LegendreQGeneral

Notations

Traditional name

Legendre function of the second kind

Traditional notation

$Q_\nu(z)$

Mathematica StandardForm notation

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LegendreQ[ν, z]
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Primary definition

07.10.02.0001.01

$$Q_\nu(z) = Q_\nu^0(z)$$

Specific values

Specialized values

For fixed ν

07.10.03.0001.01

$$Q_\nu(0) = -\frac{\pi^{3/2}\tan\left(\frac{\pi\nu}{2}\right)}{\nu\,\Gamma\left(\frac{1-\nu}{2}\right)\Gamma\left(\frac{\nu}{2}\right)}$$

07.10.03.0002.01

$$Q_\nu(1) = \tilde{\infty}$$

07.10.03.0003.01

$$Q_\nu(-1) = \tilde{\infty}$$

For fixed z

07.10.03.0004.01

$$Q_{-\frac{1}{2}}(z) = K\left(\frac{z+1}{2}\right) /;\ z \notin (1, \infty)$$

07.10.03.0005.01

$$Q_{\frac{1}{2}}(z) = K\left(\frac{z+1}{2}\right) - 2\,E\left(\frac{z+1}{2}\right) /;\ z \notin (1, \infty)$$

07.10.03.0006.01

$$Q_0(z) = \frac{1}{2}(\log(1+z) - \log(1-z))$$

07.10.03.0007.01

$$Q_1(z) = \frac{z}{2}(\log(z+1) - \log(1-z)) - 1$$

07.10.03.0008.01

$$Q_2(z) = \frac{1}{4}\left(\left(3z^2-1\right)(\log(z+1) - \log(1-z)) - 6z\right)$$

07.10.03.0009.01

$$Q_3(z) = \frac{1}{12}\left(8 - 30z^2 - 3z\left(5z^2-3\right)(\log(1-z) - \log(z+1))\right)$$

07.10.03.0010.01

$$Q_4(z) = \frac{1}{48}\left(-210z^3 + 110z + 3\left(35z^4 - 30z^2 + 3\right)(\log(z+1) - \log(1-z))\right)$$

07.10.03.0011.01

$$Q_5(z) = \frac{1}{240}\left(-2\left(945z^4 - 735z^2 + 64\right) - 15z\left(63z^4 - 70z^2 + 15\right)(\log(1-z) - \log(z+1))\right)$$

07.10.03.0012.01

$$Q_6(z) = \frac{1}{160}\left(5\left(21z^2\left(11z^4 - 15z^2 + 5\right) - 5\right)(\log(z+1) - \log(1-z)) - 14z\left(165z^4 - 170z^2 + 33\right)\right)$$

07.10.03.0013.01

$$Q_7(z) = \frac{1}{1120}\left(512 - 14\left(55\left(39z^2 - 50\right)z^2 + 849\right)z^2 - 35z\left(429z^6 - 693z^4 + 315z^2 - 35\right)(\log(1-z) - \log(z+1))\right)$$

07.10.03.0014.01

$$Q_8(z) = \frac{1}{8960}\left(-450\,450z^7 + 690\,690z^5 - 294\,910z^3 + 30\,318z + 35\left(6435z^8 - 12\,012z^6 + 6930z^4 - 1260z^2 + 35\right)(\log(z+1) - \log(1-z))\right)$$

07.10.03.0015.01

$$Q_9(z) = \frac{1}{80\,640}\left(-2\left(165\left(91z^2\left(255z^4 - 455z^2 + 249\right) - 3867\right)z^2 + 16\,384\right) - 315z\left(11\left(13z^2\left(85z^4 - 180z^2 + 126\right) - 420\right)z^2 + 315\right)(\log(1-z) - \log(z+1))\right)$$

07.10.03.0016.01

$$Q_{10}(z) = \frac{1}{161\,280}\left(315\left(11z^2\left(13\left(323z^6 - 765z^4 + 630z^2 - 210\right)z^2 + 315\right) - 63\right)(\log(z+1) - \log(1-z)) - 22z\left(39\left(7z^2\left(4845z^4 - 9860z^2 + 6594\right) - 11\,220\right)z^2 + 27\,985\right)\right)$$

07.10.03.0017.01

$$Q_n(z) = \frac{1}{2}((\log(1+z) - \log(1-z)))P_n(z) + \sum_{k=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \frac{2n-4k-1}{(2k+1)(n-k)} P_{n-2k-1}(z) \;/;\; n \in \mathbb{N}$$

07.10.03.0018.01

$$Q_n(z) = \left(\frac{1}{2}(\log(1+z) - \log(1-z)) - \psi(n+1)\right)\sum_{k=0}^{n} \frac{(-n)_k (n+1)_k}{k!^2}\left(\frac{1-z}{2}\right)^k + \sum_{k=0}^{n} \frac{(k+n)!\,\psi(k+1)}{k!^2 (n-k)!}\left(\frac{z-1}{2}\right)^k \;/;\; n \in \mathbb{N}$$

07.10.03.0019.01

$$Q_n(z) = \left(\frac{1}{2}(\log(1+z) - \log(1-z)) - \psi(n+1)\right) P_n(z) + \sum_{k=0}^{n} \frac{(k+n)!\,\psi(k+1)}{k!^2\,(n-k)!} \left(\frac{z-1}{2}\right)^k /;\, n \in \mathbb{N}$$

07.10.03.0020.01

$$Q_n(z) = (-1)^n \left(\frac{1}{2}(\log(1+z) - \log(1-z)) + \psi(n+1)\right) \sum_{k=0}^{n} \frac{(-n)_k\,(n+1)_k}{k!^2} \left(\frac{z+1}{2}\right)^k - (-1)^n \sum_{k=0}^{n} \frac{(k+n)!\,\psi(k+1)}{k!^2\,(n-k)!} \left(-\frac{z+1}{2}\right)^k /;\, n \in \mathbb{N}$$

07.10.03.0021.01

$$Q_n(z) = 2^{-n-1}\,(z+1)^n \sum_{k=0}^{n} \frac{(-n)_k^{\,2}}{k!^2} (\log(1+z) - \log(1-z) - 2\,\psi(n-k+1) + 2\,\psi(k+1)) \left(\frac{z-1}{z+1}\right)^k /;\, n \in \mathbb{N}$$

07.10.03.0022.01

$$Q_{-n}(z) = \tilde{\infty} /;\, n \in \mathbb{N}^+$$

General characteristics

Domain and analyticity

$Q_\nu(z)$ is an analytical function of ν and z which is defined over $\mathbb{C}^2$.

07.10.04.0001.01

$$(\nu * z) \longrightarrow Q_\nu(z) :: (\mathbb{C} \otimes \mathbb{C}) \longrightarrow \mathbb{C}$$

Symmetries and periodicities

Mirror symmetry

07.10.04.0002.01

$$Q_{\bar{\nu}}(\bar{z}) = \overline{Q_\nu(z)} /;\, z \notin (-\infty, -1) \bigwedge z \notin (1, \infty)$$

Periodicity

No periodicity

Poles and essential singularities

With respect to z

For fixed ν, the function $Q_\nu(z)$ does not have poles and essential singularities.

07.10.04.0003.01

$$\mathcal{S}ing_z(Q_\nu(z)) = \{\}$$

With respect to ν

For fixed z, the function $Q_\nu(z)$ has an infinite set of singular points with respect to ν:

a) the points $\nu = -k /;\, k \in \mathbb{N}^+$, are the simple poles with residues $P_{k-1}(z)$;

b) $\nu = \tilde{\infty}$ is the point of convergence of poles, which is an essential singular point.

07.10.04.0004.01

$$\mathcal{Sing}_{\nu}(Q_{\nu}(z)) = \{\{\{-k, 1\} /; k \in \mathbb{N}^+\}, \{\tilde{\infty}, \infty\}\}$$

07.10.04.0005.01

$$\operatorname{res}_{\nu}(Q_{\nu}(z))(-k) = P_{k-1}(z) /; k \in \mathbb{N}^+$$

Branch points

With respect to *z*

For fixed ν, the function $Q_{\nu}(z)$ has three singular branch points: $z = -1$, $z = 1$ and $z = \tilde{\infty}$.

07.10.04.0006.01

$$\mathcal{BP}_z(Q_{\nu}(z)) = \{-1, 1, \tilde{\infty}\}$$

07.10.04.0007.01

$$\mathcal{R}_z(Q_{\nu}(z), -1) = \log$$

07.10.04.0008.01

$$\mathcal{R}_z(Q_{\nu}(z), 1) = \log$$

07.10.04.0009.01

$$\mathcal{R}_z(Q_{\nu}(z), \tilde{\infty}) = \log$$

With respect to *ν*

For fixed z, the function $Q_{\nu}(z)$ does not have branch points.

07.10.04.0010.01

$$\mathcal{BP}_{\nu}(Q_{\nu}(z)) = \{\}$$

Branch cuts

With respect to *z*

For fixed ν, the function $Q_{\nu}(z)$ is a single-valued function on the z-plane cut along the intervals $(-\infty, -1)$ and $(1, \infty)$. The function $Q_{\nu}(z)$ is continuous from above on the interval $(-\infty, -1]$ and from below on the interval $[1, \infty)$.

07.10.04.0011.01

$$\mathcal{BC}_z(Q_{\nu}(z)) = \{\{(-\infty, -1), -i\}, \{(1, \infty), i\}\}$$

07.10.04.0012.01

$$\lim_{\epsilon \to +0} Q_{\nu}(x + i\,\epsilon) = Q_{\nu}(x) /; x < -1$$

07.10.04.0013.01

$$\lim_{\epsilon \to +0} Q_{\nu}(x - i\,\epsilon) = -2\,i\,\pi\cos(\pi\,\nu)\,P_{\nu}(-x) + i\,\pi\,P_{\nu}(x) + Q_{\nu}(x) + 2\,i\,Q_{\nu}(-x)\sin(\pi\,\nu) /; x < -1$$

07.10.04.0014.01

$$\lim_{\epsilon \to +0} Q_{\nu}(x + i\,\epsilon) = Q_{\nu}(x) + i\,\pi\,P_{\nu}(x) /; x > 1$$

07.10.04.0015.01

$$\lim_{\epsilon \to +0} Q_{\nu}(x - i\,\epsilon) = Q_{\nu}(x) /; x > 1$$

With respect to *ν*

For fixed z, the function $Q_\nu(z)$ does not have branch cuts.

07.10.04.0016.01

$$\mathcal{BC}_\nu(Q_\nu(z)) = \{\}$$

Series representations

Generalized power series

Expansions at $z = 0$

07.10.06.0001.01

$$Q_\nu(z) = \frac{\sin(\pi\nu)\,\psi(\nu+1)}{2\pi^{3/2}} \sum_{j=0}^{\infty} \frac{(-1)^j\, 2^j}{j!}\, \Gamma\left(\frac{j-\nu}{2}\right)\Gamma\left(\frac{j+\nu+1}{2}\right) z^j + \sum_{j=0}^{\infty} \frac{(-\nu)_j\,(\nu+1)_j}{j!} \sum_{k=0}^{\infty} \frac{(-1)^j\,(j-\nu)_k\,(j+\nu+1)_k\,\psi(j+k+1)\,z^j}{(j+k)!\,k!\,2^{j+k}} + \sum_{m=0}^{\infty} \frac{\left(\frac{1}{2}\right)_m (1)_m}{m!\left(\frac{3}{2}\right)_m} \sum_{j=0}^{\infty} \frac{(-\nu)_j\,(\nu+1)_j}{j!} \sum_{k=0}^{\infty} \frac{(-1)^j\,(j-\nu)_k\,(j+\nu+1)_k}{(j+k)!\,k!\,2^{j+k}}\, z^{j+2m+1} \;/;\; |z| < 1$$

07.10.06.0002.01

$$Q_\nu(z) = -\psi(\nu+1)\,\tilde{F}_{1\times0\times1}^{2\times0\times1}\left(\begin{matrix}-\nu,\,\nu+1;;\,1;\\ 1;;\,1;\end{matrix}\;\frac{1}{2},\,-\frac{z}{2}\right) - \frac{\sin(\pi\nu)\,z}{2\pi} \sum_{j=0}^{\infty} \frac{1+(-1)^j}{2^j}\,\Gamma(j-\nu)\,\Gamma(j+\nu+1)\,\tilde{F}_{1\times0\times3}^{2\times0\times1}\left(\begin{matrix}j-\nu,\,j+\nu+1;;\,j+2;\\ j+1;;\,1,\,j+1,\,j+2;\end{matrix}\;\frac{1}{2},\,-\frac{z}{2}\right) z^j + \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} \frac{(-1)^j\,(-\nu)_{j+k}\,(\nu+1)_{j+k}\,\psi(j+k+1)\,z^j}{(j+k)!\,k!\,j!\,2^{j+k}} \;/;\; |z| < 1$$

07.10.06.0003.01

$$Q_\nu(z) \propto -\frac{\pi^{3/2}\tan\left(\frac{\pi\nu}{2}\right)}{\nu\,\Gamma\left(\frac{1}{2}-\frac{\nu}{2}\right)\Gamma\left(\frac{\nu}{2}\right)} + \left(\frac{\sin(\pi\nu)}{2\pi^{3/2}}\left(\Gamma\left(\frac{\nu+1}{2}\right)\left(-\Gamma\left(-\frac{\nu}{2}\right)\right) - \nu\,\Gamma\left(\frac{1-\nu}{2}\right)\Gamma\left(\frac{\nu}{2}\right)\psi(\nu+1)\right) + \nu\,(\nu+1)\sum_{k=0}^{\infty}\frac{(1-\nu)_k\,(\nu+2)_k\,\psi(k+2)}{(k+1)!\,k!\,2^{k+1}}\right) z + O(z^2) \;/;\; (z\to 0)$$

07.10.06.0004.01

$$Q_\nu(z) \propto -\frac{\pi^{3/2}\tan\left(\frac{\pi\nu}{2}\right)}{\nu\,\Gamma\left(\frac{1}{2}-\frac{\nu}{2}\right)\Gamma\left(\frac{\nu}{2}\right)}\,(1+O(z)) \;/;\; (z\to 0)$$

Expansions at $z = 1$

07.10.06.0005.01

$$Q_\nu(z) = \left(\frac{1}{2}(\log(1+z)-\log(1-z)) - \psi(\nu+1)\right)\left(1 + \frac{\nu\,(\nu+1)}{2}(z-1) - \frac{(1-\nu)\,\nu\,(\nu+1)\,(\nu+2)}{16}(z-1)^2 + \ldots\right) - \gamma + \frac{\nu\,(\nu+1)}{2}(1-\gamma)(z-1) - \frac{(1-\nu)\,\nu\,(\nu+1)\,(\nu+2)}{16}\left(\frac{3}{2}-\gamma\right)(z-1)^2 + \ldots \;/;\; \left|\frac{1-z}{2}\right| < 1 \bigwedge \nu \notin \mathbb{Z}$$

07.10.06.0006.01

$$Q_\nu(z) = \left(\frac{1}{2}(\log(1+z)-\log(1-z))-\psi(\nu+1)\right)\sum_{k=0}^{\infty}\frac{(-\nu)_k\,(\nu+1)_k}{k!^2}\left(\frac{1-z}{2}\right)^k+\sum_{k=0}^{\infty}\frac{(-\nu)_k\,(\nu+1)_k\,\psi(k+1)}{k!^2}\left(\frac{1-z}{2}\right)^k\;/;\;\left|\frac{1-z}{2}\right|<1\bigwedge\nu\notin\mathbb{Z}$$

07.10.06.0007.01

$$Q_\nu(z) = \left(\frac{1}{2}(\log(1+z)-\log(1-z))-\psi(\nu+1)\right){}_2F_1\!\left(-\nu,\,\nu+1;\,1;\,\frac{1-z}{2}\right)+\sum_{k=0}^{\infty}\frac{(-\nu)_k\,(\nu+1)_k\,\psi(k+1)}{k!^2}\left(\frac{1-z}{2}\right)^k\;/;\;\left|\frac{1-z}{2}\right|<1\bigwedge\nu\notin\mathbb{Z}$$

07.10.06.0008.01

$$Q_\nu(z)\propto\left(\frac{1}{2}(\log(2)-\log(1-z))-\psi(\nu+1)-\gamma\right)(1+O(z-1))\;/;\;(z\to1)\bigwedge\nu\notin\mathbb{Z}$$

07.10.06.0009.01

$$Q_n(z) = \sum_{k=0}^{n}\frac{(k+n)!\,\psi(k+1)}{k!^2\,(n-k)!}\left(\frac{z-1}{2}\right)^k+\left(\frac{1}{2}(\log(1+z)-\log(1-z))-\psi(n+1)\right)\sum_{k=0}^{\infty}\frac{(-n)_k\,(n+1)_k}{k!^2}\left(\frac{1-z}{2}\right)^k\;/;\;n\in\mathbb{N}$$

07.10.06.0010.01

$$Q_n(z) = \frac{1}{2}((\log(1+z)-\log(1-z)))\,P_n(z)+\sum_{k=0}^{\left\lfloor\frac{n-1}{2}\right\rfloor}\frac{2n-4k-1}{(2k+1)(n-k)}\,P_{n-2k-1}(z)\;/;\;n\in\mathbb{N}$$

Expansions at $z = -1$

07.10.06.0011.01

$$Q_\nu(z) = \left(\frac{\cos(\pi\nu)}{2}(2\psi(-\nu)-\pi\cot(\pi\nu)-\log(1-z)+\log(1+z))-\frac{\pi}{2}\csc(\pi\nu)\right)\left(1-\frac{\nu(1+\nu)}{2}(z+1)-\frac{(1-\nu)\,\nu\,(1+\nu)\,(2+\nu)}{16}(z+1)^2+\ldots\right)-\cos(\pi\nu)\left(-\gamma-\frac{\nu(1+\nu)}{2}(1-\gamma)(z+1)-\frac{(1-\nu)\,\nu\,(1+\nu)\,(2+\nu)}{16}\left(\frac{3}{2}-\gamma\right)(z+1)^2+\ldots\right)\;/;\;\left|\frac{z+1}{2}\right|<1\bigwedge\nu\notin\mathbb{Z}$$

07.10.06.0012.01

$$Q_\nu(z) = \left(\frac{\cos(\pi\nu)}{2}(2\psi(-\nu)-\pi\cot(\pi\nu)-\log(1-z)+\log(1+z))-\frac{\pi}{2}\csc(\pi\nu)\right)\sum_{k=0}^{\infty}\frac{(-\nu)_k\,(\nu+1)_k}{k!^2}\left(\frac{z+1}{2}\right)^k-\cos(\pi\nu)\sum_{k=0}^{\infty}\frac{(-\nu)_k\,(\nu+1)_k\,\psi(k+1)}{k!^2}\left(\frac{z+1}{2}\right)^k\;/;\;\left|\frac{z+1}{2}\right|<1\bigwedge\nu\notin\mathbb{Z}$$

07.10.06.0013.01

$$Q_\nu(z)\propto\frac{1}{2}(\cos(\pi\nu)\log(z+1)-\pi\csc(\pi\nu)+\cos(\pi\nu)(-\pi\cot(\pi\nu)-\log(2)+2\psi(-\nu)+2\gamma))(1+O(z+1))\;/;\;(z\to-1)\bigwedge\nu\notin\mathbb{Z}$$

07.10.06.0014.01

$$Q_n(z) = (-1)^n\left(\frac{1}{2}(\log(1+z)-\log(1-z))+\psi(n+1)\right)\sum_{k=0}^{n}\frac{(-n)_k\,(n+1)_k}{k!^2}\left(\frac{z+1}{2}\right)^k-(-1)^n\sum_{k=0}^{n}\frac{(k+n)!\,\psi(k+1)}{k!^2\,(n-k)!}\left(-\frac{z+1}{2}\right)^k\;/;\;n\in\mathbb{N}$$

07.10.06.0015.01

$$Q_n(z) \propto (-1)^n \left(\frac{1}{2}(\log(z+1) - \log(2)) + \psi(n+1) + \gamma\right)(1 + O(z+1)) \;/;\; (z \to -1) \land n \in \mathbb{N}$$

Expansions at $z == \infty$

07.10.06.0016.01

$$Q_\nu(z) == \frac{2^{-\nu-2}}{\sqrt{\pi}} (z-1)^{-\nu-1} \left(\frac{2^{2\nu+1}\Gamma\left(\nu+\frac{1}{2}\right)}{\Gamma(\nu+1)} (z-1)^{2\nu+1} (\log(1+z) - \log(-z-1)) \left(1 - \frac{\nu}{1-z} - \frac{(1-\nu)^2\,\nu}{(1-2\nu)(1-z)^2} + \ldots\right) + \frac{\Gamma\left(-\nu-\frac{1}{2}\right)}{\Gamma(-\nu)} (2\pi\cot(\pi\nu) - \log(-z-1) + \log(1+z)) \left(1 + \frac{1+\nu}{1-z} + \frac{(1+\nu)(2+\nu)^2}{(3+2\nu)(1-z)^2} + \ldots\right)\right) \;/;\; |1-z| > 2 \bigwedge \nu \notin \mathbb{Z}$$

07.10.06.0017.01

$$Q_\nu(z) == \frac{2^{-\nu-2}}{\sqrt{\pi}} (z-1)^{-\nu-1} \left(\frac{2^{2\nu+1}\Gamma\left(\nu+\frac{1}{2}\right)}{\Gamma(\nu+1)} (z-1)^{2\nu+1} (\log(1+z) - \log(-z-1)) \sum_{k=0}^{\infty} \frac{(-\nu)_k^{\,2}}{k!\,(-2\nu)_k} \left(\frac{2}{1-z}\right)^k + \frac{\Gamma\left(-\nu-\frac{1}{2}\right)}{\Gamma(-\nu)} (2\pi\cot(\pi\nu) - \log(-z-1) + \log(1+z)) \sum_{k=0}^{\infty} \frac{(\nu+1)_k^{\,2}}{k!\,(2\nu+2)_k} \left(\frac{2}{1-z}\right)^k \right) \;/;\; |1-z| > 2 \bigwedge \nu \notin \mathbb{Z}$$

07.10.06.0018.01

$$Q_\nu(z) == \frac{2^{-\nu-2}}{\sqrt{\pi}} (z-1)^{-\nu-1} \left(\frac{2^{2\nu+1}\Gamma\left(\nu+\frac{1}{2}\right)}{\Gamma(\nu+1)} (\log(1+z) - \log(-z-1)) (z-1)^{2\nu+1} \, {}_2F_1\left(-\nu, -\nu; -2\nu; \frac{2}{1-z}\right) + \frac{\Gamma\left(-\nu-\frac{1}{2}\right)}{\Gamma(-\nu)} (2\pi\cot(\pi\nu) - \log(-z-1) + \log(1+z)) \, {}_2F_1\left(\nu+1, \nu+1; 2\nu+2; \frac{2}{1-z}\right)\right) \;/;\; z \notin (-1, 1) \bigwedge \nu \notin \mathbb{Z}$$

07.10.06.0019.01

$$Q_\nu(z) \propto \frac{2^{-\nu-2}}{\sqrt{\pi}} \left(\frac{2^{2\nu+1}\Gamma\left(\nu+\frac{1}{2}\right)}{\Gamma(\nu+1)} z^\nu (\log(1+z) - \log(-z-1)) \left(1 + O\left(\frac{1}{z}\right)\right) + \frac{\Gamma\left(-\nu-\frac{1}{2}\right)}{\Gamma(-\nu)} z^{-\nu-1} (2\pi\cot(\pi\nu) - \log(-z-1) + \log(1+z)) \left(1 + O\left(\frac{1}{z}\right)\right)\right) \;/;\; (|z| \to \infty) \bigwedge \nu \notin \mathbb{Z}$$

07.10.06.0020.01

$$Q_n(z) == \frac{(-1)^{n-1}\, 2^n\, n!^2}{(2n+1)!} (1-z)^{-n-1} \sum_{k=0}^{\infty} \frac{(n+1)_k^{\,2}}{k!\,(2n+2)_k} \left(\frac{2}{1-z}\right)^k + 2^{-n-1} (z-1)^n (\log(1+z) - \log(-z-1)) \sum_{k=0}^{n} \frac{(2n-k)!}{k!\,(n-k)!^2} \left(\frac{2}{z-1}\right)^k \;/;\; n \in \mathbb{N}$$

07.10.06.0021.01

$$Q_n(z) \propto \frac{(-1)^{n-1}\, 2^n\, n!^2}{(2n+1)!} (-z)^{-n-1} \left(1 + O\left(\frac{1}{z}\right)\right) + \frac{2^{-n-1}\,(2n)!}{n!^2} z^n (\log(1+z) - \log(-z-1)) \left(1 + O\left(\frac{1}{z}\right)\right) \;/;\; (|z| \to \infty) \land n \in \mathbb{N}$$

Integral representations

On the real axis

Of the direct function

07.10.07.0001.01

$$Q_n(x) = \frac{1}{2^{n+1}\, n!} \int_{-1}^{1} \frac{1}{x-t} \frac{\partial^n (t^2-1)^n}{\partial t^n} \, dt \;/;\; n \in \mathbb{N}$$

07.10.07.0002.01

$$Q_\nu(z) = \int_0^\infty \left(z + \sqrt{z^2-1} \cosh(t)\right)^{-\nu-1} dt + \frac{1}{2} (\log(z+1) - \log(-z-1)) P_\nu(z) \;/;\; \mathrm{Re}(z) > 1 \land \mathrm{Re}(\nu) > -1$$

07.10.07.0003.01

$$Q_n(x) = -\frac{1}{2} \mathcal{P} \int_{-1}^{1} \frac{P_n(t)}{t-x} \, dt \;/;\; -1 < x < 1 \land n \in \mathbb{N}$$

Integral representations of negative integer order

07.10.07.0004.01

$$Q_n\left(\frac{z}{r}\right) = \frac{(-1)^n r^{n+1}}{n!} \frac{\partial^n \frac{\log(r+z)-\log(r-z)}{2r}}{\partial z^n} \;/;\; r = \sqrt{x^2+y^2+z^2} \bigwedge n \in \mathbb{N}$$

Differential equations

Ordinary linear differential equations and wronskians

For the direct function itself

07.10.13.0001.01

$$(1-z^2)\, w''(z) - 2 z\, w'(z) + (\nu+1)\, \nu\, w(z) = 0 \;/;\; w(z) = c_1 P_\nu(z) + c_2 Q_\nu(z)$$

07.10.13.0002.02

$$W_z(P_\nu(z), Q_\nu(z)) = \frac{1}{1-z^2}$$

07.10.13.0003.01

$$g'(z)\, w''(z) - \left(\frac{2\, g(z)\, g'(z)^2}{1-g(z)^2} + g''(z)\right) w'(z) + \frac{\nu(\nu+1)\, g'(z)^3}{1-g(z)^2} w(z) = 0 \;/;\; w(z) = c_1 P_\nu(g(z)) + c_2 Q_\nu(g(z))$$

07.10.13.0004.01

$$W_z(P_\nu(g(z)), Q_\nu(g(z))) = \frac{g'(z)}{1-g(z)^2}$$

07.10.13.0005.01

$$g'(z)\, h(z)^2\, w''(z) - \left(\left(\frac{2\, g(z)\, g'(z)^2}{1-g(z)^2} + g''(z)\right) h(z)^2 + 2\, g'(z)\, h'(z)\, h(z)\right) w'(z) + \left(\frac{\nu\,(\nu+1)\, h(z)^2\, g'(z)^3}{1-g(z)^2} + 2\, h'(z)^2\, g'(z) + h(z)\left(h'(z)\left(\frac{2\, g(z)\, g'(z)^2}{1-g(z)^2} + g''(z)\right) - g'(z)\, h''(z)\right)\right) w(z) = 0 \;/;\; w(z) = c_1\, h(z)\, P_\nu(g(z)) + c_2\, h(z)\, Q_\nu(g(z))$$

07.10.13.0006.01

$$W_z\big(h(z)\, P_\nu(g(z)),\, h(z)\, Q_\nu(g(z))\big) == \frac{h(z)^2\, g'(z)}{1-g(z)^2}$$

07.10.13.0007.01

$$z^2\, w''(z) - z\left(2\, s + \frac{r\left(a^2\, z^{2r}+1\right)}{1-a^2\, z^{2r}} - 1\right) w'(z) + \left(-\frac{a^2\, \nu\,(\nu+1)\, r^2\left(a^2\, z^{2r}-1\right) z^{2r}}{\left(1-a^2\, z^{2r}\right)^2} + s^2 + \frac{r\, s\left(a^2\, z^{2r}+1\right)}{1-a^2\, z^{2r}}\right) w(z) = 0 \;/;\; w(z) = c_1\, z^s\, P_\nu(a\, z^r) + c_2\, z^s\, Q_\nu(a\, z^r)$$

07.10.13.0008.01

$$W_z\big(z^s\, P_\nu(a\, z^r),\, z^s\, Q_\nu(a\, z^r)\big) == \frac{a\, r\, z^{r+2s-1}}{1-a^2\, z^{2r}}$$

07.10.13.0009.01

$$w''(z) - \frac{a^2\,(\log(r) - 2\log(s))\, r^{2z} + \log(r) + 2\log(s)}{1-a^2\, r^{2z}}\, w'(z) + \left(\frac{a^2\, \nu\,(\nu+1)\log^2(r)\, r^{2z}}{1-a^2\, r^{2z}} + \log^2(s) + \frac{\left(a^2\, r^{2z}+1\right)\log(r)\log(s)}{1-a^2\, r^{2z}}\right) w(z) = 0 \;/;\; w(z) = c_1\, s^z\, P_\nu(a\, r^z) + c_2\, s^z\, Q_\nu(a\, r^z)$$

07.10.13.0010.01

$$W_z\big(s^z\, P_\nu(a\, r^z),\, s^z\, Q_\nu(a\, r^z)\big) == \frac{a\, r^z\, s^{2z}\log(r)}{1-a^2\, r^{2z}}$$

Identities

Recurrence identities

Consecutive neighbors

07.10.17.0001.01

$$Q_\nu(z) == \frac{(2\,\nu+3)\, z}{\nu+1}\, Q_{\nu+1}(z) - \frac{\nu+2}{\nu+1}\, Q_{\nu+2}(z)$$

07.10.17.0002.01

$$Q_\nu(z) == \frac{(2\,\nu-1)\, z}{\nu}\, Q_{\nu-1}(z) - \frac{\nu-1}{\nu}\, Q_{\nu-2}(z)$$

Distant neighbors

07.10.17.0005.01

$$Q_\nu(z) = C_n(\nu, z)\, Q_{\nu+n}(z) - \frac{n+\nu+1}{n+\nu} C_{n-1}(\nu, z)\, Q_{\nu+n+1}(z) \;/;$$
$$C_0(\nu, z) = 1 \bigwedge C_1(\nu, z) = \frac{(2\nu+3)z}{\nu+1} \bigwedge C_n(\nu, z) = \frac{z(2n+2\nu+1)}{n+\nu} C_{n-1}(\nu, z) - \frac{n+\nu}{n+\nu-1} C_{n-2}(\nu, z) \bigwedge n \in \mathbb{N}^+$$

07.10.17.0006.01

$$Q_\nu(z) = C_n(\nu, z)\, Q_{\nu-n}(z) + \frac{n-\nu}{\nu-n+1} C_{n-1}(\nu, z)\, Q_{\nu-n-1}(z) \;/;$$
$$C_0(\nu, z) = 1 \bigwedge C_1(\nu, z) = \frac{(2\nu-1)z}{\nu} \bigwedge C_n(\nu, z) = \frac{z(2n-2\nu-1)}{n-\nu-1} C_{n-1}(\nu, z) - \frac{\nu-n+1}{\nu-n+2} C_{n-2}(\nu, z) \bigwedge n \in \mathbb{N}^+$$

Functional identities

Relations between contiguous functions

07.10.17.0003.01

$$\nu\, Q_{\nu-1}(z) + (\nu+1)\, Q_{\nu+1}(z) = (2\nu+1)\, z\, Q_\nu(z)$$

07.10.17.0004.01

$$Q_\nu(z) = \frac{1}{(2\nu+1)z} \left(\nu\, Q_{\nu-1}(z) + (\nu+1)\, Q_{\nu+1}(z)\right)$$

Differentiation

Low-order differentiation

With respect to ν

07.10.20.0001.01

$$\frac{\partial Q_\nu(z)}{\partial \nu} = \pi \cot(\pi\nu)\, Q_\nu(z) - \psi^{(1)}(\nu+1)\, P_\nu(z) + \sum_{k=0}^{\infty} \frac{(-\nu)_k\, (\nu+1)_k}{k!^2}$$
$$\left(\frac{1}{2}(\log(1+z) - \log(1-z)) + \psi(k+1) - \psi(\nu+1)\right)(\psi(k+\nu+1) - \psi(k-\nu)) \left(\frac{1-z}{2}\right)^k \;/;\; \left|\frac{1-z}{2}\right| < 1 \bigwedge \nu \notin \mathbb{Z}$$

07.10.20.0002.01

$$\frac{\partial^2 Q_\nu(z)}{\partial \nu^2} = -\pi^2\, Q_\nu(z) - \left(2\pi \cot(\pi\nu)\, \psi^{(1)}(\nu+1) + \psi^{(2)}(\nu+1)\right) P_\nu(z) +$$
$$\sum_{k=0}^{\infty} \frac{(-\nu)_k\, (\nu+1)_k}{k!^2} \left(\pi \cot(\pi\nu)\, (\psi(k+\nu+1) - \psi(k-\nu)) \left(\frac{1}{2}(\log(1+z) - \log(1-z)) + \psi(k+1) - \psi(\nu+1)\right) + \right.$$
$$(\psi(k+\nu+1) - \psi(k-\nu))\, (\psi(k+\nu+1) - \psi(\nu+1)) \left(\frac{1}{2}(\log(1+z) - \log(1-z)) + \psi(k+1) - \psi(\nu+1)\right) +$$
$$\left(\psi^{(1)}(k-\nu) + \psi^{(1)}(k+\nu+1)\right) \left(\frac{1}{2}(\log(1+z) - \log(1-z)) + \psi(k+1) - \psi(\nu+1)\right) +$$
$$\frac{1}{2}(\psi(k-\nu) - \psi(-\nu))\, (\log(1+z) - \log(1-z) + 2\psi(k+1) - 2\psi(\nu+1))\, (\psi(k-\nu) - \psi(k+\nu+1)) +$$
$$\left. 2\, (\psi(k-\nu) - \psi(k+\nu+1))\, \psi^{(1)}(\nu+1)\right) \left(\frac{1-z}{2}\right)^k \;/;\; \left|\frac{1-z}{2}\right| < 1 \bigwedge \nu \notin \mathbb{Z}$$

With respect to z

07.10.20.0003.01

$$\frac{\partial Q_\nu(z)}{\partial z} = \frac{\nu}{z^2-1}\left(z\,Q_\nu(z) - Q_{\nu-1}(z)\right)$$

07.10.20.0004.01

$$\frac{\partial^2 Q_\nu(z)}{\partial z^2} = \frac{\nu}{\left(z^2-1\right)^2}\left(2\,z\,Q_{\nu-1}(z) + \left((\nu-1)\,z^2-\nu-1\right)Q_\nu(z)\right)$$

Symbolic differentiation

With respect to ν

07.10.20.0005.01

$$\frac{\partial^m Q_\nu(z)}{\partial \nu^m} = \sum_{k=0}^{\infty}\frac{1}{k!^2}\left(\left(\frac{1}{2}(\log(1+z)-\log(1-z)) + \psi(k+1)\right)\sum_{j=0}^{m}\binom{m}{j}\frac{\partial^j(-\nu)_k}{\partial\nu^j}\,\frac{\partial^{m-j}(\nu+1)_k}{\partial\nu^{m-j}} - \sum_{q=0}^{m}\sum_{r=0}^{m}\sum_{s=0}^{m}\frac{\partial^q(-\nu)_k}{\partial\nu^q}\,\frac{\partial^r(\nu+1)_k}{\partial\nu^r}\,\delta_{q+r+s-m}\,(q+r+s;\,q,\,r,\,s)\,\psi^{(s)}(\nu+1)\right)\left(\frac{1-z}{2}\right)^k \;/;\; \left|\frac{1-z}{2}\right| < 1 \bigwedge \nu \notin \mathbb{Z} \bigwedge m \in \mathbb{N}$$

07.10.20.0007.01

$$\frac{\partial^m Q_\nu(z)}{\partial z^m} = \frac{\left(1-z^2\right)^{-\frac{m}{2}}\,\Gamma(m+\nu+1)}{\Gamma(1-m+\nu)}\,Q_\nu^{-m}(z) \;/;\; m \in \mathbb{N}$$

With respect to z

07.10.20.0008.01

$$\frac{\partial^m Q_\nu(z)}{\partial z^m} = (-1)^m\left(1-z^2\right)^{-\frac{m}{2}}\,Q_\nu^m(z) \;/;\; m \in \mathbb{N}$$

07.10.20.0009.01

$$\frac{\partial^m Q_\nu(z)}{\partial z^m} = \frac{\Gamma(\nu+m+1)}{\Gamma(\nu-m+1)}\left(1-z^2\right)^{-\frac{m}{2}}\,Q_\nu^{-m}(z) \;/;\; m \in \mathbb{N}$$

Fractional integro-differentiation

With respect to z

07.10.20.0006.01

$$\frac{\partial^\alpha Q_\nu(z)}{\partial z^\alpha} = -\psi(\nu+1)\,z^{-\alpha}\,\tilde{F}_{1\times0\times1}^{2\times0\times1}\left(\begin{matrix}-\nu,\ \nu+1;;\ 1; \\ 1;;\ 1-\alpha;\end{matrix}\ \frac{1}{2},\ -\frac{z}{2}\right) - \frac{z^{1-\alpha}\sin(\pi\,\nu)}{2\,\pi}\sum_{j=0}^{\infty}\frac{1+(-1)^j}{2^j}\,\Gamma(j-\nu)\,\Gamma(j+\nu+1)\,\tilde{F}_{1\times0\times3}^{2\times0\times1}\left(\begin{matrix}j-\nu,\ j+\nu+1;;\ j+2; \\ j+1;;\ 1,\ j+1,\ j-\alpha+2;\end{matrix}\ \frac{1}{2},\ -\frac{z}{2}\right)z^j + \sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^j\,(-\nu)_{j+k}\,(\nu+1)_{j+k}\,\psi(j+k+1)\,z^{j-\alpha}}{(j+k)!\,k!\,\Gamma(j-\alpha+1)\,2^{j+k}} \;/;\; |z| < 1$$

Summation

Infinite summation

07.10.23.0001.01

$$\sum_{n=0}^{\infty} Q_n(z)\, w^n = \frac{1}{\sqrt{w^2 - 2\, z\, w + 1}} \left(\cosh^{-1}\left(\frac{w - z}{\sqrt{z^2 - 1}} \right) + \frac{1}{2} (\log(1 + z) - \log(-z - 1)) \right) /;\ 0 < z < 1 \land |w| < 1$$

Representations through more general functions

Through hypergeometric functions

Involving $_2F_1$

07.10.26.0001.01

$$Q_\nu(z) = \left(\frac{1}{2} (\log(1 + z) - \log(1 - z)) - \psi(\nu + 1) \right) {}_2F_1\left(-\nu, \nu + 1; 1; \frac{1 - z}{2}\right) + \sum_{k=0}^{\infty} \frac{(-\nu)_k\, (\nu + 1)_k\, \psi(k + 1)}{k!^2} \left(\frac{1 - z}{2} \right)^k /;$$
$$\left| \frac{1 - z}{2} \right| < 1 \bigwedge \nu \notin \mathbb{Z}$$

07.10.26.0002.01

$$Q_n(z) = \left(\frac{1}{2} (\log(1 + z) - \log(1 - z)) - \psi(n + 1) \right) {}_2F_1\left(-n, n + 1; 1; \frac{1 - z}{2}\right) + \sum_{k=0}^{n} \frac{(k + n)!\, \psi(k + 1)}{k!^2\, (n - k)!} \left(\frac{z - 1}{2} \right)^k /;\ n \in \mathbb{N}$$

07.10.26.0003.01

$$Q_\nu(z) = \left(\frac{\cos(\pi\,\nu)}{2} (2\, \psi(-\nu) - \pi \cot(\pi\,\nu) - \log(1 - z) + \log(z + 1)) - \frac{\pi}{2} \csc(\pi\,\nu) \right) {}_2F_1\left(-\nu, \nu + 1; 1; \frac{z + 1}{2}\right) -$$
$$\cos(\pi\,\nu) \sum_{k=0}^{\infty} \frac{(-\nu)_k\, (\nu + 1)_k\, \psi(k + 1)}{k!^2} \left(\frac{z + 1}{2} \right)^k /;\ \left| \frac{z + 1}{2} \right| < 1 \bigwedge \nu \notin \mathbb{Z}$$

07.10.26.0004.01

$$Q_n(z) = (-1)^n \left(\frac{1}{2} (\log(1 + z) - \log(1 - z)) + \psi(n + 1) \right) {}_2F_1\left(-n, n + 1; 1; \frac{z + 1}{2}\right) - (-1)^n \sum_{k=0}^{n} \frac{(k + n)!\, \psi(k + 1)}{k!^2\, (n - k)!} \left(-\frac{z + 1}{2} \right)^k /;\ z \in \mathbb{N}$$

07.10.26.0005.01

$$Q_\nu(z) = \frac{2^{-\nu-2}}{\sqrt{\pi}} (z - 1)^{-\nu-1} \left(\frac{2^{2\,\nu+1}\, \Gamma\left(\nu + \frac{1}{2}\right)}{\Gamma(\nu + 1)} (z - 1)^{2\,\nu+1} (\log(1 + z) - \log(-z - 1))\, {}_2F_1\left(-\nu, -\nu; -2\,\nu; \frac{2}{1 - z}\right) + \right.$$
$$\left. \frac{\Gamma\left(-\nu - \frac{1}{2}\right)}{\Gamma(-\nu)} (2\,\pi \cot(\pi\,\nu) - \log(-z - 1) + \log(1 + z))\, {}_2F_1\left(\nu + 1, \nu + 1; 2\,\nu + 2; \frac{2}{1 - z}\right) \right) /;\ \nu \notin \mathbb{Z} \bigwedge z \notin (-1, 1)$$

07.10.26.0006.01

$$Q_n(z) = \frac{2^{-n-1}\, n!\, \Gamma\left(-n - \frac{1}{2}\right)}{\sqrt{\pi}} (1 - z)^{-n-1}\, {}_2F_1\left(n + 1, n + 1; 2\,n + 2; \frac{2}{1 - z}\right) + 2^{-n-1}\, (z - 1)^n\, (\log(1 + z) - \log(-z - 1))$$
$$\sum_{k=0}^{n} \frac{(2\,n - k)!}{k!\, (n - k)!^2} \left(\frac{2}{z - 1} \right)^k /;\ n \in \mathbb{N}$$

07.10.26.0011.01

$$Q_\nu(z) = \sqrt{\pi}\left(\frac{\cos\left(\frac{\pi\nu}{2}\right)\Gamma\left(\frac{\nu}{2}+1\right)}{\Gamma\left(\frac{\nu+1}{2}\right)} z\, {}_2F_1\left(\frac{1-\nu}{2}, \frac{\nu}{2}+1; \frac{3}{2}; z^2\right) - \frac{\sin\left(\frac{\pi\nu}{2}\right)\Gamma\left(\frac{\nu+1}{2}\right)}{2\,\Gamma\left(\frac{\nu}{2}+1\right)}\, {}_2F_1\left(-\frac{\nu}{2}, \frac{\nu+1}{2}; \frac{1}{2}; z^2\right)\right)$$

Through Meijer *G*

Classical cases for the direct function itself

07.10.26.0007.01

$$Q_\nu(z) = -\frac{\sin(\pi\nu)}{2\pi^2 i}\int_{\gamma-i\infty}^{i\infty+\gamma}\frac{\Gamma(s)\,\Gamma(\nu+1-s)\,\Gamma(-\nu-s)\,\psi(1-s)}{\Gamma(1-s)}\left(\frac{z-1}{2}\right)^{-s} ds - \frac{\sin(\pi\nu)}{2\pi}\left(\log(1+z)-\log(1-z)-2\,\psi(\nu+1)\right) G_{2,2}^{1,2}\!\left(\frac{z-1}{2}\,\middle|\,\begin{matrix}\nu+1, -\nu\\ 0, 0\end{matrix}\right) /;\ \gamma > 0 \bigwedge \gamma < -\mathrm{Re}(\nu) \bigwedge \gamma < \mathrm{Re}(\nu)+1$$

07.10.26.0008.01

$$Q_\nu(z) = \frac{1}{4}\left(2 - \frac{\tan(\pi\nu)}{\pi}\left(\log(-z-1)-\log(1+z)\right)\right) G_{2,2}^{1,2}\!\left(\frac{2}{z-1}\,\middle|\,\begin{matrix}1, 1\\ \nu+1, -\nu\end{matrix}\right) + \frac{\tan(\pi\nu)}{4\pi}\left(\log(-z-1)-\log(1+z)\right) G_{2,2}^{1,2}\!\left(\frac{2}{z-1}\,\middle|\,\begin{matrix}1, 1\\ -\nu, \nu+1\end{matrix}\right) /;\ \nu \notin \mathbb{Z}$$

Through other functions

Involving some hypergeometric-type functions

07.10.26.0009.01

$$Q_\nu(z) = Q_\nu^0(z)$$

07.10.26.0010.01

$$Q_\nu(z) = \mathfrak{Q}_\nu^0(z) + \frac{1}{2}\left(\log(1+z)-\log(-z-1)\right) P_\nu(z)$$

Involving spheroidal functions

07.10.26.0012.01

$$Q_\nu(z) = QS_{\nu,0}(0, z)$$

Representations through equivalent functions

With related functions

07.10.27.0001.01

$$\frac{Q_{\nu-\frac{1}{2}}(z)}{P_{\nu-\frac{1}{2}}(z)} - \frac{Q_{\nu+\frac{1}{2}}(z)}{P_{\nu+\frac{1}{2}}(z)} = \frac{1}{\left(\nu+\frac{1}{2}\right) P_{\nu-\frac{1}{2}}(z)\, P_{\nu+\frac{1}{2}}(z)}$$

History

– D. Bernoulli (1748)
– A. M. Legendre (1782)
– E. Heine (1842)
– L. Schläfli (1881)

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