PolyGamma2

Notations

Traditional name

Polygamma function

Traditional notation

$\psi^{(\nu)}(z)$

Mathematica StandardForm notation

`PolyGamma[ν, z]`

Primary definition

06.15.02.0001.01

$$\psi^{(\nu)}(z) = -\mathcal{FC}_{\exp}^{(\nu)}(z, -1)\, z^{-\nu-1} - \frac{\gamma\, z^{-\nu}}{\Gamma(1-\nu)} + z^{1-\nu} \sum_{k=1}^{\infty} \frac{1}{k^2}\, {}_2\tilde{F}_1\!\left(1, 2; 2-\nu; -\frac{z}{k}\right)$$

Above formula presents (not unique) continuation of the classical definition of $\psi^{(\nu)}(z)$ from positive integer values of ν to its arbitrary complex values. Below four formulas are the particular cases of the above general definition.

06.15.02.0003.01

$$\psi^{(\nu)}(z) = \frac{\nu \log(z) - \gamma\,(z+\nu) - \nu\,\psi(-\nu)}{\Gamma(1-\nu)}\, z^{-\nu-1} + \sum_{k=1}^{\infty} \frac{1}{k^2}\, {}_2\tilde{F}_1\!\left(1, 2; 2-\nu; -\frac{z}{k}\right) z^{1-\nu} \;/;\; \neg\,(\nu \in \mathbb{Z} \land \nu \geq 0)$$

The classical definition of $\psi^{(n)}(z)$ for positive integer n is the following:

06.15.02.0002.02

$$\psi^{(n)}(z) = (-1)^{n+1}\, n! \sum_{k=0}^{\infty} \frac{1}{(k+z)^{n+1}} \;/;\; n \in \mathbb{N}^+$$

06.15.02.0004.01

$$\psi^{(0)}(z) = z \sum_{k=1}^{\infty} \frac{1}{k\,(k+z)} - \frac{1}{z} - \gamma$$

06.15.02.0005.01

$$\psi^{(0)}(z) = \sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{k+z-1}\right) - \gamma$$

06.15.02.0006.01

$$\psi^{(0)}(z) == \sum_{k=1}^{\infty}\left(\frac{1}{k} - \frac{1}{k+z-1}\right) - \gamma$$

Below formula accumulates above definitions for different values of parameter ν in one expression.

06.15.02.0007.01

$$\psi^{(\nu)}(z) = \begin{cases} -\frac{1}{z} - \gamma + z\sum_{k=1}^{\infty}\frac{1}{k\,(k+z)} & \nu = 0 \\ (-1)^{\nu+1}\,\nu!\sum_{k=0}^{\infty}\frac{1}{(k+z)^{\nu+1}} & \nu \in \mathbb{N}^+ \\ \frac{-\gamma\,(z+\nu)+\nu\log(z)-\nu\,\psi(-\nu)}{\Gamma(1-\nu)}\,z^{-\nu-1} + z^{1-\nu}\sum_{k=1}^{\infty}\frac{1}{k^2}\,{}_2\tilde{F}_1\!\left(1, 2; 2-\nu; -\frac{z}{k}\right) & \text{True} \end{cases}$$

06.15.02.0008.01

$$\psi^{(\nu)}(z) = \frac{\partial^{\nu}\psi(z)}{\partial z^{\nu}}$$

Above formula reflects the main functional property of classical polygamma function. In classical interpretation it is valid only for $\nu == 0, 1, 2, 3, \ldots$ But it is very convenient to use it as basic definition of polygamma function for any arbitrary complex values ν, where symbol $\frac{\partial^{\nu}\psi(z)}{\partial z^{\nu}}$ denotes the νth fractional integro-derivative of $\psi(z)$ with respect to z (which provides the Riemann-Liouville-Hadamard fractional left-sided integro-differentiation with beginning at the point 0). Such approach was realized in *Mathematica*.

Specific values

Specialized values

For fixed ν

06.15.03.0001.01

$$\psi^{(\nu)}(0) == \tilde{\infty} \;/;\; \mathrm{Re}(\nu) > -1$$

06.15.03.0002.01

$$\psi^{(\nu)}(0) == 0 \;/;\; \mathrm{Re}(\nu) < -1$$

06.15.03.0009.01

$$\psi^{(n)}(1) = (-1)^{n+1}\,n!\,\zeta(n+1) \;/;\; n \in \mathbb{N}^+$$

06.15.03.0010.01

$$\psi^{(-n)}(1) = \frac{1}{(n-1)!}\left(\frac{\gamma\,(n-1)}{n} + \psi(n) + \sum_{k=0}^{n-1}\binom{n-1}{k}\left(\sum_{j=1}^{k}(-1)^j\binom{k}{j}\psi(k-j+1)\,\zeta(j-k, 2) - \psi(k+1)\right) - \sum_{k=0}^{n-2}\binom{n-1}{k}\zeta'(-k)\right) /;\; n \in \mathbb{N}^+$$

06.15.03.0011.01

$$\psi^{(-1)}(1) = 0$$

06.15.03.0012.01

$$\psi^{(-2)}(1) = \frac{1}{2}\log(2\,\pi)$$

06.15.03.0013.01

$$\psi^{(-3)}(1) = \log(A) + \frac{1}{4}\log(2\pi)$$

06.15.03.0014.01

$$\psi^{(-4)}(1) = \frac{1}{24}\left(2\,(6\log(A) + \log(2\pi)) + \frac{3\,\zeta(3)}{\pi^2}\right)$$

06.15.03.0015.01

$$\psi^{(-5)}(1) = \frac{720\log(A) + 90\log(2\pi) + \frac{270\,\zeta(3)}{\pi^2} - 720\,\zeta'(-3) - 11}{4320}$$

06.15.03.0016.01

$$\psi^{(-6)}(1) = \frac{180\,\pi^2\,\zeta(3) - 270\,\zeta(5) + \pi^4\,(360\log(A) + 36\log(2\pi) - 720\,\zeta'(-3) - 11)}{8640\,\pi^4}$$

06.15.03.0017.01

$$\psi^{(-7)}(1) = \frac{5040\log(A) + 420\log(2\pi) + \frac{3150\,\zeta(3)}{\pi^2} - 5040\,\zeta'(-5) - 16\,800\,\zeta'(-3) - \frac{9450\,\zeta(5)}{\pi^4} - 211}{604\,800}$$

06.15.03.0018.01

$$\psi^{(-8)}(1) = \frac{1}{1\,814\,400\,\pi^6}\left(1890\,\pi^4\,\zeta(3) - 9450\,\pi^2\,\zeta(5) + 14\,175\,\zeta(7) + 4\,\pi^6\,(630\log(A) + 45\log(2\pi) - 1890\,\zeta'(-5) - 3150\,\zeta'(-3) - 31)\right)$$

06.15.03.0019.01

$$\psi^{(-9)}(1) = \frac{\log(A)}{5040} + \frac{\log(2\pi)}{80\,640} + \frac{\zeta(3)}{5760\,\pi^2} + \frac{\zeta(7)}{256\,\pi^6} - \frac{1}{720}\,\zeta'(-5) - \frac{1}{720}\,\zeta'(-3) - \frac{\zeta'(-7)}{5040} - \frac{16\,427}{1\,524\,096\,000} - \frac{\zeta(5)}{768\,\pi^4}$$

06.15.03.0020.01

$$\psi^{(-10)}(1) = \frac{1}{3\,048\,192\,000\,\pi^8}\left(75\,600\,\pi^6\,\zeta(3) - 793\,800\,\pi^4\,\zeta(5) + 3\,969\,000\,\pi^2\,\zeta(7) - 5\,953\,500\,\zeta(9) + \pi^8\,(75\,600\log(A) + 4200\log(2\pi) - 302\,400\,\zeta'(-7) - 1\,058\,400\,\zeta'(-5) - 705\,600\,\zeta'(-3) - 4457)\right)$$

06.15.03.0021.01

$$\psi^{(n)}\left(\frac{1}{2}\right) = (-1)^{n+1}\,n!\,\left(2^{n+1} - 1\right)\zeta(n+1)\;/;\; n \in \mathbb{N}^+$$

06.15.03.0022.01

$$\psi^{(-1)}(0) = \infty$$

06.15.03.0023.01

$$\psi^{(2n-1)}\left(\frac{1}{4}\right) + \psi^{(2n-1)}\left(\frac{3}{4}\right) = \left(4^{2n} - 4^n\right)(2n-1)!\,\zeta(2n)\;/;\; n \in \mathbb{N}^+$$

06.15.03.0024.01

$$\psi^{(\nu)}(-n) = \tilde{\infty}\;/;\; \mathrm{Re}(\nu) > -1 \land n \in \mathbb{N}$$

06.15.03.0025.01

$$\psi^{(\nu)}(-n) = ¿\;/;\; \mathrm{Re}(\nu) = -1 \land n \in \mathbb{N}$$

For fixed z

06.15.03.0004.01

$$\psi^{(0)}(z) == \psi(z)$$

06.15.03.0005.01

$$\psi^{(n)}(z) = (-1)^{n+1}\, n!\, \zeta(n+1, z) /; n \in \mathbb{N}^+ \bigwedge \operatorname{Re}(z) > 0$$

06.15.03.0026.01

$$\psi^{(n)}(z) = \frac{\partial^n \frac{\frac{\partial \Gamma(z)}{\partial z}}{\Gamma(z)}}{\partial z^n} /; n \in \mathbb{N}$$

06.15.03.0006.02

$$\psi^{(n)}(z) = \frac{\partial^n \psi(z)}{\partial z^n} /; n \in \mathbb{N}$$

06.15.03.0007.01

$$\psi^{(n)}(z) = \frac{\partial^{n+1} \log\Gamma(z)}{\partial z^{n+1}} /; n \in \mathbb{N}$$

06.15.03.0003.01

$$\psi^{(-1)}(z) = \log\Gamma(z)$$

06.15.03.0027.01

$$\psi^{(-1)}(z) = \frac{1}{2}\log(2\pi) + \zeta^{(1,0)}(0, z) /; \operatorname{Re}(z) > 0$$

06.15.03.0028.01

$$\psi^{(-1)}(z) = \log(\Gamma(z+1)) - \log(z)$$

06.15.03.0029.01

$$\psi^{(-1)}(z) = \zeta^{(1,0)}(0, z) + \frac{1}{2}\log(2\pi) + (2\,\theta(\operatorname{Im}(z)) - 1)\, i\, \pi \operatorname{Re}(\lfloor z \rfloor) + \frac{1}{2}\left(1 + (-1)^{\lfloor -\operatorname{Re}(z) \rfloor + \lfloor \operatorname{Re}(z) \rfloor}\right) i\, \pi\, \theta(-\operatorname{Im}(z))\, \theta(-\operatorname{Re}(z)) /; \operatorname{Re}(z) > 0$$

06.15.03.0030.01

$$\psi^{(-2)}(z) = \frac{1}{2}\left(-z^2 + \log(2\pi)\, z + z - 2\,\zeta'(-1) + 2\,\zeta^{(1,0)}(-1, z)\right) /; -\frac{\pi}{2} < \arg(z) \leq \frac{\pi}{2}$$

06.15.03.0031.01

$$\psi^{(-2)}(z) = \frac{1}{12}\,(12\log(A) + 6\, z\,(-z + \log(2\pi) + 1) - 12\, z \log(z) - 1) + \zeta^{(1,0)}(-1, z+1)$$

06.15.03.0032.01

$$\psi^{(-3)}(z) = \frac{1}{24}\left(-6\, z^3 + 3\,(2\log(2\pi) + 3)\, z^2 - (24\,\zeta'(-1) + 1)\, z + \frac{3\,\zeta(3)}{\pi^2} + 12\,\zeta^{(1,0)}(-2, z)\right)$$

06.15.03.0033.01

$$\psi^{(-3)}(z) = \frac{1}{8\,\pi^2}\left(-\pi^2\, z\,(-8\log(A) + z\,(2\, z - 2\log(2\pi) - 3) + 4\, z\log(z) + 1) + \zeta(3) + 4\,\pi^2\,\zeta^{(1,0)}(-2, z+1)\right)$$

06.15.03.0034.01

$$\psi^{(-4)}(z) = \frac{1}{144}\left(-24\log(z)\, z^3 + (72\log(A) + z\,(-11\, z + 12\log(2\pi) + 22) - 11)\, z^2 + \frac{18\,\zeta(3)\, z}{\pi^2} - 24\,\zeta'(-3) + 24\,\zeta^{(1,0)}(-3, z+1)\right)$$

06.15.03.0035.01

$$\psi^{(-5)}(z) = \frac{1}{8640\,\pi^4}\left(-360\,\pi^4\log(z)\, z^4 + \pi^2\left(540\, z\,\zeta(3) + \pi^2\left(5\,(288\log(A) + 3\, z\,(-10\, z + 12\log(2\pi) + 25) - 50)\, z^2 - 1440\,\zeta'(-3) + 3\right)\right) z - 270\,\zeta(5) + 360\,\pi^4\,\zeta^{(1,0)}(-4, z+1)\right)$$

06.15.03.0036.01

$$\psi^{(-6)}(z) = \frac{1}{86\,400\,\pi^4}\left(\pi^4\left(\left(-274\,(z-3)\,z + 360\log(2\,\pi)\,z + 3600\log(A) - 685\right)z^2 + 27\right)z^2 - 180\left(4\,\pi^4\log(z)\,z^5 - 10\,\pi^2\,\zeta(3)\,z^3 + 15\,\zeta(5)\,z + 4\,\pi^4\left(10\,\zeta'(-3)\,z^2 + \zeta'(-5)\right)\right) + 720\,\pi^4\,\zeta^{(1,0)}(-5,\,z+1)\right)$$

06.15.03.0037.01

$$\psi^{(-7)}(z) = \frac{1}{1\,814\,400\,\pi^6}\left(\pi^6\,z\left(7\,z^2\left(9\,(240\log(A) + z\,(-14\,z + 20\log(2\,\pi) + 49) - 49)\,z^2 + 37\right) - 10\right) + 14\,175\,\zeta(7) - 630\,\pi^2\,z\left(4\,\pi^4\log(z)\,z^5 - 15\,\pi^2\,\zeta(3)\,z^3 + 45\,\zeta(5)\,z + 8\,\pi^4\left(10\,\zeta'(-3)\,z^2 + 3\,\zeta'(-5)\right)\right) + 2520\,\pi^6\,\zeta^{(1,0)}(-6,\,z+1)\right)$$

06.15.03.0038.01

$$\psi^{(-8)}(z) = -\frac{1}{50\,803\,200\,\pi^6}\left(\pi^6\left(\left(9\,z^2\,(363\,(z-4)\,z - 560\log(2\,\pi)\,z - 7840\log(A) + 1694) - 2233\right)z^2 + 260\right)z^2 + 1260\left(8\,\pi^6\log(z)\,z^7 - 42\,\pi^4\,\zeta(3)\,z^5 + 210\,\pi^2\,\zeta(5)\,z^3 - 315\,\zeta(7)\,z + 8\,\pi^6\left(35\,\zeta'(-3)\,z^4 + 21\,\zeta'(-5)\,z^2 + \zeta'(-7)\right)\right) - 10\,080\,\pi^6\,\zeta^{(1,0)}(-7,\,z+1)\right)$$

06.15.03.0039.01

$$\psi^{(-9)}(z) = \frac{1}{3\,048\,192\,000\,\pi^8}\left(\pi^8\,z\left(-22\,830\,z^8 + 135\,(280\log(2\,\pi) + 761)\,z^7 + 180\,(3360\log(A) - 761)\,z^6 + 31\,206\,z^4 - 7300\,z^2 + 315\right) - 5\,953\,500\,\zeta(9) - 37\,800\,\pi^2\,z\left(2\,\pi^6\log(z)\,z^7 - 14\,\pi^4\,\zeta(3)\,z^5 + 105\,\pi^2\,\zeta(5)\,z^3 - 315\,\zeta(7)\,z + 16\,\pi^6\left(7\,\zeta'(-3)\,z^4 + 7\,\zeta'(-5)\,z^2 + \zeta'(-7)\right)\right) + 75\,600\,\pi^8\,\zeta^{(1,0)}(-8,\,z+1)\right)$$

06.15.03.0040.01

$$\psi^{(-10)}(z) = -\frac{1}{18\,289\,152\,000\,\pi^8}\left(\pi^8\left(z^2\left(\left(z^2\,(14\,258\,(z-5)\,z - 25\,200\log(2\,\pi)\,z - 453\,600\log(A) + 106\,935) - 35\,126\right)z^2 + 13\,750\right) - 1785\right)z^2 + 12\,600\left(4\,\pi^8\log(z)\,z^9 - 36\,\pi^6\,\zeta(3)\,z^7 + 378\,\pi^4\,\zeta(5)\,z^5 - 1890\,\pi^2\,\zeta(7)\,z^3 + 2835\,\zeta(9)\,z + 4\,\pi^8\left(84\,\zeta'(-3)\,z^6 + 126\,\zeta'(-5)\,z^4 + 36\,\zeta'(-7)\,z^2 + \zeta'(-9)\right)\right) - 50\,400\,\pi^8\,\zeta^{(1,0)}(-9,\,z+1)\right)$$

06.15.03.0041.01

$$\psi^{(-n)}(z) = \frac{1}{(n-1)!}\left(\zeta^{(1,0)}(1-n,\,z+\max(\lfloor -\mathrm{Re}(z)\rfloor,\,0)+1) - \sum_{i=0}^{\lfloor -\mathrm{Re}(z)\rfloor -1}\frac{\log(i+z+1)}{(i+z+1)^{1-n}}\right) + \frac{z^{n-1}}{(n-1)!}\left(-\frac{\gamma\,z}{n} - \log(z) + \psi(n) + \sum_{k=0}^{n-1}\frac{1}{z^k}\binom{n-1}{k}\left(\sum_{j=0}^{k}\binom{k}{j}\psi(k-j+1)\left(\sum_{i=0}^{\lfloor -\mathrm{Re}(z)\rfloor -1}(i+z+1)^{k-j} + \zeta(j-k,\,z+\max(\lfloor -\mathrm{Re}(z)\rfloor,\,0)+1)\right)(-z)^j - \psi(k+1)\,\zeta(-k) - \zeta'(-k)\right) + \gamma\right) /;\ n\in\mathbb{N}^+$$

06.15.03.0042.01

$$\psi^{(-n)}(z) = \frac{(-\gamma\,(z-n) - n\log(z) + n\,\psi(n))\,z^{n-1}}{n!} + \frac{1}{(n-1)!}\sum_{k=1}^{\infty}\left(\frac{z^n}{k}\sum_{m=0}^{n-1}\frac{(1-n)_m}{m!}\sum_{j=1}^{m+1}\frac{1}{j}\left(-\frac{z}{k}\right)^{j-m-1} - (k+z)^{n-1}\log\left(\frac{z}{k}+1\right)\right) /;\ n\in\mathbb{N}^+$$

06.15.03.0043.01

$$\psi^{\left(n+\frac{1}{2}\right)}(z) = \frac{z^{-n-\frac{3}{2}}}{2\,\Gamma\left(\frac{1}{2}-n\right)}\left(2\,z\,\psi(z+1)-(2\,n+1)\left(-\log(z)+\psi\left(-n-\frac{1}{2}\right)+\gamma\right)\right)+ \frac{2\sqrt{z}}{\Gamma\left(-n-\frac{1}{2}\right)}\sum_{k=1}^{\infty}(k+z)^{-n-2}\left(\frac{1}{z}\sqrt{-\frac{(k+z)^2}{k^2}}\;k\sqrt{\frac{z}{k+z}}\;\sin^{-1}\left(\sqrt{-\frac{z}{k}}\right)+\sum_{j=1}^{|n|}\frac{1}{2\,j-1}\left(\frac{z}{k+z}\right)^{j\,(-\operatorname{sgn}(n))-\theta(-n)}\right)\;/;\; n\in\mathbb{Z}$$

Values at fixed points

06.15.03.0008.01

$$\psi^{(1)}\left(\frac{1}{4}\right) == 8\,C+\pi^2$$

06.15.03.0044.01

$$\psi^{(1)}\left(\frac{3}{4}\right) = \pi^2-8\,C$$

06.15.03.0045.01

$$\psi^{(2)}\left(\frac{5}{6}\right) = 4\sqrt{3}\;\pi^3-182\,\zeta(3)$$

06.15.03.0046.01

$$\psi^{(0)}(-n) = \tilde{\infty}\;/;\; n\in\mathbb{N}$$

06.15.03.0047.01

$$\psi^{(-1)}(0) = \infty$$

06.15.03.0048.01

$$\psi^{(-1)}(n) = \log((n-1)!)\;/;\; n\in\mathbb{N}^+$$

06.15.03.0049.01

$$\psi^{(-1)}(-n) = \tilde{\infty}\;/;\; n\in\mathbb{N}$$

06.15.03.0050.01

$$\psi^{(-n)}(-m) = \frac{1}{(n-1)!}\,\zeta^{(1,0)}(1-n,\,1-m)-\frac{1}{(n-1)!}\sum_{i=0}^{m-2}(i-m+1)^{n-1}\left(i\,\pi+(1+(-1)^n)\log(m-i-1)\right)+ \frac{(-m)^{n-1}}{(n-1)!}\left(\gamma+\frac{\gamma\,m}{n}-i\,\pi-\log(m)+\psi(n)+\sum_{k=0}^{n-1}(-1)^k\,m^{-k}\binom{n-1}{k}\left(\sum_{j=0}^{k}m^j\binom{k}{j}\psi(k-j+1)\left(\sum_{i=0}^{m-2}(i-m+1)^{k-j}+\zeta(j-k)\right)-\psi(k+1)\,\zeta(-k)-\zeta'(-k)\right)\right)\;/;\; n\in\mathbb{Z}\bigwedge n>1\bigwedge m\in\mathbb{N}^+$$

06.15.03.0051.01

$$\psi^{(-n)}(m) = \frac{1}{(n-1)!}\,\zeta^{(1,0)}(1-n,\,m+1)+\frac{m^{n-1}}{(n-1)!}\left(-\frac{\gamma\,m}{n}-\log(m)+\psi(n)+\sum_{k=0}^{n-1}\frac{\binom{n-1}{k}\left(\sum_{j=0}^{k}\binom{k}{j}\psi(-j+k+1)\,\zeta(j-k,\,m+1)\,(-m)^j-\psi(k+1)\,\zeta(-k)-\operatorname{Zeta}'(-k)\right)}{m^k}+\gamma\right)\;/;\; \left(n\in\mathbb{N}^+\bigwedge m\in\mathbb{N}^+\right)\bigvee\left(n\in\mathbb{Z}\land n>1\land m=-1\right)$$

06.15.03.0052.01

$$\psi^{(-2n)}\left(\frac{p}{q}\right) = \frac{1}{(2n-1)!}\left(-\frac{i(2n-1)\pi}{2n}\left(\frac{p}{q}\right)^{2n} + \left(-\frac{p}{q}\right)^{2n-1}\left(\log\left(\frac{p}{\pi q}\right) + \pi i + \log\left(\sin\left(\frac{p\pi}{q}\right)\right) - 2\psi(2n) - 2\gamma + \sum_{j=0}^{2n-2}\frac{(-1)^j}{2n-j-1}\binom{2n-1}{j} + \sum_{j=0}^{2n-2}(-1)^j\binom{2n-1}{j}\sum_{k=0}^{2n-j-1}\left(-\frac{2\pi i p}{q}\right)^{-k}\binom{2n-j-1}{k}k!\,\text{Li}_{k+1}\left(e^{\frac{2ip\pi}{q}}\right)\right) - (2i\pi)^{1-2n}\sum_{j=0}^{2n-2}\left(\frac{2\pi i p}{q}\right)^j\binom{2n-1}{j}(2n-j-1)!\,\zeta(2n-j) + \left((-1)^{n+1}2^{-2n}\pi^{1-2n}q^{-2n}\sum_{j=1}^{q-1}\psi^{(2n-1)}\left(\frac{j}{q}\right)\sin\left(\frac{2j\pi(q-p)}{q}\right) + (-1)^{n+1}2^{1-2n}\pi^{-2n}(2n-1)!\,q^{-2n}\sum_{j=1}^{q-1}\cos\left(\frac{2j\pi(q-p)}{q}\right)\zeta^{(1,0)}\left(2n,\frac{j}{q}\right) + \zeta'(1-2n)q^{-2n} - \frac{(\psi(2n)-\log(2\pi))q^{-2n}}{2n}B_{2n} + \frac{(\psi(2n)-\log(2\pi q))}{2n}B_{2n}\left(\frac{q-p}{q}\right) - \left(\frac{p}{q}\right)^{2n-1}\left(\gamma + \frac{\gamma p}{2nq} - \pi i - \log\left(\frac{p}{q}\right) + \psi(2n) + \sum_{k=0}^{2n-1}\left(-\frac{p}{q}\right)^{-k}\binom{2n-1}{k}\left(\sum_{j=0}^{k}\left(\frac{p}{q}\right)^j\binom{k}{j}\psi(k-j+1)\left(2\zeta\left(j-k,1-\frac{p}{q}\right) - \zeta\left(j-k,1-\frac{p}{q}\right)\right) - \psi(k+1)\zeta(-k) - \zeta'(-k)\right)\right)\right)\right) /;\, n\in\mathbb{N}^+ \land p\in\mathbb{Z} \land 0<p<q \land q\in\mathbb{Z} \land q>1$$

06.15.03.0053.01

$$\psi^{(-2n)}\left(-\frac{p}{q}\right) = \frac{1}{(2n-1)!}\left((-1)^{n+1}2^{-2n}\pi^{1-2n}q^{-2n}\sum_{j=1}^{q-1}\psi^{(2n-1)}\left(\frac{j}{q}\right)\sin\left(\frac{2j\pi(q-p)}{q}\right) + (-1)^{n+1}2^{1-2n}\pi^{-2n}(2n-1)!\,q^{-2n}\sum_{j=1}^{q-1}\cos\left(\frac{2j\pi(q-p)}{q}\right)\zeta^{(1,0)}\left(2n,\frac{j}{q}\right) + \zeta'(1-2n)q^{-2n} - \frac{(\psi(2n)-\log(2\pi))q^{-2n}}{2n}B_{2n} + \frac{\psi(2n)-\log(2\pi q)}{2n}B_{2n}\left(\frac{q-p}{q}\right) - \left(\frac{p}{q}\right)^{2n-1}\left(\frac{\gamma p}{2nq} - \pi i - \log\left(\frac{p}{q}\right) + \psi(2n) + \sum_{k=0}^{2n-1}\left(-\frac{p}{q}\right)^{-k}\binom{2n-1}{k}\left(\sum_{j=0}^{k}\left(\frac{p}{q}\right)^j\binom{k}{j}\psi(-j+k+1)\left(2\zeta\left(j-k,1-\frac{p}{q}\right) - \zeta\left(j-k,1-\frac{p}{q}\right)\right) - \psi(k+1)\zeta(-k) - \zeta'(-k)\right) + \gamma\right)\right) /;\, n\in\mathbb{N}^+ \land p\in\mathbb{Z} \land 0<p<q \land q\in\mathbb{Z} \land q>1$$

06.15.03.0054.01

$$\psi^{(-2n)}\left(m+\frac{p}{q}\right) = \frac{1}{(2n-1)!}\left(\zeta'(1-2n)\,q^{-2n} + \frac{\psi(2n)-\log(2\pi q)}{2n}\,B_{2n}\left(\frac{p}{q}\right) + \frac{1}{2}\sum_{k=0}^{m+\max\left(0,\left\lfloor -m-\frac{p}{q}\right\rfloor\right)} \frac{\log\left(\left(k-\frac{p+m\,q+q\max\left(0,\left\lfloor -\frac{p+m\,q}{q}\right\rfloor\right)}{q}\right)^2\right)}{\left(\left(k-\frac{p+m\,q+q\max\left(0,\left\lfloor -\frac{p+m\,q}{q}\right\rfloor\right)}{q}\right)^2\right)^{\frac{1}{2}-n}} - \sum_{i=0}^{\left\lfloor -m-\frac{p}{q}\right\rfloor -1}\left(i+m+\frac{p}{q}+1\right)^{2n-1}\log\left(i+m+\frac{p}{q}+1\right) + \frac{(-1)^{n+1}\pi}{(2\pi q)^{2n}}\sum_{j=1}^{q-1}\sin\left(\frac{2\pi p j}{q}\right)\psi^{(2n-1)}\left(\frac{j}{q}\right) + \left(m+\frac{p}{q}\right)^{2n-1}\left(-\frac{\gamma(m\,q+p)}{2n\,q} - \log\left(m+\frac{p}{q}\right) + \psi(2n) + \sum_{k=0}^{2n-1}\left(m+\frac{p}{q}\right)^{-k}\binom{2n-1}{k}\left(\sum_{j=0}^{k}\left(-m-\frac{p}{q}\right)^{j}\binom{k}{j}\psi(k-j+1)\left(\sum_{i=0}^{\left\lfloor -m-\frac{p}{q}\right\rfloor -1}\left(i+m+\frac{p}{q}+1\right)^{k-j} + \zeta\left(j-k,\,m+\max\left(0,\left\lfloor -m-\frac{p}{q}\right\rfloor\right)+\frac{p}{q}+1\right)\right) - \psi(k+1)\,\zeta(-k) - \zeta'(-k)\right) + \gamma\right) + \frac{(-1)^{n+1}\,2\,(2n-1)!}{(2\pi q)^{2n}}\sum_{j=1}^{q-1}\cos\left(\frac{2\pi p j}{q}\right)\zeta^{(1,0)}\left(2n,\,\frac{j}{q}\right) - \frac{\psi(2n)-\log(2\pi)}{2\,q^{2n}\,n}\,B_{2n}\right) /;\, m\in\mathbb{Z}\land n\in\mathbb{N}^+\land p\in\mathbb{Z}\land 0<p<q\land q\in\mathbb{Z}\land q>1$$

06.15.03.0055.01

$$\psi^{(-2n)}\left(-\frac{3}{4}\right) = \frac{1}{(2n-1)!}\left(\frac{\pi\left(4^n-4^{2n}\right)+\left(2^{2n+1}-8\right)\log(2)}{4^{2n+1}\,n}\,B_{2n} - \frac{2^{2n-1}-1}{2^{4n-1}}\,\zeta'(1-2n) - \frac{(-1)^n}{4\,(8\pi)^{2n-1}}\,\psi^{(2n-1)}\left(\frac{1}{4}\right) - \left(\frac{3}{4}\right)^{2n-1}\left(\gamma + \frac{3\gamma}{8n} - i\pi + \log\left(\frac{4}{3}\right) + \psi(2n) + \sum_{k=0}^{2n-1}\left(-\frac{4}{3}\right)^k\binom{2n-1}{k}\sum_{j=0}^{k}\left(\frac{3}{4}\right)^j\binom{k}{j}\psi(-j+k+1)\,\zeta\left(j-k,\,\frac{1}{4}\right) - \psi(k+1)\,\zeta(-k) - \zeta'(-k)\right)\right) /;\, n\in\mathbb{N}^+$$

06.15.03.0056.01

$$\psi^{(-2n)}\left(-\frac{2}{3}\right) = -\frac{1}{(2n-1)!}\left(\frac{2}{3}\right)^{2n-1}\left(\gamma + \frac{\gamma}{3n} - i\pi + \log\left(\frac{3}{2}\right) + \psi(2n) + \sum_{k=0}^{2n-1}\left(-\frac{3}{2}\right)^k \binom{2n-1}{k} \sum_{j=0}^{k}\left(\frac{2}{3}\right)^j \binom{k}{j} \psi(-j+k+1)\,\zeta\left(j-k, \frac{1}{3}\right) - \psi(k+1)\,\zeta(-k) - \zeta'(-k)\right) - \frac{9^{-n}\left(\sqrt{3}\,\pi(-1+9^n) + 6\log(3)\right)}{8n(2n-1)!} B_{2n} + \frac{(-1)^{n-1}\, 3^{\frac{1}{2}-2n}\, 4^{-n}\, \pi^{1-2n}}{(2n-1)!} \psi^{(2n-1)}\left(\frac{1}{3}\right) + \frac{9^{-n}(3-9^n)}{2(2n-1)!} \zeta'(1-2n) /; n \in \mathbb{N}^+$$

06.15.03.0057.01

$$\psi^{(-2n)}\left(-\frac{1}{2}\right) = -\frac{B_{2n}\log(2)}{2^{2n}\, n\,(2n-1)!} - \frac{1}{2^{2n-1}(2n-1)!}\left(\gamma + \frac{\gamma}{4n} - i\pi + \log(2) + \psi(2n) + \sum_{k=0}^{2n-1}(-1)^k \binom{2n-1}{k} 2^k \sum_{j=0}^{k} 2^{-j}\left(-1+2^{j-k}\right)\binom{k}{j}\psi(-j+k+1)\,\zeta(j-k) - \psi(k+1)\,\zeta(-k) - \zeta'(-k)\right) - \frac{2^{2n-1}-1}{2^{2n-1}(2n-1)!}\zeta'(1-2n) /; n \in \mathbb{N}^+$$

06.15.03.0058.01

$$\psi^{(-2n)}\left(-\frac{1}{3}\right) = \frac{9^{-n}\left(\sqrt{3}\,(-1+9^n)\pi - 6\log(3)\right)}{8n(2n-1)!} B_{2n} + \frac{(-1)^n\, 3^{\frac{1}{2}-2n}\, 4^{-n}\,\pi^{1-2n}}{(2n-1)!}\psi^{(2n-1)}\left(\frac{1}{3}\right) - \frac{1}{3^{2n-1}(2n-1)!}\left(\gamma + \frac{\gamma}{6n} - i\pi + \log(3) + \psi(2n) + \sum_{k=0}^{2n-1}(-1)^k \binom{2n-1}{k} 3^k \left(\sum_{j=0}^{k} 3^{-j}\binom{k}{j}\psi(-j+k+1)\,\zeta\left(j-k, \frac{2}{3}\right) - \psi(k+1)\,\zeta(-k) - \zeta'(-k)\right)\right) + \frac{9^{-n}(3-9^n)}{2(2n-1)!}\zeta'(1-2n) /; n \in \mathbb{N}^+$$

06.15.03.0059.01

$$\psi^{(-2n)}\left(-\frac{1}{4}\right) = \frac{1}{(2n-1)!}\left(\left(\frac{\left(2^{2n+1}-2^3\right)\log(2) - \left(4^n - 4^{2n}\right)\pi}{4^{2n+1}\, n} B_{2n} + \frac{(-1)^n}{4(8\pi)^{2n-1}}\psi^{(2n-1)}\left(\frac{1}{4}\right) - \frac{2^{2n-1}-1}{2^{4n-1}}\zeta'(1-2n)\right) - 4^{1-2n}\left(\gamma + \frac{\gamma}{8n} - i\pi + \log(4) + \psi(2n) + \sum_{k=0}^{2n-1}(-4)^k\binom{2n-1}{k}\sum_{j=0}^{k}4^{-j}\binom{k}{j}\psi(-j+k+1)\,\zeta\left(j-k, \frac{3}{4}\right) - \psi(k+1)\,\zeta(-k) - \zeta'(-k)\right)\right) /; n \in \mathbb{N}^+$$

06.15.03.0060.01

$$\psi^{(-2n)}\left(\frac{1}{4}\right)=\frac{1}{(2n-1)!}\Bigg((-1)^n\,2^{1-6n}\,\pi^{1-2n}\,\psi^{(2n-1)}\left(\frac{1}{4}\right)-\sum_{j=0}^{2n-2}\left(\frac{i\pi}{2}\right)^j\binom{2n-1}{j}(2n-j-1)!\,\zeta(2n-j)\,(2i\pi)^{1-2n}+$$
$$\frac{4^{-2n-1}(\pi(-4^n+16^n)+(-4+4^n)\log(4))}{n}B_{2n}+\frac{2^{-4n-1}\left(-\gamma+i\pi-2in\pi+n\log\left(16\pi^8\right)+8n(\psi(2n)+\gamma)\right)}{n}-$$
$$4^{1-2n}\sum_{j=0}^{2n-2}\frac{(-1)^j}{2n-j-1}\binom{2n-1}{j}-4^{1-2n}\sum_{j=0}^{2n-2}(-1)^j\binom{2n-1}{j}\sum_{k=0}^{2n-j-1}\left(-\frac{\pi i}{2}\right)^{-k}\binom{2n-j-1}{k}k!\,\mathrm{Li}_{k+1}(i)-$$
$$4^{1-2n}\sum_{k=0}^{2n-1}(-4)^k\binom{2n-1}{k}\left(\sum_{j=0}^{k}4^{-j}\binom{k}{j}\psi(k-j+1)\,\zeta\left(j-k,\frac{3}{4}\right)-\psi(k+1)\,\zeta(-k)-\zeta'(-k)\right)-$$
$$16^{-n}(-2+4^n)\,\zeta'(1-2n)\Bigg)\,/;\,n\in\mathbb{N}^+$$

06.15.03.0061.01

$$\psi^{(-2n)}\left(\frac{1}{3}\right)=-3^{1-2n}\sum_{j=1}^{2n-1}\frac{\left(\frac{2\pi i}{3}\right)^{-j}\zeta(j+1)}{(-j+2n-1)!}-$$
$$\frac{3^{1-2n}}{(2n-1)!}\left(\sum_{j=0}^{2n-2}\frac{(-1)^j\binom{2n-1}{j}}{-j+2n-1}+\sum_{j=0}^{2n-2}(-1)^j\binom{2n-1}{j}\sum_{k=0}^{-j+2n-1}\left(-\frac{1}{3}(2\pi i)\right)^{-k}\binom{-j+2n-1}{k}k!\,\mathrm{Li}_{k+1}\left(e^{\frac{2i\pi}{3}}\right)+\right.$$
$$\left.\sum_{k=0}^{2n-1}(-1)^k\binom{2n-1}{k}3^k\left(\sum_{j=0}^{k}3^{-j}\binom{k}{j}\psi(-j+k+1)\,\zeta\left(j-k,\frac{2}{3}\right)-\psi(k+1)\,\zeta(-k)-\zeta'(-k)\right)\right)+\frac{3^{-4n}\pi^{-2n}}{8n(2n-1)!}$$
$$\Bigg(-4\pi i(2n-1)(3\pi)^{2n}+B_{2n}\left(\sqrt{3}\,(-1+9^n)\pi-6\log(3)\right)(3\pi)^{2n}+3n\log\left(\frac{256\pi^8}{81}\right)(3\pi)^{2n}+24n\,\psi(2n)\,(3\pi)^{2n}+$$
$$4(3-9^n)\,n\,\zeta'(1-2n)\,(3\pi)^{2n}+4(6n-1)\,\gamma\,(3\pi)^{2n}+(-1)^n\,2^{3-2n}\,3^{2n+\frac{1}{2}}\,n\,\pi\,\psi^{(2n-1)}\left(\frac{1}{3}\right)\Bigg)\,/;\,n\in\mathbb{N}^+$$

06.15.03.0062.01

$$\psi^{(-2n)}\left(\frac{1}{2}\right) = -\frac{\log(2)}{2^{2n}\, n\,(2n-1)!} B_{2n} + \frac{2^{-2n-1}\left(\gamma\,(4n-1) + 4n\log(\pi) + 4n\,\psi(2n) + i\,(1-2n)\,\pi\right)}{n\,(2n-1)!} - \frac{2^{1-2n}}{(2n-1)!}\sum_{j=0}^{2n-2}\frac{(-1)^j}{-j+2n-1}\binom{2n-1}{j} - 2^{1-2n}\sum_{j=1}^{2n-1}\frac{(i\,\pi)^{-j}\,\zeta(j+1)}{(-j+2n-1)!} + \frac{2^{1-2n}}{(2n-1)!}\sum_{j=0}^{2n-2}(-1)^j\binom{2n-1}{j}\left(\log(2) + \sum_{k=1}^{-j+2n-1}\left(1-2^{-k}\right)(-i\,\pi)^{-k}\binom{-j+2n-1}{k} k!\,\zeta(k+1)\right) - \frac{2^{1-2n}}{(2n-1)!}\sum_{k=0}^{2n-1}(-2)^k\binom{2n-1}{k}\left(\sum_{j=0}^{k} 2^{-j}\left(-1+2^{j-k}\right)\binom{k}{j}\psi(-j+k+1)\,\zeta(j-k) - \psi(k+1)\,\zeta(-k) - \zeta'(-k)\right) - \frac{2^{2n-1}-1}{2^{2n-1}\,(2n-1)!}\zeta'(1-2n)\;/;\; n \in \mathbb{N}^+$$

06.15.03.0063.01

$$\psi^{(-2n)}\left(\frac{2}{3}\right) = \frac{1}{(2n-1)!}\left(-3^{\frac{1}{2}-2n}\,\pi^{1-2n}\,\psi^{(2n-1)}\left(\frac{1}{3}\right)\left(-\frac{1}{4}\right)^n + 3^{1-2n}\,4^{n-1}\log\left(\frac{4\pi^2}{3}\right) + \left(\frac{3}{2}\right)^{1-2n}\psi(2n) - \left(\frac{3}{2}\right)^{1-2n}\sum_{j=0}^{2n-2}\frac{(-1)^j\binom{2n-1}{j}}{-j+2n-1} - \left(\frac{3}{2}\right)^{1-2n}(2n-1)!\sum_{j=1}^{2n-1}\frac{\left(-\frac{1}{3}(4\,i\,\pi)\right)^{-j}\zeta(j+1)}{(-j+2n-1)!} - \left(\frac{3}{2}\right)^{1-2n}\sum_{j=0}^{2n-2}(-1)^j\binom{2n-1}{j}\sum_{k=0}^{-j+2n-1}\left(\frac{4\,i\,\pi}{3}\right)^{-k}\binom{-j+2n-1}{k}k!\,\mathrm{Li}_{k+1}\left(e^{\frac{2i\pi}{3}}\right) - \left(\frac{3}{2}\right)^{1-2n}\sum_{k=0}^{2n-1}(-1)^k\left(\frac{3}{2}\right)^k\binom{2n-1}{k}\sum_{j=0}^{k}\left(\frac{2}{3}\right)^j\binom{k}{j}\psi(-j+k+1)\,\zeta\left(j-k,\frac{1}{3}\right) - \psi(k+1)\,\zeta(-k) - \zeta'(-k) - \frac{1}{2}9^{-n}\left(-3+9^n\right)\zeta'(1-2n) - \frac{i\,2^{2n-1}\,9^{-n}\,\pi}{n} - \frac{2^{2n-1}\,9^{-n}\,\gamma}{n} - \frac{9^{-n}\,B_{2n}\left(\sqrt{3}\,\pi\left(-1+9^n\right)+\log(729)\right)}{8n} + \left(\frac{4}{9}\right)^n i\,\pi + \left(\frac{3}{2}\right)^{1-2n}\gamma\right)\;/;\; n \in \mathbb{N}^+$$

06.15.03.0064.01

$$\psi^{(-2n)}\left(\frac{3}{4}\right) = \frac{1}{(2n-1)!}\left(-\frac{(-1)^n}{4(8\pi)^{2n-1}}\psi^{(2n-1)}\left(\frac{1}{4}\right) + \left(\frac{3}{4}\right)^{2n-1}\sum_{j=0}^{2n-2}\frac{(-1)^{j-1}}{-j+2n-1}\binom{2n-1}{j} - \left(\frac{3}{4}\right)^{2n-1}\sum_{j=0}^{2n-2}(-1)^j\binom{2n-1}{j}\sum_{k=0}^{-j+2n-1}\left(-\frac{1}{2}(3i\pi)\right)^{-k}\binom{-j+2n-1}{k}k!\,\mathrm{Li}_{k+1}(-i) - \left(\frac{3}{4}\right)^{2n-1}\sum_{k=0}^{2n-1}\left(-\frac{4}{3}\right)^k\binom{2n-1}{k}\left(\sum_{j=0}^{k}\left(\frac{3}{4}\right)^j\binom{k}{j}\psi(-j+k+1)\,\zeta\left(j-k,\frac{1}{4}\right) - \psi(k+1)\,\zeta(-k) - \zeta'(-k)\right) + \frac{\left(\pi\left(4^n-4^{2n}\right)+\left(2^{2n+1}-8\right)\log(2)\right)B_{2n}}{4^{2n+1}\,n} + \frac{\left(2^{-4n-1}\,3^{2n-1}\right)\left(n\log\left(16\pi^8\right)+8n(\psi(2n)+\gamma)-3\gamma\right)}{n} - (2i\pi)^{1-2n}\sum_{j=0}^{2n-2}\left(\frac{3i\pi}{2}\right)^j\binom{2n-1}{j}(-j+2n-1)!\,\zeta(2n-j) - \frac{2^{2n-1}-1}{2^{4n-1}}\zeta'(1-2n) - \frac{i\,2^{-4n-1}\,3^{2n}(2n-1)\pi}{n}\right) /;\ n\in\mathbb{N}^+$$

General characteristics

Domain and analyticity

$\psi^{(\nu)}(z)$ is an analytical function of ν and z which is defined in $\mathbb{C}^2$.

06.15.04.0001.01

$$(\nu * z) \longrightarrow \psi^{(\nu)}(z) :: (\mathbb{C}\otimes\mathbb{C}) \longrightarrow \mathbb{C}$$

Symmetries and periodicities

Mirror symmetry

06.15.04.0002.02

$$\psi^{(\bar{\nu})}(\bar{z}) = \overline{\psi^{(\nu)}(z)} /;\ z \notin (-\infty, 0)$$

Periodicity

No periodicity

Poles and essential singularities

With respect to z

For fixed noninteger ν or fixed negative integer ν, the function $\psi^{(\nu)}(z)$ does not have poles.

06.15.04.0018.01

$$\mathcal{S}ing_z\left(\psi^{(\nu)}(z)\right) = \{\} /;\ \nu \notin \mathbb{N}$$

For fixed nonnegative integer ν, the function $\psi^{(\nu)}(z)$ has an infinite set of singular points:

a) $z = -m /;\ \nu = 0 \bigwedge m \in \mathbb{N}$, are the simple poles with residues -1;

b) $z = -m /;\ \nu > 0 \bigwedge m \in \mathbb{N}$, are the poles of order $\nu + 1$ with residues 0;

c) $z = \tilde{\infty}$ is the point of convergence of poles, which is an essential singular point.

06.15.04.0003.01

$$\mathcal{S}ing_z\left(\psi^{(\nu)}(z)\right) = \{\{\{-m, \nu+1\} /; m \in \mathbb{N}\}, \{\tilde{\infty}, \infty\}\} /; \nu \in \mathbb{N}^+$$

06.15.04.0004.01

$$\operatorname{res}_z\left(\psi^{(\nu)}(z)\right)(-m) = -\delta_\nu /; \nu \in \mathbb{N} \land m \in \mathbb{N}$$

With respect to ν

For fixed z, the function $\psi^{(\nu)}(z)$ does not have poles.

06.15.04.0005.01

$$\mathcal{S}ing_\nu\left(\psi^{(\nu)}(z)\right) = \{\}$$

Branch points

With respect to z

For fixed noninteger ν or fixed negative integer ν, the function $\psi^{(\nu)}(z)$ has infinitely many branch points: $z = -n /; n \in \mathbb{N}$ and $z = \tilde{\infty}$. All these are power-logarithmic type branch points.
For nonnegative integer ν, the function $\psi^{(\nu)}(z)$ does not have branch points.

06.15.04.0006.01

$$\mathcal{BP}_z\left(\psi^{(\nu)}(z)\right) = \{\{-n /; n \in \mathbb{N}\}, \tilde{\infty}\}$$

06.15.04.0007.01

$$\mathcal{BP}_z\left(\psi^{(\nu)}(z)\right) = \{\} /; \nu \in \mathbb{N}$$

06.15.04.0019.01

$$\mathcal{R}_z\left(\psi^{(\nu)}(z), -n\right) = \log /; \neg\, \nu \in \mathbb{N} \land n \in \mathbb{N}$$

06.15.04.0010.01

$$\mathcal{R}_z\left(\psi^{(\nu)}(z), \tilde{\infty}\right) = \log /; \neg\, \nu \in \mathbb{N}$$

With respect to ν

For fixed z, the function $\psi^{(\nu)}(z)$ does not have branch points.

06.15.04.0012.01

$$\mathcal{BP}_\nu\left(\psi^{(\nu)}(z)\right) = \{\}$$

Branch cuts

With respect to z

For fixed noninteger ν or fixed negative integer ν, the function $\psi^{(\nu)}(z)$ is a single-valued function on the z-plane cut along the interval $(-\infty, 0)$, where $\psi^{(\nu)}(z)$ is continuous from above. This interval includes an infinite set of branch cut lines of combined power-logarithmic type along $(-\infty, -n) /; n \in \mathbb{N}$.

For nonnegative integer ν, the function $\psi^{(\nu)}(z)$ does not have branch cuts.

06.15.04.0013.01

$$\mathcal{BC}_z\left(\psi^{(\nu)}(z)\right) = \{\{(-\infty, 0), -i\}\} /; \nu \notin \mathbb{N}$$

06.15.04.0014.01

$$\mathcal{BC}_z\left(\psi^{(\nu)}(z)\right) = \{\} \,/;\, \nu \in \mathbb{N}$$

06.15.04.0015.01

$$\lim_{\epsilon \to +0} \psi^{(\nu)}(x - i\,\epsilon) = \psi^{(\nu)}(x) \,/;\, x < 0 \bigwedge x \notin \mathbb{Z}$$

06.15.04.0016.01

$$\lim_{\epsilon \to +0} \psi^{(\nu)}(x - i\,\epsilon) = e^{2\,i\,\pi\,\nu} \left(\psi^{(\nu)}(x) + \frac{2\,\pi\,i}{\Gamma(-\nu)} \sum_{k=0}^{\lfloor -x \rfloor} (k+x)^{-\nu-1}\right) /;\, x < 0 \bigwedge x \notin \mathbb{Z} \bigwedge \nu \notin \mathbb{N}$$

With respect to ν

For fixed z, the function $\psi^{(\nu)}(z)$ does not have branch cuts.

06.15.04.0017.01

$$\mathcal{BC}_\nu\left(\psi^{(\nu)}(z)\right) = \{\}$$

Series representations

Generalized power series

Expansions at generic point $z = z_0$

06.15.06.0014.01

$$\psi^{(\nu)}(z) \propto \psi^{(\nu)}(z_0) + \psi^{(\nu+1)}(z_0)\,(z - z_0) + \frac{1}{2}\,\psi^{(\nu+2)}(z_0)\,(z - z_0)^2 + \ldots \,/;\, (z \to z_0)$$

06.15.06.0015.01

$$\psi^{(\nu)}(z) \propto \psi^{(\nu)}(z_0) + \psi^{(\nu+1)}(z_0)\,(z - z_0) + \frac{1}{2}\,\psi^{(\nu+2)}(z_0)\,(z - z_0)^2 + O\left((z - z_0)^3\right)$$

06.15.06.0016.01

$$\psi^{(\nu)}(z) = \sum_{k=0}^{\infty} \frac{1}{k!}\,\psi^{(0,k)}(\nu,\, z_0)\,(z - z_0)^k$$

06.15.06.0017.01

$$\psi^{(\nu)}(z) = \sum_{k=0}^{\infty} \frac{1}{k!}\,\psi^{(k+\nu)}(z_0)\,(z - z_0)^k$$

06.15.06.0018.01

$$\psi^{(\nu)}(z) \propto \psi^{(\nu)}(z_0)\,(1 + O(z - z_0))$$

Expansions on branch cuts

General case

06.15.06.0019.01

$$\psi^{(\nu)}(z) \propto e^{-2\,i\,\pi\,\nu \left\lfloor \frac{\arg(z-x)}{2\pi} \right\rfloor} \left(\psi^{(\nu)}(x) + \left(1 + \left\lfloor \frac{\arg(z-x)}{2\,\pi} \right\rfloor\right)(z - x)\left(\psi^{(\nu+1)}(x) + \frac{1}{2}\,\psi^{(\nu+2)}(x)\,(z - x) + \frac{1}{6}\,\psi^{(\nu+3)}(x)\,(z - x)^2 + \ldots\right) - \left\lfloor \frac{\arg(z-x)}{2\,\pi} \right\rfloor \frac{2\,\pi\,i}{\Gamma(-\nu)} \sum_{k=0}^{\lfloor -x \rfloor} (k+x)^{-\nu-1}\right) /;\, (z \to x) \land x \in \mathbb{R} \land x < 0$$

06.15.06.0020.01

$$\psi^{(\nu)}(z) \propto e^{-2 i \pi \nu \left\lfloor \frac{\arg(z-x)}{2\pi} \right\rfloor} \left(\psi^{(\nu)}(x) + \left(1 + \left\lfloor \frac{\arg(z-x)}{2\pi} \right\rfloor\right) (z-x) \left(\psi^{(\nu+1)}(x) + \frac{1}{2} \psi^{(\nu+2)}(x) (z-x) + \frac{1}{6} \psi^{(\nu+3)}(x) (z-x)^2 + O\left((z-x)^3\right)\right) - \left\lfloor \frac{\arg(z-x)}{2\pi} \right\rfloor \frac{2 \pi i}{\Gamma(-\nu)} \sum_{k=0}^{\lfloor -x \rfloor} (k+x)^{-\nu-1} \right) /; x \in \mathbb{R} \land x < 0$$

06.15.06.0021.01

$$\psi^{(\nu)}(z) = e^{-2 i \pi \nu \left\lfloor \frac{\arg(z-x)}{2\pi} \right\rfloor} \left(\psi^{(\nu)}(x) + \left(\left\lfloor \frac{\arg(z-x)}{2\pi} \right\rfloor + 1 \right) \sum_{k=1}^{\infty} \frac{\psi^{(k+\nu)}(x) (z-x)^k}{k!} - \left\lfloor \frac{\arg(z-x)}{2\pi} \right\rfloor \frac{2 \pi i}{\Gamma(-\nu)} \sum_{k=0}^{\lfloor -x \rfloor} (k+x)^{-\nu-1} \right) /; x \in \mathbb{R} \land x < 0$$

06.15.06.0022.01

$$\psi^{(\nu)}(z) \propto e^{-2 i \pi \nu \left\lfloor \frac{\arg(z-x)}{2\pi} \right\rfloor} \left(\psi^{(\nu)}(x) - \left\lfloor \frac{\arg(z-x)}{2\pi} \right\rfloor \frac{2 \pi i}{\Gamma(-\nu)} \sum_{k=0}^{\lfloor -x \rfloor} (k+x)^{-\nu-1} \right) (1 + O(z-x)) /; x \in \mathbb{R} \land x < 0$$

Expansions at $z == 0$

For the function itself

General case

06.15.06.0023.01

$$\psi^{(\nu)}(z) \propto -\mathcal{FC}_{\exp}^{(\nu)}(z, -1)\, z^{-\nu-1} - \frac{\gamma z^{-\nu}}{\Gamma(1-\nu)} + \frac{\pi^2 z^{1-\nu}}{6\,\Gamma(2-\nu)} \left(1 - \frac{12\,\zeta(3)\, z}{\pi^2 (2-\nu)} + \frac{2 \pi^2 z^2}{5 (2-\nu)(3-\nu)} + \ldots\right) /; (z \to 0) \land \mathrm{Re}(\nu) > 0$$

06.15.06.0024.01

$$\psi^{(\nu)}(z) \propto -\mathcal{FC}_{\exp}^{(\nu)}(z, -1)\, z^{-\nu-1} - \frac{\gamma z^{-\nu}}{\Gamma(1-\nu)} + \frac{\pi^2 z^{1-\nu}}{6\,\Gamma(2-\nu)} \left(1 - \frac{12\,\zeta(3)\, z}{\pi^2 (2-\nu)} + \frac{2 \pi^2 z^2}{5 (2-\nu)(3-\nu)} + O\left(z^3\right)\right) /; \mathrm{Re}(\nu) > 0$$

06.15.06.0005.01

$$\psi^{(\nu)}(z) == -\mathcal{FC}_{\exp}^{(\nu)}(z, -1)\, z^{-\nu-1} - \frac{\gamma z^{-\nu}}{\Gamma(1-\nu)} + z^{1-\nu} \sum_{k=1}^{\infty} \frac{1}{k^2}\, {}_2\tilde{F}_1\left(1, 2; 2-\nu; -\frac{z}{k}\right)$$

06.15.06.0001.01

$$\psi^{(\nu)}(z) == -\mathcal{FC}_{\exp}^{(\nu)}(z, -1)\, z^{-\nu-1} - \frac{\gamma z^{-\nu}}{\Gamma(1-\nu)} + \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{(-1)^j (j+1)!\, z^{j-\nu+1}}{(k+1)^{j+2}\, \Gamma(j-\nu+2)} /; \mathrm{Re}(\nu) > 0 \land \mathrm{Re}(z) > 0$$

06.15.06.0025.01

$$\psi^{(\nu)}(z) = \frac{(\psi(-\nu) + \gamma - \log(z))\, z^{-\nu-1}}{\Gamma(-\nu)} + \sum_{k=0}^{\infty} \frac{\psi^{(k)}(1)\, z^{k-\nu}}{\Gamma(k-\nu+1)}$$

06.15.06.0006.01

$$\psi^{(\nu)}(z) == -\mathcal{FC}_{\exp}^{(\nu)}(z, -1)\, z^{-\nu-1} - \frac{\gamma z^{-\nu}}{\Gamma(1-\nu)} + \sum_{j=0}^{\infty} \frac{(-1)^j (j+1)!\, \zeta(j+2)\, z^{j-\nu+1}}{\Gamma(j-\nu+2)} /; \mathrm{Re}(\nu) > 0 \land \mathrm{Re}(z) > 0$$

06.15.06.0002.02

$$\psi^{(\nu)}(z) \propto -\mathcal{FC}_{\exp}^{(\nu)}(z, -1)\, z^{-\nu-1} (1 + O(z)) /; \nu \notin \mathbb{N}$$

Special cases

06.15.06.0026.01

$$\psi^{(n)}(z) \propto \frac{(-1)^{n-1}\, n!}{z^{n+1}} + \psi^{(n)}(1) + \psi^{(n+1)}(1)\, z + \frac{1}{2}\, \psi^{(n+2)}(1)\, z^2 + \ldots \;/;\; (z \to 0) \bigwedge n \in \mathbb{N}^+$$

06.15.06.0027.01

$$\psi^{(n)}(z) \propto \frac{(-1)^{n-1}\, n!}{z^{n+1}} + \psi^{(n)}(1) + \psi^{(n+1)}(1)\, z + \frac{1}{2}\, \psi^{(n+2)}(1)\, z^2 + O\left(z^3\right) \;/;\; n \in \mathbb{N}^+$$

06.15.06.0003.02

$$\psi^{(n)}(z) = \frac{(-1)^{n-1}\, n!}{z^{n+1}} + \sum_{k=0}^{\infty} \frac{\psi^{(k+n)}(1)\, z^k}{k!} \;/;\; |z| < 1 \bigwedge n \in \mathbb{N}^+$$

06.15.06.0004.02

$$\psi^{(n)}(z) \propto \frac{(-1)^{n-1}\, n!}{z^{n+1}} + (-1)^{n-1}\, n!\, \zeta(n+1)\, (1 + O(z)) \;/;\; n \in \mathbb{N}^+$$

Expansions at $z = -m$

For the function itself

General case

06.15.06.0028.01

$$\psi^{(\nu)}(z) = -\frac{(\log(z) - \psi(-\nu) - \gamma)\, z^{-\nu-1}}{\Gamma(-\nu)} + \frac{\Gamma(\nu+1)}{\Gamma(1-\nu)}\, z^{-\nu} \sum_{k=1}^{\infty} \frac{1}{k}\, {}_2\tilde{F}_1\left(1, \nu; \nu+2; 1 + \frac{k}{z}\right) + \Gamma(\nu+1)\, (-z)^{\nu}\, (m+z)^{-\nu-1}\, z^{-\nu} +$$
$$\Gamma(\nu+1)\, z^{-\nu} \sum_{k=1}^{m-1} (-z)^{\nu}\, (k+z)^{-\nu-1} + \Gamma(\nu+1)\, z^{-\nu} \sum_{k=m+1}^{\infty} (-z)^{\nu}\, (k+z)^{-\nu-1} - \frac{\gamma\, z^{-\nu}}{\Gamma(1-\nu)} \;/;\; \nu \notin \mathbb{Z} \bigwedge m \in \mathbb{N}$$

Special cases

06.15.06.0029.01

$$\psi^{(n)}(z) \propto \frac{(-1)^{n-1}\, n!}{(z+m)^{n+1}} + \Gamma\,(n+1)\, H_m^{(n+1)} + \psi^{(n)}(1) + \left(\Gamma(n+2)\, H_m^{(n+2)} + \psi^{(n+1)}(1)\right)(z+m) +$$
$$\frac{1}{2}\left(\Gamma(n+3)\, H_m^{(n+3)} + \psi^{(n+2)}(1)\right)(z+m)^2 + \ldots \;/;\; (z \to -m) \bigwedge m \in \mathbb{N} \bigwedge n \in \mathbb{N}^+$$

06.15.06.0030.01

$$\psi^{(n)}(z) \propto \frac{(-1)^{n-1}\, n!}{(z+m)^{n+1}} + \Gamma\,(n+1)\, H_m^{(n+1)} + \psi^{(n)}(1) + \left(\Gamma(n+2)\, H_m^{(n+2)} + \psi^{(n+1)}(1)\right)(z+m) +$$
$$\frac{1}{2}\left(\Gamma(n+3)\, H_m^{(n+3)} + \psi^{(n+2)}(1)\right)(z+m)^2 + O\left((z+m)^3\right) \;/;\; m \in \mathbb{N} \bigwedge n \in \mathbb{N}^+$$

06.15.06.0031.01

$$\psi^{(n)}(z) \propto \frac{(-1)^{n-1}\, n!}{(z+m)^{n+1}} + (1 - (-1)^n)\,(1 - \delta_n)\, \psi^{(n)}(1) + (-1)^n\, \psi^{(n)}(m+1) +$$
$$\sum_{k=1}^{\infty} \frac{\left(\left(1 - (-1)^{k+n}\right) \psi^{(k+n)}(1) + (-1)^{k+n}\, \psi^{(k+n)}(m+1)\right)}{k!}\, (z+m)^k \;/;\; (z \to -m) \land m \in \mathbb{N} \land n \in \mathbb{N}$$

06.15.06.0032.01

$$\psi^{(n)}(z) \propto \frac{(-1)^{n-1}\, n!}{(z+m)^{n+1}} + \sum_{k=0}^{\infty} \frac{\left(\left(1-(-1)^{k+n}\right)\psi^{(k+n)}(1) + (-1)^{k+n}\,\psi^{(k+n)}(m+1)\right)}{k!}\,(z+m)^k \;/;\; (z\to -m)\bigwedge m\in\mathbb{N}\bigwedge n\in\mathbb{N}^+$$

06.15.06.0007.02

$$\psi^{(n)}(z) = \frac{(-1)^{n-1}\, n!}{(m+z)^{n+1}} + \sum_{k=0}^{\infty} \frac{\psi^{(k+n)}(1) + \Gamma(k+n+1)\, H_m^{(k+n+1)}}{k!}\,(m+z)^k \;/;\; |m+z|<1 \bigwedge m\in\mathbb{N}\bigwedge n\in\mathbb{N}^+$$

06.15.06.0033.01

$$\psi^{(-n)}(z) = (-1)^n\,\psi^{(-n)}(-z) + \frac{(m+z)^{n-1}}{n\,(n-1)!}\left(-i\,\pi\,(m+z) + 2\,i\,n\,\pi\left\lfloor -\frac{\arg(m+z)}{2\,\pi}\right\rfloor + 2\,i\,n\,\pi\left\lfloor \frac{3}{4} - \frac{\arg(-m-z)}{2\,\pi}\right\rfloor + n\log(-2\,i\,\pi) + i\,n\,\pi\right) +$$
$$\frac{(m+z-1)^{n-1}}{(n-1)!}\left(\gamma - i\,\pi - 2\,i\,\pi\left\lfloor -\frac{\arg(-m-z+1)}{2\,\pi}\right\rfloor - \log(-m-z+1) + \psi(n)\right) +$$
$$\sum_{p=1}^{m-1}\left(\frac{(p+z)^{n-1}}{(n-1)!}\left(\log(-p-z) - \psi(n) - \gamma\right) + \frac{(p+z-1)^{n-1}}{(n-1)!}\right.$$
$$\left(-2\,i\,\pi\left\lfloor \frac{1}{2} - \frac{1}{2\,\pi}\arg\left(1+\frac{m+z}{-m+p-1}\right) - \frac{\arg(-m+p-1)}{2\,\pi}\right\rfloor - \log\left(1+\frac{m+z}{-m+p-1}\right) - \log(-m+p-1) + \psi(n) + \gamma\right) +$$
$$\left.\sum_{k=0}^{n-2}\frac{-(-1)^{k+n}\,(p+z-1)^k - (p+z)^k}{k!}\sum_{j=0}^{-k+n-2}\frac{(-1)^j\,\psi^{(j+k-n)}(1)}{j!}\right) +$$
$$\sum_{k=0}^{n-2}\frac{-(-1)^{k+n}\,(m+z-1)^k - (m+z)^k}{k!}\sum_{j=0}^{-k+n-2}\frac{(-1)^j\,\psi^{(j+k-n)}(1)}{j!} -$$
$$\frac{1}{(n-1)!}\left(\sum_{k=0}^{n-2}(-2\,\pi\,i)^{k-n+1}\,(m+z)^k\binom{n-1}{k}(-k+n-1)!\,\mathrm{Li}_{n-k}(1) +\right.$$
$$\left.\sum_{k=0}^{n-1}(-1)^{n-k}\binom{n-1}{k}\sum_{j=0}^{-k+n-1}(2\,\pi\,i)^{j+k-n+1}\,(m+z)^{j+k}\binom{-k+n-1}{j}(-j-k+n-1)!\,\mathrm{Li}_{-j-k+n}\left(e^{-2\,i\,\pi\,z}\right)\right) /;$$
$$(z\to -m)\bigwedge m\in\mathbb{N}^+\bigwedge\left((n\in\mathbb{Z}\land n>1)\bigvee\left(n=1\bigwedge -\pi<\arg(m+z)\le\frac{\pi}{2}\right)\right)$$

06.15.06.0008.01

$$\psi^{(n)}(z) \propto \frac{(-1)^{n-1}\, n!}{(z+m)^{n+1}} - (-1)^n\, n!\,\zeta(n+1) + n!\,H_m^{(n+1)}\,(1+O(z+m)) \;/;\; (z\to -m)\land m\in\mathbb{N}\bigwedge n\in\mathbb{N}^+$$

Expansions of $\psi^{(\nu)}(-m+\epsilon)$ at $\epsilon = 0$

For the function itself

Special cases

06.15.06.0034.01

$$\psi^{(n)}(-m+\epsilon) \propto \frac{(-1)^{n-1}\, n!}{\epsilon^{n+1}} + \Gamma\,(n+1)\,H_m^{(n+1)} + \psi^{(n)}(1) + \left(\Gamma(n+2)\,H_m^{(n+2)} + \psi^{(n+1)}(1)\right)\epsilon + \frac{1}{2}\left(\Gamma(n+3)\,H_m^{(n+3)} + \psi^{(n+2)}(1)\right)\epsilon^2 + \ldots /;$$
$$(\epsilon\to 0)\land m\in\mathbb{N}\bigwedge n\in\mathbb{N}^+$$

06.15.06.0035.01

$$\psi^{(n)}(-m+\epsilon) \propto \frac{(-1)^{n-1}\, n!}{\epsilon^{n+1}} + \Gamma(n+1)\, H_m^{(n+1)} + \psi^{(n)}(1) + \left(\Gamma(n+2)\, H_m^{(n+2)} + \psi^{(n+1)}(1)\right)\epsilon + \frac{1}{2}\left(\Gamma(n+3)\, H_m^{(n+3)} + \psi^{(n+2)}(1)\right)\epsilon^2 + O\left(\epsilon^3\right) /;\ m \in \mathbb{N} \bigwedge n \in \mathbb{N}^+$$

06.15.06.0036.01

$$\psi^{(n)}(-m+\epsilon) = \frac{(-1)^{n-1}\, n!}{\epsilon^{n+1}} + \sum_{k=0}^{\infty} \frac{\psi^{(k+n)}(1) + \Gamma(k+n+1)\, H_m^{(k+n+1)}}{k!}\, \epsilon^k /;\ |\epsilon| < 1 \bigwedge m \in \mathbb{N} \bigwedge n \in \mathbb{N}^+$$

06.15.06.0037.01

$$\psi^{(n)}(-m+\epsilon) \propto \frac{(-1)^{n-1}\, n!}{\epsilon^{n+1}} - (-1)^n\, n!\, \zeta(n+1) + n!\, H_m^{(n+1)}\, (1 + O(\epsilon)) /;\ m \in \mathbb{N} \bigwedge n \in \mathbb{N}^+$$

Asymptotic series expansions

For the function itself

General case

06.15.06.0038.01

$$\psi^{(\nu)}(z) \propto z^{-\nu}\left(-\sum_{k=1}^{\infty} \frac{(k\, \zeta'(1-k) - B_k\, H_{k-1})\, z^{-k}}{k!\, \Gamma(1-k-\nu)} - \sum_{k=0}^{\infty} \frac{B_k\, (\psi(1-k-\nu) + \gamma)\, z^{-k}}{k!\, \Gamma(1-k-\nu)} + \log(z) \sum_{k=0}^{\infty} \frac{B_k\, z^{-k}}{k!\, \Gamma(1-k-\nu)}\right) /;\ (|z| \to \infty) \bigwedge |\arg(z)| < \pi$$

06.15.06.0039.01

$$\psi^{(\nu)}(z) \propto z^{-\nu} \sum_{k=0}^{\infty} \frac{z^{-k}}{k!\, \Gamma(1-k-\nu)} \left(\frac{k+\nu}{z\,(k+1)}\, ((k+1)\, \zeta'(-k) - B_{k+1}\, H_k) + B_k \log(z) - B_k\, (\psi(1-k-\nu) + \gamma)\right) /;\ (|z| \to \infty) \bigwedge |\arg(z)| < \pi$$

06.15.06.0040.01

$$\psi^{(\nu)}(z) \propto -\frac{(2\, z\, H_{-\nu} + \nu \log(2\pi) - 2\, z \log(z))\, z^{-\nu-1}}{2\, \Gamma(1-\nu)} + \frac{(z\, (-\log(z) + \psi(-\nu) + \gamma) - 2\, (\nu+1) \log(A))\, z^{-\nu-2}}{2\, \Gamma(-\nu)} + z^{-\nu} \sum_{k=1}^{\infty} \frac{z^{-2k}}{(2k)!\, \Gamma(1-2k-\nu)} \left(B_{2k}\, (\log(z) - \psi(1-2k-\nu) - \gamma) + \frac{(-1)^k\, (2k)!\, (2k+\nu)}{2^{2k+1}\, \pi^{2k}\, z}\, \zeta(2k+1)\right) + \frac{1}{2}\, z^{-\nu-2} \sum_{k=1}^{\infty} \frac{(2k+\nu+1)\, z^{-2k}}{(2k+1)!\, (k+1)\, \Gamma(-2k-\nu)}\, (2\, (k+1)\, \zeta'(-2k-1) - B_{2k+2}\, H_{2k+1}) /;\ (|z| \to \infty) \bigwedge |\arg(z)| < \pi$$

Special cases

06.15.06.0009.01

$$\psi^{(n)}(z) \propto \frac{(-1)^{n-1}\, (n-1)!\, (n+2z)}{2\, z^{n+1}} - (-1)^n \sum_{k=1}^{\infty} \frac{(2k+n-1)!}{(2k)!\, z^{2k+n}}\, B_{2k} /;\ n \in \mathbb{N}^+ \bigwedge |\arg(z)| < \pi \bigwedge (|z| \to \infty)$$

06.15.06.0041.01

$$\psi^{(n)}(z) \propto \frac{(-1)^{n-1}(n-1)!\,(n+2z)}{2z^{n+1}} - (-1)^n \sum_{k=1}^{\infty} \frac{(2k+n-1)!}{(2k)!\,z^{2k+n}} B_{2k} - \left\lfloor \frac{|\arg(z)|}{\pi} \right\rfloor (-i\pi)^{n+1} 2^n (i\cot(\pi z)-1) \sum_{k=0}^{n} \frac{(-1)^k k!\, \mathcal{S}_n^{(k)} (i\cot(\pi z)+1)^k}{2^k} \;/;\; n \in \mathbb{N}^+ \bigwedge \neg (z \in \mathbb{Z} \land z < 0) \bigwedge (|z| \to \infty)$$

06.15.06.0042.01

$$\psi^{(n)}(z) \propto \frac{\partial^{n+1}\left(-z+\frac{1}{2}\log(2\pi)+\left(z-\frac{1}{2}\right)\log(z)\right)}{\partial z^{n+1}} - (-1)^n \sum_{k=1}^{\infty} \frac{(2k+n-1)!}{(2k)!\,z^{2k+n}} B_{2k} - \left\lfloor \frac{|\arg(z)|}{\pi} \right\rfloor (-i\pi)^{n+1} 2^n (i\cot(\pi z)-1) \sum_{k=0}^{n} \frac{(-1)^k k!\, \mathcal{S}_n^{(k)} (i\cot(\pi z)+1)^k}{2^k} \;/;\; n \in \mathbb{Z} \land n \geq -1 \land \neg (z \in \mathbb{Z} \land z < 0) \land (|z| \to \infty)$$

06.15.06.0043.01

$$\psi^{(-n)}(z) \propto \frac{1}{n!}\left(\log(z+1)-\frac{1}{n}\right) B_n(z+1) + (-1)^{n-1} \sum_{k=n+1}^{\infty} \frac{B_k (z+1)^{n-k}}{(k-n)_{n+1}} - \frac{1}{n!} \sum_{k=2}^{n} B_k \binom{n}{k} (z+1)^{n-k} \sum_{j=0}^{k-1} \frac{(-1)^j}{k-j} + \frac{z^{n-1}}{(n-1)!} \left(\gamma\left(1-\frac{z}{n}\right) - \log(z) + \psi(n) + \sum_{k=0}^{n-1} \binom{n-1}{k} z^{-k} \left(\sum_{j=0}^{k} \binom{k}{j} \psi(-j+k+1)\,\zeta(j-k, z+1)\,(-z)^j - \psi(k+1)\,\zeta(-k) - \zeta'(-k)\right)\right) - \frac{(z+1)^{n-1}}{2\,n!} \;/;\; |\arg(z)| \leq \frac{\pi}{2} \bigwedge (|z| \to \infty) \bigwedge n \in \mathbb{N}^+$$

06.15.06.0044.01

$$\psi^{(-n)}(z) \propto -\frac{z^{n-1}}{2\,n!}(2z H_n - n\log(2\pi) - 2z\log(z)) + z^n \sum_{k=1}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{z^{-2k}}{(2k)!\,\Gamma(1-2k+n)} \left(\frac{(-1)^k (2k)!\,(2k-n)}{2^{2k+1}\pi^{2k} z} \zeta(2k+1) + B_{2k}(\log(z) - \psi(1-2k+n) - \gamma)\right) + z^n \sum_{k=\left\lfloor \frac{n}{2} \right\rfloor+1}^{\infty} \frac{(-1)^{n-1}(2k-n-1)!}{(2k)!} B_{2k} z^{-2k} + \frac{z^{n-2}}{2(n-1)!}(z(-\log(z)+\psi(n)+\gamma) - 2(1-n)\log(A)) + \frac{1}{2} z^{n-2} \sum_{k=1}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \frac{(2k-n+1)\,z^{-2k}}{(2k+1)!\,(k+1)\,(n-2k-1)!} (2(k+1)\,\zeta'(-2k-1) - B_{2k+2} H_{2k+1}) \;/;\; (|z| \to \infty) \land n \in \mathbb{N}^+$$

06.15.06.0010.01

$$\psi^{(n)}(z) \propto \frac{(-1)^{n-1}(n-1)!}{z^n} + \frac{(-1)^{n-1}\,n!}{2z^{n+1}} + \frac{(-1)^{n-1}(n+1)!}{12z^{n+2}}\left(1+O\left(\frac{1}{z^2}\right)\right) \;/;\; n \in \mathbb{N}^+ \bigwedge |\arg(z)| < \pi \bigwedge (|z| \to \infty)$$

Residue representations

06.15.06.0011.01

$$\psi^{(n)}(z) = (-1)^{n+1}\, n! \sum_{j=0}^{\infty} \operatorname{res}_s\left(\frac{\Gamma(1-s)\,\Gamma(z-s)^{n+1}\,(-1)^{-s}}{\Gamma(1+z-s)^{n+1}}\,\Gamma(s)\right)(-j) \;/;\; n \in \mathbb{N}^+$$

Other series representations

06.15.06.0045.01

$$\psi^{(\nu)}(z) = \frac{z^{-\nu-1}}{\Gamma(1-\nu)}\left(-\gamma\,(z+\nu)+\nu\log(z)-\nu\,\psi(-\nu)\right)+z^{1-\nu}\sum_{k=1}^{\infty}\frac{(k+z)^{-\nu-1}}{k^{1-\nu}}\,{}_2\tilde{F}_1\!\left(1-\nu,\,-\nu;\,2-\nu;\,-\frac{z}{k}\right)$$

06.15.06.0046.01

$$\psi^{(\nu)}(z) = \frac{z^{-\nu-1}}{\Gamma(1-\nu)}\left(-\gamma\,(z+\nu)+\nu\log(z)-\nu\,\psi(-\nu)\right)+z^{1-\nu}\sum_{k=1}^{\infty}\frac{1}{k\,(k+z)}\,{}_2\tilde{F}_1\!\left(1,\,-\nu;\,2-\nu;\,\frac{z}{k+z}\right)$$

06.15.06.0047.01

$$\psi^{(\nu)}(z) = \frac{\left(-\gamma\,(z+\nu)+\nu\log(z)-\nu\,\psi(-\nu)\right)z^{-\nu-1}}{\Gamma(1-\nu)}+z^{1-\nu}\sum_{k=1}^{\infty}\frac{1}{(k+z)^2}\,{}_2\tilde{F}_1\!\left(2,\,1-\nu;\,2-\nu;\,\frac{z}{k+z}\right)$$

06.15.06.0048.01

$$\psi^{(\nu)}(z) = -\frac{\left(\log(z)-\psi(-\nu)-\gamma\right)z^{-\nu-1}}{\Gamma(-\nu)}+\Gamma(\nu+1)\left(\sum_{k=1}^{\infty}\left((-z)^{\nu-1}\,(k+z)^{-\nu-1}-\frac{1}{k^2\,\Gamma(1-\nu)}\,{}_2\tilde{F}_1\!\left(1,\,2;\,\nu+2;\,\frac{z}{k}+1\right)\right)\right)z^{1-\nu}-\frac{\gamma\,z^{-\nu}}{\Gamma(1-\nu)}\,/;\,\nu\notin\mathbb{Z}$$

06.15.06.0049.01

$$\psi^{(\nu)}(z) = -\frac{\left(\log(z)-\psi(-\nu)-\gamma\right)z^{-\nu-1}}{\Gamma(-\nu)}+\Gamma(\nu+1)\left(\sum_{k=1}^{\infty}\left((-z)^{\nu-1}\,(k+z)^{-\nu-1}+\frac{1}{z\,k\,\Gamma(1-\nu)}\,{}_2\tilde{F}_1\!\left(1,\,\nu;\,\nu+2;\,\frac{k}{z}+1\right)\right)\right)z^{1-\nu}-\frac{\gamma\,z^{-\nu}}{\Gamma(1-\nu)}\,/;\,\nu\notin\mathbb{Z}$$

06.15.06.0050.01

$$\psi^{(\nu)}(z) = -\frac{\left(\log(z)-\psi(-\nu)-\gamma\right)z^{-\nu-1}}{\Gamma(-\nu)}+\frac{\Gamma(\nu+1)}{\Gamma(1-\nu)}\,z^{-\nu}\sum_{k=1}^{\infty}\frac{1}{k}\,{}_2\tilde{F}_1\!\left(1,\,\nu;\,\nu+2;\,1+\frac{k}{z}\right)+\Gamma(\nu+1)\,(-z)^{\nu}\,(m+z)^{-\nu-1}\,z^{-\nu}+\Gamma(\nu+1)\,z^{-\nu}\sum_{k=1}^{m-1}(-z)^{\nu}\,(k+z)^{-\nu-1}+\Gamma(\nu+1)\,z^{-\nu}\sum_{k=m+1}^{\infty}(-z)^{\nu}\,(k+z)^{-\nu-1}-\frac{\gamma\,z^{-\nu}}{\Gamma(1-\nu)}\,/;\,\nu\notin\mathbb{Z}$$

06.15.06.0051.01

$$\psi^{(\nu)}(z) = -\frac{\left(\log(z)-\psi(-\nu)-\gamma\right)z^{-\nu-1}}{\Gamma(-\nu)}-\Gamma(\nu+1)\,(-z)^{\nu}\,\zeta(\nu+1,\,z+1)\,z^{-\nu}-\frac{\gamma\,z^{-\nu}}{\Gamma(1-\nu)}-\frac{\Gamma(\nu+1)\,z^{1-\nu}}{\Gamma(1-\nu)}\sum_{k=1}^{\infty}\frac{1}{k^2}\,{}_2\tilde{F}_1\!\left(1,\,2;\,\nu+2;\,\frac{z}{k}+1\right)/;\,\nu\notin\mathbb{Z}$$

06.15.06.0052.01

$$\psi^{(\nu)}(z) = \frac{\left(z\,\psi(z)-\nu\left(-\log(z)+\psi(1-\nu)+\gamma\right)\right)z^{-\nu-1}}{\Gamma(1-\nu)}-\frac{z^{1-\nu}}{\Gamma(-\nu)}\sum_{k=0}^{\infty}\frac{1}{k-\nu+1}\,\zeta(k+2,\,z+1)\,z^k\,/;\,\neg\,\nu\in\mathbb{N}^+\bigwedge|z|<1$$

06.15.06.0053.01

$$\psi^{(\nu)}(z) = -\frac{\left(\gamma\,(z+\nu)-\nu\log(z)+\nu\,\psi(1-\nu)+1\right)z^{-\nu-1}}{\Gamma(1-\nu)}+\frac{z^{1-\nu}}{\Gamma(2-\nu)}\sum_{k=0}^{\infty}\frac{(k+1)!}{(2-\nu)_k}\,\zeta(k+2)\,(-z)^k\,/;\,\neg\,\nu\in\mathbb{N}^+\bigwedge|z|<1$$

06.15.06.0054.01

$$\psi^{(n)}(z)\propto\frac{(-1)^{n-1}\,n!}{(m+z)^{n+1}}-n!\,(-1)^n\left(\sum_{k=0}^{m-1}(k+z)^{-n-1}+\zeta(n+1,\,m+z+1)\right)/;\,(z\to-m)\bigwedge m\in\mathbb{N}\bigwedge n\in\mathbb{N}^+$$

06.15.06.0012.01

$$\psi^{(n)}(z) = (-1)^{n+1}\, n!\left(z^{-n-1} + (z+1)^{-n-1} + (z+2)^{-n-1} + \ldots\right) /;\, n \in \mathbb{N}^+$$

06.15.06.0013.01

$$\psi^{(n)}(z) = (-1)^{n+1}\, n! \sum_{k=0}^{\infty} \frac{1}{(k+z)^{n+1}} /;\, n \in \mathbb{N}^+$$

Integral representations

On the real axis

Of the direct function

06.15.07.0007.01

$$\psi^{(\nu)}(z) = \frac{1}{\Gamma(-\nu)} \int_0^z \psi(t+1)\,(z-t)^{-\nu-1}\,dt - \frac{z^{-\nu-1}}{\Gamma(-\nu)}\,(\log(z) - \psi(-\nu) - \gamma) /;\, \mathrm{Re}(\nu) < 0$$

06.15.07.0008.01

$$\psi^{(\nu)}(z) = \frac{1}{\Gamma(n-\nu)} \int_0^z \frac{\partial^n \psi(t+1)}{\partial t^n}\,(z-t)^{n-\nu-1}\,dt - \lim_{\mu\to\nu} \frac{z^{-\mu-1}}{\Gamma(-\mu)}\,(\log(z) - \psi(-\mu) - \gamma) + \sum_{k=0}^{n-1} \frac{\psi^{(k)}(1)\, z^{k-\nu}}{\Gamma(k-\nu+1)} /;\, \mathrm{Re}(\nu) < n \land n \in \mathbb{N}^+$$

06.15.07.0001.01

$$\psi^{(\nu)}(z) = \int_0^1 \frac{1}{1-t}\left(\frac{z^{-\nu}}{\Gamma(1-\nu)} - t^{z-1}\log^{\nu}(t)\, Q(-\nu,\,0,\,z\log(t))\right)dt - \frac{\gamma\, z^{-\nu}}{\Gamma(1-\nu)} /;\, \mathrm{Re}(z) > 0$$

06.15.07.0002.01

$$\psi^{(\nu)}(z) = \int_0^{\infty}\left(\frac{e^{-t}\, z^{-\nu}}{t\,\Gamma(1-\nu)} - \frac{(t+1)^{-z}}{t}\,(-\log(t+1))^{\nu}\, Q(-\nu,\,0,\,-z\log(t+1))\right)dt /;\, \mathrm{Re}(z) > 0$$

06.15.07.0003.01

$$\psi^{(\nu)}(z) = \int_0^{\infty} \frac{1}{1-e^{-t}}\left(\frac{e^{-t}\, z^{-\nu}}{\Gamma(1-\nu)} - (-t)^{\nu}\, e^{-z t}\, Q(-\nu,\,0,\,-t z)\right)dt - \frac{\gamma\, z^{-\nu}}{\Gamma(1-\nu)} /;\, \mathrm{Re}(z) > 0$$

06.15.07.0004.01

$$\psi^{(n)}(z) = (-1)^{n+1} \int_0^{\infty} \frac{t^n\, e^{-t z}}{1-e^{-t}}\,dt /;\, n \in \mathbb{N}^+ \land \mathrm{Re}(z) > 0$$

Contour integral representations

06.15.07.0009.01

$$\psi^{(\nu)}(z) = -\frac{\gamma\, z^{-\nu}}{\Gamma(1-\nu)} + \frac{1}{\Gamma(-\nu)}\left(-\log(z) + \frac{\Gamma'(-\nu)}{\Gamma(-\nu)} + \gamma\right) z^{-\nu-1} - \frac{z^{-\nu-1}}{2\pi i} \int_{\gamma-i\infty}^{i\infty+\gamma} \frac{z^s\,\Gamma(s)\,\zeta(s)\,\pi}{\Gamma(s-\nu)\sin(\pi s)}\,ds /;\, 1 < \gamma < 2$$

Nathaniel Grossman, SIAM J.Math.Anal.,Vol.7,No.3, May 1976, pp 366-372

06.15.07.0005.01

$$\psi^{(n)}(z) = \frac{(-1)^{n+1}\, n!}{2\pi i} \int_{\mathcal{L}} \frac{\Gamma(s)\,\Gamma(1-s)\,\Gamma(z-s)^{n+1}\,(-1)^{-s}}{\Gamma(1+z-s)^{n+1}}\,ds /;\, n \in \mathbb{N}^+$$

06.15.07.0006.01

$$\psi^{(n)}(z) = \frac{(-1)^{n+1}\, n!}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\Gamma(s)\,\Gamma(1-s)\,\Gamma(z-s)^{n+1}\,(-1)^{-s}}{\Gamma(1+z-s)^{n+1}}\,ds /;\, 0 < \gamma < 1 \land n \in \mathbb{N}^+$$

Continued fraction representations

06.15.10.0001.01

$$\psi^{(1)}(z) = 1 \Bigg/ \left(z - \frac{1}{2} + \cfrac{1/12}{z - \frac{1}{2} + \cfrac{4/15}{z - \frac{1}{2} + \cfrac{81/140}{z - \frac{1}{2} + \cfrac{64/63}{z - \frac{1}{2} + \cfrac{625/396}{z - \frac{1}{2} + \cfrac{324/143}{\cfrac{1}{z - \frac{1}{2} + \ldots}}}}}}} \right) /; \operatorname{Re}(z) > 0$$

06.15.10.0002.01

$$\psi^{(1)}(z) = \frac{1}{z - \frac{1}{2} + K_k\left(\frac{k^4}{4(4k^2-1)}, z - \frac{1}{2}\right)_1^{\infty}} /; \operatorname{Re}(z) > 0$$

06.15.10.0003.01

$$\psi^{(1)}(z) = \frac{1}{2z^2} + \frac{1}{z} + \frac{1}{2z^2} \cfrac{1}{3z + \cfrac{3}{5z + \cfrac{18}{7z + \cfrac{60}{9z + \cfrac{150}{11z + \cfrac{315}{13z + \cfrac{588}{15z + \cfrac{1008}{17z + \ldots}}}}}}}} /; \operatorname{Re}(z) > 0$$

06.15.10.0004.01

$$\psi^{(1)}(z) = \frac{1}{2z^2} + \frac{1}{z} + \frac{1}{2z^2\left(3z + K_k\left(\frac{1}{4}k(k+1)^2(k+2), (2k+3)z\right)_1^{\infty}\right)} /; \operatorname{Re}(z) > 0$$

06.15.10.0005.01

$$\psi^{(2)}(z) = -\frac{1}{z^3} - \frac{1}{z^2} - \frac{1}{2 z^3} \frac{1}{z + \frac{1/3}{z + \frac{2/3}{z + \frac{6/5}{z + \frac{9/5}{z + \frac{18/7}{z + \frac{24/7}{z + \frac{40/9}{z + \ldots}}}}}}}} \;/;\; \operatorname{Re}(z) > 0$$

06.15.10.0006.01

$$\psi^{(2)}(z) = -\frac{1}{z^3} - \frac{1}{z^2} - \frac{1}{2 z^3 \left(z + \operatorname{K}_k\left(\frac{\left(\left\lfloor \frac{k}{2} \right\rfloor + 1\right) \left\lfloor \frac{k+1}{2} \right\rfloor \left\lfloor \frac{k+3}{2} \right\rfloor}{2 k - (-1)^k + 3}, z\right)_1^\infty\right)} \;/;\; \operatorname{Re}(z) > 0$$

Transformations

Transformations and argument simplifications

Argument involving basic arithmetic operations

06.15.16.0001.01

$$\psi^{(n)}(1 - z) = (-1)^n \psi^{(n)}(z) + (-1)^n \pi \frac{\partial^n \cot(\pi z)}{\partial z^n} \;/;\; n \in \mathbb{N}$$

06.15.16.0014.01

$$\psi^{(n)}(1 - z) = (-1)^n \psi^{(n)}(z) - (\pi i)^{n+1} 2^n (i \cot(\pi z) - 1) \sum_{k=0}^{n} \frac{(-1)^k k! \mathcal{S}_n^{(k)} (i \cot(\pi z) + 1)^k}{2^k} + (-1)^{n-1} i \pi \delta_n \;/;\; n \in \mathbb{N}$$

06.15.16.0015.01

$$\psi^{(\nu)}(-z) = z^\nu (-z)^{-\nu} \psi^{(\nu)}(z) - \frac{(-z)^{-\nu-1}}{\Gamma(-\nu)} (\log(-z) + \log(z) - 2 \psi(-\nu) - 2 \gamma) + 2 \pi^2 (-z)^{1-\nu} \sum_{j=0}^{\infty} \frac{(-1)^j B_{2 j+2} (2 \pi z)^{2 j}}{(j + 1) \Gamma(2 j - \nu + 2)}$$

06.15.16.0016.01

$$\psi^{(\nu)}(-z) = z^\nu (-z)^{-\nu} \psi^{(\nu)}(z) + \pi z^\nu (-z)^{-\nu} \cot^{(\nu)}(\pi z) + \frac{(-z)^{-\nu-1}}{\Gamma(-\nu)} (-\log(-z) + \psi(-\nu) + \gamma)$$

06.15.16.0002.01

$$\psi^{(n)}(-z) = (-1)^n \psi^{(n)}(z) + n! z^{-n-1} + (-1)^n \pi \frac{\partial^n \cot(\pi z)}{\partial z^n} \;/;\; n \in \mathbb{N}$$

06.15.16.0003.02

$$\psi^{(n)}(-z) = (-1)^n \psi^{(n)}(z) + n! z^{-n-1} - \pi i \delta_n - \frac{1}{2} (2 \pi i)^{n+1} (i \cot(\pi z) - 1) \sum_{k=0}^{n} \frac{(-1)^k k! \mathcal{S}_n^{(k)} (i \cot(\pi z) + 1)^k}{2^k} \;/;\; n \in \mathbb{N}$$

06.15.16.0017.01

$$\psi^{(-n)}(-z) = (-1)^n \psi^{(-n)}(z) + \frac{(-z)^{n-1}}{\Gamma(n)} (-\log(-z) + \psi(n) + \gamma) + \pi (-z)^n \cot^{(-n)}(\pi z) \;/;\; n \in \mathbb{N}^+$$

06.15.16.0018.01

$$\psi^{(-n)}(-z) = (-1)^n \psi^{(-n)}(z) + \frac{(-z)^{n-1}}{\Gamma(n)} (-\log(-z) + \psi(n) + \gamma) + \frac{\pi (-z)^n}{\Gamma(n)} \left(\text{Integrate}\left[\frac{(z-t)^{n-1}}{\pi t}, \{t, 0, z\}, \text{GenerateConditions} \to \text{False}\right] + \text{Integrate}\left[(z-t)^{n-1} \left(\cot(\pi t) - \frac{1}{\pi t}\right), \{t, 0, z\}, \text{GenerateConditions} \to \text{False}\right]\right) /; n \in \mathbb{N}^+$$

06.15.16.0019.01

$$\psi^{(-n)}(-z) = (-1)^n \psi^{(-n)}(z) + \frac{(-z)^{n-1}}{(n-1)!} \left(\frac{i \pi z}{n} + \gamma + 2 i \pi \left\lfloor \frac{3}{4} - \frac{\arg(z)}{2\pi} \right\rfloor + \log(-2 \pi i) - \log(-z) + \psi(n) - \sum_{k=1}^{n-1} (2 \pi i z)^{-k} \binom{n-1}{k} k!\, \text{Li}_{k+1}(1) + \sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} \sum_{j=0}^{k} (-2 \pi i z)^{-j} \binom{k}{j} j!\, \text{Li}_{j+1}\left(e^{2 i \pi z}\right)\right) /; n \in \mathbb{N}^+ \land (0 < \arg(z) \leq \pi \lor -1 < \text{Re}(z) \leq 1)$$

06.15.16.0020.01

$$\psi^{(-n)}(-z) = (-1)^n \psi^{(-n)}(z) + \frac{1}{(n-1)!} \left(z^{n-1} \left((-1)^n \log\left(-\frac{z}{\pi}\right) + (-1)^n \log(\sin(\pi z)) - 2 (-1)^n \psi(n) - \sum_{j=0}^{n-2} \frac{(-1)^{j-1}}{n-j-1} \binom{n-1}{j} + \sum_{j=0}^{n-2} (-1)^j \binom{n-1}{j} \sum_{k=0}^{n-j-1} (2 \pi i z)^{-k} \binom{n-j-1}{k} k!\, \text{Li}_{k+1}\left(e^{-2 i \pi z}\right) - 2 (-1)^n \gamma \right) - (2 \pi i)^{1-n} \sum_{j=0}^{n-2} \binom{n-1}{j} (n-j-1)!\, (-2 \pi i z)^j \zeta(n-j) - \frac{\pi i (n-1) (-z)^n}{n} \right) /; n \in \mathbb{N}^+ \land |\text{Re}(z)| < 1$$

06.15.16.0021.01

$$\psi^{(-n)}(z) = (-1)^n \psi^{(-n)}(-z) + \sum_{p=1}^{\lfloor \text{Re}(z) \rfloor} \left(\frac{(-\log(p-z-1) + \psi(n) + \gamma) (-p+z+1)^{n-1} + (z-p)^{n-1} (\log(z-p) - \psi(n) - \gamma)}{(n-1)!} + \sum_{k=0}^{n-2} \frac{(-p+z+1)^k + (-1)^{k+n} (z-p)^k}{k!} \sum_{j=0}^{n-k-2} \frac{(-1)^j \psi^{(j+k-n)}(1)}{j!} \right) + \frac{(z - \lfloor \text{Re}(z) \rfloor)^{n-1}}{(n-1)!} \left(2 i \pi \left\lfloor \frac{3}{4} - \frac{\arg(z - \lfloor \text{Re}(z) \rfloor)}{2 \pi} \right\rfloor + \gamma + \frac{i \pi (z - \lfloor \text{Re}(z) \rfloor)}{n} + \log(-2 \pi i) - \log(\lfloor \text{Re}(z) \rfloor - z) + \psi(n) - \sum_{k=1}^{n-1} (2 \pi i (z - \lfloor \text{Re}(z) \rfloor))^{-k} \binom{n-1}{k} k!\, \text{Li}_{k+1}(1) + \sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} \sum_{j=0}^{k} (2 \pi i (\lfloor \text{Re}(z) \rfloor - z))^{-j} \binom{k}{j} j!\, \text{Li}_{j+1}\left(e^{2 i \pi z}\right) \right) /; n \in \mathbb{N}^+ \land \text{Re}(z) \geq 0$$

06.15.16.0022.01

$$\psi^{(-n)}(z)=(-1)^n\,\psi^{(-n)}(-z)+\sum_{p=1}^{\lfloor -\mathrm{Re}(z)\rfloor}\Bigg(\frac{(-\log(p+z-1)+\psi(n)+\gamma)\,(p+z-1)^{n-1}+(p+z)^{n-1}\,(\log(-p-z)-\psi(n)-\gamma)}{(n-1)!}+\sum_{k=0}^{n-2}\frac{-(-1)^{k+n}\,(p+z-1)^k-(p+z)^k}{k!}\sum_{j=0}^{-k+n-2}\frac{(-1)^j\,\psi^{(j+k-n)}(1)}{j!}\Bigg)+\frac{(z+\lfloor -\mathrm{Re}(z)\rfloor)^{n-1}}{(n-1)!}\Bigg(2\,i\,\pi\left\lfloor\frac{3}{4}-\frac{\arg(-z-\lfloor -\mathrm{Re}(z)\rfloor)}{2\,\pi}\right\rfloor+\gamma+\frac{i\,\pi\,(-z-\lfloor -\mathrm{Re}(z)\rfloor)}{n}+\log(-2\,\pi\,i)-\log(z+\lfloor -\mathrm{Re}(z)\rfloor)+\psi(n)-\sum_{k=1}^{n-1}(2\,\pi\,i\,(-z-\lfloor -\mathrm{Re}(z)\rfloor))^{-k}\binom{n-1}{k}k!\,\mathrm{Li}_{k+1}(1)+\sum_{k=0}^{n-1}(-1)^k\binom{n-1}{k}\sum_{j=0}^{k}(2\,\pi\,i\,(z+\lfloor -\mathrm{Re}(z)\rfloor))^{-j}\binom{k}{j}j!\,\mathrm{Li}_{j+1}\!\left(e^{-2\,i\,\pi\,z}\right)\Bigg)\;/;\;n\in\mathbb{N}^+\bigwedge\mathrm{Re}(z)\leq 0$$

06.15.16.0023.01

$$\psi^{(-n)}(z)=(-1)^n\,\psi^{(-n)}(-z)+\sum_{p=1}^{\lfloor|\mathrm{Re}(z)|\rfloor}\Bigg(\frac{1}{(n-1)!}\Big((-\log(p-(2\,\theta(\mathrm{Re}(z))-1)\,z-1)+\psi(n)+\gamma)\,(-p\,(2\,\theta(\mathrm{Re}(z))-1)+(2\,\theta(\mathrm{Re}(z))-1)+z)^{n-1}+(z-(2\,\theta(\mathrm{Re}(z))-1)\,p)^{n-1}\,(\log((2\,\theta(\mathrm{Re}(z))-1)\,z-p)-\psi(n)-\gamma)\Big)+(2\,\theta(\mathrm{Re}(z))-1)\sum_{k=0}^{n-2}\frac{1}{k!}\Big((-1)^{(k+n)\,\theta(-\mathrm{Re}(z))}\,(-(2\,\theta(\mathrm{Re}(z))-1)\,p+z+2\,\theta(\mathrm{Re}(z))-1)^k+(-1)^{(k+n)\,\theta(\mathrm{Re}(z))}\,(z-(2\,\theta(\mathrm{Re}(z))-1)\,p)^k\Big)\sum_{j=0}^{-k+n-2}\frac{(-1)^j\,\psi^{(j+k-n)}(1)}{j!}\Bigg)+\frac{(z-(2\,\theta(\mathrm{Re}(z))-1)\,\lfloor|\mathrm{Re}(z)|\rfloor)^{n-1}}{(n-1)!}\Bigg(2\,i\,\pi\left\lfloor\frac{3}{4}-\frac{\arg((2\,\theta(\mathrm{Re}(z))-1)\,z-\lfloor|\mathrm{Re}(z)|\rfloor)}{2\,\pi}\right\rfloor+\frac{i\,\pi\,((2\,\theta(\mathrm{Re}(z))-1)\,z-\lfloor|\mathrm{Re}(z)|\rfloor)}{n}+\log(-2\,\pi\,i)-\log(\lfloor|\mathrm{Re}(z)|\rfloor-(2\,\theta(\mathrm{Re}(z))-1)\,z)+\psi(n)-\sum_{k=1}^{n-1}(2\,\pi\,i\,((2\,\theta(\mathrm{Re}(z))-1)\,z-\lfloor|\mathrm{Re}(z)|\rfloor))^{-k}\binom{n-1}{k}k!\,\mathrm{Li}_{k+1}(1)+\sum_{k=0}^{n-1}(-1)^k\binom{n-1}{k}\sum_{j=0}^{k}(2\,\pi\,i\,(\lfloor|\mathrm{Re}(z)|\rfloor-(2\,\theta(\mathrm{Re}(z))-1)\,z))^{-j}\binom{k}{j}j!\,\mathrm{Li}_{j+1}\!\left(e^{2\,i\,\pi\,(2\,\theta(\mathrm{Re}(z))-1)\,z}\right)+\gamma\Bigg)\;/;\;n\in\mathbb{N}^+$$

06.15.16.0004.01

$$\psi^{(n)}(z+1)=\psi^{(n)}(z)+(-1)^n\,n!\,z^{-n-1}\;/;\;n\in\mathbb{N}$$

06.15.16.0005.01

$$\psi^{(n)}(z-1)=\psi^{(n)}(z)-(-1)^n\,n!\,(z-1)^{-n-1}\;/;\;n\in\mathbb{N}$$

06.15.16.0006.01

$$\psi^{(n)}(z+m)=\psi^{(n)}(z)+(-1)^n\,n!\sum_{k=0}^{m-1}\frac{1}{(z+k)^{n+1}}\;/;\;n\in\mathbb{N}$$

06.15.16.0007.01

$$\psi^{(n)}(z-m) = \psi^{(n)}(z) - (-1)^n\, n! \sum_{k=1}^{m} \frac{1}{(z-k)^{n+1}} \;/;\; n \in \mathbb{N}$$

06.15.16.0024.01

$$\psi^{(-n)}(z+m) = \psi^{(-n)}(z) + \sum_{p=1}^{m} \left(\frac{(p+z-1)^{n-1}}{(n-1)!} (\log(p+z-1) - \psi(n) - \gamma) + \sum_{k=0}^{n-2} \frac{(p+z)^k}{k!} \sum_{j=0}^{n-k-2} \frac{(-1)^j}{j!} \psi^{(j+k-n)}(1) \right) /;\; n \in \mathbb{N}^+ \bigwedge m \in \mathbb{N}^+$$

06.15.16.0025.01

$$\psi^{(-n)}(z+1) = \psi^{(-n)}(z) + \frac{z^{n-1} (\log(z) - \psi(n) - \gamma)}{(n-1)!} + \sum_{k=0}^{n-2} \frac{(z+1)^k}{k!} \sum_{j=0}^{n-k-2} \frac{(-1)^j}{j!} \psi^{(j+k-n)}(1) \;/;\; n \in \mathbb{N}^+$$

06.15.16.0026.01

$$\psi^{(-n)}(z-m) = \psi^{(-n)}(z) - \sum_{p=1}^{m} \left(\frac{(z-p)^{n-1}}{(n-1)!} (\log(z-p) - \psi(n) - \gamma) + \sum_{k=0}^{n-2} \frac{(z-p+1)^k}{k!} \sum_{j=0}^{n-k-2} \frac{(-1)^j}{j!} \psi^{(j+k-n)}(1) \right) /;\; n \in \mathbb{N}^+ \bigwedge m \in \mathbb{N}^+$$

06.15.16.0027.01

$$\psi^{(-n)}(z-1) = \psi^{(-n)}(z) - \frac{(z-1)^{n-1} (\log(z-1) - \psi(n) - \gamma)}{(n-1)!} - \sum_{k=0}^{n-2} \frac{z^k}{k!} \sum_{j=0}^{n-k-2} \frac{(-1)^j}{j!} \psi^{(j+k-n)}(1) \;/;\; n \in \mathbb{N}^+$$

06.15.16.0028.01

$$\psi^{(-2)}(z+1) = \log(z)\, z - z + \frac{1}{2} \log(2\pi) + \psi^{(-2)}(z)$$

Multiple arguments

Argument involving numeric multiples of variable

06.15.16.0008.01

$$\psi^{(0)}(2z) = \log(2) + \frac{1}{2} \left(\psi\left(z + \frac{1}{2}\right) + \psi(z) \right)$$

06.15.16.0009.01

$$\psi^{(n)}(2z) = 2^{-n-1} \left(\psi^{(n)}(z) + \psi^{(n)}\left(z + \frac{1}{2}\right) \right) /;\; n \in \mathbb{N}^+$$

06.15.16.0029.01

$$\psi^{(\nu)}(2z) = -\frac{\log(2)\,(2z)^{-\nu-1}}{\Gamma(-\nu)} + \frac{\log(2)\,(2z)^{-\nu}}{\Gamma(1-\nu)} + 2^{-\nu-1} \psi^{(\nu)}(z) + \frac{(2z)^{-\nu}}{2} \sum_{j=0}^{\infty} \frac{\psi^{(j)}\left(\frac{1}{2}\right) z^j}{\Gamma(j-\nu+1)}$$

06.15.16.0030.01

$$\psi^{(0)}(3z) = \log(3) + \frac{1}{3} \left(\psi(z) + \psi\left(z + \frac{1}{3}\right) + \psi\left(z + \frac{2}{3}\right) \right)$$

06.15.16.0031.01

$$\psi^{(n)}(3z) = 3^{-n-1} \left(\psi^{(n)}(z) + \psi^{(n)}\left(z + \frac{1}{3}\right) + \psi^{(n)}\left(z + \frac{2}{3}\right) \right)$$

Argument involving symbolic multiples of variable

06.15.16.0010.01

$$\psi^{(0)}(m\,z) == \log(m) + \frac{1}{m}\sum_{k=0}^{m-1}\psi\left(z+\frac{k}{m}\right) /; m \in \mathbb{N}^+$$

06.15.16.0011.01

$$\psi^{(n)}(m\,z) == m^{-n-1}\sum_{k=0}^{m-1}\psi^{(n)}\left(z+\frac{k}{m}\right) /; n \in \mathbb{N}^+ \bigwedge m \in \mathbb{N}^+$$

Products, sums, and powers of the direct function

Sums of the direct function

06.15.16.0012.01

$$\psi^{(0)}\left(z+\frac{1}{2}\right) + \psi^{(0)}(z) == 2\,\psi^{(0)}(2\,z) - 2\log(2)$$

06.15.16.0013.01

$$\psi^{(n)}\left(z+\frac{1}{2}\right) + \psi^{(n)}(z) == 2^{n+1}\,\psi^{(n)}(2\,z) /; n \in \mathbb{N}^+$$

Identities

Recurrence identities

Consecutive neighbors

06.15.17.0001.01

$$\psi^{(n)}(z) == \psi^{(n)}(z+1) - (-1)^n\, n!\, z^{-n-1} /; n \in \mathbb{N}$$

06.15.17.0002.01

$$\psi^{(n)}(z) == \psi^{(n)}(z-1) + (-1)^n\, n!\, (z-1)^{-n-1} /; n \in \mathbb{N}$$

Distant neighbors

06.15.17.0003.02

$$\psi^{(n)}(z) == \psi^{(n)}(z+m) - (-1)^n\, n!\sum_{k=0}^{m-1}\frac{1}{(z+k)^{n+1}} /; n \in \mathbb{N} \bigwedge m \in \mathbb{N}^+$$

06.15.17.0004.02

$$\psi^{(n)}(z) == \psi^{(n)}(z-m) + (-1)^n\, n!\sum_{k=1}^{m}\frac{1}{(z-k)^{n+1}} /; n \in \mathbb{N} \bigwedge m \in \mathbb{N}^+$$

Functional identities

Relations of special kind

06.15.17.0005.01

$$\psi^{(n)}(-z) == (-1)^n\,\psi^{(n)}(z) + n!\, z^{-n-1} + (-1)^n\,\pi\,\frac{\partial^n \cot(\pi\,z)}{\partial z^n} /; n \in \mathbb{N}$$

06.15.17.0006.01

$$\psi^{(n)}(-z) = (-1)^n \psi^{(n)}(z) + n!\, z^{-n-1} + (-1)^n \pi^{n+1} \left(\cot(\pi z)\, \delta_n - \delta_{n-1} \csc^2(\pi z) - n \sum_{k=0}^{n-1} \sum_{j=0}^{k-1} \frac{(-1)^j (k-j)^{n-1}}{k+1} \binom{n-1}{k} \sin^{-2k-2}(\pi z)\, 2^{n-2k} \binom{2k}{j} \sin\left(\frac{\pi n}{2} + 2(k-j)\pi z\right) \right) /;\, n \in \mathbb{N}$$

Differentiation

Low-order differentiation

With respect to ν

06.15.20.0001.01

$$\frac{\partial \psi^{(\nu)}(z)}{\partial \nu} = \frac{z^{-\nu-1}}{\Gamma(1-\nu)} \left(-\nu \log^2(z) + \gamma z \log(z) + \gamma \nu \log(z) - \nu \psi(-\nu)^2 - \gamma z \psi(1-\nu) - \nu (\gamma - 2\log(z)) \psi(-\nu) + \nu \psi^{(1)}(-\nu)\right) - \sum_{j=0}^{\infty} \frac{(-1)^j (j+1)!\, z^{j-\nu+1} (\log(z) - \psi(j-\nu+2))\, \zeta(j+2)}{\Gamma(j-\nu+2)} /;\, \mathrm{Re}(\nu) > 0 \land \mathrm{Re}(z) > 0$$

06.15.20.0002.01

$$\frac{\partial^2 \psi^{(\nu)}(z)}{\partial \nu^2} = \frac{z^{-\nu-1}}{\Gamma(1-\nu)} \left(-\gamma z \left(\log^2(z) - 2\psi(1-\nu)\log(z) + \psi(1-\nu)^2 - \psi^{(1)}(1-\nu)\right) - \nu \left(-\log^3(z) + \gamma \log^2(z) + \psi(-\nu)^3 + (\gamma - 3\log(z))\, \psi(-\nu)^2 - (\gamma - 3\log(z))\, \psi^{(1)}(-\nu) + \psi(-\nu)\left(\log(z)(3\log(z) - 2\gamma) - 3\psi^{(1)}(-\nu)\right) + \psi^{(2)}(-\nu)\right)\right) + \sum_{j=0}^{\infty} \frac{(-1)^j (j+1)!\, z^{j-\nu+1}}{\Gamma(j-\nu+2)} \left(\log^2(z) - 2\psi(j-\nu+2)\log(z) + \psi(j-\nu+2)^2 - \psi^{(1)}(j-\nu+2)\right) \zeta(j+2) /;\, \mathrm{Re}(\nu) > 0 \land \mathrm{Re}(z) > 0$$

With respect to z

06.15.20.0003.01

$$\frac{\partial \psi^{(\nu)}(z)}{\partial z} = \psi^{(\nu+1)}(z)$$

06.15.20.0004.01

$$\frac{\partial^2 \psi^{(\nu)}(z)}{\partial z^2} = \psi^{(\nu+2)}(z)$$

Symbolic differentiation

With respect to z

06.15.20.0005.02

$$\frac{\partial^m \psi^{(\nu)}(z)}{\partial z^m} = \psi^{(\nu+m)}(z) /;\, m \in \mathbb{N}$$

Fractional integro-differentiation

With respect to z

06.15.20.0006.01

$$\frac{\partial^{\alpha} \psi^{(\nu)}(z)}{\partial z^{\alpha}} = \psi^{(\alpha+\nu)}(z)$$

06.15.20.0007.01

$$\frac{\partial^{\alpha} \psi^{(\nu)}(z)}{\partial z^{\alpha}} = -\mathcal{FC}_{\exp}^{(\nu+\alpha)}(z, -1)\, z^{-\alpha-\nu-1} - \frac{\gamma\, z^{-\alpha-\nu}}{\Gamma(1-\alpha-\nu)} + z^{1-\alpha-\nu} \sum_{k=1}^{\infty} \frac{1}{k^2}\, {}_2\tilde{F}_1\left(1, 2; 2-\alpha-\nu; -\frac{z}{k}\right)$$

Integration

Indefinite integration

Involving only one direct function

06.15.21.0001.01

$$\int \psi^{(\nu)}(z)\, dz = \psi^{(\nu-1)}(z)$$

Involving one direct function and elementary functions

Involving power function

06.15.21.0002.01

$$\int z^{\alpha-1}\, \psi^{(\nu)}(z)\, dz = -\frac{\gamma\, z^{\alpha-\nu}}{\Gamma(1-\nu)\,(\alpha-\nu)} + \frac{z^{\alpha-\nu-1}}{(\alpha-\nu-1)\,\Gamma(-\nu)} \left(\psi(-\nu) + \frac{1}{\alpha-\nu-1} + \gamma - \log(z)\right) + \frac{z^{\alpha-\nu+1}}{(\alpha-\nu+1)\,\Gamma(2-\nu)} \sum_{k=0}^{\infty} \frac{1}{(k+1)^2}\, {}_3F_2\left(1, 2, \alpha-\nu+1; 2-\nu, \alpha-\nu+2; -\frac{z}{k+1}\right)$$

06.15.21.0003.01

$$\int z^n\, \psi^{(-m)}(z)\, dz = \sum_{j=0}^{n} (-1)^j\, (n-j+1)_j\, z^{n-j}\, \psi^{(-j-m-1)}(z) \;/;\; n \in \mathbb{N} \land m \in \mathbb{N}$$

Integral transforms

Laplace transforms

06.15.22.0001.01

$$\mathcal{L}_t\left[\psi^{(n)}(t)\right](z) = (-1)^{n-1}\, n!\, z^n \sum_{k=0}^{\infty} e^{k z}\, \Gamma(-n, k z) \;/;\; \mathrm{Re}(z) > 0 \bigwedge n \in \mathbb{N}^+$$

Representations through more general functions

Through hypergeometric functions

Involving ${}_pF_q$

06.15.26.0001.01

$$\psi^{(1)}(z) = (z-1)^2\,{}_3F_2(1,1,2-z;2,2;1)^2 - 2\,(z-1)\,{}_4F_3(1,1,1,2-z;2,2,2;1) + \frac{\pi^2}{6}$$

06.15.26.0002.01

$$\psi^{(2)}(z) = 2\,(z-1)^3\,{}_3F_2(1,1,2-z;2,2;1)^3 - 6\,(z-1)^2\,{}_4F_3(1,1,1,2-z;2,2,2;1)\,{}_3F_2(1,1,2-z;2,2;1) + 6\,(z-1)\,{}_5F_4(1,1,1,1,2-z;2,2,2,2;1) - 2\,\zeta(3)$$

06.15.26.0003.01

$$\psi^{(n)}(z) = (-1)^{n+1}\,n!\,z^{-n-1}\,{}_{n+2}F_{n+1}(1,a_1,a_2,\ldots,a_{n+1};a_1+1,a_2+1,\ldots,a_{n+1}+1;1)\;/;\;a_1 = a_2 = \ldots = a_{n+1} = z \bigwedge n \in \mathbb{N}^+$$

Through Meijer *G*

Classical cases for the direct function itself

06.15.26.0004.01

$$\psi^{(n)}(z) = (-1)^{n+1}\,n!\,G_{n+2,n+2}^{1,n+2}\left(-1 \,\middle|\, \begin{matrix} 0,1-z,\ldots,1-z \\ 0,-z,\ldots,-z \end{matrix}\right)\;/;\;n \in \mathbb{N}^+$$

Representations through equivalent functions

With related functions

06.15.27.0001.01

$$\psi^{(n)}(z) = (-1)^{n+1}\,n!\,\zeta(n+1,z)\;/;\;n \in \mathbb{N}^+ \bigwedge \operatorname{Re}(a) > 0$$

06.15.27.0002.01

$$\psi^{(n)}(z) = (-1)^{n+1}\,n!\left(\sum_{k=0}^{\lfloor -\operatorname{Re}(z)\rfloor} \frac{1}{(k+z)^{n+1}} + \zeta(n+1,z+(\lfloor -\operatorname{Re}(z)\rfloor+1)\,\theta(\lfloor -\operatorname{Re}(z)\rfloor+1))\right)\;/;\;n \in \mathbb{N}^+$$

06.15.27.0003.01

$$\psi^{(n)}(z) = \frac{\partial^n \frac{\frac{\partial \Gamma(z)}{\partial z}}{\Gamma(z)}}{\partial z^n}\;/;\;n \in \mathbb{N}^+$$

06.15.27.0004.01

$$\psi^{(n)}(z) = \frac{\partial^{n+1} \log\Gamma(z)}{\partial z^{n+1}}\;/;\;n \in \mathbb{N}^+$$

06.15.27.0005.01

$$\psi^{(n)}(z) = (-1)^{n+1}\,n!\left(\zeta(n+1) - H_{z-1}^{(n+1)}\right)\;/;\;n \in \mathbb{N}^+$$

History

– B. Ross (1974)

– N. Grossman (1976)

– R.W. Gosper (1997) defined and studed cases $\nu == -2, -3$ of function $\psi^{(\nu)}(z)$

– V. S. Adamchik (1998) suggested the idea to define $\psi^{(\nu)}(z)$ for complex ν via Liouville's fractional integration operator

-O. I. Marichev (1998) derived representation of $\psi^{(\nu)}(z)$ for complex ν through sums with hypergeometric or zeta functions

using fractional integro differentiation

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