(\frac{1}{z}\right)\right)$$

01.02.06.0079.01

$$\frac{1}{1+z} = F_\infty(z) /; \left(\left(F_n(z) = \frac{1}{z}\sum_{k=0}^{\infty}(-1)^k z^{-k} = \frac{z^{n+1}+(-1)^n}{z^n\left(z^2+z\right)}\right) \bigwedge n \in \mathbb{N}\right)$$

Summed form of the truncated series expansion.

Asymptotic series expansions

01.02.06.0017.01

$$z^a \propto z^a /; (z \to 0)$$

01.02.06.0018.01

$$z^a \propto z^a /; (|z| \to \infty)$$

01.02.06.0019.01

$$z^a \propto z^a /; (|a| \to \infty)$$

Residue representations

Representations of z^a

01.02.06.0080.01

$$z^a = \sum_{j=0}^{\infty} \operatorname{res}\left((-a \log(z))^{-s}\, \Gamma(s), \{s, -j\}\right)$$

01.02.06.0081.01

$$z^a = \frac{1}{\Gamma(-a)} \sum_{j=0}^{\infty} \text{res}_s((\Gamma(-a-s)(z-1)^{-s})\Gamma(s))(-j) \text{ /; } |z-1| < 1$$

01.02.06.0082.01

$$z^a = -\frac{1}{\Gamma(-a)} \sum_{j=0}^{\infty} \text{res}_s((\Gamma(s)\ (z-1)^{-s})\Gamma(-a-s))(j-a) \text{ /; } |z-1| > 1$$

Representations of $(1+z)^a$

01.02.06.0083.01

$$(1+z)^a = \sum_{j=0}^{\infty} \text{res}\left((-a\log(z))^{-s}\,\Gamma(s+1),\ \{s,\ -j\}\right)$$

01.02.06.0020.01

$$(1+z)^a = \frac{1}{\Gamma(-a)} \sum_{j=0}^{\infty} \text{res}_s((\Gamma(-a-s)\,z^{-s})\Gamma(s))(-j) \text{ /; } |z| < 1$$

01.02.06.0021.01

$$(1+z)^a = -\frac{1}{\Gamma(-a)} \sum_{j=0}^{\infty} \text{res}_s((\Gamma(s)\ z^{-s})\Gamma(-a-s))(j-a) \text{ /; } |z| > 1$$

Integral representations

Contour integral representations

01.02.07.0001.01

$$(1+z)^a = \frac{1}{\Gamma(-a)\,2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \Gamma(s)\,\Gamma(-a-s)\,z^{-s}\,ds \text{ /; } 0 < \gamma < -\text{Re}(a) \bigwedge |\arg(z)| < \pi$$

01.02.07.0002.01

$$(1+z)^a = \frac{1}{\Gamma(-a)\,2\pi i} \int_{\mathcal{L}} \Gamma(s)\,\Gamma(-a-s)\,z^{-s}\,ds$$

Continued fraction representations

01.02.10.0001.01

$$(1+z)^a = 1 + \cfrac{a\,z}{1 + \cfrac{(1-a)\,z}{3 + \cfrac{(1+a)\,z}{5 + \cfrac{(2-a)\,z}{5 + \cfrac{(2+a)\,z}{9 + \cfrac{(3-a)\,z}{7 + \ldots}}}}}} \text{ /; } z \notin (-\infty, -1)$$

01.02.10.0002.01

$$(1+z)^a = 1 + \text{K}_k\left(\left(\left\lfloor \frac{k}{2} \right\rfloor - (-1)^k a\right) z,\ \frac{1}{2}\left(3 - (-1)^k\right) k + (-1)^k\right)_1^{\infty} \text{ /; } z \notin (-\infty, -1)$$

01.02.10.0003.01

$$(1+z)^a = 1+\cfrac{a\,z}{1+\cfrac{(1-a)\,z}{3+\cfrac{(1+a)\,z}{5+\cfrac{(2-a)\,z}{5+\cfrac{(2+a)\,z}{9+\cfrac{(3-a)\,z}{7+\ldots}}}}}} \;/;\; z\notin(-\infty,-1)$$

01.02.10.0004.01

$$(1+z)^a = 1+\mathrm{K}_k\left(\left(\left\lfloor\frac{k}{2}\right\rfloor-(-1)^k a\right)z,\ \frac{1}{2}\left(3-(-1)^k\right)k+(-1)^k\right)_1^{\infty} \;/;\; z\notin(-\infty,-1)$$

01.02.10.0005.01

$$(1+z)^a = \cfrac{1}{1-\cfrac{a\,z}{1+\cfrac{(1+a)\,z}{3+\cfrac{(1-a)\,z}{5+\cfrac{(2+a)\,z}{5+\cfrac{(2-a)\,z}{9+\cfrac{(3+a)\,z}{7+\ldots}}}}}}} \;/;\; z\notin(-\infty,-1)$$

01.02.10.0006.01

$$(1+z)^a = \frac{1}{\left(1+\mathrm{K}_k\left(z\left((-1)^k a+\left\lfloor\frac{k}{2}\right\rfloor\right),\ \frac{1}{2}\left(3-(-1)^k\right)k+(-1)^k\right)_1^{\infty}\right)} \;/;\; z\notin(-\infty,-1)$$

01.02.10.0007.01

$$(1+z)^a = \cfrac{1}{1-\cfrac{a\,z}{1+(1+a)\,z-\cfrac{(1+a)\,z\,(1+z)}{2+(3+a)\,z-\cfrac{2\,(2+a)\,z\,(1+z)}{3+(5+a)\,z-\ldots}}}} \;/;\; \mathrm{Re}(z)>-\frac{1}{2}$$

01.02.10.0008.01

$$(1+z)^a = \cfrac{1}{1-\cfrac{a\,z}{1+(a+1)\,z+\mathrm{K}_k(-k\,(a+k)\,z\,(z+1),k+(a+2\,k+1)\,z+1)_1^{\infty}}} \;/;\; \mathrm{Re}(z)>-\frac{1}{2}$$

01.02.10.0009.01

$$(1+z)^a = \cfrac{1}{1-\cfrac{a\,z}{1+z+\cfrac{(-1+a)\,z}{2+\cfrac{(-1-a)\,z}{3\,(1+z)+\cfrac{(-2+a)\,z}{2+\cfrac{(-2-a)\,z}{5\,(1+z)+\cfrac{(-3+a)\,z}{2+\ldots}}}}}}} \;/;\; z\notin(-\infty,-1)$$

01.02.10.0010.01

$$(1+z)^a = \frac{1}{\left(1+\underset{k}{\mathrm{K}}\left(-\left(\left\lfloor \frac{k}{2} \right\rfloor - (-1)^k a\right) z,\ \frac{1}{2}\left(1-(-1)^k\right) k (z+1) + (-1)^k + 1\right)_1^\infty\right)} \;/;\; z \notin (-\infty, -1)$$

01.02.10.0011.01

$$(1+z)^a = \cfrac{1}{1-\cfrac{a z}{1+a z+\cfrac{(1-a) z}{2-(1-a) z+\cfrac{2(2-a) z}{3-(2-a) z+\cfrac{3(3-a) z}{4-(3-a) z+\ldots}}}}} \;/;\; |z| < 1$$

01.02.10.0012.01

$$(1+z)^a = \cfrac{1}{1-\cfrac{a z}{1+a z+\mathrm{K}_k(k(k-a) z, k-(k-a) z+1)_1^\infty}} \;/;\; |z| < 1$$

01.02.10.0013.01

$$(1+z)^a = 1+\cfrac{a z}{1+\cfrac{(1-a) z}{2\left(1-\frac{1}{2}(1-a) z+\cfrac{(2-a) z}{3\left(1-\frac{1}{3}(2-a) z+\ldots\right)}\right)}} \;/;\; |z| < 1$$

01.02.10.0014.01

$$(1+z)^a = 1+\frac{z a}{1+\mathrm{K}_k\left(\frac{(k-a) z}{k+1},\ 1-\frac{(k-a) z}{k+1}\right)_1^\infty} \;/;\; |z| < 1$$

Differential equations

Ordinary linear differential equations and wronskians

For the direct function itself

With respect to *z*

01.02.13.0001.01

$$z\, w'(z) - a\, w(z) = 0 \;/;\; w(z) = z^a \;/;\; w(1) = 1$$

01.02.13.0003.01

$$z\, w'(z) - a\, w(z) = 0 \;/;\; w(z) = c_1 z^a$$

01.02.13.0004.01

$$w'(z) - \frac{a\, g'(z)}{g(z)} w(z) = 0 \;/;\; w(z) = c_1\, g(z)^a$$

01.02.13.0005.01

$$w'(z) - \left(\frac{a\, g'(z)}{g(z)} + \frac{h'(z)}{h(z)}\right) w(z) = 0 \;/;\; w(z) = c_1\, h(z)\, g(z)^a$$

With respect to *a*

01.02.13.0006.01

$$w'(a) - \log(z)\, w(a) = 0 \;/;\; w(a) = c_1\, z^a$$

01.02.13.0007.01

$$w'(a) - \log(z)\, w(a) = 0 \;/;\; w(a) = z^a \bigwedge w(1) = z$$

01.02.13.0008.01

$$w'(a) - \log(z)\, g'(a)\, w(a) = 0 \;/;\; w(a) = c_1\, z^{g(a)}$$

01.02.13.0009.01

$$w'(a) - \left(\log(z)\, g'(a) + \frac{h'(a)}{h(a)}\right) w(a) = 0 \;/;\; w(a) = c_1\, h(a)\, z^{g(a)}$$

Ordinary nonlinear differential equations

01.02.13.0002.01

$$w(a)\, w''(a) - w'(a)^2 == 0 \;/;\; (w(a) == c_1\, z^a \;/;\; z == e^{c_2})$$

Transformations

Transformations and argument simplifications

Argument involving basic arithmetic operations

With respect to *z*

01.02.16.0006.01

$$(-z)^n == (-1)^n\, z^n \;/;\; n \in \mathbb{Z}$$

01.02.16.0001.01

$$(-z)^a == e^{-i a \pi}\, z^a \;/;\; \mathrm{Im}(z) > 0 \lor \mathrm{Im}(z) == 0 \land z < 0$$

01.02.16.0002.01

$$(-z)^a == e^{i a \pi}\, z^a \;/;\; \mathrm{Im}(z) < 0 \lor \mathrm{Im}(z) == 0 \land z > 0$$

01.02.16.0047.01

$$(-z)^a == e^{-i a \pi}\, z^a \;/;\; \arg(z) > 0$$

01.02.16.0048.01

$$(-z)^a == e^{i a \pi}\, z^a \;/;\; \arg(z) \leq 0$$

01.02.16.0005.01

$$(-z)^a == \exp\left(\frac{a \pi \sqrt{-z^2}}{z}\right) z^a$$

01.02.16.0003.01

$$(-z)^a == \exp\left(i\, a\, \pi \left(1 + 2 \left\lfloor -\frac{\arg(z)}{2 \pi} \right\rfloor\right)\right) z^a$$

01.02.16.0049.01

$$(i\, z)^a == i^a\, z^a \;/;\; \arg(z) \leq \frac{\pi}{2}$$

01.02.16.0050.01

$$(i\,z)^a = i^a\, e^{-2\,i\,a\,\pi}\, z^a \;/;\; \arg(z) > \frac{\pi}{2}$$

01.02.16.0051.01

$$(i\,z)^a = i^a\, e^{2\,i\,a\,\pi \left\lfloor \frac{1}{4} - \frac{\arg(z)}{2\pi} \right\rfloor}\, z^a$$

01.02.16.0052.01

$$(-i\,z)^a = (-i)^a\, z^a \;/;\; \arg(z) > -\frac{\pi}{2}$$

01.02.16.0053.01

$$(-i\,z)^a = (-i)^a\, e^{2\,i\,a\,\pi}\, z^a \;/;\; \arg(z) \le -\frac{\pi}{2}$$

01.02.16.0054.01

$$(-i\,z)^a = (-i)^a\, e^{2\,i\,a\,\pi \left\lfloor \frac{3}{4} - \frac{\arg(z)}{2\pi} \right\rfloor}\, z^a$$

01.02.16.0007.01

$$\left(\frac{1}{z}\right)^a = z^{-a} \;/;\; z \notin (-\infty, 0) \lor a \in \mathbb{Z}$$

01.02.16.0055.01

$$\left(\frac{1}{z}\right)^a = z^{-a} \;/;\; \arg(z) \neq \pi \lor a \in \mathbb{Z}$$

01.02.16.0056.01

$$\left(\frac{1}{z}\right)^a = e^{2\,\pi\,i\,a}\, z^{-a} \;/;\; z \in \mathbb{R} \land z < 0$$

01.02.16.0057.01

$$\left(\frac{1}{z}\right)^a = z^{-a}\, e^{2\,\pi\,i\,a \left\lfloor \frac{\arg(z)+\pi}{2\pi} \right\rfloor}$$

01.02.16.0058.01

$$\left(-\frac{1}{z}\right)^a = (-1)^a\, z^{-a} \;/;\; \operatorname{Im}(z) \ge 0$$

01.02.16.0059.01

$$\left(-\frac{1}{z}\right)^a = e^{-\pi\,i\,a}\, z^{-a} \;/;\; \operatorname{Im}(z) < 0$$

01.02.16.0060.01

$$\left(-\frac{1}{z}\right)^a = (-1)^a\, e^{2\,\pi\,i\,a \left\lfloor \frac{\arg(z)}{2\pi} \right\rfloor}\, z^{-a}$$

01.02.16.0061.01

$$\left(\frac{i}{z}\right)^a = i^a\, z^{-a} \;/;\; \arg(z) \ge -\frac{\pi}{2}$$

01.02.16.0062.01

$$\left(\frac{i}{z}\right)^a = i^a\, e^{-2\,i\,a\,\pi}\, z^{-a} \;/;\; \arg(z) < -\frac{\pi}{2}$$

01.02.16.0063.01

$$\left(\frac{i}{z}\right)^a = i^a\, e^{2\,i\,a\,\pi \left\lfloor \frac{\arg(z)}{2\pi} + \frac{1}{4} \right\rfloor}\, z^{-a}$$

01.02.16.0064.01

$$\left(-\frac{i}{z}\right)^a = (-i)^a z^{-a} /; \arg(z) < \frac{\pi}{2}$$

01.02.16.0065.01

$$\left(-\frac{i}{z}\right)^a = (-i)^a e^{2 i a \pi} z^{-a} /; \arg(z) \geq \frac{\pi}{2}$$

01.02.16.0066.01

$$\left(-\frac{i}{z}\right)^a = (-i)^a e^{2 i a \pi \left\lfloor \frac{\arg(z)}{2\pi} + \frac{3}{4} \right\rfloor} z^{-a}$$

With respect to *a*

01.02.16.0067.01

$$z^{-a} = \frac{1}{z^a}$$

01.02.16.0068.01

$$z^{i a} = (z^a)^i /; -\pi < \operatorname{Im}(a \log(z)) \leq \pi$$

01.02.16.0069.01

$$z^{i a} = (z^a)^i e^{2 \pi k} /; -2 \pi k - \pi < \operatorname{Im}(a \log(z)) \leq \pi - 2 \pi k \bigwedge k \in \mathbb{Z}$$

01.02.16.0070.01

$$z^{i a} = (z^a)^i e^{2 \pi \left\lfloor \frac{\pi - \operatorname{Im}(a \log(z))}{2\pi} \right\rfloor}$$

01.02.16.0071.01

$$z^{-i a} = (z^a)^{-i} /; -\pi < \operatorname{Im}(a \log(z)) \leq \pi$$

01.02.16.0072.01

$$z^{-i a} = (z^a)^{-i} e^{-2 \pi k} /; -2 \pi k - \pi < \operatorname{Im}(a \log(z)) \leq \pi - 2 \pi k \bigwedge k \in \mathbb{Z}$$

01.02.16.0073.01

$$z^{-i a} = (z^a)^{-i} e^{-2 \pi \left\lfloor \frac{\pi - \operatorname{Im}(a \log(z))}{2\pi} \right\rfloor}$$

Addition formulas

With respect to *z*

01.02.16.0009.01

$$(z_1 + z_2)^n = \sum_{k=0}^{n} \binom{n}{k} z_1^k z_2^{n-k} /; n \in \mathbb{N}^+$$

01.02.16.0074.01

$$(z_1 + z_2 + z_3)^n = \sum_{n_1=0}^{n} \sum_{n_2=0}^{n} \sum_{n_3=0}^{n} \delta_{n, n_1+n_2+n_3} (n_1 + n_2 + n_3; n_1, n_2, n_3) z_1^{n_1} z_2^{n_2} z_3^{n_3} /; n \in \mathbb{N}^+$$

01.02.16.0010.01

$$(z_1 + z_2 + \ldots + z_m)^n = \sum_{n_1=0}^{n} \sum_{n_2=0}^{n} \ldots \sum_{n_m=0}^{n} \delta_{n, \sum_{k=1}^{m} n_k} (n_1 + n_2 + \ldots + n_m; n_1, n_2, \ldots, n_m) \prod_{k=1}^{m} z_k^{n_k} /; n \in \mathbb{N}^+$$

With respect to *a*

01.02.16.0008.01

$$z^{a_1+a_2} = z^{a_1} z^{a_2}$$

01.02.16.0075.01

$$z^{a_1+a_2+\ldots+a_m} = \prod_{k=1}^{m} z^{a_k}$$

Half-angle formulas

With respect to *z*

01.02.16.0011.01

$$\left(\frac{z}{2}\right)^a = 2^{-a} z^a$$

With respect to *a*

01.02.16.0076.01

$$z^{a/2} = \sqrt{z^a} \;/;\; a \in \mathbb{R} \bigwedge -\pi < a \arg(z) \leq \pi$$

01.02.16.0077.01

$$z^{a/2} = \sqrt{z^a} \;/;\; -\pi < \operatorname{Im}(a \log(z)) \leq \pi$$

01.02.16.0078.01

$$z^{a/2} = (-1)^k \sqrt{z^a} \;/;\; -2\pi k - \pi < \operatorname{Im}(a \log(z)) \leq \pi - 2\pi k \bigwedge k \in \mathbb{Z}$$

01.02.16.0079.01

$$z^{a/2} = \sqrt{z^a}\, e^{-\pi i \left\lfloor \frac{\pi - \operatorname{Im}(a \log(z))}{2\pi} \right\rfloor}$$

Multiple arguments

With respect to *z*

For products

01.02.16.0012.01

$$(c z)^a = c^a z^a \;/;\; c > 0$$

01.02.16.0013.01

$$(z_1 z_2)^a = z_1^a z_2^a \;/;\; z_1 + z_2 \geq 0 \bigvee a \in \mathbb{Z}$$

01.02.16.0014.01

$$\left(z - z^2\right)^a = (1 - z)^a z^a$$

01.02.16.0015.01

$$\left(-z^2 - z\right)^a = (-z)^a (z + 1)^a$$

01.02.16.0080.01

$$(z_1 z_2)^a = z_1^a z_2^a \;/;\; \arg(z_1) \leq 0 \bigwedge -\arg(z_1) - \pi < \arg(z_2) \bigvee \arg(z_1) \geq 0 \bigwedge \arg(z_2) \leq \pi - \arg(z_1)$$

01.02.16.0081.01

$$(z_1 z_2)^a = z_1^a z_2^a e^{-2\pi i a} \;/;\; \arg(z_1) \geq 0 \bigwedge \arg(z_2) > \pi - \arg(z_1)$$

01.02.16.0082.01

$$(z_1 z_2)^a = z_1^a z_2^a e^{2\pi i a} \;/;\; \arg(z_1) \leq 0 \bigwedge \arg(z_2) \leq -\arg(z_1) - \pi$$

01.02.16.0016.01

$$(z_1\, z_2)^a = z_1^a\, z_2^a \exp\left(2\,\pi\, i\, a \left\lfloor \frac{\pi - \arg(z_1) - \arg(z_2)}{2\,\pi} \right\rfloor\right)$$

01.02.16.0083.01

$$\left(\prod_{k=1}^{n} z_k\right)^a = e^{2\,\pi\, i\, a \left\lfloor \frac{\pi - \sum_{k=1}^{n} \arg(z_k)}{2\,\pi} \right\rfloor} \prod_{k=1}^{n} z_k^a \;/;\; n \in \mathbb{N}^+$$

For quotients

01.02.16.0019.01

$$\left(\frac{z}{z+1}\right)^a = \frac{z^a}{(z+1)^a}$$

01.02.16.0020.01

$$\left(\frac{z}{z-1}\right)^a = \frac{(-z)^a}{(1-z)^a}$$

01.02.16.0021.01

$$\left(\frac{z_1}{z_2}\right)^a = \frac{1}{\left(\frac{z_2}{z_2-z_1}\right)^a} \left(\frac{z_1}{z_2-z_1}\right)^a \;/;\; z_2 - z_1 \in \mathbb{R} \bigwedge z_2 \neq z_1$$

01.02.16.0022.01

$$\left(\frac{z_1}{z_2}\right)^a = \frac{z_1^a}{z_2^a} \;/;\; z_2 - z_1 \geq 0$$

01.02.16.0023.01

$$\left(\frac{z_1}{z_2}\right)^a = \frac{(-z_1)^a}{(-z_2)^a} \;/;\; z_2 - z_1 < 0$$

01.02.16.0084.01

$$\left(\frac{z_1}{z_2}\right)^a = z_1^a\, z_2^{-a} \;/;\; \arg(z_1) \leq 0 \bigwedge \arg(z_2) < \arg(z_1) + \pi \bigvee \arg(z_1) > 0 \bigwedge \arg(z_2) \geq \arg(z_1) - \pi$$

01.02.16.0085.01

$$\left(\frac{z_1}{z_2}\right)^a = z_1^a\, z_2^{-a}\, e^{-2\, i\, a\, \pi} \;/;\; \arg(z_1) \geq 0 \bigwedge \arg(z_2) < \arg(z_1) - \pi$$

01.02.16.0086.01

$$\left(\frac{z_1}{z_2}\right)^a = z_1^a\, z_2^{-a}\, e^{2\, i\, a\, \pi} \;/;\; \arg(z_1) \leq 0 \bigwedge \arg(z_2) \geq \arg(z_1) + \pi$$

01.02.16.0087.01

$$\left(\frac{z_1}{z_2}\right)^a = z_1^a\, z_2^{-a}\, e^{2\, i\, a\, \pi \left\lfloor \frac{\pi - \arg(z_1) + \arg(z_2)}{2\,\pi} \right\rfloor}$$

01.02.16.0024.01

$$\left(\frac{a+z}{b+z}\right)^c = \frac{1}{\left(\frac{b+z}{b-a}\right)^c} \left(\frac{a+z}{b-a}\right)^c \;/;\; a - b \in \mathbb{R} \bigwedge a \neq b$$

01.02.16.0025.01

$$\left(\frac{z}{1-z}\right)^a = z^a \left(\frac{1}{1-z}\right)^a$$

01.02.16.0026.01

$$\left(-\frac{z}{z+1}\right)^a == (-z)^a \left(\frac{1}{z+1}\right)^a$$

01.02.16.0027.01

$$\left(\frac{z_1}{z_2}\right)^a == \left(\frac{z_1}{z_1+z_2}\right)^a \left(\frac{z_1+z_2}{z_2}\right)^a \text{ /; } z_1+z_2 \in \mathbb{R} \bigwedge z_1+z_2 \neq 0$$

01.02.16.0028.01

$$\left(\frac{z_1}{z_2}\right)^a == z_1^a \left(\frac{1}{z_2}\right)^a \text{ /; } z_1+z_2 \geq 0$$

01.02.16.0029.01

$$\left(\frac{z_1}{z_2}\right)^a == (-z_1)^a \left(-\frac{1}{z_2}\right)^a \text{ /; } z_1+z_2 < 0$$

01.02.16.0030.01

$$\left(\frac{a+z}{b-z}\right)^c == \left(\frac{a+z}{a+b}\right)^c \left(\frac{a+b}{b-z}\right)^c \text{ /; } a+b \in \mathbb{R} \bigwedge a+b \neq 0$$

With respect to a

For products

01.02.16.0017.01

$$z^{a_1 a_2} == (z^{a_1})^{a_2} \text{ /; } -\pi < \mathrm{Im}(a_1 \log(z)) \leq \pi \bigvee -1 < a_1 \leq 1 \bigvee a_2 \in \mathbb{Z}$$

01.02.16.0088.01

$$z^{a_1 a_2} == (z^{a_1})^{a_2} e^{-2 i \pi a_2 k} \text{ /; } -2\pi k - \pi < \mathrm{Im}(a_1 \log(z)) \leq \pi - 2\pi k \bigwedge k \in \mathbb{Z}$$

01.02.16.0018.01

$$z^{a_1 a_2} == (z^{a_1})^{a_2} \exp\left(-2 i \pi a_2 \left\lfloor \frac{\pi - \mathrm{Im}(a_1 \log(z))}{2\pi} \right\rfloor\right)$$

Power of arguments

With respect to z

01.02.16.0089.01

$$\sqrt{z}^a == z^{a/2}$$

01.02.16.0090.01

$$\sqrt{z^2} == z \text{ /; } -\frac{\pi}{2} < \arg(z) \leq \frac{\pi}{2}$$

01.02.16.0091.01

$$\sqrt{z^2} == -z \text{ /; } -\pi < \arg(z) \leq -\frac{\pi}{2} \bigvee \frac{\pi}{2} < \arg(z) \leq \pi$$

01.02.16.0031.01

$$\sqrt{z^2} == \sqrt{-i z} \sqrt{i z}$$

01.02.16.0092.01

$$\sqrt{-z^2} == \sqrt{z} \sqrt{-z}$$

01.02.16.0032.01

$$(z^n)^{1/n} == z \;/;\; -\frac{\pi}{n} < \arg(z) \leq \frac{\pi}{n} \bigwedge n \in \mathbb{N}^+$$

01.02.16.0093.01

$$(z^n)^{1/n} == z \;/;\; \frac{\pi}{n} \leq \arg(z) < -\frac{\pi}{n} \bigwedge n \in \mathbb{Z} \bigwedge n < 0$$

01.02.16.0033.01

$$\left(z^{1/n}\right)^n == z \;/;\; n \in \mathbb{Z} \land n \neq 0$$

01.02.16.0034.01

$$\left(z^2\right)^a == z^{2a} \;/;\; -\frac{\pi}{2} < \arg(z) \leq \frac{\pi}{2}$$

01.02.16.0094.01

$$\left(z^2\right)^a == e^{2 i a \pi} z^{2a} \;/;\; -\pi < \arg(z) \leq -\frac{\pi}{2}$$

01.02.16.0035.01

$$\left(z^2\right)^a == e^{-2 i a \pi} z^{2a} \;/;\; \frac{\pi}{2} < \arg(z) \leq \pi$$

01.02.16.0095.01

$$\left(z^2\right)^a == (-i z)^a (i z)^a$$

01.02.16.0096.01

$$\left(-z^2\right)^a == (-z)^a z^a$$

01.02.16.0097.01

$$(z^{a_1})^{a_2} == z^{a_1 a_2} \;/;\; a_1 \in \mathbb{R} \bigwedge -\pi < a_1 \arg(z) \leq \pi$$

01.02.16.0098.01

$$(z^{a_1})^{a_2} == z^{a_1 a_2} e^{2 \pi i a_2 k} \;/;\; a_1 \in \mathbb{R} \bigwedge -2 \pi k - \pi < a_1 \arg(z) \leq \pi - 2 \pi k \bigwedge k \in \mathbb{Z}$$

01.02.16.0036.01

$$(z^{a_1})^{a_2} == z^{a_1 a_2} \;/;\; -\pi < \operatorname{Im}(a_1 \log(z)) \leq \pi \bigvee -1 < a_1 \leq 1 \bigvee a_2 \in \mathbb{Z}$$

01.02.16.0099.01

$$(z^{a_1})^{a_2} == z^{a_1 a_2} e^{2 \pi i a_2 k} \;/;\; -2 \pi k - \pi < \operatorname{Im}(a_1 \log(z)) \leq \pi - 2 \pi k \bigwedge k \in \mathbb{Z}$$

01.02.16.0037.01

$$(z^{a_1})^{a_2} == z^{a_1 a_2} \exp\left(2 \pi i a_2 \left\lfloor \frac{\pi - \operatorname{Im}(a_1 \log(z))}{2 \pi} \right\rfloor\right)$$

Exponent of arguments

With respect to z

01.02.16.0100.01

$$(e^z)^a == e^{a z} \;/;\; -\pi < \operatorname{Im}(z) \leq \pi$$

01.02.16.0101.01

$$(e^z)^a == e^{a (2 i \pi k + z)} \;/;\; -2 \pi k - \pi < \operatorname{Im}(z) \leq \pi - 2 \pi k \bigwedge k \in \mathbb{Z}$$

01.02.16.0102.01

$$(e^z)^a == e^{a z} e^{2 i \pi a \left\lfloor \frac{\pi - \operatorname{Im}(z)}{2 \pi} \right\rfloor}$$

Some functions of arguments

With respect to z

01.02.16.0103.01

$$(c\, z^m)^a = c^a\, z^{m a}\, e^{2 \pi i a \left\lfloor \frac{\pi - \arg(c) - \operatorname{Im}(m \log(z))}{2 \pi} \right\rfloor}$$

01.02.16.0104.01

$$(c\, e^z)^a = c^a\, e^{a z}\, e^{2 \pi i a \left\lfloor \frac{\pi - \arg(c) - \operatorname{Im}(z)}{2 \pi} \right\rfloor}$$

01.02.16.0105.01

$$(x^{a_1}\, y^{a_2})^b = e^{2 i b \pi \left\lfloor \frac{\pi - \operatorname{Im}(a_1 \log(x)) - \operatorname{Im}(a_2 \log(y))}{2 \pi} \right\rfloor}\, x^{b a_1}\, y^{b a_2}$$

01.02.16.0106.01

$$(x^{a_1}\, y^{a_2}\, z^{a_3})^b = e^{2 i b \pi \left\lfloor \frac{\pi - \operatorname{Im}(a_1 \log(x)) - \operatorname{Im}(a_2 \log(y)) - \operatorname{Im}(a_3 \log(z))}{2 \pi} \right\rfloor}\, x^{b a_1}\, y^{b a_2}\, z^{b a_3}$$

01.02.16.0107.01

$$\left(\prod_{k=1}^{n} z_k^{a_k}\right)^b = e^{2 i b \pi \left\lfloor \frac{\pi - \sum_{k=1}^{n} \operatorname{Im}(a_k \log(z_k))}{2 \pi} \right\rfloor} \prod_{k=1}^{n} z_k^{b a_k}$$

Products, sums, and powers of the direct function

Products of the direct function

01.02.16.0038.01

$$z^{a_1}\, z^{a_2} = z^{a_1 + a_2}$$

01.02.16.0040.01

$$x_1^a\, x_2^a = (x_1\, x_2)^a \;/;\; x_1 > 0 \land x_2 > 0 \lor x_1\, x_2 < 0$$

01.02.16.0041.01

$$x_1^a\, x_2^a = (-1)^{2 a}\, (x_1\, x_2)^a \;/;\; x_1 < 0 \land x_2 < 0$$

01.02.16.0042.01

$$z_1^a\, z_2^a = (z_1\, z_2)^a \;/;\; z_1 + z_2 \geq 0 \lor a \in \mathbb{Z}$$

01.02.16.0108.01

$$z_1^a\, z_2^a = (z_1\, z_2)^a \;/;\; \arg(z_1) \leq 0 \land -\arg(z_1) - \pi < \arg(z_2) \lor \arg(z_1) \geq 0 \land \arg(z_2) \leq \pi - \arg(z_1)$$

01.02.16.0109.01

$$z_1^a\, z_2^a = (z_1\, z_2)^a\, e^{2 \pi i a} \;/;\; \arg(z_1) \geq 0 \land \arg(z_2) > \pi - \arg(z_1)$$

01.02.16.0110.01

$$z_1^a\, z_2^a = (z_1\, z_2)^a\, e^{-2 \pi i a} \;/;\; \arg(z_1) \leq 0 \land \arg(z_2) \leq -\arg(z_1) - \pi$$

01.02.16.0043.01

$$z_1^a\, z_2^a = (z_1\, z_2)^a \exp\left(-2 \pi i a \left\lfloor \frac{\pi - \arg(z_1) - \arg(z_2)}{2 \pi} \right\rfloor\right)$$

01.02.16.0111.01

$$z_1^a\, z_2^{m a} = (z_1\, z_2^m)^a\, e^{-2 \pi i a \left\lfloor \frac{-\arg(z_1) - \operatorname{Im}(m \log(z_2)) + \pi}{2 \pi} \right\rfloor}$$

01.02.16.0112.01

$$z_1^{a_1}\, z_2^{a_2} = e^{\log(z_1)\, a_1 + \log(z_2)\, a_2}$$

01.02.16.0113.01

$$\prod_{k=1}^{n} z_k^{a_k} = e^{\sum_{k=1}^{n} a_k \log(z_k)} \; /;\; n \in \mathbb{N}^+$$

01.02.16.0114.01

$$x^{b\, a_1}\, y^{b\, a_2} = e^{-2\, i\, b\, \pi \left\lfloor \frac{-\operatorname{Im}(a_1 \log(x)) - \operatorname{Im}(a_2 \log(y)) + \pi}{2\pi} \right\rfloor} \left(x^{a_1}\, y^{a_2}\right)^b$$

01.02.16.0115.01

$$x^{b\, a_1}\, y^{b\, a_2}\, z^{b\, a_3} = e^{-2\, i\, b\, \pi \left\lfloor \frac{-\operatorname{Im}(a_1 \log(x)) - \operatorname{Im}(a_2 \log(y)) - \operatorname{Im}(a_3 \log(z)) + \pi}{2\pi} \right\rfloor} \left(x^{a_1}\, y^{a_2}\, z^{a_3}\right)^b$$

01.02.16.0116.01

$$\prod_{k=1}^{n} z_k^{b\, a_k} = e^{-2\, i\, b\, \pi \left\lfloor \frac{\pi - \sum_{k=1}^{n} \operatorname{Im}(a_k \log(z_k))}{2\pi} \right\rfloor} \left(\prod_{k=1}^{n} z_k^{a_k}\right)^b$$

Quotients of the direct function

01.02.16.0039.01

$$\frac{z^{a_1}}{z^{a_2}} = z^{a_1 - a_2}$$

01.02.16.0117.01

$$\frac{z_1^a}{z_2^a} = \left(\frac{z_1}{z_2}\right)^a \; /;\; \arg(z_1) \leq 0 \bigwedge \arg(z_2) < \arg(z_1) + \pi \bigvee \arg(z_1) > 0 \bigwedge \arg(z_2) \geq \arg(z_1) - \pi$$

01.02.16.0118.01

$$\frac{z_1^a}{z_2^a} = \left(\frac{z_1}{z_2}\right)^a e^{2\, i\, a\, \pi} \; /;\; \arg(z_1) \geq 0 \bigwedge \arg(z_2) < \arg(z_1) - \pi$$

01.02.16.0119.01

$$\frac{z_1^a}{z_2^a} = \left(\frac{z_1}{z_2}\right)^a e^{-2\, i\, a\, \pi} \; /;\; \arg(z_1) \leq 0 \bigwedge \arg(z_2) \geq \arg(z_1) + \pi$$

01.02.16.0120.01

$$\frac{z_1^a}{z_2^a} = \left(\frac{z_1}{z_2}\right)^a e^{-2\, i\, a\, \pi \left\lfloor \frac{\pi - \arg(z_1) + \arg(z_2)}{2\pi} \right\rfloor}$$

01.02.16.0121.01

$$\frac{z_1^{a_1}}{z_2^{a_2}} = e^{a_1 \log(z_1) - a_2 \log(z_2)}$$

01.02.16.0122.01

$$\frac{\prod_{k=1}^{n} z_k^{a_k}}{\prod_{k=1}^{m} w_k^{b_k}} = e^{\sum_{k=1}^{n} a_k \log(z_k) - \sum_{k=1}^{m} b_k \log(w_k)} \; /;\; n \in \mathbb{N}^+ \bigwedge m \in \mathbb{N}^+$$

Power of the direct function

01.02.16.0044.01

$$\left(z^{a_1}\right)^{a_2} = z^{a_1\, a_2} \; /;\; -\pi < \operatorname{Im}(a_1 \log(z)) \leq \pi \bigvee -1 < a_1 \leq 1 \bigvee a_2 \in \mathbb{Z}$$

01.02.16.0123.01

$$\left(z^{a_1}\right)^{a_2} = z^{a_1 a_2} e^{2 \pi i a_2 k} /; -2 \pi k - \pi < \operatorname{Im}(a_1 \log(z)) \leq \pi - 2 \pi k \land k \in \mathbb{Z}$$

01.02.16.0045.01

$$\left(z^{a_1}\right)^{a_2} = z^{a_1 a_2} \exp\left(2 \pi i a_2 \left\lfloor \frac{\pi - \operatorname{Im}(a_1 \log(z))}{2 \pi} \right\rfloor\right)$$

01.02.16.0124.01

$$\left(z^m\right)^{1/n} = z^{m/n} \left(e^{\frac{i (2 k - 1) \pi}{m}}\right)^{-\frac{m}{n}} (-1)^{1/n} /; m \in \mathbb{N}^+ \bigwedge n \in \mathbb{N}^+ \bigwedge \frac{(2 k - 3) \pi}{m} < \arg(z) \leq \frac{(2 k - 1) \pi}{m} \bigwedge 1 \leq k \leq \left\lfloor \frac{m + 1}{2} \right\rfloor$$

01.02.16.0125.01

$$\left(z^m\right)^{1/n} = z^{m/n} \left(e^{-\frac{i (2 k - 1) \pi}{m}}\right)^{-\frac{m}{n}} (-1)^{1/n} /; m \in \mathbb{N}^+ \bigwedge n \in \mathbb{N}^+ \bigwedge -\frac{(2 k + 1) \pi}{m} < \arg(z) \leq -\frac{(2 k - 1) \pi}{m} \bigwedge 1 \leq k \leq \left\lfloor \frac{m - 1}{2} \right\rfloor$$

01.02.16.0126.01

$$\left(z^m\right)^{1/n} = z^{m/n} (-1)^{-\frac{m}{n}} /; \frac{m}{2} \in \mathbb{N}^+ \bigwedge n \in \mathbb{N}^+ \bigwedge \pi - \frac{\pi}{m} < \arg(z) \leq \pi \bigvee -\pi < \arg(z) \leq -\pi + \frac{\pi}{m}$$

Sum of the direct function

01.02.16.0127.01

$$z^a + (-z)^a = 2 \left(-z^2\right)^{a/2} \cos\left(\frac{a \pi}{2}\right)$$

01.02.16.0128.01

$$z^a - (-z)^a = 2 z \left(-z^2\right)^{\frac{a-1}{2}} \sin\left(\frac{a \pi}{2}\right)$$

Power of the product of the direct function

01.02.16.0129.01

$$\left(x^{a_1} y^{a_2}\right)^b = e^{2 i b \pi \left\lfloor \frac{\pi - \operatorname{Im}(a_1 \log(x)) - \operatorname{Im}(a_2 \log(y))}{2 \pi} \right\rfloor} x^{b a_1} y^{b a_2}$$

01.02.16.0130.01

$$\left(x^{a_1} y^{a_2} z^{a_3}\right)^b = e^{2 i b \pi \left\lfloor \frac{\pi - \operatorname{Im}(a_1 \log(x)) - \operatorname{Im}(a_2 \log(y)) - \operatorname{Im}(a_3 \log(z))}{2 \pi} \right\rfloor} x^{b a_1} y^{b a_2} z^{b a_3}$$

01.02.16.0131.01

$$\left(\prod_{k=1}^n z_k^{a_k}\right)^b = e^{2 i b \pi \left\lfloor \frac{\pi - \sum_{k=1}^n \operatorname{Im}(a_k \log(z_k))}{2 \pi} \right\rfloor} \prod_{k=1}^n z_k^{b a_k}$$

01.02.16.0132.01

$$\left((-1)^r z^r\right)^b = (-1)^{b r} z^{b r} /; r \in \mathbb{R} \bigwedge 0 < r \leq \frac{1}{2}$$

Related transformations

01.02.16.0046.01

$$\left(z^{\frac{1}{z-1}}\right)^{z^{\frac{z}{z-1}}} = \left(z^{\frac{z}{z-1}}\right)^{z^{\frac{1}{z-1}}}$$

Identities

Functional identities

Univariate functional identities

01.02.17.0001.01

$$z^{-a} = \frac{1}{z^a}$$

01.02.17.0005.01

$$z^a = \left(\frac{z}{b}\right)^a \left(\frac{1}{b}\right)^{a\left\lfloor \frac{\arg(z-b)}{2\pi} \right\rfloor} b^{a\left(\left\lfloor \frac{\arg(z-b)}{2\pi} \right\rfloor+1\right)}$$

01.02.17.0006.01

$$z^a = x^a \left(\frac{z}{x}\right)^a e^{2 a \pi i \left\lfloor \frac{\arg(z-x)}{2\pi} \right\rfloor} /; x < 0$$

Biivariate functional identities

01.02.17.0002.01

$$g\left(\frac{x+y}{x-y}\right) = \frac{g(x)+g(y)}{g(x)-g(y)} /; g(x) = x^1 \bigwedge x \in \mathbb{R} \bigwedge y \in \mathbb{R} \bigwedge x \neq y$$

01.02.17.0003.01

$$|g(x + i y)| = |g(x) + g(i y)| /; g(z) = c z^1 \bigwedge x \in \mathbb{R} \bigwedge y \in \mathbb{R}$$

Other identities

01.02.17.0004.01

$$\sum_{k=1}^{n} \frac{1}{z_k} \left(-1 \left(1 + z_k^{n-1}\right) \left(\prod_{j=1}^{k-1} \frac{z_j}{z_j - z_k}\right) \left(\prod_{j=k+1}^{n} \frac{z_j}{z_j - z_k}\right)\right) = 0 /; n \in \mathbb{N}^+ \bigwedge z_j \neq 0$$

Complex characteristics

Real part

01.02.19.0001.01

$$\operatorname{Re}(x^a) = x^{\operatorname{Re}(a)} \cos(\operatorname{Im}(a) \log(x)) /; x \in \mathbb{R} \land x > 0$$

01.02.19.0002.01

$$\operatorname{Re}(z^a) = |z|^a \cos\left(a \tan^{-1}(\operatorname{Re}(z), \operatorname{Im}(z))\right) /; a \in \mathbb{R}$$

01.02.19.0003.01

$$\operatorname{Re}(z^a) = |z|^a \cos(a \arg(z)) /; a \in \mathbb{R}$$

01.02.19.0004.01

$$\operatorname{Re}(z^a) = \frac{1}{2} \left(\left(\operatorname{Re}(z) - \operatorname{Re}(z) \sqrt{-\frac{\operatorname{Im}(z)^2}{\operatorname{Re}(z)^2}}\right)^a + \left(\operatorname{Re}(z) + \operatorname{Re}(z) \sqrt{-\frac{\operatorname{Im}(z)^2}{\operatorname{Re}(z)^2}}\right)^a\right) /; a \in \mathbb{R} \land \operatorname{Re}(z) \neq 0$$

01.02.19.0005.01

$$\operatorname{Re}((i y)^a) = |y|^a \cos\left(\frac{\pi a}{2}\right) /; a \in \mathbb{R} \bigwedge y \in \mathbb{R}$$

01.02.19.0006.01

$$\operatorname{Re}(z^a) = \sum_{j=0}^{\infty}\sum_{l=0}^{\infty} \frac{(-a)_{j+l}}{(l-j)!\, j!\, \left(\frac{1}{2}\right)_j} (1-\operatorname{Re}(z))^l \left(\frac{\operatorname{Im}(z)^2}{4(\operatorname{Re}(z)-1)}\right)^j /;\ a \in \mathbb{R}$$

01.02.19.0007.01

$$\operatorname{Re}(z^a) = F_{0\times 1\times 1}^{1\times 0\times 0}\left(\begin{matrix} -a;;; \\ ;\frac{1}{2};\frac{1}{2}; \end{matrix} \frac{1}{2}\left(1-\operatorname{Re}(z)+\sqrt{\operatorname{Im}(z)^2+(1-\operatorname{Re}(z))^2}\right), \frac{1}{2}\left(1-\operatorname{Re}(z)-\sqrt{\operatorname{Im}(z)^2+(1-\operatorname{Re}(z))^2}\right)\right) /;\ a \in \mathbb{R}$$

01.02.19.0008.01

$$\operatorname{Re}(z^n) = \sum_{j=0}^{\left\lfloor \frac{n}{2} \right\rfloor} (-1)^j \binom{n}{2j} \operatorname{Im}(z)^{2j} \operatorname{Re}(z)^{n-2j} /;\ n \in \mathbb{N}^+$$

01.02.19.0009.01

$$\operatorname{Re}(z^a) = |z|^{\operatorname{Re}(a)} e^{-\operatorname{Im}(a)\arg(z)} \cos(\operatorname{Im}(a)\log(|z|) + \arg(z)\operatorname{Re}(a))$$

01.02.19.0010.01

$$\operatorname{Re}(z^a) = \exp\left(-\tan^{-1}(\operatorname{Re}(z), \operatorname{Im}(z))\operatorname{Im}(a)\right) |z|^{\operatorname{Re}(a)} \cos\left(\operatorname{Im}(a)\log(|z|) + \tan^{-1}(\operatorname{Re}(z), \operatorname{Im}(z))\operatorname{Re}(a)\right)$$

Imaginary part

01.02.19.0011.01

$$\operatorname{Im}(x^a) = x^{\operatorname{Re}(a)} \sin(\operatorname{Im}(a)\log(x)) /;\ x \in \mathbb{R} \land x > 0$$

01.02.19.0012.01

$$\operatorname{Im}(z^a) = |z|^a \sin\left(a \tan^{-1}(\operatorname{Re}(z), \operatorname{Im}(z))\right) /;\ a \in \mathbb{R}$$

01.02.19.0013.01

$$\operatorname{Im}(z^a) = |z|^a \sin(a \arg(z)) /;\ a \in \mathbb{R}$$

01.02.19.0014.01

$$\operatorname{Im}(z^a) = \frac{\operatorname{Re}(z)}{2\operatorname{Im}(z)} \sqrt{-\frac{\operatorname{Im}(z)^2}{\operatorname{Re}(z)^2}} \left(\left(\operatorname{Re}(z) - \operatorname{Re}(z)\sqrt{-\frac{\operatorname{Im}(z)^2}{\operatorname{Re}(z)^2}}\right)^a - \left(\sqrt{-\frac{\operatorname{Im}(z)^2}{\operatorname{Re}(z)^2}}\operatorname{Re}(z) + \operatorname{Re}(z)\right)^a\right) /;\ a \in \mathbb{R} \land \operatorname{Re}(z) \neq 0$$

01.02.19.0015.01

$$\operatorname{Im}((i\, y)^a) = \operatorname{sgn}(y)\, |y|^a \sin\left(\frac{\pi a}{2}\right) /;\ a \in \mathbb{R} \bigwedge y \in \mathbb{R}$$

01.02.19.0016.01

$$\operatorname{Im}(z^n) = \sum_{j=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} (-1)^j \binom{n}{2j+1} \operatorname{Im}(z)^{2j+1} \operatorname{Re}(z)^{n-2j-1} /;\ n \in \mathbb{N}^+$$

01.02.19.0017.01

$$\operatorname{Im}(z^a) = |z|^{\operatorname{Re}(a)} e^{-\operatorname{Im}(a)\arg(z)} \sin(\operatorname{Im}(a)\log(|z|) + \arg(z)\operatorname{Re}(a))$$

01.02.19.0018.01

$$\operatorname{Im}(z^a) = \exp\left(-\operatorname{Im}(a)\tan^{-1}(\operatorname{Re}(z), \operatorname{Im}(z))\right) |z|^{\operatorname{Re}(a)} \sin\left(\operatorname{Im}(a)\log(|z|) + \tan^{-1}(\operatorname{Re}(z), \operatorname{Im}(z))\operatorname{Re}(a)\right)$$

Absolute value

01.02.19.0019.01

$$|x^a| = x^{\operatorname{Re}(a)} /;\ x > 0$$

01.02.19.0020.01

$$|z^a| = |z|^a \;/;\; a \in \mathbb{R}$$

01.02.19.0021.01

$$|z^a| = \exp(i\, a\, \mathrm{Im}(\log(z))) \;/;\; i\, a \in \mathbb{R}$$

01.02.19.0022.01

$$|z^a| = \exp(i\, a \arg(z)) \;/;\; i\, a \in \mathbb{R}$$

01.02.19.0023.01

$$|z^a| = \exp(\mathrm{Re}(a \log(z)))$$

01.02.19.0024.01

$$|z^a| = \exp(\mathrm{Re}(a) \log(|z|) - \mathrm{Im}(a) \arg(z))$$

01.02.19.0025.01

$$|z^a| = |z|^{\mathrm{Re}(a)} \exp\left(-\mathrm{Im}(a) \tan^{-1}(\mathrm{Re}(z), \mathrm{Im}(z))\right)$$

01.02.19.0026.01

$$|z^a| = |z|^{\mathrm{Re}(a)} \exp(-\mathrm{Im}(a) \arg(z))$$

Argument

01.02.19.0027.01

$$\arg(x^a) = \tan^{-1}(\cos(\mathrm{Im}(a) \log(x)), \sin(\mathrm{Im}(a) \log(x))) \;/;\; x > 0$$

01.02.19.0028.01

$$\arg(z^a) = a \arg(z) \;/;\; a \in \mathbb{R} \bigwedge -\pi < a \arg(z) \leq \pi$$

01.02.19.0029.01

$$\arg(z^a) = \arg\left(e^{i\, a \arg(z)}\right) \;/;\; a \in \mathbb{R}$$

01.02.19.0030.01

$$\arg(z^a) = \tan^{-1}\left(\cos\left(a \tan^{-1}(\mathrm{Re}(z), \mathrm{Im}(z))\right), \sin\left(a \tan^{-1}(\mathrm{Re}(z), \mathrm{Im}(z))\right)\right) \;/;\; a \in \mathbb{R}$$

01.02.19.0031.01

$$\arg(z^a) = \mathrm{Im}(a \log(z)) \;/;\; -\frac{\pi}{\log(z)} < \mathrm{Im}(a) \leq \frac{\pi}{\log(z)} \bigwedge z > 0$$

01.02.19.0032.01

$$\arg(z^a) = \mathrm{Im}(a) \log(|z|) + \arg(z) \mathrm{Re}(a) \;/;\; -\frac{\pi}{\log(z)} < \mathrm{Im}(a) \leq \frac{\pi}{\log(z)} \bigwedge z > 0$$

01.02.19.0033.01

$$\arg(z^a) = a \arg(z) + 2\pi \left\lfloor \frac{\pi - a \arg(z)}{2\pi} \right\rfloor$$

01.02.19.0034.01

$$\arg(z^a) = 2\pi \left\lfloor \frac{\pi - \mathrm{Im}(a \log(z))}{2\pi} \right\rfloor + \mathrm{Im}(a \log(z))$$

01.02.19.0035.01

$$\arg(z^a) = \mathrm{Im}(a) \log(|z|) + \arg(z) \mathrm{Re}(a) + 2\pi \left\lfloor \frac{\pi - \mathrm{Im}(a) \log(|z|) - \arg(z) \mathrm{Re}(a)}{2\pi} \right\rfloor$$

01.02.19.0036.01

$$\arg(z^a) = \tan^{-1}\left(\cos\left(\mathrm{Im}(a) \log(|z|) + \tan^{-1}(\mathrm{Re}(z), \mathrm{Im}(z)) \mathrm{Re}(a)\right), \sin\left(\mathrm{Im}(a) \log(|z|) + \tan^{-1}(\mathrm{Re}(z), \mathrm{Im}(z)) \mathrm{Re}(a)\right)\right)$$

Conjugate value

01.02.19.0037.01

$$\overline{x^a} = x^{\mathrm{Re}(a)} \left(\cos(\mathrm{Im}(a) \log(x)) - i \sin(\mathrm{Im}(a) \log(x))\right) /; x \in \mathbb{R} \land x > 0$$

01.02.19.0049.01

$$z^{\overline{a}} = z^{a\, e^{-2 i \arg(a)}}$$

01.02.19.0050.01

$$\overline{z}^a = z^a\, e^{2 i a \left(\pi \left\lfloor \frac{\arg(z)+\pi}{2\pi} \right\rfloor - \arg(z)\right)}$$

01.02.19.0038.01

$$\overline{z^a} = |z|^a \left(\cos\left(a \tan^{-1}(\mathrm{Re}(z), \mathrm{Im}(z))\right) - i \sin\left(a \tan^{-1}(\mathrm{Re}(z), \mathrm{Im}(z))\right)\right) /; a \in \mathbb{R}$$

01.02.19.0039.01

$$\overline{z^a} = |z|^{\mathrm{Re}(a)} \exp(-\arg(z) \mathrm{Im}(a) - i (\mathrm{Im}(a) \log(|z|) + \arg(z) \mathrm{Re}(a)))$$

01.02.19.0040.01

$$\overline{z^a} = \exp\left(-\mathrm{Im}(a) \tan^{-1}(\mathrm{Re}(z), \mathrm{Im}(z))\right) |z|^{\mathrm{Re}(a)} \left(\cos\left(\mathrm{Im}(a) \log(|z|) + \tan^{-1}(\mathrm{Re}(z), \mathrm{Im}(z)) \mathrm{Re}(a)\right) - i \sin\left(\mathrm{Im}(a) \log(|z|) + \tan^{-1}(\mathrm{Re}(z), \mathrm{Im}(z)) \mathrm{Re}(a)\right)\right)$$

01.02.19.0051.01

$$\overline{z^a} = z^{a \cos(2 \arg(a)) - i a \sin(2 \arg(a))}\, e^{-2 a \arg(z) (i \cos(2 \arg(a)) + \sin(2 \arg(a)))}$$

Signum value

01.02.19.0041.01

$$\mathrm{sgn}(x^a) = x^{i\, \mathrm{Im}(a)} /; x > 0$$

01.02.19.0042.01

$$\mathrm{sgn}(z^a) = \mathrm{sgn}(z)^a /; a \in \mathbb{R}$$

01.02.19.0043.01

$$\mathrm{sgn}(z^a) = \exp(a\, \mathrm{Re}(\log(z))) /; i\, a \in \mathbb{R}$$

01.02.19.0044.01

$$\mathrm{sgn}(z^a) = |z|^a /; i\, a \in \mathbb{R}$$

01.02.19.0045.01

$$\mathrm{sgn}(z^a) = z^a \exp(-\mathrm{Re}(a \log(z)))$$

01.02.19.0046.01

$$\mathrm{sgn}(z^a) = |z|^{i\, \mathrm{Im}(a)} \exp(i\, \mathrm{Re}(a) \arg(z))$$

01.02.19.0047.01

$$\mathrm{sgn}(z^a) = |z|^{i\, \mathrm{Im}(a)} \exp\left(i\, \mathrm{Re}(a) \tan^{-1}(\mathrm{Re}(z), \mathrm{Im}(z))\right)$$

01.02.19.0048.01

$$\mathrm{sgn}(z^a) = \exp(i (\mathrm{Im}(a) \log(|z|) + \arg(z) \mathrm{Re}(a)))$$

Differentiation

Low-order differentiation

With respect to *z*

01.02.20.0001.01

$$\frac{\partial z^a}{\partial z} == a\, z^{a-1}$$

01.02.20.0002.01

$$\frac{\partial^2 z^a}{\partial z^2} == a\,(a-1)\, z^{a-2}$$

With respect to *a*

01.02.20.0003.01

$$\frac{\partial z^a}{\partial a} == z^a \log(z)$$

01.02.20.0004.01

$$\frac{\partial^2 z^a}{\partial a^2} == z^a \log^2(z)$$

Symbolic differentiation

With respect to *z*

01.02.20.0005.02

$$\frac{\partial^n z^a}{\partial z^n} == (a-n+1)_n\, z^{a-n} \;/;\; n \in \mathbb{N}$$

01.02.20.0021.01

$$\frac{\partial^n (b+c\,z)^a}{\partial z^n} == c^n\,(a-n+1)_n\,(b+c\,z)^{a-n} \;/;\; n \in \mathbb{N}$$

01.02.20.0022.01

$$\frac{\partial^n \left(c\,z^2+b\right)^a}{\partial z^n} == \sum_{k=0}^{n} \frac{(2\,k-n+1)_{2\,(n-k)}\,(a-k+1)_k\, c^k \left(c\,z^2+b\right)^{a-k}}{(n-k)!\,(2\,z)^{n-2\,k}} \;/;\; n \in \mathbb{N}$$

01.02.20.0006.02

$$\frac{\partial^n \left(c\,z^2+b\right)^a}{\partial z^n} == 2^n\,(c\,z)^n \left(c\,z^2+b\right)^{a-n} (a-n+1)_n\; {}_2F_1\!\left(\frac{1-n}{2}, -\frac{n}{2};\, a-n+1;\, \frac{b}{c\,z^2}+1\right) /;\; n \in \mathbb{N}$$

01.02.20.0023.01

$$\frac{\partial^n \left(b+c\,\sqrt{z}\,\right)^a}{\partial z^n} == \sum_{k=0}^{n} \frac{(-1)^{n-k}\,(k)_{2\,(n-k)}\,(a-k+1)_k\, c^k \left(b+c\,\sqrt{z}\,\right)^{a-k}}{(n-k)!\left(2\,\sqrt{z}\,\right)^{2\,n-k}} \;/;\; n \in \mathbb{N}$$

01.02.20.0024.01

$$\frac{\partial^n (c\,z^\nu+b)^a}{\partial z^n} == \sum_{k=0}^{n}\sum_{j=0}^{k} \frac{(-1)^j\,(-n-\nu\,j+k\,\nu+1)_n\,(a-k+1)_k\, c^k\,(c\,z^\nu+b)^{a-k}}{j!\,(k-j)!\,z^{n-\nu\,k}} \;/;\; n \in \mathbb{N}$$

01.02.20.0007.02

$$\frac{\partial^n g(z)^a}{\partial z^n} == a \binom{n-a}{n} \sum_{k=0}^{n} \frac{(-1)^k}{a-k} \binom{n}{k} g(z)^{a-k}\, \frac{\partial^n g(z)^k}{\partial z^n} \;/;\; n \in \mathbb{N}$$

01.02.20.0008.02

$$\frac{\partial^n \frac{1}{g(z)}}{\partial z^n} == (n+1)\sum_{k=0}^{n} \frac{(-1)^k}{k+1} \binom{n}{k} g(z)^{-k-1} \frac{\partial^n g(z)^k}{\partial z^n} /; n \in \mathbb{N}$$

01.02.20.0025.01

$$\frac{\partial^n \frac{f(z)}{g(z)}}{\partial z^n} == n! \sum_{k=0}^{n} \frac{\partial^{n-k} f(z)}{\partial z^{n-k}} \sum_{j=0}^{k} \frac{(-1)^j (k+1)\, g(z)^{-j-1}}{(j+1)!\,(n-k)!\,(k-j)!} \frac{\partial^k g(z)^j}{\partial z^k} /; n \in \mathbb{N}$$

01.02.20.0009.02

$$\frac{\partial^n f(z^a)}{\partial z^n} == \sum_{k=0}^{n}\sum_{j=0}^{k} \frac{(-1)^j (a\,k - a\,j - n + 1)_n\, f^{(k)}(z^a)}{j!\,(k-j)!\, z^{n-a\,k}} /; n \in \mathbb{N}$$

01.02.20.0010.02

$$\frac{\partial^n f(z^2)}{\partial z^n} == \sum_{k=0}^{n} \frac{(2\,k-n+1)_{2\,(n-k)}}{(n-k)!\,(2\,z)^{n-2\,k}} f^{(k)}(z^2) /; n \in \mathbb{N}$$

01.02.20.0011.01

$$\frac{\partial^n f\left(\frac{1}{z}\right)}{\partial z^n} == (-1)^n (n-1)! \sum_{k=1}^{n} \frac{1}{(k-1)!\, z^{k+n}} \binom{n}{k} f^{(k)}\left(\frac{1}{z}\right) /; n \in \mathbb{N}^+$$

01.02.20.0012.02

$$\frac{\partial^n f\left(\sqrt{z}\right)}{\partial z^n} == \sum_{k=0}^{n} \frac{(-1)^{n-k} (k)_{2\,(n-k)}}{(n-k)!\left(2\sqrt{z}\right)^{2\,n-k}} f^{(k)}\left(\sqrt{z}\right) /; n \in \mathbb{N}$$

With respect to *a*

01.02.20.0013.02

$$\frac{\partial^n z^a}{\partial a^n} == z^a \log^n(z) /; n \in \mathbb{N}$$

01.02.20.0014.02

$$\frac{\partial^n f(z^a)}{\partial a^n} == \log^n(z) \sum_{k=0}^{n} z^{k\,a}\, \mathcal{S}_n^{(k)}\, f^{(k)}(z^a) /; n \in \mathbb{N}$$

Fractional integro-differentiation

With respect to *z*

01.02.20.0015.01

$$\frac{\partial^\alpha z^a}{\partial z^\alpha} == \frac{\Gamma(a+1)\, z^{a-\alpha}}{\Gamma(a-\alpha+1)} /; -a \notin \mathbb{N}^+$$

01.02.20.0016.01

$$\frac{\partial^\alpha z^a}{\partial z^\alpha} == \mathcal{FC}_{\exp}^{(\alpha)}(z, a)\, z^{a-\alpha}$$

01.02.20.0026.01

$$\frac{\partial^\alpha z^a}{\partial z^\alpha} = \begin{cases} (-1)^\alpha (-a)_\alpha z^{a-\alpha} & \alpha \in \mathbb{Z} \land a \in \mathbb{Z} \land a < 0 \land a < \alpha \\ \frac{(-1)^{a-1} (\log(z)+\psi(-a)-\psi(a-\alpha+1))}{(-a-1)!\, \Gamma(a-\alpha+1)} z^{a-\alpha} & a \in \mathbb{Z} \land a < 0 \\ \frac{\Gamma(a+1)}{\Gamma(a-\alpha+1)} z^{a-\alpha} & \text{True} \end{cases}$$

01.02.20.0027.01

$$\frac{\partial^\alpha (b\, z^c)^a}{\partial z^\alpha} = \begin{cases} (-1)^\alpha (-a\, c)_\alpha (b\, z^c)^a z^{-\alpha} & \alpha \in \mathbb{Z} \land a\, c \in \mathbb{Z} \land a\, c < 0 \land a\, c < \alpha \\ \frac{(-1)^{a\, c-1} (\log(b\, z^c)+c\, \psi(-a\, c)-c\, \psi(a\, c-\alpha+1))}{c\, (-a\, c-1)!\, \Gamma(a\, c-\alpha+1)} (b\, z^c)^a z^{-\alpha} & a\, c \in \mathbb{Z} \land a\, c < 0 \\ \frac{\Gamma(a\, c+1)}{\Gamma(a\, c-\alpha+1)} (b\, z^c)^a z^{-\alpha} & \text{True} \end{cases}$$

01.02.20.0017.01

$$\frac{\partial^\alpha (b + c\, z)^a}{\partial z^\alpha} = b\, z^{-\alpha} (b + c\, z)^{a-1} \left(\frac{c\, z}{b} + 1\right)^{1-a} {}_2\tilde{F}_1\left(-a, 1; 1-\alpha; -\frac{c\, z}{b}\right)$$

01.02.20.0018.01

$$\frac{\partial^\alpha \left(c\, z^2 + b\right)^a}{\partial z^\alpha} = \frac{b^a z^{-\alpha}}{\Gamma(1-\alpha)} \left(\frac{\sqrt{-\frac{b}{c}}}{\sqrt{-\frac{b}{c}} - z}\right)^{-a} \left(\frac{\sqrt{-\frac{b}{c}}}{z + \sqrt{-\frac{b}{c}}}\right)^{-a} F_1\left(-\alpha; -a, -a; 1-\alpha; \frac{z}{z + \sqrt{-\frac{b}{c}}}, -\frac{z}{\sqrt{-\frac{b}{c}} - z}\right) /; \alpha \notin \mathbb{N}^+$$

01.02.20.0019.01

$$\frac{\partial^\alpha \left(c\, z^2 + b\right)^a}{\partial z^\alpha} = b^a z^{-\alpha} \left(\frac{\sqrt{-\frac{b}{c}}}{\sqrt{-\frac{b}{c}} - z}\right)^{-a} \left(\frac{\sqrt{-\frac{b}{c}}}{z + \sqrt{-\frac{b}{c}}}\right)^{-a} \tilde{F}^{1\times1\times1}_{1\times0\times0}\left(\begin{matrix} -\alpha; a; a; \\ 1-\alpha;;; \end{matrix} \frac{z}{z + \sqrt{-\frac{b}{c}}}, -\frac{z}{\sqrt{-\frac{b}{c}} - z}\right)$$

01.02.20.0028.01

$$\frac{\partial^\alpha f(z)^a}{\partial z^\alpha} = \frac{a}{\Gamma(1-a)} e^{2\, i\, a\, \pi \left\lfloor \frac{1}{2} - \frac{\arg\left(\frac{f(z)}{f(0)}\right)}{2\pi} - \frac{\arg(f(0))}{2\pi} \right\rfloor} f(0)^a \sum_{k=0}^{\infty} \frac{\Gamma(k-a+1)}{\Gamma(k-\alpha+1)} \sum_{j=0}^{k} \frac{(-1)^j}{a-j} \binom{k}{j} p_{j,k}\, z^{k-\alpha} /;$$

$$p_{j,0} = 1 \bigwedge p_{j,k} = \frac{1}{f(0)\, k} \sum_{m=1}^{k} \frac{j\, m + m - k}{m!} f^{(m)}(0)\, p_{j,k-m} \bigwedge k \in \mathbb{N}^+ \bigwedge f(0) \neq 0$$

With respect to *a*

01.02.20.0020.01

$$\frac{\partial^\alpha z^a}{\partial a^\alpha} = a^{-\alpha} z^a (a \log(z))^\alpha Q(-\alpha, 0, a \log(z))$$

Integration

Indefinite integration

For the direct function with respect to *z*

01.02.21.0001.01

$$\int z^a \, da z = \frac{z^{a+1}}{a+1} \; /; \; a \neq -1$$

01.02.21.0002.01

$$\int \frac{1}{z} \, dz = \log(z)$$

For the direct function with respect to *a*

01.02.21.0003.01

$$\int z^a \, da = \frac{z^a}{\log(z)}$$

01.02.21.0004.01

$$\int a^{\alpha-1} z^a \, da = -a^\alpha \, (-a \log(z))^{-\alpha} \, \Gamma(\alpha, -a \log(z))$$

Definite integration

For the direct function itself

01.02.21.0005.01

$$\int_0^1 t^a \, dt = \frac{1}{a+1} \; /; \; \mathrm{Re}(a) > -1$$

01.02.21.0006.01

$$\int_0^1 z^t \, dt = \frac{z-1}{\log(z)}$$

Involving the direct function

01.02.21.0007.01

$$\int_0^1 t^a \, (t+1)^b \, dt = \frac{1}{a+1} \, {}_2F_1(a+1, -b; a+2; -1) \; /; \; \mathrm{Re}(a) > -1$$

01.02.21.0009.01

$$\int_0^1 \frac{1}{\sqrt{1-t^2} \, \sqrt{1-m\,t^2}} \, dt = K(m)$$

01.02.21.0010.01

$$\int_0^1 \frac{\sqrt{1-m\,t^2}}{\sqrt{1-t^2}} \, dt = E(m)$$

01.02.21.0011.01

$$\int_0^1 \frac{1}{\left(1-n\,t^2\right) \sqrt{1-t^2} \, \sqrt{1-m\,t^2}} \, dt = \Pi(n \mid m)$$

01.02.21.0012.01

$$\int_1^\infty \frac{1}{\sqrt{1-t^2} \, \sqrt{1-m\,t^2}} \, dt = -i\, K(1-m) - \frac{i \sqrt{m}}{\sqrt{-m}} \, K(m)$$

01.02.21.0013.01

$$\int_1^{\infty} \frac{1}{\sqrt{1-t^2}\,\sqrt{1-m\,t^2}}\,dt = -i\left(1-\sqrt{\frac{m-1}{m}}\,\sqrt{\frac{m}{m-1}}\right)K(1-m) - \frac{i}{\sqrt{-m}}\,K\!\left(\frac{1}{m}\right)$$

01.02.21.0014.01

$$\int_1^{\infty}\left(\frac{\sqrt{1-m\,t^2}}{\sqrt{1-t^2}} - \frac{\sqrt{-m\,t^2}}{\sqrt{-t^2}}\right)dt =$$

$$-i\,\sqrt{-m}\;-\sqrt{m}\,\left(E\!\left(\frac{1}{m}\right)+\left(1-\sqrt{m-1}\,\sqrt{\frac{1}{m-1}}\right)i\left(\frac{1}{m}\,K\!\left(1-\frac{1}{m}\right)-E\!\left(1-\frac{1}{m}\right)\right)+\frac{1-m}{m}\,K\!\left(\frac{1}{m}\right)\right)$$

01.02.21.0015.01

$$\int_1^{\infty}\frac{1}{\left(1-n\,t^2\right)\sqrt{1-t^2}\,\sqrt{1-m\,t^2}}\,dt =$$

$$-\frac{i}{\sqrt{-m}}\left(K\!\left(\frac{1}{m}\right)+i\left(1-\sqrt{\frac{m}{m-1}}\,\sqrt{\frac{m-1}{m}}\right)\left(K\!\left(1-\frac{1}{m}\right)+\frac{n}{m-n}\,\Pi\!\left(\frac{m-1}{m-n}\,\middle|\,\frac{m-1}{m}\right)\right)-\Pi\!\left(\frac{1}{n}\,\middle|\,\frac{1}{m}\right)\right)$$

01.02.21.0016.01

$$\int_0^{\infty}\frac{1}{\sqrt{1-t^2}\,\sqrt{1-m\,t^2}}\,dt = \left(1-\frac{i\,\sqrt{m}}{\sqrt{-m}}\right)K(m) - i\,K(1-m)$$

01.02.21.0017.01

$$\int_0^{\infty}\frac{1}{\sqrt{1-t^2}\,\sqrt{1-m\,t^2}}\,dt = K(m) - \frac{i}{\sqrt{-m}}\,K\!\left(\frac{1}{m}\right)+\left(1-\sqrt{\frac{m-1}{m}}\,\sqrt{\frac{m}{m-1}}\right)\frac{\sqrt{m}}{\sqrt{-m}}\,K(1-m)$$

01.02.21.0018.01

$$\int_0^{\infty}\frac{1}{\sqrt{1-t^2}\,\sqrt{1-m\,t^2}}\,dt = K(m) - \frac{i}{\sqrt{-m}}\,K\!\left(\frac{1}{m}\right)+\frac{1}{\sqrt{-m}}\left(1-\sqrt{\frac{m-1}{m}}\,\sqrt{\frac{m}{m-1}}\right)K\!\left(\frac{m-1}{m}\right)$$

01.02.21.0019.01

$$\int_0^{\infty}\left(\frac{\sqrt{1-m\,t^2}}{\sqrt{1-t^2}} - \frac{\sqrt{-m\,t^2}}{\sqrt{-t^2}}\right)dt =$$

$$E(m)-\sqrt{m}\,\left(E\!\left(\frac{1}{m}\right)+\left(1-\sqrt{m-1}\,\sqrt{\frac{1}{m-1}}\right)i\left(\frac{1}{m}\,K\!\left(1-\frac{1}{m}\right)-E\!\left(1-\frac{1}{m}\right)\right)+\frac{1-m}{m}\,K\!\left(\frac{1}{m}\right)\right)$$

Involving related functions

01.02.21.0008.01

$$\int_0^1 t^a \log(t)\,dt = -\frac{1}{(a+1)^2} \;/;\; \mathrm{Re}(a) > -1$$

Multiple integration

01.02.21.0020.01

$$\int_0^1 \int_0^1 \frac{1}{(x^2 y^2 + 1)\sqrt{\sqrt{\log^2(x y) + \frac{\pi^2}{4}} - \frac{\pi}{2}}} \, dy \, dx == \frac{1}{2} \sqrt{\pi} \left(1 - \frac{\sqrt{2}}{4}\right) \zeta\left(\frac{3}{2}\right)$$

01.02.21.0021.01

$$\int_0^1 \int_0^1 \frac{1}{(x^2 y^2 + 1)\sqrt{\sqrt{\log^2(x y) + \frac{\pi^2}{4}} - \frac{\pi}{2}} \sqrt{\log^2(x y) + \frac{\pi^2}{4}}} \, dy \, dx == \sqrt{\pi} \left(\frac{\sqrt{2}}{2} - 1\right) \zeta\left(\frac{1}{2}\right)$$

Integral transforms

Fourier exp transforms

01.02.22.0001.01

$$\mathcal{F}_t[t^n](x) == (-i)^n \sqrt{2\pi} \, \delta^{(n)}(x) /; n \in \mathbb{N}$$

01.02.22.0002.01

$$\mathcal{F}_t[t^n \theta(t)](x) == \frac{i^{n+1} n! + i^{-n} \pi x^{n+1} \delta^{(n)}(x)}{\sqrt{2\pi} \, x^{n+1}} /; n \in \mathbb{N}$$

01.02.22.0029.01

$$\mathcal{F}_t\left[\frac{1}{t}\right](x) = i \sqrt{\frac{\pi}{2}} \operatorname{sgn}(x)$$

01.02.22.0003.01

$$\mathcal{F}_t\left[\frac{1}{t^n}\right](x) == \frac{i^n}{(n-1)!} \sqrt{\frac{\pi}{2}} \, x^{n-1} \operatorname{sgn}(x) /; n \in \mathbb{N}^+$$

01.02.22.0004.01

$$\mathcal{F}_t\left[\frac{\theta(t)}{t^n}\right](x) == \frac{x^{n-1} i^{n-1}}{2\sqrt{2\pi} \, (n-1)!} \left(2\psi(n) - \log(x^2) + i \pi \operatorname{sgn}(x)\right) /; n \in \mathbb{N}^+$$

01.02.22.0005.01

$$\mathcal{F}_t[t^a](z) == -\frac{i}{\sqrt{2\pi}} (z^2)^{-\frac{1+a}{2}} \Gamma(a+1) \left((-1 + (-1)^a) \cos\left(\frac{a\pi}{2}\right) \operatorname{sgn}(z) - i (1 + (-1)^a) \sin\left(\frac{a\pi}{2}\right)\right) /; -1 < \operatorname{Re}(a) < 0 \land \operatorname{Im}(z) == 0$$

Inverse Fourier exp transforms

01.02.22.0006.01

$$\mathcal{F}_t^{-1}[t^n](x) == i^n \sqrt{2\pi} \, \delta^{(n)}(x) /; n \in \mathbb{N}$$

01.02.22.0007.01

$$\mathcal{F}_t^{-1}[t^n \theta(t)](x) == \frac{i^{-n-1} n! + i^n \pi x^{n+1} \delta^{(n)}(x)}{\sqrt{2\pi} \, x^{n+1}} /; n \in \mathbb{N}$$

01.02.22.0008.01

$$\mathcal{F}_t^{-1}\left[\frac{1}{t^n}\right](x) = \frac{(-i)^n}{(n-1)!}\sqrt{\frac{\pi}{2}}\, x^{n-1}\operatorname{sgn}(x) \,/;\, n \in \mathbb{N}^+$$

01.02.22.0009.01

$$\mathcal{F}_t^{-1}\left[\frac{\theta(t)}{t^n}\right](x) = \frac{x^{n-1}(-i)^{n-1}}{2\sqrt{2\pi}\,(n-1)!}\left(2\psi(n) - \log(x^2) - i\pi\operatorname{sgn}(x)\right) /;\, n \in \mathbb{N}^+$$

Fourier cos transforms

01.02.22.0010.01

$$\mathcal{F}c_t\left[t^{2n}\right](x) = (-1)^n\sqrt{2\pi}\,\delta^{(2n)}(x) \,/;\, n \in \mathbb{N}$$

01.02.22.0011.01

$$\mathcal{F}c_t\left[t^{2n-1}\right](x) = \sqrt{\frac{2}{\pi}}\,\frac{(-1)^n(2n-1)!}{x^{2n}} \,/;\, n \in \mathbb{N}^+$$

01.02.22.0012.01

$$\mathcal{F}c_t[t^n](x) = \frac{(-i)^{-n}}{\sqrt{2\pi}}\left((1-(-1)^n)\, i\, n!\, x^{-n-1} + ((-1)^n + 1)\,\pi\,\delta^{(n)}(x)\right) /;\, n \in \mathbb{N}$$

01.02.22.0013.01

$$\mathcal{F}c_t\left[\frac{1}{t^{2n}}\right](x) = \sqrt{\frac{\pi}{2}}\,\frac{(-1)^n x^{2n-1}\operatorname{sgn}(x)}{(2n-1)!} \,/;\, n \in \mathbb{N}^+$$

01.02.22.0014.01

$$\mathcal{F}c_t\left[\frac{1}{t^{2n-1}}\right](x) = \frac{(-1)^n x^{2n-2}}{\sqrt{2\pi}\,(2n-2)!}\left(\log(x^2) - 2\psi(2n-1)\right) /;\, n \in \mathbb{N}^+$$

01.02.22.0015.01

$$\mathcal{F}c_t\left[\frac{1}{t^n}\right](x) = \frac{i^n x^{n-1}}{2\sqrt{2\pi}\,(n-1)!}\left((1-(-1)^n)\, i\left(\log(x^2) - 2\psi(n)\right) + (1+(-1)^n)\,\pi\operatorname{sgn}(x)\right) /;\, n \in \mathbb{N}^+$$

01.02.22.0016.01

$$\mathcal{F}c_t[t^a](x) = -\sqrt{\frac{2}{\pi}}\,(x^2)^{-\frac{a+1}{2}}\sin\left(\frac{a\pi}{2}\right)\Gamma(a+1) \,/;\, -1 < \operatorname{Re}(a) < 0 \land \operatorname{Im}(x) = 0$$

01.02.22.0017.01

$$\mathcal{F}c_t\left[z^t\right](p) = -\sqrt{\frac{2}{\pi}}\,\frac{\log(z)}{p^2 + \log^2(z)} \,/;\, \operatorname{Re}(\log(z)) < |\operatorname{Im}(p)|$$

Fourier sin transforms

01.02.22.0018.01

$$\mathcal{F}s_t\left[t^{2n}\right](x) = \sqrt{\frac{2}{\pi}}\,\frac{(-1)^n(2n)!}{x^{2n+1}} \,/;\, n \in \mathbb{N}$$

01.02.22.0019.01

$$\mathcal{F}s_t\left[t^{2n-1}\right](x) = (-1)^n\sqrt{2\pi}\,\delta^{(2n-1)}(x) \,/;\, n \in \mathbb{N}^+$$

01.02.22.0020.01

$$\mathcal{F}s_t[t^n](x) = \frac{i^{-n} x^{-n-1}}{\sqrt{2\pi}} \left(((-1)^n - 1)\, i\, \pi\, \delta^{(n)}(x)\, x^{n+1} + (1 + (-1)^n)\, n!\right) /; n \in \mathbb{N}$$

01.02.22.0021.01

$$\mathcal{F}s_t\left[\frac{1}{t^{2n}}\right](x) = \frac{(-1)^n x^{2n-1}}{\sqrt{2\pi}\, (2n-1)!} \left(\log(x^2) - 2\psi(n)\right) /; n \in \mathbb{N}^+$$

01.02.22.0022.01

$$\mathcal{F}s_t\left[\frac{1}{t^{2n-1}}\right](x) = -\sqrt{\frac{\pi}{2}}\, \frac{(-1)^n x^{2n-2} \operatorname{sgn}(x)}{(2n-2)!} /; n \in \mathbb{N}^+$$

01.02.22.0023.01

$$\mathcal{F}s_t\left[\frac{1}{t^n}\right](x) = \frac{i^n x^{n-1}}{2\sqrt{2\pi}\, (n-1)!} \left((1 + (-1)^n)\left(\log(x^2) - 2\psi(n)\right) + ((-1)^n - 1)\, i\, \pi \operatorname{sgn}(x)\right) /; n \in \mathbb{N}^+$$

01.02.22.0024.01

$$\mathcal{F}s_t[t^a](x) = \sqrt{\frac{2}{\pi}} \operatorname{sgn}(x) \left(x^2\right)^{-\frac{a+1}{2}} \cos\left(\frac{a\pi}{2}\right) \Gamma(a+1) /; -2 < \operatorname{Re}(a) < 0 \land \operatorname{Im}(x) = 0$$

01.02.22.0025.01

$$\mathcal{F}s_t\left[z^t\right](p) = \sqrt{\frac{2}{\pi}}\, \frac{p}{p^2 + \log^2(z)} /; \operatorname{Re}(\log(z)) < |\operatorname{Im}(p)|$$

Laplace transforms

01.02.22.0026.01

$$\mathcal{L}_t[t^a](z) = z^{-a-1}\, \Gamma(a+1) /; \operatorname{Re}(a) > -1 \land \operatorname{Re}(z) > 0$$

01.02.22.0027.01

$$\mathcal{L}_t\left[z^t\right](p) = \frac{1}{p - \log(z)} /; \operatorname{Re}(\log(z) - p) < 0$$

Hankel transforms

01.02.22.0028.01

$$\mathcal{H}_{t;\nu}[t^a](z) = \frac{2^{a+\frac{1}{2}} z^{-a-1}}{\Gamma\left(\frac{1}{4}(2\nu - 2a + 1)\right)} \Gamma\left(\frac{1}{4}(2a + 2\nu + 3)\right) /; z > 0 \bigwedge \operatorname{Re}(a + \nu) > -\frac{3}{2} \bigwedge \operatorname{Re}(a) < 0$$

Summation

Finite summation

Various sums

01.02.23.0001.01

$$\sum_{k=1}^{m} k^n = \frac{B_{n+1}(m+1) - B_{n+1}(0)}{n+1} /; m \in \mathbb{N} \bigwedge n \in \mathbb{N}^+$$

01.02.23.0002.01

$$\sum_{k=1}^{m}(-1)^k k^n = \frac{1}{2}\left(E_n(0)+(-1)^m E_n(m+1)\right) /; n\in\mathbb{N}^+$$

01.02.23.0019.01

$$\sum_{k=0}^{n} k^i (n-k)^j = \frac{(i!\,j!)\,n^{i+j+1}}{(i+j+1)!} + \frac{1}{2}\left(\delta_{i\,j,0}+\delta_{i+j,0}\right) n^{i+j} + \sum_{k=\min(i,j)+1}^{i+j} \delta_{k \bmod 2,0} \frac{B_k}{k}\left((-1)^j \binom{i}{i+j-k+1} + (-1)^i \binom{j}{i+j-k+1}\right) n^{i+j-k+1} /; n\in\mathbb{N}\land i\in\mathbb{N}\land j\in\mathbb{N}$$

01.02.23.0003.01

$$\sum_{k=0}^{n}(a+k\,z) = a\,(n+1) + \frac{1}{2}\,n\,(n+1)\,z$$

01.02.23.0004.01

$$\sum_{k=0}^{n} a\,z^k = \frac{a\left(1-z^{n+1}\right)}{1-z}$$

01.02.23.0020.01

$$\sum_{k=1}^{\lfloor x\rfloor}(x-k)^{n-1} = \frac{1}{n}\left(B_n(x)+B_n(x-\lfloor x\rfloor)\right) /; x\in\mathbb{R}\land x>0\land n\in\mathbb{N}^+$$

01.02.23.0023.01

$$\sum_{k=0}^{m}\frac{z^k}{(a+k)^s} = -\Phi(z,s,a+m+1)\,e^{-\theta(\lfloor-\mathrm{Re}(a+m+1)\rfloor)(2\theta(\mathrm{Im}(a+m+1))-1)\pi i s}\,z^{m+1} - \theta(\lfloor-\mathrm{Re}(a+m+1)\rfloor)\left(1-e^{-(2\theta(\mathrm{Im}(a+m+1))-1)\pi i s}\right)\left(z\,\Phi(z,s,a+\lfloor-\mathrm{Re}(a)\rfloor+1)+\frac{(\lfloor-\mathrm{Re}(a)\rfloor+\lfloor\mathrm{Re}(a)\rfloor+1)\,\theta(\mathrm{Im}(a+m+1))}{\left((a+\lfloor-\mathrm{Re}(a)\rfloor)^2\right)^{s/2}}\right) z^{\lfloor-\mathrm{Re}(a)\rfloor} + \left(1-e^{-(2\theta(\mathrm{Im}(a))-1)\pi i s}\right)\left(z\,\Phi(z,s,a+\lfloor-\mathrm{Re}(a)\rfloor+1)+\frac{(\lfloor-\mathrm{Re}(a)\rfloor+\lfloor\mathrm{Re}(a)\rfloor+1)\,\theta(\mathrm{Im}(a))}{\left((a+\lfloor-\mathrm{Re}(a)\rfloor)^2\right)^{s/2}}\right)\theta(\lfloor-\mathrm{Re}(a)\rfloor)\,z^{\lfloor-\mathrm{Re}(a)\rfloor} + e^{-\theta(\lfloor-\mathrm{Re}(a)\rfloor)(2\theta(\mathrm{Im}(a))-1)\pi i s}\,\Phi(z,s,a)$$

01.02.23.0024.01

$$\sum_{k=0}^{m}\frac{z^k}{\left((a+k)^2\right)^{s/2}} = \Phi(z,s,a) - z^{m+1}\,\Phi(z,s,a+m+1)$$

01.02.23.0025.01

$$\sum_{k=0}^{\lfloor-\mathrm{Re}(a)\rfloor}\left(\frac{z^k}{\left((a+k)^2\right)^{s/2}} - \frac{z^k}{(a+k)^s}\right) = \theta(\lfloor-\mathrm{Re}(a)\rfloor)\left(-\left(1-e^{-(2\theta(\mathrm{Im}(a))-1)\pi i s}\right)\Phi(z,s,a+\lfloor-\mathrm{Re}(a)\rfloor+1)\,z^{\lfloor-\mathrm{Re}(a)\rfloor+1} + \left(\Phi(z,s,a)-\frac{z^{\lfloor-\mathrm{Re}(a)\rfloor}}{\left((a+\lfloor-\mathrm{Re}(a)\rfloor)^2\right)^{s/2}}\right)\left(1-e^{-(2\theta(\mathrm{Im}(a))-1)\pi i s}\right) + \frac{\left(\left(1-e^{-(2\theta(\mathrm{Im}(a))-1)\pi i s}\right)z^{\lfloor-\mathrm{Re}(a)\rfloor}\right)\left(1-\theta(\mathrm{Im}(a))\left(\lfloor-\mathrm{Re}(a)\rfloor+\lfloor\mathrm{Re}(a)\rfloor+1\right)\right)}{\left((a+\lfloor-\mathrm{Re}(a)\rfloor)^2\right)^{s/2}}\right)$$

01.02.23.0026.01

$$\sum_{k=0}^{\lfloor -\mathrm{Re}(a)\rfloor}\left(\frac{1}{\left((a+k)^2\right)^{s/2}}-\frac{1}{(a+k)^s}\right)=$$
$$\theta(\lfloor -\mathrm{Re}(a)\rfloor)\left(\left(1-e^{-(2\,\theta(\mathrm{Im}(a))-1)\,\pi\,i\,s}\right)\left(\zeta(s,a)-\frac{1}{\left((a+\lfloor -\mathrm{Re}(a)\rfloor)^2\right)^{s/2}}\right)-\zeta(s,a+\lfloor -\mathrm{Re}(a)\rfloor+1)\left(1-e^{-(2\,\theta(\mathrm{Im}(a))-1)\,\pi\,i\,s}\right)+\right.$$
$$\left.\frac{\left(1-e^{-(2\,\theta(\mathrm{Im}(a))-1)\,\pi\,i\,s}\right)(1-\theta(\mathrm{Im}(a))\,(\lfloor -\mathrm{Re}(a)\rfloor+\lfloor \mathrm{Re}(a)\rfloor+1))}{\left((a+\lfloor -\mathrm{Re}(a)\rfloor)^2\right)^{s/2}}\right)$$

Summed truncated generalized hypergeometric series

General case

01.02.23.0027.01

$$\sum_{k=0}^{n}\frac{\prod_{j=1}^{p}(a_j)_k\,z^k}{k!\prod_{j=1}^{q}(b_j)_k}={}_pF_q(a_1,\ldots,a_p;b_1,\ldots,b_q;z)-$$
$$\frac{z^{n+1}\prod_{j=1}^{p}(a_j)_{n+1}}{(n+1)!\prod_{j=1}^{q}(b_j)_{n+1}}\;{}_{p+1}F_{q+1}(1,a_1+n+1,\ldots,a_p+n+1;n+2,b_1+n+1,\ldots,b_q+n+1;z)\;/;\;n\in\mathbb{N}$$

Case ${}_0F_0$

01.02.23.0028.01

$$\sum_{k=0}^{n}\frac{z^k}{k!}=e^z\,Q(n+1,z)\;/;\;n\in\mathbb{N}$$

Case ${}_1F_0$

01.02.23.0029.01

$$\sum_{k=0}^{n}\frac{(a)_k\,z^k}{k!}=(1-z)^{-a}-z^{n+1}\,(a)_{n+1}\,{}_2\tilde{F}_1(1,a+n+1;n+2;z)\;/;\;n\in\mathbb{N}$$

Case ${}_2F_1$

01.02.23.0030.01

$$\sum_{k=0}^{n}\frac{(a)_k\,(b)_k\,z^k}{k!\,(c)_k}={}_2F_1(a,b;c;z)-\frac{z^{n+1}\,(a)_{n+1}\,(b)_{n+1}}{(n+1)!\,(c)_{n+1}}\,{}_3F_2(1,a+n+1,b+n+1;n+2,c+n+1;z)\;/;\;n\in\mathbb{N}$$

01.02.23.0031.01

$$\sum_{k=0}^{n}\frac{z^k}{k+1}=-\frac{\mathrm{B}_z(n+2,0)+\log(1-z)}{z}\;/;\;n\in\mathbb{N}$$

01.02.23.0032.01

$$\sum_{k=0}^{n} \frac{z^k}{2k+1} = \frac{1}{2\sqrt{z}} \left(2 \tanh^{-1}\left(\sqrt{z}\right) - B_z\left(n+\frac{3}{2}, 0\right)\right) /; n \in \mathbb{N}$$

01.02.23.0033.01

$$\sum_{k=0}^{n} \frac{\left(\frac{1}{2}\right)_k z^k}{(2k+1)\, k!} = \frac{\sin^{-1}\left(\sqrt{z}\right)}{\sqrt{z}} - \frac{z^{n+1}}{2\sqrt{\pi}} \Gamma\left(n+\frac{3}{2}\right)^2 {}_3\tilde{F}_2\left(1, n+\frac{3}{2}, n+\frac{3}{2}; n+2, n+\frac{5}{2}; z\right) /; n \in \mathbb{N}$$

Case $_3F_2$

01.02.23.0034.01

$$\sum_{k=0}^{n} \frac{(a_1)_k (a_2)_k (a_3)_k z^k}{k!\, (b_1)_k (b_2)_k} = {}_3F_2(a_1, a_2, a_3; b_1, b_2; z) - \frac{z^{n+1} (a_1)_{n+1} (a_2)_{n+1} (a_3)_{n+1}}{(n+1)!\, (b_1)_{n+1} (b_2)_{n+1}} {}_4F_3(1, n+a_1+1, n+a_2+1, n+a_3+1; n+2, n+b_1+1, n+b_2+1; z) /; n \in \mathbb{N}$$

01.02.23.0035.01

$$\sum_{k=0}^{n} \frac{2^{2k} k!^2 z^k}{(2k+1)!\, (k+1)} = \frac{\sin^{-1}\left(\sqrt{z}\right)^2}{z} - \frac{1}{2} \sqrt{\pi}\, z^{n+1} \Gamma(n+2)^2 {}_3\tilde{F}_2\left(1, n+2, n+2; n+\frac{5}{2}, n+3; z\right) /; n \in \mathbb{N}$$

Case $_1F_1$

01.02.23.0036.01

$$\sum_{k=0}^{n} \frac{(a)_k z^k}{k!\, (c)_k} = {}_1F_1(a; c; z) - \frac{z^{n+1} (a)_{n+1}}{(n+1)!\, (c)_{n+1}} {}_2F_2(1, a+n+1; n+2, c+n+1; z) /; n \in \mathbb{N}$$

Case $_0F_1$

01.02.23.0037.01

$$\sum_{k=0}^{n} \frac{z^k}{k!\, (c)_k} = {}_0F_1(; c; z) - \frac{z^{n+1}}{(n+1)!\, (c)_{n+1}} {}_1F_2(1; n+2, c+n+1; z) /; n \in \mathbb{N}$$

01.02.23.0038.01

$$\sum_{k=0}^{n} \frac{z^k}{k!\left(\frac{1}{2}\right)_k} = \cosh\left(2\sqrt{z}\right) - \sqrt{\pi}\, z^{n+1} {}_1\tilde{F}_2\left(1; n+\frac{3}{2}, n+2; z\right) /; n \in \mathbb{N}$$

01.02.23.0039.01

$$\sum_{k=0}^{n} \frac{z^k}{k!\left(\frac{3}{2}\right)_k} = \frac{1}{2\sqrt{z}} \left(\sinh\left(2\sqrt{z}\right) - \sqrt{\pi}\, z^{n+\frac{3}{2}} {}_1\tilde{F}_2\left(1; n+2, n+\frac{5}{2}; z\right)\right) /; n \in \mathbb{N}$$

Infinite summation

01.02.23.0005.01

$$\sum_{k=1}^{\infty} q^{k^2} = \frac{1}{2} (\vartheta_3(0, q) - 1)$$

01.02.23.0006.01

$$\sum_{k=1}^{\infty}(-1)^k q^{k^2} = \frac{1}{2}(\vartheta_4(0, q) - 1)$$

01.02.23.0007.01

$$\sum_{k=0}^{\infty} q^{k^2+k} = \frac{\vartheta_2(0, q)}{2\sqrt[4]{q}}$$

01.02.23.0008.01

$$\sum_{k=0}^{\infty} q^{k(k+1)} = \frac{1}{2\sqrt[4]{q}}\vartheta_1\left(\frac{\pi}{2}, q\right)$$

01.02.23.0009.01

$$\sum_{k=0}^{\infty} q^{\frac{1}{2}k(k+1)} = \frac{1}{2\sqrt[8]{q}}\vartheta_1\left(\frac{\pi}{2}, \sqrt{q}\right)$$

01.02.23.0010.01

$$\sum_{k=1}^{\infty}(-1)^k k\, q^{k^2} = -\frac{1}{4}\vartheta_3'\left(\frac{\pi}{2}, q\right)$$

01.02.23.0011.01

$$\sum_{k=0}^{\infty}(-1)^k (2k+1)\, q^{k(k+1)} = \frac{\vartheta_1'(0, q)}{2\sqrt[4]{q}}$$

01.02.23.0012.01

$$\sum_{k=0}^{\infty}(-1)^k (2k+1)\, q^{k(k+1)} = -\frac{1}{2\sqrt[4]{q}}\vartheta_2'\left(\frac{\pi}{2}, q\right)$$

01.02.23.0013.01

$$\sum_{k=0}^{\infty}(-1)^k (2k+1)\, q^{\frac{1}{2}k(k+1)} = \frac{1}{\sqrt[8]{q}}\eta\left(-\frac{i\log(q)}{2\pi}\right)^3$$

01.02.23.0014.01

$$\sum_{k=0}^{\infty}(-1)^k (2k+1)\, q^{(2k+1)^2} = -\frac{1}{4}\vartheta_3'\left(\frac{\pi}{4}, q\right)$$

01.02.23.0015.01

$$\sum_{k=0}^{\infty}(-1)^k (2k+1)\, q^{(2k+1)^2} = \frac{1}{4}\vartheta_4'\left(\frac{\pi}{4}, q\right)$$

Gauss' identity

01.02.23.0016.01

$$\sum_{k=-\infty}^{\infty} q^{k^2} = \vartheta_3(0, q)$$

01.02.23.0017.01

$$\sum_{k=-\infty}^{\infty} (-1)^k q^{k^2} = \vartheta_4(0, q)$$

01.02.23.0018.01

$$\sum_{k=-\infty}^{\infty} (-1)^k q^{\frac{1}{2} k (3k-1)} = \frac{1}{\sqrt[24]{q}} \eta\left(-\frac{i \log(q)}{2\pi}\right)$$

Gauss' identity

01.02.23.0040.01

$$\sum_{k=1}^{\infty} \frac{1}{(a+k)^n (b+k)^m} = \frac{(-1)^{m+n} (b-a)^{-n}}{(m-1)!} \sum_{k=0}^{m-1} (b-a)^{-k} \binom{m-1}{k} (-k-n+1)_k \psi^{(-k+m-1)}(b+1) - \frac{(-1)^{m+n} (b-a)^{-m}}{(m-1)!(n-1)!} \sum_{k=0}^{n-1} (b-a)^{-k} \binom{n-1}{k} k! (-k-m+1)_{m-1} \psi^{(-k+n-1)}(a+1) /; m \in \mathbb{N}^+ \wedge n \in \mathbb{N}^+$$

Multidimensional summation

01.02.23.0021.01

$$\sum_{k_0=1}^{n} \sum_{k_1=1}^{n} \sum_{k_2=1}^{n} \ldots \sum_{k_r=1}^{n} \frac{\theta(n-(k_0+k_1+k_2+\ldots+k_r))}{k_0 k_1 k_2 \ldots k_r} = -\frac{(-1)^{n+r}}{n!} \left(\frac{\log^{r+1}(t)}{t}\right)^{(n)} \Big|_{t=1} /; n \in \mathbb{N}^+ \wedge r \in \mathbb{N}$$

01.02.23.0022.01

$$\sum_{k_1=1}^{\infty} \sum_{k_2=k_1+1}^{\infty} \ldots \sum_{k_m=k_{m-1}+1}^{\infty} \left(\prod_{j=1}^{m} \frac{1}{k_j^2}\right) = \frac{\pi^{2m}}{(2m+1)!} /; m \in \mathbb{N}^+$$

Products

Finite products

01.02.24.0001.01

$$\prod_{k=0}^{n-1} \Gamma\left(\frac{k+z}{n}\right) = n^{\frac{1}{2}-z} (2\pi)^{\frac{n-1}{2}} \Gamma(z) /; n \in \mathbb{N}^+$$

Infinite products

01.02.24.0002.01

$$\prod_{k=1}^{\infty} (1-q^k) = \frac{1}{\sqrt[24]{q}} \eta\left(-\frac{i \log(q)}{2\pi}\right)$$

Gauss' identity

01.02.24.0003.01

$$\prod_{k=1}^{\infty} (1-q^k)^a = \frac{1}{q^{a/24}} \eta\left(-\frac{i \log(q)}{2\pi}\right)^a /; a \in \mathbb{R}$$

Gauss' identity

01.02.24.0004.01

$$\prod_{k=1}^{\infty} \frac{1-q^k}{1+q^k} = \vartheta_4(0, q)$$

01.02.24.0005.01

$$\prod_{k=1}^{\infty} \frac{1-q^{2k}}{1-q^{2k-1}} = \frac{\vartheta_1\left(\frac{\pi}{2}, \sqrt{q}\right)}{2\sqrt[8]{q}}$$

01.02.24.0013.01

$$\prod_{k=1}^{\infty} k^{\frac{1}{k^2}} = \left(\frac{\text{Glaisher}^{12}}{2\pi e^{\gamma}}\right)^{\frac{\pi^2}{6}}$$

01.02.24.0014.01

$$\prod_{k=1}^{\infty} (2k+1)^{\frac{1}{(2k+1)^2}} = \left(\frac{\text{Glaisher}^{12}}{2^{4/3}\pi e^{\gamma}}\right)^{\frac{\pi^2}{8}}$$

01.02.24.0015.01

$$\frac{\prod_{k=1}^{\infty} (4k+1)^{\frac{1}{(4k+1)^3}}}{\prod_{j=1}^{\infty} (4j+3)^{\frac{1}{(4j+3)^3}}} = \left(\frac{\text{Glaisher}}{2^{9/32}\sqrt[32]{\pi}} \exp\left(-\frac{3\zeta(3)}{64\pi^2} - \zeta'(-1) + 2\zeta^{(1,0)}\left(-2, \frac{1}{4}\right) - \frac{29}{192} - \frac{\gamma}{96}\right)\right)^{\pi^3}$$

Regularized infinite products

01.02.24.0006.01

$$\prod_{k=0}^{\infty} (k+z) = \frac{\sqrt{2\pi}}{\Gamma(z)}$$

01.02.24.0007.01

$$\prod_{k=0}^{\infty} \left(k^2+z^2\right) = 2z\sinh(\pi z)$$

01.02.24.0008.01

$$\prod_{k=0}^{\infty} \left(k^4+z^4\right) = 2z^2\left(\cosh\left(\sqrt{2}\,\pi z\right) - \cos\left(\sqrt{2}\,\pi z\right)\right)$$

01.02.24.0009.01

$$\prod_{k=0}^{\infty} \left(w^2+(k+z)^2\right) = \frac{2\pi}{\Gamma(i w+z)\,\Gamma(z-i w)}$$

01.02.24.0010.01

$$\prod_{k=0}^{\infty} \left((k+z)^n - w^n\right) = \frac{(2\pi)^{n/2}}{\prod_{j=0}^{n-1} \Gamma\left(z - e^{\frac{j 2\pi i}{n}} w\right)} \;/;\; n \in \mathbb{N}^+$$

01.02.24.0011.01

$$\prod_{k=0}^{\infty} \left((k+z)^n + w^n\right) = \frac{(2\pi)^{n/2}}{\prod_{j=0}^{n-1} \Gamma\left(z + e^{\frac{j 2\pi i}{n}} w\right)} \;/;\; \frac{n-1}{2} \in \mathbb{N}$$

01.02.24.0012.01

$$\prod_{k=0}^{\infty} \left((k+z)^n + w^n\right) = \frac{(2\pi)^{n/2}}{\prod_{j=0}^{n-1} \Gamma\left(z + e^{\frac{j \pi i}{n}} w\right)} \;/;\; \frac{n}{2} \in \mathbb{N}$$

Operations

Limit operation

01.02.25.0001.01

$$\lim_{\varepsilon \to 0} \frac{z^{\varepsilon} - 1}{\varepsilon} == \log(z)$$

01.02.25.0002.01

$$\lim_{\omega \to \infty} \omega \left(z^{1/\omega} - 1\right) == \log(z)$$

01.02.25.0003.01

$$\lim_{x \to \infty} x^a b^{-x} == 0 \;/;\; b > 1$$

01.02.25.0004.01

$$\lim_{a \to \infty} \frac{z^a}{a!} == 0$$

01.02.25.0005.01

$$\lim_{z \to 0} z^a \log(z) == 0 \;/;\; \operatorname{Re}(a) > 0$$

01.02.25.0006.01

$$\lim_{z \to \infty} z^{-a} \log(z) == 0 \;/;\; \operatorname{Re}(a) > 0$$

Representations through more general functions

Through hypergeometric functions

Involving ${}_p\tilde{F}_q$

01.02.26.0001.01

$$z^a == {}_1\tilde{F}_0(-a;\, ;\, 1 - z)$$

Involving ${}_pF_q$

01.02.26.0002.01

$$z^a == {}_1F_0(-a;\, ;\, 1 - z)$$

01.02.26.0003.01

$$z^a == {}_0F_0(;\, ;\, a \log(z))$$

Involving ${}_2F_1$

01.02.26.0004.01

$$z^a == {}_2F_1(-a, b;\, b;\, 1 - z)$$

Through Meijer *G*

Classical cases for the direct function itself

01.02.26.0048.01

$$z^a == G_{0,1}^{1,0}\left(z \,\middle|\, a\right) + G_{1,2}^{1,1}\left(z \,\middle|\, \begin{matrix} a+1 \\ a+1,\, a \end{matrix}\right)$$

01.02.26.0005.01

$$z^a = \frac{1}{\Gamma(-a)} G_{1,1}^{1,1}\left(z-1 \,\middle|\, \begin{matrix} a+1 \\ 0 \end{matrix}\right)$$

01.02.26.0006.01

$$z^a = G_{0,1}^{1,0}(-a \log(z) \mid 0)$$

01.02.26.0007.01

$$(1+z)^a = \frac{1}{\Gamma(-a)} G_{1,1}^{1,1}\left(z \,\middle|\, \begin{matrix} a+1 \\ 0 \end{matrix}\right)$$

01.02.26.0008.01

$$\frac{1}{1-z} = \pi\, G_{2,2}^{1,1}\left(z \,\middle|\, \begin{matrix} 0, \frac{1}{2} \\ 0, \frac{1}{2} \end{matrix}\right)$$

01.02.26.0009.01

$$\frac{z^a - z^b}{z-1} = \frac{\sin((a-b)\pi)}{\pi} G_{2,2}^{2,2}\left(z \,\middle|\, \begin{matrix} a, b \\ b, a \end{matrix}\right)$$

Classical cases for the direct function itself minus parts of its series expansion

01.02.26.0047.01

$$(z+1)^a - \sum_{k=0}^{n} \binom{a}{k} z^k = \frac{(-1)^{n-1}}{\Gamma(-a)} G_{2,2}^{1,2}\left(z \,\middle|\, \begin{matrix} n+1, a+1 \\ n+1, 0 \end{matrix}\right) /; n \in \mathbb{N}$$

01.02.26.0049.01

$$\frac{1}{1-z} - \sum_{k=0}^{n} z^k = \pi\,(-1)^{n-1} G_{3,3}^{1,2}\left(z \,\middle|\, \begin{matrix} n+1, 0, \frac{1}{2} \\ n+1, 0, \frac{1}{2} \end{matrix}\right) /; n \in \mathbb{N}$$

Classical cases involving unit step θ

01.02.26.0010.01

$$\theta(1-|z|)\,(1-z)^a = \Gamma(a+1)\, G_{1,1}^{1,0}\left(z \,\middle|\, \begin{matrix} a+1 \\ 0 \end{matrix}\right) /; \operatorname{Re}(a) > -1$$

01.02.26.0011.01

$$\theta(|z|-1)\,(z-1)^a = \Gamma(a+1)\, G_{1,1}^{0,1}\left(z \,\middle|\, \begin{matrix} a+1 \\ 0 \end{matrix}\right) /; \operatorname{Re}(a) > -1$$

Classical cases involving sgn

01.02.26.0012.01

$$((1-z)\operatorname{sgn}(1-|z|))^\nu = \frac{\pi}{\Gamma(-\nu)} \sec\left(\frac{\pi\nu}{2}\right) G_{2,2}^{1,1}\left(z \,\middle|\, \begin{matrix} \nu+1, \frac{\nu+1}{2} \\ 0, \frac{\nu+1}{2} \end{matrix}\right)$$

01.02.26.0013.01

$$\operatorname{sgn}(1-|z|)\,((1-z)\operatorname{sgn}(1-|z|))^\nu = -\frac{\pi}{\Gamma(-\nu)} \csc\left(\frac{\nu\pi}{2}\right) G_{2,2}^{1,1}\left(z \,\middle|\, \begin{matrix} \nu+1, \frac{\nu}{2}+1 \\ 0, \frac{\nu}{2}+1 \end{matrix}\right)$$

Classical cases involving sqrt in the arguments

01.02.26.0014.01

$$\left(\sqrt{z+1}+1\right)^{\beta}=-\frac{\beta}{2\sqrt{\pi}}\,G_{2,2}^{1,2}\!\left(z\,\middle|\,\begin{matrix}\frac{\beta+1}{2},\frac{\beta}{2}+1\\0,\beta\end{matrix}\right)$$

01.02.26.0015.01

$$\left(\sqrt{z+1}-1\right)^{\beta}=\frac{\beta}{2\sqrt{\pi}}\,G_{2,2}^{1,2}\!\left(z\,\middle|\,\begin{matrix}\frac{\beta+1}{2},\frac{\beta}{2}+1\\\beta,0\end{matrix}\right)$$

01.02.26.0016.01

$$\left(\sqrt{z+1}+\sqrt{z}\right)^{\beta}=-\frac{\beta}{2\sqrt{\pi}}\,G_{2,2}^{2,1}\!\left(z\,\middle|\,\begin{matrix}\frac{\beta}{2}+1,1-\frac{\beta}{2}\\0,\frac{1}{2}\end{matrix}\right)/;\,z\notin(-1,0)$$

01.02.26.0017.01

$$\left(\sqrt{z+1}-\sqrt{z}\right)^{\beta}=\frac{\beta}{2\sqrt{\pi}}\,G_{2,2}^{2,1}\!\left(z\,\middle|\,\begin{matrix}1-\frac{\beta}{2},\frac{\beta}{2}+1\\0,\frac{1}{2}\end{matrix}\right)/;\,z\notin(-1,0)$$

01.02.26.0018.01

$$\frac{\left(\sqrt{z+1}+1\right)^{\beta}}{\sqrt{z+1}}=\frac{1}{\sqrt{\pi}}\,G_{2,2}^{1,2}\!\left(z\,\middle|\,\begin{matrix}\frac{\beta+1}{2},\frac{\beta}{2}\\0,\beta\end{matrix}\right)$$

01.02.26.0019.01

$$\frac{\left(\sqrt{z+1}-1\right)^{\beta}}{\sqrt{z+1}}=\frac{1}{\sqrt{\pi}}\,G_{2,2}^{1,2}\!\left(z\,\middle|\,\begin{matrix}\frac{\beta+1}{2},\frac{\beta}{2}\\\beta,0\end{matrix}\right)$$

01.02.26.0020.01

$$\frac{\left(\sqrt{z+1}+\sqrt{z}\right)^{\beta}}{\sqrt{z+1}}=\frac{1}{\sqrt{\pi}}\,G_{2,2}^{2,1}\!\left(z\,\middle|\,\begin{matrix}\frac{\beta+1}{2},\frac{1-\beta}{2}\\0,\frac{1}{2}\end{matrix}\right)/;\,z\notin(-1,0)$$

01.02.26.0021.01

$$\frac{\left(\sqrt{z+1}-\sqrt{z}\right)^{\beta}}{\sqrt{z+1}}=\frac{1}{\sqrt{\pi}}\,G_{2,2}^{2,1}\!\left(z\,\middle|\,\begin{matrix}\frac{1-\beta}{2},\frac{\beta+1}{2}\\0,\frac{1}{2}\end{matrix}\right)/;\,z\notin(-1,0)$$

Classical cases involving sqrt in the arguments and unit step θ

01.02.26.0022.01

$$\theta(1-|z|)\left(\left(1+\sqrt{1-z}\right)^{\beta}-\left(1-\sqrt{1-z}\right)^{\beta}\right)=\sqrt{\pi}\,\beta\,G_{2,2}^{2,0}\!\left(z\,\middle|\,\begin{matrix}\frac{\beta+1}{2},\frac{\beta}{2}+1\\0,\beta\end{matrix}\right)/;\,z\notin(-1,0)$$

01.02.26.0023.01

$$\theta(|z|-1)\left(\left(\sqrt{z}+\sqrt{z-1}\right)^{\beta}-\left(\sqrt{z}-\sqrt{z-1}\right)^{\beta}\right)=\sqrt{\pi}\,\beta\,G_{2,2}^{0,2}\!\left(z\,\middle|\,\begin{matrix}\frac{\beta}{2}+1,1-\frac{\beta}{2}\\0,\frac{1}{2}\end{matrix}\right)$$

01.02.26.0024.01

$$\frac{\theta(1-|z|)}{\sqrt{1-z}}\left(\left(1-\sqrt{1-z}\right)^{\beta}+\left(1+\sqrt{1-z}\right)^{\beta}\right)=2\sqrt{\pi}\,G_{2,2}^{2,0}\!\left(z\,\middle|\,\begin{matrix}\frac{\beta}{2},\frac{\beta+1}{2}\\0,\beta\end{matrix}\right)/;\,z\notin(-1,0)$$

01.02.26.0025.01

$$\frac{\theta(|z|-1)}{\sqrt{z-1}}\left(\left(\sqrt{z}-\sqrt{z-1}\right)^{\beta}+\left(\sqrt{z}+\sqrt{z-1}\right)^{\beta}\right)=2\sqrt{\pi}\,G_{2,2}^{0,2}\left(z\,\middle|\,\begin{matrix}\frac{\beta+1}{2},\frac{1-\beta}{2}\\0,\frac{1}{2}\end{matrix}\right)$$

01.02.26.0026.01

$$\theta(1-|z|)\left(\left(\sqrt{1+\sqrt{z}}-\sqrt{1-\sqrt{z}}\right)^{\beta}-\left(\sqrt{1+\sqrt{z}}+\sqrt{1-\sqrt{z}}\right)^{\beta}\right)=-2^{\frac{\beta}{2}-1}\beta\sqrt{\pi}\,G_{2,2}^{2,0}\left(z\,\middle|\,\begin{matrix}\frac{\beta+2}{4},\frac{\beta}{4}+1\\0,\frac{\beta}{2}\end{matrix}\right)/;\,z\notin(-1,0)$$

01.02.26.0027.01

$$\theta(|z|-1)\left(\left(\sqrt{\sqrt{z}+1}+\sqrt{\sqrt{z}-1}\right)^{\beta}-\left(\sqrt{\sqrt{z}+1}-\sqrt{\sqrt{z}-1}\right)^{\beta}\right)=2^{\frac{\beta}{2}-1}\sqrt{\pi}\,\beta\,G_{2,2}^{0,2}\left(z\,\middle|\,\begin{matrix}\frac{\beta}{4}+1,1-\frac{\beta}{4}\\0,\frac{1}{2}\end{matrix}\right)$$

01.02.26.0028.01

$$\frac{\theta(1-|z|)}{\sqrt{1-z}}\left(\left(\sqrt{1+\sqrt{z}}-\sqrt{1-\sqrt{z}}\right)^{\beta}+\left(\sqrt{1+\sqrt{z}}+\sqrt{1-\sqrt{z}}\right)^{\beta}\right)=2^{\frac{\beta}{2}+1}\sqrt{\pi}\,G_{2,2}^{2,0}\left(z\,\middle|\,\begin{matrix}\frac{\beta}{4},\frac{\beta+2}{4}\\0,\frac{\beta}{2}\end{matrix}\right)/;\,z\notin(-1,0)$$

01.02.26.0029.01

$$\frac{\theta(|z|-1)}{\sqrt{z-1}}\left(\left(\sqrt{\sqrt{z}+1}-\sqrt{\sqrt{z}-1}\right)^{\beta}+\left(\sqrt{\sqrt{z}+1}+\sqrt{\sqrt{z}-1}\right)^{\beta}\right)=2^{\frac{\beta}{2}+1}\sqrt{\pi}\,G_{2,2}^{0,2}\left(z\,\middle|\,\begin{matrix}\frac{\beta+2}{4},1-\frac{\beta+2}{4}\\0,\frac{1}{2}\end{matrix}\right)$$

Generalized cases involving sqrt in the arguments

01.02.26.0030.01

$$\left(\sqrt{z^2+1}+1\right)^{\beta}=-\frac{\beta}{2\sqrt{\pi}}\,G_{2,2}^{1,2}\left(z,\frac{1}{2}\,\middle|\,\begin{matrix}\frac{\beta+1}{2},\frac{\beta}{2}+1\\0,\beta\end{matrix}\right)$$

01.02.26.0031.01

$$\left(\sqrt{z^2+1}-1\right)^{\beta}=\frac{\beta}{2\sqrt{\pi}}\,G_{2,2}^{1,2}\left(z,\frac{1}{2}\,\middle|\,\begin{matrix}\frac{\beta+1}{2},\frac{\beta}{2}+1\\\beta,0\end{matrix}\right)/;\,\mathrm{Re}(z)>0$$

01.02.26.0032.01

$$\left(\sqrt{z^2+1}+z\right)^{\beta}=-\frac{\beta}{2\sqrt{\pi}}\,G_{2,2}^{2,1}\left(z,\frac{1}{2}\,\middle|\,\begin{matrix}\frac{\beta}{2}+1,1-\frac{\beta}{2}\\0,\frac{1}{2}\end{matrix}\right)/;\,\mathrm{Re}(z)>0$$

01.02.26.0033.01

$$\left(\sqrt{z^2+1}-z\right)^{\beta}=\frac{\beta}{2\sqrt{\pi}}\,G_{2,2}^{2,1}\left(z,\frac{1}{2}\,\middle|\,\begin{matrix}1-\frac{\beta}{2},\frac{\beta}{2}+1\\0,\frac{1}{2}\end{matrix}\right)/;\,\mathrm{Re}(z)>0$$

01.02.26.0034.01

$$\frac{\left(\sqrt{z^2+1}+1\right)^{\beta}}{\sqrt{z^2+1}}=\frac{1}{\sqrt{\pi}}\,G_{2,2}^{1,2}\left(z,\frac{1}{2}\,\middle|\,\begin{matrix}\frac{\beta+1}{2},\frac{\beta}{2}\\0,\beta\end{matrix}\right)$$

01.02.26.0035.01

$$\frac{\left(\sqrt{z^2+1}-1\right)^{\beta}}{\sqrt{z^2+1}}=\frac{1}{\sqrt{\pi}}\,G_{2,2}^{1,2}\left(z,\frac{1}{2}\,\middle|\,\begin{matrix}\frac{\beta+1}{2},\frac{\beta}{2}\\\beta,0\end{matrix}\right)/;\,\mathrm{Re}(z)>0$$

01.02.26.0036.01

$$\frac{\left(\sqrt{z^2+1}+z\right)^{\beta}}{\sqrt{z^2+1}} = \frac{1}{\sqrt{\pi}} G_{2,2}^{2,1}\left(z, \frac{1}{2} \,\middle|\, \begin{matrix} \frac{\beta+1}{2}, \frac{1-\beta}{2} \\ 0, \frac{1}{2} \end{matrix}\right) /; \operatorname{Re}(z) > 0$$

01.02.26.0037.01

$$\frac{\left(\sqrt{z^2+1}-z\right)^{\beta}}{\sqrt{z^2+1}} = \frac{1}{\sqrt{\pi}} G_{2,2}^{2,1}\left(z, \frac{1}{2} \,\middle|\, \begin{matrix} \frac{1-\beta}{2}, \frac{\beta+1}{2} \\ 0, \frac{1}{2} \end{matrix}\right) /; \operatorname{Re}(z) > 0$$

Generalized cases involving sqrt in the arguments and unit step θ

01.02.26.0038.01

$$\theta(1-|z|)\left(\left(1+\sqrt{1-z^2}\right)^{\beta} - \left(1-\sqrt{1-z^2}\right)^{\beta}\right) = \sqrt{\pi}\, \beta\, G_{2,2}^{2,0}\left(z, \frac{1}{2} \,\middle|\, \begin{matrix} \frac{\beta+1}{2}, \frac{\beta}{2}+1 \\ 0, \beta \end{matrix}\right) /; \operatorname{Re}(z) > 0$$

01.02.26.0039.01

$$\theta(|z|-1)\left(\left(z+\sqrt{z^2-1}\right)^{\beta} - \left(z-\sqrt{z^2-1}\right)^{\beta}\right) = \sqrt{\pi}\, \beta\, G_{2,2}^{0,2}\left(z, \frac{1}{2} \,\middle|\, \begin{matrix} \frac{\beta}{2}+1, 1-\frac{\beta}{2} \\ 0, \frac{1}{2} \end{matrix}\right) /; \operatorname{Re}(z) > 0$$

01.02.26.0040.01

$$\frac{\theta(1-|z|)}{\sqrt{1-z^2}}\left(\left(1-\sqrt{1-z^2}\right)^{\beta} + \left(1+\sqrt{1-z^2}\right)^{\beta}\right) = 2\sqrt{\pi}\, G_{2,2}^{2,0}\left(z, \frac{1}{2} \,\middle|\, \begin{matrix} \frac{\beta}{2}, \frac{\beta+1}{2} \\ 0, \beta \end{matrix}\right) /; \operatorname{Re}(z) > 0$$

01.02.26.0041.01

$$\frac{\theta(|z|-1)}{\sqrt{z^2-1}}\left(\left(z-\sqrt{z^2-1}\right)^{\beta} + \left(z+\sqrt{z^2-1}\right)^{\beta}\right) = 2\sqrt{\pi}\, G_{2,2}^{0,2}\left(z, \frac{1}{2} \,\middle|\, \begin{matrix} \frac{\beta+1}{2}, \frac{1-\beta}{2} \\ 0, \frac{1}{2} \end{matrix}\right) /; \operatorname{Re}(z) > 0$$

01.02.26.0042.01

$$\theta(1-|z|)\left(\left(\sqrt{1+z}-\sqrt{1-z}\right)^{\beta} - \left(\sqrt{1+z}+\sqrt{1-z}\right)^{\beta}\right) = -2^{\frac{\beta}{2}-1}\, \beta\sqrt{\pi}\, G_{2,2}^{2,0}\left(z, \frac{1}{2} \,\middle|\, \begin{matrix} \frac{\beta+2}{4}, \frac{\beta}{4}+1 \\ 0, \frac{\beta}{2} \end{matrix}\right) /; z \notin (-1, 0)$$

01.02.26.0043.01

$$\theta(|z|-1)\left(\left(\sqrt{z-1}+\sqrt{z+1}\right)^{\beta} - \left(\sqrt{z+1}-\sqrt{z-1}\right)^{\beta}\right) = 2^{\frac{\beta}{2}-1}\sqrt{\pi}\, \beta\, G_{2,2}^{0,2}\left(z, \frac{1}{2} \,\middle|\, \begin{matrix} \frac{\beta}{4}+1, 1-\frac{\beta}{4} \\ 0, \frac{1}{2} \end{matrix}\right)$$

01.02.26.0044.01

$$\frac{\theta(1-|z|)}{\sqrt{1-z^2}}\left(\left(\sqrt{1+z}-\sqrt{1-z}\right)^{\beta} + \left(\sqrt{1+z}+\sqrt{1-z}\right)^{\beta}\right) = 2^{\frac{\beta}{2}+1}\sqrt{\pi}\, G_{2,2}^{2,0}\left(z, \frac{1}{2} \,\middle|\, \begin{matrix} \frac{\beta}{4}, \frac{\beta+2}{4} \\ 0, \frac{\beta}{2} \end{matrix}\right) /; z \notin (-1, 0)$$

01.02.26.0045.01

$$\frac{\theta(|z|-1)}{\sqrt{z^2-1}}\left(\left(\sqrt{z+1}-\sqrt{z-1}\right)^{\beta} + \left(\sqrt{z-1}+\sqrt{z+1}\right)^{\beta}\right) = 2^{\frac{\beta}{2}+1}\sqrt{\pi}\, G_{2,2}^{0,2}\left(z, \frac{1}{2} \,\middle|\, \begin{matrix} \frac{\beta+2}{4}, 1-\frac{\beta+2}{4} \\ 0, \frac{1}{2} \end{matrix}\right) /; \operatorname{Re}(z) > 0$$

Through other functions

01.02.26.0046.01

$$z^{1/n} == (x;\, x^n - z)_n^{-1} \,/;\, n == 2 \bigvee n - 3 \in \mathbb{N} \bigwedge z \notin (0, \infty)$$

Representations through equivalent functions

With inverse function

01.02.27.0013.01

$$\left(z^{1/a}\right)^a = e^{2 i a \pi \left\lfloor \frac{\pi - \mathrm{Im}\left(\frac{\log(z)}{a}\right)}{2\pi} \right\rfloor} z$$

The left side of the above formula corresponds to the composition $f\left(f^{(-1)}(z)\right) /;\; f(z) == z^a$, which generically does not equal z. This relation also reflects the invertibility of the power function z^a with respect to the variable z because its inverse $f^{(-1)}$ can coincide with the original function f, but for another value of the parameter a.

01.02.27.0014.01

$$\left(z^{1/a}\right)^a = z \,/;\, -\pi < \mathrm{Im}\left(\frac{\log(z)}{a}\right) \leq \pi$$

The left side of above formula corresponds to composition $f\left(f^{(-1)}(z)\right) /;\; f(z) == z^a$, which equal to z under restriction $-\pi < \mathrm{Im}\left(\frac{\log(z)}{a}\right) \leq \pi$. This relation also reflects invertibility of power function z^a with respect to variable z because its inversion $f^{(-1)}$ can coincides with original f but for other values of parameter a.

01.02.27.0015.01

$$(z^a)^{1/a} = e^{\frac{2 i \pi}{a} \left\lfloor \frac{\pi - \mathrm{Im}(a \log(z))}{2\pi} \right\rfloor} z$$

The left side of above formula corresponds to composition $f^{(-1)}(f(z)) /;\; f(z) == z^a$, which generically does not equal to z. This relation also reflects invertibility of power function z^a with respect to variable z because its inversion $f^{(-1)}$ can coincides with original f but for other values of parameter a.

01.02.27.0016.01

$$(z^a)^{1/a} = z \,/;\, -\pi < \mathrm{Im}(a \log(z)) \leq \pi$$

The left side of above formula corresponds to composition $f^{(-1)}(f(z)) /;\; f(z) == z^a$, which equal to z under restriction $-\pi < \mathrm{Im}(a \log(z)) \leq \pi$. This relation also reflects invertibility of power function z^a with respect to variable z because its inversion $f^{(-1)}$ can coincides with original f but for other values of parameter a.

01.02.27.0017.01

$$z^{\log_z(a)} = a$$

The left side of above formula corresponds to composition $f\left(f^{(-1)}(a)\right) /;\; f(a) == z^a$, which generically equal to a.

01.02.27.0018.01

$$\log_z(z^a) = a + \frac{2 i \pi}{\log(z)} \left\lfloor \frac{\pi - \mathrm{Im}(a \log(z))}{2\pi} \right\rfloor$$

The left side of above formula corresponds to composition $f^{(-1)}(f(a)) /;\; f(a) == z^a$, which generically does not equal to a.

01.02.27.0019.01

$$\log_z(z^a) = a \;/;\; -\pi < \operatorname{Im}(a \log(z)) \leq \pi$$

The left side of above formula corresponds to composition $f^{(-1)}(f(a))$ /; $f(a) == z^a$, which equal to a under restriction $-\pi < \operatorname{Im}(a \log(z)) \leq \pi$.

01.02.27.0020.01

$$\log_a(a^z) == z + \frac{2 i \pi}{\log(a)} \left\lfloor \frac{\pi - \operatorname{Im}(z \log(a))}{2 \pi} \right\rfloor$$

The left side of above formula corresponds to composition $f\left(f^{(-1)}(z)\right)$ /; $f(z) == \log_a(z)$, which generically does not equal to z.

01.02.27.0021.01

$$\log_a(a^z) == z \;/;\; a \in \mathbb{R} \land z > 0 \bigvee a > 0 \bigwedge -\pi < \operatorname{Im}(z \log(a)) \leq \pi$$

The left side of above formula corresponds to composition $f\left(f^{(-1)}(z)\right)$ /; $f(z) == \log_a(z)$, which equal to z under restriction $-\pi < \operatorname{Im}(z \log(a)) \leq \pi$.

01.02.27.0022.01

$$z^{\log_z(a)} == a$$

The left side of above formula corresponds to composition $f^{(-1)}(f(z))$ /; $f(z) == \log_a(z)$, which generically equal to z.

01.02.27.0023.01

$$\log_{z^a}(z) = \frac{\log(z)}{2 i \pi \left\lfloor \frac{\pi - \operatorname{Im}(a \log(z))}{2 \pi} \right\rfloor + a \log(z)}$$

01.02.27.0024.01

$$\log_{z^a}(z) = \frac{1}{a} \;/;\; -\pi < \operatorname{Im}(a \log(z)) \leq \pi$$

With related functions

01.02.27.0025.01

$$z = |z| \, e^{i \arg(z)}$$

01.02.27.0004.01

$$z^a == e^{a \log(z)}$$

01.02.27.0005.01

$$z^{\log(a)} == a^{\log(z)}$$

01.02.27.0006.01

$$z^{\frac{\log(a)}{\log(z)}} == a$$

With orthogonal polynomials

01.02.27.0007.01

$$z^n == n! \sum_{k=0}^{n} (-1)^k \binom{n + \lambda}{n - k} L_k^{\lambda}(z) \;/;\; n \in \mathbb{N}$$

01.02.27.0008.01

$$z^n = \frac{n!}{2^n}\sum_{k=0}^{\lfloor n/2\rfloor}\frac{1}{k!\,(n-2k)!}\,H_{n-2k}(z)\;/;\;n\in\mathbb{N}$$

01.02.27.0009.01

$$z^n = 2^{-n}\sum_{k=0}^{\left\lfloor \frac{n}{2}\right\rfloor}\left(\binom{n}{k}-\binom{n}{k-1}\right)U_{n-2k}(z)\;/;\;n\in\mathbb{N}^+$$

01.02.27.0010.01

$$z^n = \frac{n!}{2^n}\sum_{k=0}^{\lfloor n/2\rfloor}\frac{(-2k+n+\lambda)}{k!\,(\lambda)_{-k+n+1}}\,C_{n-2k}^{(\lambda)}(z)\;/;\;n\in\mathbb{N}$$

01.02.27.0011.01

$$(1-z)^n = 2^n\,\Gamma(a+n+1)\sum_{k=0}^{n}\frac{(a+b+2k+1)\,\Gamma(a+b+k+1)\,(-n)_k}{\Gamma(a+k+1)\,\Gamma(a+b+k+n+2)}\,P_k^{(a,b)}(z)\;/;\;n\in\mathbb{N}$$

01.02.27.0012.01

$$(1-z)^n = 2^n\,\Gamma(b+n+1)\sum_{k=0}^{n}\frac{(-1)^k\,(a+b+2k+1)\,\Gamma(a+b+k+1)\,(-n)_k}{\Gamma(b+k+1)\,\Gamma(a+b+k+n+2)}\,P_k^{(a,b)}(z)\;/;\;n\in\mathbb{N}$$

Inequalities

01.02.29.0001.01

$$\frac{y^{x^y}}{x^{y^x}} > \frac{y}{x} > \frac{y^x}{x^y}\;/;\;0<x<y<1\lor 1<x<y$$

01.02.29.0002.01

$$\left(\frac{y}{x}\right)^{x\,y} > \frac{y^x}{x^y}\;/;\;0<x<y<1\lor 1<x<y$$

01.02.29.0003.01

$$0 < x^{1/x}-1-\frac{\log(x)}{x} \le e\left(-1-e+e^{1+\frac{1}{e}}\right)\left(\frac{\log(x)}{x}\right)^2\;/;\;x\in\mathbb{R}\land x\ge 1$$

Zeros

01.02.30.0001.01

$$z^a = 0\;/;\;z=0\land \mathrm{Re}(a)>0$$

Theorems

Gelfond-Schneider theorem

The number α^β, where α, β are algebraic numbers, $\alpha \neq 0,\ 1, \beta \notin \mathbb{Q}$, is transcendental.

Mellin transformation and Parseval relation

$$\hat{f}(s) = \int_0^\infty x^{s-1} f(x)\, dx \Leftrightarrow f(x) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \hat{f}(s)\, x^{-s}\, ds;\ \int_0^\infty f_1\left(\frac{x}{t}\right) f_2(t)\frac{dt}{t} = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \hat{f}_1(s)\,\hat{f}_2(s)\, x^{-s}\, ds.$$

Hilbert transformation

$$\hat{f}(y) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{f(x)}{x-y}\, dx \Leftrightarrow f(x) = -\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\hat{f}(y)}{y-x}\, dy.$$

The eigenfunctions of the Hilbert transformation are given by the following:

$$\frac{1}{\pi}\mathcal{P}\int_{-\infty}^{\infty}\frac{f(x,n)}{x-y}\, dx = \text{sgn}(n)\, f(y,n)\ /;\ f(x,n) = \frac{(1+i\,x)^n}{(1-i\,x)^{n+1}} \bigwedge n \in \mathbb{Z}$$

Zipf's law

The frequency f versus the rank r in a text has the form $f = r^{-\beta}$, where $\beta \approx 0.6$ for natural languages.

The quantum mechanical density of states of a d-dimensional lattice

The quantum mechanical density of states $\mathcal{D}(\varepsilon)$ of a d-dimensional lattice exhibits van Hove singularities of the form $\mathcal{D}(\varepsilon) \propto \varepsilon^{\frac{d-1}{2}}$, where ε is the energy.

Infinite tetration

The sequence

$x^x,\ x^{x^x},\ x^{x^{x^x}},\ \ldots$

converges for real x only if $e^{-e} \le x \le e^{1/e}$.

Differential-Algebraic Constants

To ensure the correctness of many formulas given in this collection over the whole complex plane, it is often necessary to work with expressions of the form $\left(z^2\right)^a = (i\,z)^a\,(-i\,z)^a$, $(z^a)^{1/a}$, $(z^a)^{1/a}/z$, $z\,(z^{-a})^{1/a}$, $(-z^a)^{1/a}$, $\left(-z^2\right)^a = z^a(-z)^a$,etc. (where a is a generic complex number). While in a textbook-mathematics setting these expressions are often simplified to z, ± 1, 1, $(-1)^{1/a}\,z$, etc, this cannot be done inside *Mathematica*. From a complex function point of view the Riemann surface of such functions are made from disconnected sheets. Inside *Mathematica* all branch cuts of all functions (that have branch cuts) follow uniquely from the branch cut of the single power function (the logarithm function respectively). As a result the branch cuts related to more complicated functions such as $\left(z^2\right)^a$, $(z^a)^{1/a}$, $(z^a)^{1/a}/z$, $z\,(z^{-a})^{1/a}$, $(-z^a)^{1/a}$, $\left(-z^2\right)^a$ exist, although they do not start and end at branch points, but mostly extend from the origin to $\tilde{\infty}$ along opposite rays.
For instance we have:

$$\mathcal{BC}_z\left(\left(z^2\right)^a\right) = \mathcal{BC}_z\left((i\,z)^a\,(-i\,z)^a\right) = \{\{(-i\,\infty,\,0),\,1\},\,\{(0,\,i\,\infty),\,-1\}\}$$

$$\lim_{\epsilon\to+0}\left((x+\epsilon)^2\right)^a == e^{-2 i \pi a}\left(x^2\right)^a \;/;\; i\,x>0$$

$$\lim_{\epsilon\to+0}\left((x-\epsilon)^2\right)^a == e^{-2 i \pi a}\left(x^2\right)^a \;/;\; i\,x<0$$

An expression of the form $(z^a)^{1/a}/z$, $z\,(z^{-a})^{1/a}$ are called differential-algebraic constants because their derivative vanishes generically everywhere as a complex function (but not as a generalized function).

History

– Ancient Egyptian, Babylonian, and Greek mathematicians
– Euclid knew $a^m a^n == a^{m+n}$ for integers m and n
– N. Oresme (14th century) knew this rule even for fractional positive m and n, and also knew $(a\,b)^{1/n} == a^{1/n}\,b^{1/n}$ and $(a^m)^{p/q} == (a^{m\,p})^{1/q}$
– F. Viete (1593) derived the famous geometric series for 1/(1+x)
– Joh. Bernoulli (1697) defined arbitrary powers as $z^a == \left(e^{\log(z)}\right)^a == e^{a\log(z)}$
– L. Euler (1728) proved that $i^i == e^{-\pi/2}$ and in 1748 used a series expansion for $(1+x)^a$ with rational a
– Martin Ohm (1823) was the first who fully develop the theory of the exponential z^a, when both z and a are complex numbers
– N. H. Abel (1826) first proved correctness of the series expansion of $(1+x)^a$ in the unit circle for arbitrary a

The function z^a is encountered often in mathematics and the natural sciences.

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