MoebiusMu

Notations

Traditional name

Möbius function

Traditional notation

$\mu(n)$

Mathematica StandardForm notation

`MoebiusMu[n]`

Primary definition

13.07.02.0001.01

$$\mu(1) == 1$$

13.07.02.0002.01

$$\mu(n) == 0 \;/;\; \exists_m \frac{n}{m^2} \in \mathbb{Z}$$

13.07.02.0003.01

$$\mu(n) == (-1)^k \;/;\; n == \prod_{j=1}^{k} p_j \bigwedge p_{j-1} < p_j \bigwedge p_j == \text{prime}(j)$$

13.07.02.0004.01

$$\mu(-n) == \mu(n) \;/;\; n \in \mathbb{N}^+$$

13.07.02.0005.01

$$\mu(n) == 0 \;/;\; \exists_{k \in \mathbb{Z}} \frac{n}{k^2} \in \mathbb{Z}$$

13.07.02.0006.01

$$\mu(n) == (-1)^k \;/;\; n = \prod_{j=1}^{k} p_j \bigwedge p_j \in \mathbb{P} \bigwedge p_j \neq p_i$$

For integer n, the Moebius μ function $\mu(n)$ is the function that equals 0 if n has repeated integer factors. Otherwise, if n is the product of k distinct primes, the Moebius μ function $\mu(n)$ equals $(-1)^k$.

So, the condition $\mu(n) \neq 0$ indicates that n is squarefree (its prime decomposition contains no repeated factors). It is evidently that $\mu(n) == -1$ for all prime numbers $n \in \mathbb{P}$ and for n equal to products of an odd number of different primes. If n equal to products of two or other even different single primes, $\mu(n) == 1$.

Examples: The numbers 4, 8, 9, 12, contain at least one square (for example, $12 == 3\times 2^2$); so $\mu(4) == \mu(8) == \mu(9) == \mu(12) == 0$. The numbers 2, 3, 5, 6, 7, are squarefree numbers (for example, $5 == \prod_{j=1}^{1} p_j == 5$ or $6 == \prod_{j=1}^{2} p_j == 2\times 3$); so $\mu(2) == \mu(3) == \mu(5) == \mu(7) == (-1)^1 == -1$ and $\mu(6) == (-1)^2 == 1$.

Specific values

Specialized values

13.07.03.0055.01

$$\mu(p) == -1 \;/;\; p \in \mathbb{P}$$

13.07.03.0001.01

$$\mu(m\,n) == \mu(n)\,\mu(m) \;/;\; \gcd(m, n) == 1$$

13.07.03.0002.01

$$\mu(m\,n) == 0 \;/;\; \gcd(m, n) > 1$$

13.07.03.0003.01

$$\mu(n) == (-1)^m\,(1 - \delta_{s,0}) \;/;\; s == \prod_{j=2}^{\lfloor \sqrt{n} \rfloor} \left(\frac{n}{j^2} - \left\lfloor \frac{n}{j^2} \right\rfloor \right) \bigwedge m == \sum_{j=1}^{n} \delta_{\frac{n}{p_j} - \left\lfloor \frac{n}{p_j} \right\rfloor, 0} \bigwedge p_j == \text{prime}(j)$$

Values at fixed points

13.07.03.0004.01

$$\mu(0) == 0$$

13.07.03.0005.01

$$\mu(1) == 1$$

13.07.03.0006.01

$$\mu(2) == -1$$

13.07.03.0007.01

$$\mu(3) == -1$$

13.07.03.0008.01

$$\mu(4) == 0$$

13.07.03.0009.01

$$\mu(5) == -1$$

13.07.03.0010.01

$$\mu(6) == 1$$

13.07.03.0011.01

$$\mu(7) == -1$$

13.07.03.0012.01

$$\mu(8) == 0$$

13.07.03.0013.01

$$\mu(9) == 0$$

13.07.03.0014.01

$$\mu(10) == 1$$

13.07.03.0015.01
$\mu(11) = -1$

13.07.03.0016.01
$\mu(12) = 0$

13.07.03.0017.01
$\mu(13) = -1$

13.07.03.0018.01
$\mu(14) = 1$

13.07.03.0019.01
$\mu(15) = 1$

13.07.03.0020.01
$\mu(16) = 0$

13.07.03.0021.01
$\mu(17) = -1$

13.07.03.0022.01
$\mu(18) = 0$

13.07.03.0023.01
$\mu(19) = -1$

13.07.03.0024.01
$\mu(20) = 0$

13.07.03.0025.01
$\mu(21) = 1$

13.07.03.0026.01
$\mu(22) = 1$

13.07.03.0027.01
$\mu(23) = -1$

13.07.03.0028.01
$\mu(24) = 0$

13.07.03.0029.01
$\mu(25) = 0$

13.07.03.0030.01
$\mu(26) = 1$

13.07.03.0031.01
$\mu(27) = 0$

13.07.03.0032.01
$\mu(28) = 0$

13.07.03.0033.01
$\mu(29) = -1$

13.07.03.0034.01
$\mu(30) = -1$

13.07.03.0035.01
$\mu(31) == -1$

13.07.03.0036.01
$\mu(32) == 0$

13.07.03.0037.01
$\mu(33) == 1$

13.07.03.0038.01
$\mu(34) == 1$

13.07.03.0039.01
$\mu(35) == 1$

13.07.03.0040.01
$\mu(36) == 0$

13.07.03.0041.01
$\mu(37) == -1$

13.07.03.0042.01
$\mu(38) == 1$

13.07.03.0043.01
$\mu(39) == 1$

13.07.03.0044.01
$\mu(40) == 0$

13.07.03.0045.01
$\mu(41) == -1$

13.07.03.0046.01
$\mu(42) == -1$

13.07.03.0047.01
$\mu(43) == -1$

13.07.03.0048.01
$\mu(44) == 0$

13.07.03.0049.01
$\mu(45) == 0$

13.07.03.0050.01
$\mu(46) == 1$

13.07.03.0051.01
$\mu(47) == -1$

13.07.03.0052.01
$\mu(48) == 0$

13.07.03.0053.01
$\mu(49) == 0$

13.07.03.0054.01
$\mu(50) == 0$

13.07.03.0056.01

$\mu(100) == 0$

13.07.03.0057.01

$\mu(1000) == 0$

13.07.03.0058.01

$\mu(10\,000) == 0$

13.07.03.0059.01

$\mu(5 + 20\, i) == -1$

13.07.03.0060.01

$\mu(-100) == 0$

General characteristics

Domain and analyticity

$\mu(n)$ is a nonanalytical function which is defined only for integers.

13.07.04.0001.01

$n \longrightarrow \mu(n) :: \mathbb{Z} \longrightarrow \{-1, 0, 1\}$

Symmetries and periodicities

Parity

$\mu(n)$ is an even function.

13.07.04.0002.01

$\mu(-n) == \mu(n) \;/;\; n \in \mathbb{Z}$

Periodicity

No periodicity

Series representations

Other series representations

13.07.06.0002.01

$$\mu(n) == \sum_{k=1}^{n} \delta_{\gcd(n,k),1} \cos\left(\frac{2\pi k}{n}\right)$$

Transformations

Multiple arguments

13.07.16.0001.01

$\mu(m\, n) == \mu(n)\, \mu(m) \;/;\; \gcd(m, n) == 1$

13.07.16.0002.01

$\mu(m\,n) = 0 \;/;\; \gcd(m, n) > 1$

Products, sums, and powers of the direct function

Products of the direct function

13.07.16.0003.01

$\mu(n)\,\mu(m) = \mu(m\,n) \;/;\; \gcd(m, n) = 1$

Summation

Finite summation

13.07.23.0001.01

$$\sum_{k=1}^{x} \mu(k) \left\lfloor \frac{x}{k} \right\rfloor = 1 \;/;\; x \geq 1$$

13.07.23.0002.01

$$\sum_{d_j | n} \mu(d_j) = \delta_{n,1} \;/;\; n \in \mathbb{N} \land \land d_j \in \text{divisors}(n)$$

13.07.23.0003.01

$$\sum_{d_j | n} \mu(d_j)^2 = 2^k \;/;\; \text{factors}(n) = \{\{p_1, n_1\}, \ldots, \{p_k, n_k\}\} \land n_j \in \mathbb{N}^+ \land p_j \in \mathbb{P} \land d_j \in \text{divisors}(n)$$

Infinite summation

13.07.23.0004.01

$$\sum_{k=1}^{\infty} \frac{\mu(k)}{k} = 0$$

13.07.23.0008.01

$$\sum_{k=1}^{\infty} \frac{\mu(k)}{k^2} = \frac{6}{\pi^2}$$

13.07.23.0005.01

$$\sum_{k=1}^{\infty} \frac{\mu(k)}{k^s} = \frac{1}{\zeta(s)} \;/;\; \text{Re}(s) > 1$$

13.07.23.0006.01

$$\sum_{k=1}^{\infty} \mu(k\,n)\,k^s = \frac{\mu(n)\,n^{-s}}{\zeta(-s)\,(n^{-s} - 1)}$$

13.07.23.0007.01

$$\sum_{k=1}^{\infty} \frac{\mu(k)\log(k)}{k} = -1$$

13.07.23.0009.01

$$\sum_{k=1}^{\infty} \frac{|\mu(k)|}{k^s} = \frac{15}{\pi^2}$$

Ed Pegg

13.07.23.0010.01

$$\sum_{k=1}^{n} \frac{\delta_{\mu(k)-1}}{k^2} = \frac{21}{2\,\pi^2}$$

Ed Pegg

13.07.23.0011.01

$$\sum_{k=1}^{\infty} \frac{\delta_{\mu(k)-1}}{k^s} = \frac{\zeta(s)}{2\,\zeta(2\,s)} + \frac{1}{2\,\zeta(s)}$$

Ed Pegg

13.07.23.0012.01

$$\sum_{k=1}^{n} \frac{\delta_{\mu(k)+1}}{k^2} = \frac{9}{2\,\pi^2}$$

Ed Pegg

13.07.23.0013.01

$$\sum_{k=1}^{\infty} \frac{\delta_{\mu(k)+1}}{k^s} = \frac{\zeta(s)}{2\,\zeta(2\,s)} - \frac{1}{2\,\zeta(s)}$$

Ed Pegg

Products

Infinite products

13.07.24.0001.01

$$\prod_{k=1}^{\infty} k^{-\frac{\mu(k)}{k}} = e$$

13.07.24.0002.01

$$\prod_{k=1}^{\infty} \left(1-q^k\right)^{\frac{\sum_{d|k} \mu(d_j)\, r^{\frac{k}{d_j}}}{k}} = 1 - r\,q \;/;\; r \in \mathbb{N} \land d_j \in \text{divisors}(k)$$

Witt formula

S.-J. Kang, M.-H. Kim: Free Lie Algebras, Generalized Witt Formula, and the Denominator Identity Journal of Algebra 183, 560-594 (1996)

Representations through equivalent functions

With related functions

13.07.27.0001.01

$\mu(n) = \delta_{n_1,\ldots,n_m,1} (-1)^m \text{ /; } \text{factors}(n) = \{\{p_1, n_1\}, \{p_2, n_2\}, \ldots, \{p_m, n_m\}\} \bigwedge p_m \in \mathbb{P} \bigwedge n \in \mathbb{N}^+$

Zeros

13.07.30.0001.01

$\mu(m\, n) = 0 \text{ /; } \gcd(m, n) > 1$

Theorems

The Möbius inversion theorem

$$F(n) = \sum_{d|n} f(d) \Longleftrightarrow f(n) = \sum_{d|n} F(d)\, \mu\!\left(\frac{n}{d}\right)$$

or

$$F(x) = \sum_{k=1}^{\infty} f(k^\alpha\, x),\ \alpha \in \mathbb{R},\ \alpha \neq 0,\ x \in \mathbb{R} \Longleftrightarrow f(x) = \sum_{k=1}^{\infty} F(k^\alpha\, x)\, \mu(k).$$

Generalized Mertens conjecture

For $\varepsilon > 0$ and sufficiently large n: $\left|\sum_{k=1}^{n} \mu(k)\right| \leq n^{\frac{1}{2}+\varepsilon}$.

History

– C. F. Gauss in his Disquisitiones Arithmeticae (1801)
– A. F. Möbius (1831)
– F. Mertens (1874) introduced the notation $\mu(n)$ and the name "Möbius function"
– P. L. Chebyshev (1851) and R. Dedekind (1857) found the Möbius inversion theorem

Copyright