PolyGamma

Notations

Traditional name

Digamma function

Traditional notation

$\psi(z)$

Mathematica StandardForm notation

`PolyGamma[z]`

Primary definition

06.14.02.0001.01

$$\psi(z) = \sum_{k=1}^{\infty}\left(\frac{1}{k} - \frac{1}{k+z-1}\right) - \gamma$$

Specific values

Specialized values

06.14.03.0001.01

$$\psi(n) = \sum_{k=1}^{n-1}\frac{1}{k} - \gamma \;/;\; n \in \mathbb{N}^+$$

06.14.03.0002.01

$$\psi(-n) = \tilde{\infty} \;/;\; n \in \mathbb{N}$$

06.14.03.0003.01

$$\psi\left(n + \frac{1}{4}\right) = 4\sum_{k=0}^{n-1}\frac{1}{4k+1} - \frac{\pi}{2} - \log(8) - \gamma \;/;\; n \in \mathbb{N}$$

06.14.03.0004.01

$$\psi\left(\frac{1}{4} - n\right) = 4\sum_{k=0}^{n-1}\frac{1}{4k+3} - \frac{\pi}{2} - \log(8) - \gamma \;/;\; n \in \mathbb{N}$$

06.14.03.0005.01

$$\psi\left(n + \frac{1}{3}\right) = 3\sum_{k=0}^{n-1}\frac{1}{3k+1} - \frac{1}{6}\left(9\log(3) + \sqrt{3}\,\pi\right) - \gamma \;/;\; n \in \mathbb{N}$$

06.14.03.0006.01

$$\psi\left(\frac{1}{3}-n\right) = 3\sum_{k=0}^{n-1}\frac{1}{3k+2} - \frac{1}{6}\left(9\log(3)+\sqrt{3}\,\pi\right) - \gamma \;/;\; n\in\mathbb{N}$$

06.14.03.0007.01

$$\psi\left(n+\frac{1}{2}\right) = \sum_{k=1}^{n-1}\frac{1}{k} + \sum_{k=n}^{2n-1}\frac{2}{k} - \log(4) - \gamma \;/;\; n\in\mathbb{N}$$

06.14.03.0008.01

$$\psi\left(\frac{1}{2}-n\right) = \sum_{k=1}^{n-1}\frac{1}{k} + \sum_{k=n}^{2n-1}\frac{2}{k} - \log(4) - \gamma \;/;\; n\in\mathbb{N}$$

06.14.03.0009.01

$$\psi\left(n+\frac{2}{3}\right) = 3\sum_{k=0}^{n-1}\frac{1}{3k+2} + \frac{1}{6}\left(\sqrt{3}\,\pi - 9\log(3)\right) - \gamma \;/;\; n\in\mathbb{N}$$

06.14.03.0010.01

$$\psi\left(\frac{2}{3}-n\right) = 3\sum_{k=0}^{n-1}\frac{1}{3k+1} + \frac{1}{6}\left(\sqrt{3}\,\pi - \log(19\,683)\right) - \gamma \;/;\; n\in\mathbb{N}$$

06.14.03.0011.01

$$\psi\left(n+\frac{3}{4}\right) = 4\sum_{k=0}^{n-1}\frac{1}{4k+3} + \frac{\pi}{2} - \log(8) - \gamma \;/;\; n\in\mathbb{N}$$

06.14.03.0012.01

$$\psi\left(\frac{3}{4}-n\right) = 4\sum_{k=0}^{n-1}\frac{1}{4k+1} + \frac{\pi}{2} - \log(8) - \gamma \;/;\; n\in\mathbb{N}$$

06.14.03.0013.01

$$\psi\left(n+\frac{p}{q}\right) = q\sum_{k=0}^{n-1}\frac{1}{p+kq} + 2\sum_{k=1}^{\left\lfloor\frac{q-1}{2}\right\rfloor}\cos\left(\frac{2\pi pk}{q}\right)\log\left(\sin\left(\frac{\pi k}{q}\right)\right) - \frac{\pi}{2}\cot\left(\frac{\pi p}{q}\right) - \log(2q) - \gamma \;/;\; n\in\mathbb{N}\land p\in\mathbb{N}^+\land q\in\mathbb{N}^+\land p<q$$

06.14.03.0014.01

$$\psi\left(\frac{p}{q}-n\right) = q\sum_{k=0}^{n-1}\frac{1}{q(k+1)-p} + 2\sum_{k=1}^{\left\lfloor\frac{q-1}{2}\right\rfloor}\cos\left(\frac{2\pi pk}{q}\right)\log\left(\sin\left(\frac{\pi k}{q}\right)\right) - \frac{\pi}{2}\cot\left(\frac{\pi p}{q}\right) - \log(2q) - \gamma \;/;$$
$$n\in\mathbb{N}\land p\in\mathbb{N}^+\land q\in\mathbb{N}^+\land p<q$$

06.14.03.0033.01

$$\psi\left(\frac{p}{q}\right) = 2\sum_{k=1}^{\left\lfloor\frac{q-1}{2}\right\rfloor}\cos\left(\frac{2\pi pk}{q}\right)\log\left(\sin\left(\frac{\pi k}{q}\right)\right) - \frac{1}{2}\pi\cot\left(\frac{\pi p}{q}\right) - \log(2q) - \gamma \;/;\; p\in\mathbb{N}^+\land q\in\mathbb{N}^+\land p<q$$

Values at fixed points

06.14.03.0015.01

$$\psi(-3) = \tilde{\infty}$$

06.14.03.0016.01
$$\psi\left(-\frac{5}{2}\right) == \frac{46}{15} - \gamma - \log(4)$$

06.14.03.0017.01
$$\psi(-2) == \tilde{\infty}$$

06.14.03.0018.01
$$\psi\left(-\frac{3}{2}\right) == \frac{8}{3} - \gamma - \log(4)$$

06.14.03.0019.01
$$\psi(-1) == \tilde{\infty}$$

06.14.03.0020.01
$$\psi\left(-\frac{1}{2}\right) == 2 - \gamma - \log(4)$$

06.14.03.0021.01
$$\psi(0) == \tilde{\infty}$$

06.14.03.0022.01
$$\psi\left(\frac{1}{2}\right) == -\gamma - \log(4)$$

06.14.03.0023.01
$$\psi(1) == -\gamma$$

06.14.03.0024.01
$$\psi\left(\frac{3}{2}\right) == 2 - \gamma - \log(4)$$

06.14.03.0025.01
$$\psi(2) == 1 - \gamma$$

06.14.03.0026.01
$$\psi\left(\frac{5}{2}\right) == \frac{8}{3} - \gamma - \log(4)$$

06.14.03.0027.01
$$\psi(3) == \frac{3}{2} - \gamma$$

Values at infinities

06.14.03.0028.01
$$\psi(\infty) == \infty$$

06.14.03.0029.01
$$\psi(-\infty) == \infty$$

06.14.03.0030.01
$$\psi(i\,\infty) == \infty$$

06.14.03.0031.01
$$\psi(-i\,\infty) == \infty$$

06.14.03.0032.01
$$\psi(\tilde{\infty}) == \infty$$

General characteristics

Domain and analyticity

$\psi(z)$ is an analytical function of z which is defined over the whole complex z-plane with the exception of countably many points $z = -k /; k \in \mathbb{N}$.

06.14.04.0001.01

$$z \longrightarrow \psi(z) :: \mathbb{C} \longrightarrow \mathbb{C}$$

Symmetries and periodicities

Mirror symmetry

06.14.04.0002.01

$$\psi(\bar{z}) == \overline{\psi(z)}$$

Periodicity

No periodicity

Poles and essential singularities

The function $\psi(z)$ has an infinite set of singular points:

a) $z == -k /; k \in \mathbb{N}$, are the simple poles with residues -1 ;

b) $z == \tilde{\infty}$ is the point of convergence of poles, which is an essential singular point.

06.14.04.0003.01

$$\mathcal{Sing}_z(\psi(z)) == \{\{\{-k, 1\} /; k \in \mathbb{N}\}, \{\tilde{\infty}, \infty\}\}$$

06.14.04.0004.01

$$\text{res}_z(\psi(z))(-k) == -1 /; k \in \mathbb{N}$$

Branch points

The function $\psi(z)$ does not have branch points.

06.14.04.0005.01

$$\mathcal{BP}_z(\psi(z)) == \{\}$$

Branch cuts

The function $\psi(z)$ does not have branch cuts.

06.14.04.0006.01

$$\mathcal{BC}_z(\psi(z)) == \{\}$$

Series representations

Generalized power series

Expansions at $z == 0$

For the function itself

06.14.06.0002.02

$$\psi(z) \propto -\frac{1}{z} - \gamma + \frac{\pi^2 z}{6} - \zeta(3) z^2 + \frac{\pi^4 z^3}{90} - \ldots /; (z \to 0)$$

06.14.06.0017.01

$$\psi(z) \propto -\frac{1}{z} - \gamma + \frac{\pi^2 z}{6} - \zeta(3) z^2 + \frac{\pi^4 z^3}{90} - O\left(z^4\right)$$

06.14.06.0003.01

$$\psi(z) == -\frac{1}{z} - \gamma + \sum_{j=0}^{\infty} (-1)^j \zeta(j+2) z^{j+1} /; |z| < 1$$

06.14.06.0001.01

$$\psi(z) == -\frac{1}{z} - \gamma + \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{(-1)^j z^{j+1}}{(k+1)^{j+2}} /; |z| < 1$$

06.14.06.0004.02

$$\psi(z) \propto -\frac{1}{z} - \gamma (1 + O(z))$$

Expansions at $z == z_0$ /; $z_0 \neq -n$

For the function itself

06.14.06.0006.02

$$\psi(z) \propto \psi(z_0) + \zeta(2, z_0)(z - z_0) - \zeta(3, z_0)(z - z_0)^2 + \ldots /; (z \to z_0) \bigwedge \neg (z_0 \in \mathbb{Z} \bigwedge z_0 \leq 0)$$

06.14.06.0018.01

$$\psi(z) \propto \psi(z_0) + \zeta(2, z_0)(z - z_0) - \zeta(3, z_0)(z - z_0)^2 + O\left((z - z_0)^3\right) /; \neg (z_0 \in \mathbb{Z} \bigwedge z_0 \leq 0)$$

06.14.06.0007.02

$$\psi(z) == \psi(z_0) + \sum_{j=0}^{\infty} (-1)^j \zeta(j+2, z_0)(z - z_0)^{j+1} /; \neg (z_0 \in \mathbb{Z} \bigwedge z_0 \leq 0)$$

06.14.06.0005.02

$$\psi(z) == \psi(z_0) + \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{(-1)^j (z - z_0)^{j+1}}{(k + z_0)^{j+2}} /; \neg (z_0 \in \mathbb{Z} \bigwedge z_0 \leq 0)$$

06.14.06.0008.02

$$\psi(z) \propto \psi(z_0) + \zeta(2, z_0)(z - z_0)(1 + O(z - z_0)) /; \neg (z_0 \in \mathbb{Z} \bigwedge z_0 \leq 0)$$

Expansions at $z == -n$

For the function itself

06.14.06.0009.02

$$\psi(z) \propto -\frac{1}{z+n} + \psi(n+1) + \left(\frac{\pi^2}{3} - \zeta(2, n+1)\right)(z+n) - \zeta(3, n+1)(z+n)^2 + \left(\frac{\pi^4}{45} - \zeta(4, n+1)\right)(z+n)^3 - \ldots /; (z \to -n) \land n \in \mathbb{N}$$

06.14.06.0019.01

$$\psi(z) \propto -\frac{1}{z+n} + \psi(n+1) + \left(\frac{\pi^2}{3} - \zeta(2, n+1)\right)(z+n) - \zeta(3, n+1)(z+n)^2 + \left(\frac{\pi^4}{45} - \zeta(4, n+1)\right)(z+n)^3 + O\left((z+n)^4\right) /; n \in \mathbb{N}$$

06.14.06.0010.02

$$\psi(z) == -\frac{1}{z+n} + \psi(n+1) + \sum_{k=1}^{\infty}\left(\frac{\psi^{(k)}(1)}{k!} + \zeta(k+1) - \zeta(k+1, n+1)\right)(z+n)^k /; n \in \mathbb{N}$$

06.14.06.0011.02

$$\psi(z) \propto -\frac{1}{z+n} + \psi(n+1)\,(1 + O(z+n)) /; n \in \mathbb{N}$$

Expansions of $\psi(z+\epsilon)$ at $\epsilon == 0$ /; $z \neq -n$

For the function itself

06.14.06.0020.01

$$\psi(z+\epsilon) \propto \psi(z) + \zeta(2, z)\,\epsilon - \zeta(3, z)\,\epsilon^2 + \ldots /; (\epsilon \to 0) \land \neg (z \in \mathbb{Z} \land z \leq 0)$$

06.14.06.0021.01

$$\psi(z+\epsilon) \propto \psi(z) + \zeta(2, z)\,\epsilon - \zeta(3, z)\,\epsilon^2 + O\left(\epsilon^3\right) /; \neg (z \in \mathbb{Z} \land z \leq 0)$$

06.14.06.0022.01

$$\psi(z+\epsilon) == \psi(z) + \sum_{j=0}^{\infty} (-1)^j\,\zeta(j+2, z)\,\epsilon^{j+1} /; \neg (z \in \mathbb{Z} \land z \leq 0)$$

06.14.06.0023.01

$$\psi(z+\epsilon) \propto \psi(z)\,(1 + O(\epsilon)) /; \neg (z \in \mathbb{Z} \land z \leq 0)$$

Expansions of $\psi(-n+\epsilon)$ at $\epsilon == 0$

For the function itself

06.14.06.0024.01

$$\psi(-n+\epsilon) \propto -\frac{1}{\epsilon} + \psi(n+1) + \left(\frac{\pi^2}{3} - \zeta(2, n+1)\right)\epsilon + \frac{1}{2}\,\psi^{(2)}(n+1)\,\epsilon^2 + \ldots /; (\epsilon \to 0) \land n \in \mathbb{N}$$

06.14.06.0025.01

$$\psi(-n+\epsilon) \propto -\frac{1}{\epsilon} + \psi(n+1) + \left(\frac{\pi^2}{3} - \zeta(2, n+1)\right)\epsilon + \frac{1}{2}\,\psi^{(2)}(n+1)\,\epsilon^2 + O\left(\epsilon^3\right) /; n \in \mathbb{N}$$

06.14.06.0026.01

$$\psi(-n+\epsilon) \propto -\frac{1}{\epsilon} + \psi(n+1) + \sum_{k=1}^{\infty}\left(\frac{\psi^{(k)}(1)}{k!} + \zeta(k+1) - \zeta(k+1, n+1)\right)\epsilon^k /; (\epsilon \to 0) \land n \in \mathbb{N}$$

06.14.06.0027.01

$$\psi(-n+\epsilon) \propto -\frac{1}{\epsilon}\,(1 + O(\epsilon)) /; n \in \mathbb{N}$$

Asymptotic series expansions

06.14.06.0012.01

$$\psi(z) \propto \log(z) - \frac{1}{2z} - \sum_{k=1}^{\infty} \frac{B_{2k}}{2k z^{2k}} \;/;\; |\arg(z)| < \pi \bigwedge (|z| \to \infty)$$

06.14.06.0028.01

$$\psi(z) \propto \log(z) - \frac{1}{2z} - \sum_{k=1}^{\infty} \frac{B_{2k}}{2k z^{2k}} + i\pi (i \cot(\pi z) - 1) \left\lfloor \frac{|\arg(z)|}{\pi} \right\rfloor \;/;\; \neg (z \in \mathbb{Z} \land z < 0) \land (|z| \to \infty)$$

06.14.06.0013.01

$$\psi(z) \propto \log(z) - \frac{1}{2z} - \frac{1}{12 z^2}\left(1 + O\left(\frac{1}{z^2}\right)\right) \;/;\; |\arg(z)| < \pi \bigwedge (|z| \to \infty)$$

06.14.06.0029.01

$$\psi(z) \propto \log(z) - \frac{1}{2z} - \frac{1}{12 z^2}\left(1 + O\left(\frac{1}{z^2}\right)\right) + i\pi (i \cot(\pi z) - 1) \left\lfloor \frac{|\arg(z)|}{\pi} \right\rfloor \;/;\; \neg (z \in \mathbb{Z} \land z < 0) \land (|z| \to \infty)$$

Residue representations

06.14.06.0014.01

$$\psi(z) = -\frac{1}{\Gamma(1-z)} \sum_{j=0}^{\infty} \operatorname{res}_s\left(\frac{\Gamma(1-s)^2\, \Gamma(2-s-z)\, (-1)^{-s}}{\Gamma(2-s)^2}\, \Gamma(s)\right)(-j) - \gamma$$

Other series representations

06.14.06.0015.01

$$\psi(z) = -\frac{1}{z} + z\left(\frac{1}{z+1} + \frac{1}{2(z+2)} + \frac{1}{3(z+3)} + \ldots\right) - \gamma$$

06.14.06.0016.01

$$\psi(z) = -\frac{1}{z} + z \sum_{k=0}^{\infty} \frac{1}{(k+1)(k+z+1)} - \gamma$$

Integral representations

On the real axis

Of the direct function

06.14.07.0001.01

$$\psi(z) = \int_0^1 \frac{1 - t^{z-1}}{1-t}\, dt - \gamma \;/;\; \operatorname{Re}(z) > 0$$

06.14.07.0002.01

$$\psi(z) = \int_0^{\infty} \left(\frac{e^{-t}}{t} - \frac{(t+1)^{-z}}{t}\right) dt \;/;\; \operatorname{Re}(z) > 0$$

06.14.07.0003.01

$$\psi(z) = \int_0^{\infty} \frac{e^{-t} - e^{-zt}}{1 - e^{-t}}\, dt - \gamma \;/;\; \operatorname{Re}(z) > 0$$

06.14.07.0004.01

$$\psi(n) = \frac{1}{2} \int_{-1}^{1} \frac{1 - P_{n-1}(t)}{1-t} \, dt - \gamma \,/;\, n \in \mathbb{N}^+$$

06.14.07.0007.01

$$\psi(z) = \int_0^1 \left(\frac{x^{z-1}}{x-1} - \frac{1}{\log(x)} \right) dx \,/;\, \mathrm{Re}(z) > 0$$

A. Radovi[redacted]

Contour integral representations

06.14.07.0005.01

$$\psi(z) = -\gamma - \frac{1}{2 \pi i \, \Gamma(1-z)} \int_{\mathcal{L}} \frac{\Gamma(s)\, \Gamma(1-s)\, \Gamma(1-s)\, \Gamma(2-z-s)}{\Gamma(2-s)\, \Gamma(2-s)} (-1)^{-s} \, ds$$

06.14.07.0006.01

$$\psi(z) = -\gamma - \frac{1}{2 \pi i \, \Gamma(1-z)} \int_{\gamma - i \infty}^{\gamma + i \infty} \frac{\Gamma(s)\, \Gamma(1-s)\, \Gamma(1-s)\, \Gamma(2-z-s)}{\Gamma(2-s)\, \Gamma(2-s)} (-1)^{-s} \, ds \,/;\, 0 < \gamma < 1$$

Limit representations

06.14.09.0001.01

$$\psi(z) = \lim_{n \to \infty} \left(\log(n) - \sum_{k=0}^{n} \frac{1}{z+k} \right)$$

06.14.09.0002.01

$$\psi(z) = -\lim_{s \to 1} \left(\zeta(s, z) - \frac{1}{s-1} \right)$$

Generating functions

06.14.11.0001.01

$$\psi(n) = -\left([t^n] \, \frac{t \log(1-t)}{1-t} \right) - \gamma \,/;\, n \in \mathbb{N}^+$$

Transformations

Transformations and argument simplifications

Argument involving basic arithmetic operations

06.14.16.0001.01

$$\psi(1-z) = \pi \cot(\pi z) + \psi(z)$$

06.14.16.0002.01

$$\psi(-z) = \psi(z) + \pi \cot(\pi z) + \frac{1}{z}$$

06.14.16.0003.01

$$\psi(z+1) = \psi(z) + \frac{1}{z}$$

06.14.16.0004.01

$$\psi(z-1) == \psi(z) - \frac{1}{z-1}$$

06.14.16.0005.01

$$\psi(z+n) == \psi(z) + \sum_{k=0}^{n-1} \frac{1}{z+k} \;/;\; n \in \mathbb{N}$$

06.14.16.0006.01

$$\psi(z-n) == \psi(z) - \sum_{k=0}^{n-1} \frac{1}{z-k-1} \;/;\; n \in \mathbb{N}$$

Multiple arguments

Argument involving numeric multiples of variable

06.14.16.0007.01

$$\psi(2\,z) == \log(2) + \frac{1}{2}\left(\psi\left(z+\frac{1}{2}\right) + \psi(z)\right)$$

06.14.16.0010.01

$$\psi(3\,z) == \frac{1}{3}\left(\psi(z) + \psi\left(z+\frac{1}{3}\right) + \psi\left(z+\frac{2}{3}\right)\right) + \log(3)$$

Argument involving symbolic multiples of variable

06.14.16.0008.01

$$\psi(m\,z) == \log(m) + \frac{1}{m}\sum_{k=0}^{m-1} \psi\left(z+\frac{k}{m}\right) \;/;\; m \in \mathbb{N}^+$$

Products, sums, and powers of the direct function

Sums of the direct function

06.14.16.0009.01

$$\psi(z) + \psi\left(z+\frac{1}{2}\right) == 2\,\psi(2\,z) - 2\log(2)$$

Identities

Recurrence identities

Consecutive neighbors

06.14.17.0001.01

$$\psi(z) == \psi(z+1) - \frac{1}{z}$$

06.14.17.0002.01

$$\psi(z) == \psi(z-1) + \frac{1}{z-1}$$

Distant neighbors

06.14.17.0003.01

$$\psi(z) == \psi(z+n) - \sum_{k=0}^{n-1} \frac{1}{z+k} /; n \in \mathbb{N}$$

06.14.17.0004.01

$$\psi(z) == \psi(z-n) + \sum_{k=1}^{n} \frac{1}{z-k} /; n \in \mathbb{N}$$

Functional identities

Relations of special kind

06.14.17.0005.01

$$\psi(-z) == \psi(z) + \pi \cot(\pi z) + \frac{1}{z}$$

Complex characteristics

Real part

06.14.19.0001.01

$$\mathrm{Re}(\psi(x + i\, y)) == \mathrm{RootSum}\left[(\#1+1)\left(x^2 + 2\, \#1\, x + y^2 + \#1^2\right) \&, -\frac{\psi^{(0)}(-\#1)\left(x^2 + (\#1 - 1)\, x + y^2 - \#1\right)}{x^2 + (4\, \#1 + 2)\, x + y^2 + \#1\, (3\, \#1 + 2)} \&\right] - \gamma$$

06.14.19.0002.01

$$\mathrm{Re}(\psi(x + i\, y)) == \frac{1}{2}\left(\psi(x - i\, y) + \psi(x + i\, y)\right)$$

Imaginary part

06.14.19.0003.01

$$\mathrm{Im}(\psi(x + i\, y)) == \frac{i}{2}\left(\psi(x - i\, y) - \psi(x + i\, y)\right)$$

Differentiation

Low-order differentiation

06.14.20.0001.01

$$\frac{\partial \psi(z)}{\partial z} == \psi^{(1)}(z)$$

06.14.20.0002.01

$$\frac{\partial^2 \psi(z)}{\partial z^2} == \psi^{(2)}(z)$$

Symbolic differentiation

06.14.20.0003.02

$$\frac{\partial^n \psi(z)}{\partial z^n} == \psi^{(n)}(z) /; n \in \mathbb{N}$$

Fractional integro-differentiation

06.14.20.0004.01

$$\frac{\partial^\alpha \psi(z)}{\partial z^\alpha} = \psi^{(\alpha)}(z)$$

06.14.20.0005.01

$$\frac{\partial^\alpha \psi(z)}{\partial z^\alpha} = -\mathcal{F}C_{\exp}^{(\nu+\alpha)}(z,-1)\, z^{-\alpha-1} - \frac{\gamma\, z^{-\alpha}}{\Gamma(1-\alpha)} + z^{1-\alpha} \sum_{k=1}^{\infty} \frac{1}{k^2}\, {}_2\tilde{F}_1\!\left(1, 2; 2-\alpha; -\frac{z}{k}\right)$$

Integration

Indefinite integration

Involving only one direct function

06.14.21.0001.01

$$\int \psi(z)\, dz = \log\Gamma(z)$$

Involving one direct function and elementary functions

Involving power function

06.14.21.0002.01

$$\int z^{\alpha-1}\, \psi(z)\, dz = \frac{z^{\alpha-1}}{1-\alpha} - \frac{\gamma\, z^\alpha}{\alpha} + \frac{z^{\alpha+1}}{\alpha+1} \sum_{k=0}^{\infty} \frac{1}{(k+1)^2}\, {}_3F_2\!\left(1, 2, \alpha+1; 2, \alpha+2; -\frac{z}{k+1}\right)$$

06.14.21.0003.01

$$\int z^n\, \psi(z)\, dz = \sum_{j=0}^{n} (-1)^j\, (n-j+1)_j\, z^{n-j}\, \psi^{(-j-1)}(z) \,/;\, n \in \mathbb{N}$$

Definite integration

Involving the direct function

06.14.21.0004.01

$$\int_0^z \psi(t+1)\,(z-t)^{n-1}\, dt = (n-1)!\, \psi^{(-n)}(z+1) - \sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} (z+1)^{-k+n-1} \sum_{j=0}^{k} (-1)^j\, (-j+k+1)_j\, \psi^{(-j-1)}(1) \,/;\, n \in \mathbb{N}$$

Summation

Finite summation

06.14.23.0001.01

$$\sum_{k=1}^{n} \psi(k) = n\,(\psi(n+1)-1) \,/;\, n \in \mathbb{N}^+$$

06.14.23.0002.01

$$\sum_{k=1}^{q} \psi\left(\frac{k}{q}\right) e^{\frac{2\pi p k i}{q}} = q \log\left(1 - e^{\frac{2\pi p i}{q}}\right) /; p \in \mathbb{N}^+ \land q \in \mathbb{N}^+ \land p < q$$

06.14.23.0003.01

$$\sum_{i=0}^{m} \frac{(-m)_i \prod_{k=1}^{p} (a_k)_i}{i!\,(n+1)_i \prod_{k=1}^{q} (b_k)_i} \left(\psi(i+1) + \psi(i+n+1) - \psi(m-i+1) - \sum_{k=\text{n1}+1}^{p} \psi(1-i-a_k) + \sum_{k=1}^{\text{m1}} \psi(1-i-b_k) - \sum_{k=1}^{\text{n1}} \psi(i+a_k) + \sum_{k=\text{m1}+1}^{q} \psi(i+b_k)\right) z^i =$$

$$-\frac{n!\,m!\left(\prod_{k=\text{n1}+1}^{p} \Gamma(1-a_k)\right) \prod_{k=\text{m1}+1}^{q} \Gamma(b_k)}{\left(\prod_{k=1}^{\text{m1}} \Gamma(1-b_k)\right) \prod_{k=1}^{\text{n1}} \Gamma(a_k)} \left((-1)^{p-\text{m1}-\text{n1}} z\right)^{-n}$$

$$\left(\frac{\left(\prod_{k=1}^{\text{m1}} \Gamma(-m-b_k)\right) \prod_{k=1}^{\text{n1}} \Gamma(m+a_k+1)}{(m+n+1)!\,(m+1)!\left(\prod_{k=\text{n1}+1}^{p} \Gamma(-m-a_k)\right) \prod_{k=\text{m1}+1}^{q} \Gamma(m+b_k+1)} (-1)^n \left((-1)^{p-\text{m1}-\text{n1}-1} z\right)^{m+n+1}\right.$$

$$_{p+2}F_{q+2}(1, 1, m+a_1+1, \ldots, m+a_p+1; m+n+2, m+2, m+b_1+1, \ldots, m+b_q+1; z) +$$

$$\sum_{j=0}^{n-1} \frac{(n-j-1)!\left(\prod_{k=1}^{\text{m1}} \Gamma(1-j+n-b_k)\right) \prod_{k=1}^{\text{n1}} \Gamma(j-n+a_k)}{j!\,(m+n-j)!\left(\prod_{k=\text{n1}+1}^{p} \Gamma(n-j-a_k+1)\right) \prod_{k=\text{m1}+1}^{q} \Gamma(j-n+b_k)} \left((-1)^{-\text{m1}-\text{n1}+p} z\right)^j +$$

$$\sum_{i=1}^{\text{m1}} \frac{\Gamma(b_i-1)\,\Gamma(b_i-n-1)\left(\prod_{k=1}^{i-1} \Gamma(b_i-b_k)\right)\left(\prod_{k=i+1}^{\text{m1}} \Gamma(b_i-b_k)\right) \prod_{k=1}^{\text{n1}} \Gamma(a_k-b_i+1)}{\Gamma(m+b_i)\left(\prod_{k=\text{n1}+1}^{p} \Gamma(b_i-a_k)\right) \prod_{k=\text{m1}+1}^{q} \Gamma(1-b_i+b_k)} \left((-1)^{-\text{m1}-\text{n1}+p+1} z\right)^{n-b_i+1}$$

$$\left. _{p+2}F_{q+2}(1, -m-b_i+1, a_1-b_i+1, \ldots, a_p-b_i+1; n-b_i+2, 2-b_i, b_1-b_i+1, \ldots, b_q-b_i+1; z)\right) +$$

$$\log\left((-1)^{p-\text{m1}-\text{n1}-1} z\right) {}_{p+1}F_{q+1}(-m, a_1, \ldots, a_p; n+1, b_1, \ldots, b_q; z) + \frac{(-1)^n\, n!\, m!\left(\prod_{k=\text{n1}+1}^{p} \Gamma(1-a_k)\right) \prod_{k=\text{m1}+1}^{q} \Gamma(b_k)}{\left(\prod_{k=1}^{\text{m1}} \Gamma(1-b_k)\right) \prod_{k=1}^{\text{n1}} \Gamma(a_k)}$$

$$G_{p+1,q+2}^{\text{m1}+2,\text{n1}}\left((-1)^{p-\text{m1}-\text{n1}+1} z \,\middle|\, \begin{array}{l} 1-a_1, \ldots, 1-a_{\text{n1}}, m+1, 1-a_{\text{n1}+1}, \ldots, 1-a_p \\ 0, -n, 1-b_1, \ldots, 1-b_{\text{m1}}, 1-b_{\text{m1}+1}, \ldots, 1-b_q \end{array}\right) /;$$

$$n \in \mathbb{N} \land m \in \mathbb{N} \land \forall_{\{j,k\},\{j,k\} \in \mathbb{Z} \land j \neq k \land 1 \leq j \leq \text{m1} \land 1 \leq k \leq \text{m1}} (b_j - b_k \notin \mathbb{Z})$$

```
{m1, n1, p, q} = {3, 1, 4, 5};
 a = {Random[Complex], Random[Complex],
   Random[Complex], Random[Complex], Random[Complex],
   Random[Complex]}; b = {Random[Complex],
   Random[Complex], Random[Complex], Random[Complex],
   Random[Complex]};
 Chop[Table[
   {Sum[((Pochhammer[-m, i]*Product[Pochhammer[
          a[[k]], i], {k, 1, p}])/
        (i!*Pochhammer[n + 1, i]*Product[
          Pochhammer[b[[k]], i], {k, 1, q}]))*z^i*
       (Sum[PolyGamma[1 - b[[k]] - i], {k, 1, m1}] -
        Sum[PolyGamma[a[[k]] + i], {k, 1, n1}] -
        Sum[PolyGamma[1 - a[[k]] - i],
```

```
    {k, n1 + 1, p}] + Sum[PolyGamma[
     b[[k]] + i], {k, m1 + 1, q}] +
   PolyGamma[n + i + 1] + PolyGamma[i + 1] -
   PolyGamma[m + 1 - i]), {i, 0, m}]/
((((-1)^n*n!*m!*Product[Gamma[1 - a[[k]]],
     {k, n1 + 1, p}]*Product[Gamma[b[[k]]],
     {k, m1 + 1, q}])/
   (Product[Gamma[1 - b[[k]]], {k, 1, m1}]*
    Product[Gamma[a[[k]]], {k, 1, n1}]))*
  MeijerG[{Table[1 - a[[r]], {r, 1, n1}],
    Join[{m + 1}, Table[1 - a[[r]],
      {r, 1 + n1, p}]]}, {Join[{0, -n},
     Table[1 - b[[r]], {r, 1, m1}]],
    Table[1 - b[[r]], {r, 1 + m1, q}]},
   (-1)^(p - n1 - m1 + 1)*z] -
 (((n!*m!*Product[Gamma[1 - a[[k]]],
      {k, n1 + 1, p}]*Product[Gamma[b[[k]]],
      {k, m1 + 1, q}])/(Product[
      Gamma[1 - b[[k]]], {k, 1, m1}]*
     Product[Gamma[a[[k]]], {k, 1, n1}]))*
   (Sum[Gamma[-1 + b[[i]]]*Gamma[-n - 1 + b[[
        i]]]*((Product[Gamma[-b[[k]] +
          b[[i]]], {k, 1, i - 1}]*Product[
        Gamma[-b[[k]] + b[[i]]], {k, i + 1,
         m1}]*Product[Gamma[1 + a[[k]] -
          b[[i]]], {k, 1, n1}])/(Gamma[
        m + b[[i]]]*Product[Gamma[-a[[k]] +
          b[[i]]], {k, n1 + 1, p}]*Product[
        Gamma[1 + b[[k]] - b[[i]]],
        {k, m1 + 1, q}]))*((-1)^(p - n1 - m1 +
         1)*z)^(1 + n - b[[i]])*
     HypergeometricPFQ[Join[{1, 1 - m -
         b[[i]]}, Table[1 + a[[r]] - b[[i]],
        {r, 1, p}]], Join[{2 + n - b[[i]],
        2 - b[[i]]}, Table[1 + b[[r]] -
         b[[i]], {r, 1, q}]], z],
    {i, 1, m1}] + Sum[(((n - j - 1)!*Product[
        Gamma[1 + n - b[[k]] - j], {k, 1, m1}]*
      Product[Gamma[-n + a[[k]] + j],
        {k, 1, n1}])/((n + m - j)!*Product[
        Gamma[1 + n - a[[k]] - j], {k, n1 + 1,
         p}]*Product[Gamma[-n + b[[k]] + j],
        {k, m1 + 1, q}]))*(((-1)^(p - n1 - m1)*
        z)^j/j!), {j, 0, n - 1}] +
   ((Product[Gamma[-b[[k]] - m], {k, 1, m1}]*
     Product[Gamma[1 + a[[k]] + m], {k, 1,
```

```
n1}])/(Product[Gamma[-a[[k]] - m], {k,
n1 + 1, p}]*Product[Gamma[1 + b[[k]] +
 m], {k, m1 + 1, q}]))*
(((-1)^n*((-1)^(p - n1 - m1 - 1)*z)^(n +
 m + 1))/((n + m + 1)!*(m + 1)!))*
HypergeometricPFQ[Join[{1, 1}, Table[1 +
 m + a[[r]], {r, 1, p}]],
 Join[{2 + n + m, 2 + m}, Table[1 + m +
 b[[r]], {r, 1, q}]], z]))/
((-1)^(p - n1 - m1)*z)^n +
Log[(-1)^(p - n1 - m1 - 1)*z]*
HypergeometricPFQ[Join[{-m}, Table[a[[r]],
 {r, 1, p}]], Join[{n + 1}, Table[b[[r]],
 {r, 1, q}]], z])} /.
{z -> Random[]*Exp[Pi*I*(ii/4)]}, {n, 0, 3},
{m, 0, 3}, {ii, 0, 7}]]
```

Infinite summation

06.14.23.0004.01

$$\sum_{i=0}^{\infty} \frac{\prod_{k=1}^{p} (a_k)_i}{i!\,(n+1)_i \prod_{k=1}^{q} (b_k)_i} \left(\psi(i+1)+\psi(i+n+1)-\sum_{k=\text{n1}+1}^{p}\psi(1-i-a_k)+\sum_{k=1}^{\text{m1}}\psi(1-i-b_k)-\sum_{k=1}^{\text{n1}}\psi(i+a_k)+\sum_{k=\text{m1}+1}^{q}\psi(i+b_k)\right) z^i =$$
$$n!\log\left((-1)^{p-\text{m1}-\text{n1}} z\right)\left(\prod_{k=1}^{q}\Gamma(b_k)\right) {}_{p+1}\tilde{F}_{q+2}\left(1, a_1, \ldots, a_p; 1, n+1, b_1, \ldots, b_q; z\right) + \frac{(-1)^n\, n!\left(\prod_{k=\text{n1}+1}^{p}\Gamma(1-a_k)\right)\prod_{k=\text{m1}+1}^{q}\Gamma(b_k)}{\left(\prod_{k=1}^{\text{m1}}\Gamma(1-b_k)\right)\prod_{k=1}^{\text{n1}}\Gamma(a_k)}$$
$$G_{p,q+2}^{\text{m1}+2,\text{n1}}\left((-1)^{p-\text{m1}-\text{n1}} z \,\middle|\, \begin{matrix} 1-a_1, \ldots, 1-a_{\text{n1}}, 1-a_{\text{n1}+1}, \ldots, 1-a_p \\ -n, 0, 1-b_1, \ldots, 1-b_{\text{m1}}, 1-b_{\text{m1}+1}, \ldots, 1-b_q \end{matrix}\right) - \frac{(-1)^n\, n!\left(\prod_{k=\text{n1}+1}^{p}\Gamma(1-a_k)\right)\prod_{k=\text{m1}+1}^{q}\Gamma(b_k)}{\left(\prod_{k=1}^{\text{m1}}\Gamma(1-b_k)\right)\prod_{k=1}^{\text{n1}}\Gamma(a_k)}$$
$$\left(\left((-1)^{p-\text{m1}-\text{n1}} z\right)^{-n} \sum_{j=0}^{n-1} \frac{(n-j-1)!\left(\prod_{k=1}^{\text{m1}}\Gamma(1-j+n-b_k)\right)\prod_{k=1}^{\text{n1}}\Gamma(j-n+a_k)}{j!\left(\prod_{k=\text{n1}+1}^{p}\Gamma(1-j+n-a_k)\right)\prod_{k=\text{m1}+1}^{q}\Gamma(j-n+b_k)} \left((-1)^{-\text{m1}-\text{n1}+p-1} z\right)^j + \right.$$
$$(-1)^n \pi^{\text{m1}+1} \sum_{i=1}^{\text{m1}} \frac{\csc^2(\pi\, b_i)\left(\prod_{k=1}^{\text{n1}}\Gamma(a_k-b_i+1)\right)\left(\prod_{k=1}^{i-1}\csc(\pi\,(b_i-b_k))\right)\prod_{k=i+1}^{\text{m1}}\csc(\pi\,(b_i-b_k))}{\prod_{k=\text{n1}+1}^{p}\Gamma(b_i-a_k)} \left((-1)^{p-\text{m1}-\text{n1}} z\right)^{1-b_i}$$
$$\left. {}_{p+1}\tilde{F}_{q+2}\left(1, a_1-b_i+1, \ldots, a_p-b_i+1; n-b_i+2, 2-b_i, b_1-b_i+1, \ldots, b_q-b_i+1; z\right)\right) /;$$

$$n \in \mathbb{N} \land \forall_{\{j,k\},\{j,k\}\in\mathbb{Z}\land j\neq k\land 1\leq j\leq \text{m1}\land 1\leq k\leq \text{m1}} \left(b_j - b_k \notin \mathbb{Z}\right)$$

```
{m1, n1, p, q} = {3, 2, 4, 5};
 aa = {Random[Complex], Random[Complex],
  Random[Complex], Random[Complex], Random[Complex],
  Random[Complex]}; bb = {Random[Complex],
  Random[Complex], Random[Complex], Random[Complex],
  Random[Complex]};
 Chop[Table[
  {Sum[(Product[Pochhammer[aa[[k]], i], {k, 1, p}]/
```

```
(i!*Pochhammer[n + 1, i]*Product[
  Pochhammer[bb[[k]], i], {k, 1, q}]))*z^i*
(Sum[PolyGamma[1 - bb[[k]] - i],
  {k, 1, m1}] - Sum[PolyGamma[aa[[k]] + i],
  {k, 1, n1}] - Sum[PolyGamma[1 - aa[[k]] -
   i], {k, n1 + 1, p}] +
 Sum[PolyGamma[bb[[k]] + i], {k, m1 + 1,
  q}] + PolyGamma[n + i + 1] +
 PolyGamma[i + 1]), {i, 0, 100}]/
((((-1)^n*n!*Product[Gamma[1 - aa[[k]]],
   {k, n1 + 1, p}]*Product[Gamma[bb[[k]]],
   {k, m1 + 1, q}])/
 (Product[Gamma[1 - bb[[k]]], {k, 1, m1}]*
  Product[Gamma[aa[[k]]], {k, 1, n1}]))*
MeijerG[{Table[1 - aa[[r]], {r, 1, n1}],
  Table[1 - aa[[r]], {r, 1 + n1, p}]},
 {Join[{-n, 0}, Table[1 - bb[[r]],
   {r, 1, m1}]], Table[1 - bb[[r]],
   {r, 1 + m1, q}]}, (-1)^(p - n1 - m1)*z] -
(((-1)^n*n!*Product[Gamma[1 - aa[[k]]],
   {k, n1 + 1, p}]*Product[Gamma[bb[[k]]],
   {k, m1 + 1, q}])/
 (Product[Gamma[1 - bb[[k]]], {k, 1, m1}]*
  Product[Gamma[aa[[k]]], {k, 1, n1}]))*
((-1)^n*Pi^(m1 + 1)*
  Sum[(Product[Gamma[1 + aa[[k]] - bb[[i]]], {
     k, 1, n1}]/Product[Gamma[-aa[[k]] +
      bb[[i]]], {k, n1 + 1, p}])*
    Csc[Pi*bb[[i]]]^2*Product[Csc[Pi*
      (-bb[[k]] + bb[[i]])], {k, 1, i - 1}]*
    Product[Csc[Pi*(-bb[[k]] + bb[[i]])],
     {k, i + 1, m1}]*((-1)^(p - n1 - m1)*z)^
     (1 - bb[[i]])*
    HypergeometricPFQRegularized[Join[{1},
      Table[1 + aa[[r]] - bb[[i]], {r, 1,
        p}]], Join[{n + 2 - bb[[i]],
       2 - bb[[i]]}, Table[1 + bb[[r]] -
        bb[[i]], {r, 1, q}]], z],
   {i, 1, m1}] +
 Sum[(((n - j - 1)!*Product[Gamma[1 -
         bb[[k]] + n - j], {k, 1, m1}]*
      Product[Gamma[aa[[k]] - n + j],
       {k, 1, n1}])/(Product[Gamma[
         1 - aa[[k]] + n - j], {k, n1 + 1, p}]*
      Product[Gamma[bb[[k]] - n + j],
       {k, m1 + 1, q}]))*
```

```
(((-1)^(p - n1 - m1 - 1)*z)^j/j!),
{j, 0, n - 1}]/((-1)^(p - n1 - m1)*z)^n) +
Log[(-1)^(p - n1 - m1)*z]*n!*
Product[Gamma[bb[[k]]], {k, 1, q}]*
HypergeometricPFQRegularized[Join[{1},
Table[aa[[r]], {r, 1, p}]],
Join[{n + 1, 1}, Table[bb[[r]], {r, 1, q}]],
z])} /. {z -> Random[]*Exp[Pi*I*(ii/4)]},
{n, 0, 3}, {ii, 0, 7}]]
```

Representations through more general functions

Through hypergeometric functions

Involving $_pF_q$

06.14.26.0001.01

$$\psi(z) == (z-1)\,{}_3F_2(1, 1, 2-z; 2, 2; 1) - \gamma$$

Through Meijer *G*

Classical cases for the direct function itself

06.14.26.0002.01

$$\psi(z) == -\frac{1}{\Gamma(1-z)}\, G_{3,3}^{1,3}\!\left(-1 \,\middle|\, \begin{matrix} 0, 0, z-1 \\ 0, -1, -1 \end{matrix}\right) - \gamma$$

Through other functions

Involving some hypergeometric-type functions

06.14.26.0003.01

$$\psi(z) == \psi^{(0)}(z)$$

Representations through equivalent functions

With related functions

06.14.27.0001.01

$$\psi(z) == \frac{1}{\Gamma(z)} \frac{\partial \Gamma(z)}{\partial z}$$

06.14.27.0002.01

$$\psi(z) == \frac{\partial \log\Gamma(z)}{\partial z}$$

06.14.27.0003.01

$$\psi(z) == H_{z-1} - \gamma$$

Zeros

06.14.30.0001.01

$$\psi(z_k) = 0 \;/;\; 1.4 < z_0 < 1.5 \bigwedge -0.6 < z_1 < -0.5 \bigwedge -1.6 < z_2 < -1.5 \bigwedge -2.7 < z_3 < -2.6 \bigwedge -3.7 < z_4 < -3.6 \bigwedge -4.7 < z_5 < -4.6 \bigwedge -5.7 < z_6 < -5.6 \bigwedge -6.7 < z_7 < -6.6 \bigwedge \ldots \bigwedge k \in \mathbb{N}$$

History

– J. Stirling (1730)
– A.-M. Legendre (1809)
– S. Poisson (1811)
– C. F. Gauss (1810)
– M.A. Stern (1847) proved convergence of the Stirling series for digamma function

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