LegendreP2

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Notations

Traditional name

Associated Legendre polynomial

Traditional notation

$P_n^\mu(z)$

Mathematica StandardForm notation

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LegendreP[n, μ, 2, z]
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Primary definition

05.07.02.0001.01

$$P_n^\mu(z) = \frac{(1+z)^{\mu/2}}{(1-z)^{\mu/2}} \sum_{k=0}^{n} \frac{(-n)_k\,(n+1)_k}{\Gamma(k-\mu+1)\,k!} \left(\frac{1-z}{2}\right)^k /;\, n \in \mathbb{N}$$

Specific values

Specialized values

For fixed *n*, *μ*

05.07.03.0001.01

$$P_n^\mu(0) = \frac{2^\mu \sqrt{\pi}}{\Gamma\left(\frac{1-\mu-n}{2}\right)\Gamma\left(1-\frac{\mu-n}{2}\right)}$$

05.07.03.0002.01

$$P_n^\mu(1) = 0 /;\, \mathrm{Re}(\mu) < 0 \bigvee \mu \in \mathbb{N}^+$$

05.07.03.0003.01

$$P_n^\mu(1) = \tilde{\infty} /;\, \mathrm{Re}(\mu) > 0 \bigwedge \mu \notin \mathbb{N}^+$$

05.07.03.0004.01

$$P_n^\mu(1) = ¿ /;\, \mathrm{Re}(\mu) = 0 \bigwedge \mu \neq 0$$

05.07.03.0005.01

$$P_n^\mu(-1) = \tilde{\infty} /;\, n \notin \mathbb{Z}$$

For fixed *n*, *z*

05.07.03.0006.01

$$P_n^0(z) == P_n(z)$$

05.07.03.0007.01

$$P_n^{-n-1}(z) == \frac{2^{n+1}}{\Gamma(n+1)} \left(1-z^2\right)^{-\frac{n+1}{2}} \mathrm{B}_{\frac{1-z}{2}}(n+1, n+1)$$

05.07.03.0008.01

$$P_n^{-n}(z) == \frac{2^{-n}}{\Gamma(n+1)} \left(1-z^2\right)^{n/2}$$

05.07.03.0090.01

$$P_n^m(z) = (-1)^m \left(1-z^2\right)^{m/2} (2\,m-1)!! \sum_{i_1=0}^{n-m} \dots \sum_{i_{2\,m+1}=0}^{n-m} \delta_{\sum_{j=1}^{2\,m+1} i_j, n-m} \prod_{j=1}^{2\,m+1} P_{i_j}(z) \;/;\; m \in \mathbb{N} \land n \in \mathbb{Z} \land n \geq m$$

05.07.03.0091.01

$$P_n^{-m}(z) = \frac{\left(1-z^2\right)^{m/2} (2\,m-1)!!}{(n-m+1)_{2\,m}} \sum_{i_1=0}^{n-m} \dots \sum_{i_{2\,m+1}=0}^{n-m} \delta_{\sum_{j=1}^{2\,m+1} i_j, n-m} \prod_{j=1}^{2\,m+1} P_{i_j}(z) \;/;\; m \in \mathbb{N} \land n \in \mathbb{Z} \land n \geq m$$

For fixed μ, z

05.07.03.0009.01

$$P_0^\mu(z) == \frac{1}{\Gamma(1-\mu)} \frac{(1+z)^{\mu/2}}{(1-z)^{\mu/2}}$$

05.07.03.0010.01

$$P_1^\mu(z) == \frac{z-\mu}{\Gamma(2-\mu)} \frac{(1+z)^{\mu/2}}{(1-z)^{\mu/2}}$$

05.07.03.0011.01

$$P_2^\mu(z) == \frac{3\,z^2 - 3\,\mu\,z + \mu^2 - 1}{\Gamma(3-\mu)} \frac{(1+z)^{\mu/2}}{(1-z)^{\mu/2}}$$

05.07.03.0012.01

$$P_3^\mu(z) == \frac{15\,z^3 - 15\,\mu\,z^2 + 3\left(2\,\mu^2 - 3\right) z - \mu^3 + 4\,\mu}{\Gamma(4-\mu)} \frac{(1+z)^{\mu/2}}{(1-z)^{\mu/2}}$$

05.07.03.0013.01

$$P_4^\mu(z) == \frac{9 + 105\,z^4 - 105\,z^3\,\mu - 10\,\mu^2 + \mu^4 + 45\,z^2\left(\mu^2 - 2\right) + z\left(55\,\mu - 10\,\mu^3\right)}{\Gamma(5-\mu)} \frac{(1+z)^{\mu/2}}{(1-z)^{\mu/2}}$$

05.07.03.0014.01

$$P_5^\mu(z) ==$$

$$\frac{1}{\Gamma(6-\mu)} \left(-945\,z^5 + 64\,\mu + 945\,z^4\,\mu - 20\,\mu^3 + \mu^5 + 105\,z^2\,\mu\left(\mu^2 - 7\right) - 210\,z^3\left(2\,\mu^2 - 5\right) - 15\,z\left(15 - 13\,\mu^2 + \mu^4\right)\right) \frac{(1+z)^{\mu/2}}{(1-z)^{\mu/2}}$$

05.07.03.0015.01

$$P_6^\mu(z) == \frac{1}{\Gamma(7-\mu)} \left(10\,395\,z^6 - 10\,395\,\mu\,z^5 + 4725\left(\mu^2 - 3\right) z^4 - 630\,\mu\left(2\,\mu^2 - 17\right) z^3 + \right.$$

$$\left. 105\left(2\,\mu^4 - 32\,\mu^2 + 45\right) z^2 - 21\,\mu\left(\mu^4 - 25\,\mu^2 + 99\right) z + (\mu-5)(\mu-3)(\mu-1)(\mu+1)(\mu+3)(\mu+5)\right) \frac{(1+z)^{\mu/2}}{(1-z)^{\mu/2}}$$

05.07.03.0016.01

$$P_7^{\mu}(z) = \frac{1}{\Gamma(8-\mu)} \left(135\,135\, z^7 - 135\,135\, \mu\, z^6 + 31\,185 \left(2\mu^2 - 7\right) z^5 - 17\,325\, \mu \left(\mu^2 - 10\right) z^4 + 1575 \left(2\mu^4 - 38\mu^2 + 63\right) z^3 - 189\, \mu \left(2\mu^4 - 60\mu^2 + 283\right) z^2 + 7 \left(4\mu^6 - 170\mu^4 + 1516\mu^2 - 1575\right) z - \mu \left(\mu^6 - 56\mu^4 + 784\mu^2 - 2304\right)\right) \frac{(1+z)^{\mu/2}}{(1-z)^{\mu/2}}$$

05.07.03.0017.01

$$P_8^{\mu}(z) = \frac{1}{\Gamma(9-\mu)} \left(2\,027\,025\, z^8 - 2\,027\,025\, \mu\, z^7 + 945\,945 \left(\mu^2 - 4\right) z^6 - 135\,135\, \mu \left(2\mu^2 - 23\right) z^5 + 51\,975 \left(\mu^4 - 22\mu^2 + 42\right) z^4 - 3465\, \mu \left(2\mu^4 - 70\mu^2 + 383\right) z^3 + 315 \left(2\mu^6 - 100\mu^4 + 1043\mu^2 - 1260\right) z^2 - 9\, \mu \left(4\mu^6 - 266\mu^4 + 4396\mu^2 - 15\,159\right) z + \mu^8 - 84\mu^6 + 1974\mu^4 - 12\,916\mu^2 + 11\,025\right) \frac{(1+z)^{\mu/2}}{(1-z)^{\mu/2}}$$

05.07.03.0018.01

$$P_9^{\mu}(z) = \frac{1}{\Gamma(10-\mu)} \left(34\,459\,425\, z^9 - 34\,459\,425\, \mu\, z^8 + 8\,108\,100 \left(2\mu^2 - 9\right) z^7 - 4\,729\,725\, \mu \left(\mu^2 - 13\right) z^6 + 945\,945 \left(\mu^4 - 25\mu^2 + 54\right) z^5 - 135\,135\, \mu \left(\mu^4 - 40\mu^2 + 249\right) z^4 + 6930 \left(2\mu^6 - 115\mu^4 + 1373\mu^2 - 1890\right) z^3 - 495\, \mu \left(2\mu^6 - 154\mu^4 + 2933\mu^2 - 11\,601\right) z^2 + 45 \left(\mu^8 - 98\mu^6 + 2674\mu^4 - 20\,217\mu^2 + 19\,845\right) z - \mu \left(\mu^8 - 120\mu^6 + 4368\mu^4 - 52\,480\mu^2 + 147\,456\right)\right) \frac{(1+z)^{\mu/2}}{(1-z)^{\mu/2}}$$

05.07.03.0019.01

$$P_{10}^{\mu}(z) = \frac{1}{\Gamma(11-\mu)} \left(654\,729\,075\, z^{10} - 654\,729\,075\, \mu\, z^9 + 310\,134\,825 \left(\mu^2 - 5\right) z^8 - 45\,945\,900\, \mu \left(2\mu^2 - 29\right) z^7 + 9\,459\,450 \left(2\mu^4 - 56\mu^2 + 135\right) z^6 - 2\,837\,835\, \mu \left(\mu^4 - 45\mu^2 + 314\right) z^5 + 315\,315 \left(\mu^6 - 65\mu^4 + 874\mu^2 - 1350\right) z^4 - 12\,870\, \mu \left(2\mu^6 - 175\mu^4 + 3773\mu^2 - 16\,830\right) z^3 + 1485 \left(\mu^8 - 112\mu^6 + 3479\mu^4 - 29\,828\mu^2 + 33\,075\right) z^2 - 55\, \mu \left(\mu^8 - 138\mu^6 + 5754\mu^4 - 78\,877\mu^2 + 251\,865\right) z + \mu^{10} - 165\mu^8 + 8778\mu^6 - 172\,810\mu^4 + 1\,057\,221\mu^2 - 893\,025\right) \frac{(1+z)^{\mu/2}}{(1-z)^{\mu/2}}$$

05.07.03.0020.01

$$P_n^{\mu}(z) = \frac{(1+z)^{\mu/2}}{(1-z)^{\mu/2}} \sum_{k=0}^{n} \frac{(-n)_k\, (n+1)_k}{\Gamma(k-\mu+1)\, k!} \left(\frac{1-z}{2}\right)^k$$

05.07.03.0021.01

$$P_n^{\mu}(z) = \frac{\Gamma(-\mu)\, (z+1)^{\mu/2}}{\Gamma(-n-\mu)\, \Gamma(n-\mu+1)\, (1-z)^{\mu/2}} \sum_{k=0}^{n} \frac{(-n)_k\, (n+1)_k}{(\mu+1)_k\, k!} \left(\frac{z+1}{2}\right)^k$$

05.07.03.0022.01

$$P_n^{\mu}(z) = \frac{(-1)^n\, 2^{-n}}{n!} \frac{(1+z)^{\mu/2}}{(1-z)^{\mu/2-n}} \sum_{k=0}^{n} \frac{(2n-k)!\, (-n)_k}{k!\, \Gamma(n-k-\mu+1)} \left(\frac{2}{1-z}\right)^k$$

05.07.03.0023.01

$$P_n^m(z) = 0 \;/;\; m \in \mathbb{N}^+ \bigwedge n < m$$

For fixed *z*

05.07.03.0024.01

$$P_0^0(z) == 1$$

05.07.03.0025.01

$$P_1^0(z) == z$$

05.07.03.0026.01

$$P_1^1(z) == -\sqrt{1-z^2}$$

05.07.03.0027.01

$$P_2^0(z) == \frac{1}{2}\left(3 z^2-1\right)$$

05.07.03.0028.01

$$P_2^1(z) == -3 z \sqrt{1-z^2}$$

05.07.03.0029.01

$$P_2^2(z) == 3-3 z^2$$

05.07.03.0030.01

$$P_3^0(z) == \frac{1}{2} z \left(5 z^2-3\right)$$

05.07.03.0031.01

$$P_3^1(z) == -\frac{3}{2} \sqrt{1-z^2} \left(5 z^2-1\right)$$

05.07.03.0032.01

$$P_3^2(z) == -15 (z-1) z (1+z)$$

05.07.03.0033.01

$$P_3^3(z) == -15 \left(1-z^2\right)^{3/2}$$

05.07.03.0034.01

$$P_4^0(z) == \frac{1}{8}\left(3-30 z^2+35 z^4\right)$$

05.07.03.0035.01

$$P_4^1(z) == -\frac{5}{2} z \sqrt{1-z^2} \left(7 z^2-3\right)$$

05.07.03.0036.01

$$P_4^2(z) == -\frac{15}{2} (z-1) (z+1) \left(7 z^2-1\right)$$

05.07.03.0037.01

$$P_4^3(z) == -105 z \left(1-z^2\right)^{3/2}$$

05.07.03.0038.01

$$P_4^4(z) == 105 \left(z^2-1\right)^2$$

05.07.03.0039.01

$$P_5^0(z) == \frac{1}{8} z \left(15-70 z^2+63 z^4\right)$$

05.07.03.0040.01

$$P_5^1(z) = -\frac{15}{8}\sqrt{1-z^2}\left(21z^4-14z^2+1\right)$$

05.07.03.0041.01

$$P_5^2(z) = -\frac{105}{2}(z-1)\,z\,(z+1)\left(3z^2-1\right)$$

05.07.03.0042.01

$$P_5^3(z) = -\frac{105}{2}\left(1-z^2\right)^{3/2}\left(9z^2-1\right)$$

05.07.03.0043.01

$$P_5^4(z) = 945\,z\left(z^2-1\right)^2$$

05.07.03.0044.01

$$P_5^5(z) = -945\left(1-z^2\right)^{5/2}$$

05.07.03.0045.01

$$P_6^0(z) = \frac{1}{16}\left(231z^6-315z^4+105z^2-5\right)$$

05.07.03.0046.01

$$P_6^1(z) = -\frac{21}{8}\,z\sqrt{1-z^2}\left(33z^4-30z^2+5\right)$$

05.07.03.0047.01

$$P_6^2(z) = -\frac{105}{8}(z-1)(z+1)\left(33z^4-18z^2+1\right)$$

05.07.03.0048.01

$$P_6^3(z) = -\frac{315}{2}\,z\left(1-z^2\right)^{3/2}\left(11z^2-3\right)$$

05.07.03.0049.01

$$P_6^4(z) = \frac{945}{2}\left(z^2-1\right)^2\left(11z^2-1\right)$$

05.07.03.0050.01

$$P_6^5(z) = -10\,395\,z\left(1-z^2\right)^{5/2}$$

05.07.03.0051.01

$$P_6^6(z) = 10\,395\left(1-z^2\right)^3$$

05.07.03.0052.01

$$P_7^0(z) = \frac{1}{16}\,z\left(429z^6-693z^4+315z^2-35\right)$$

05.07.03.0053.01

$$P_7^1(z) = -\frac{7}{16}\sqrt{1-z^2}\left(429z^6-495z^4+135z^2-5\right)$$

05.07.03.0054.01

$$P_7^2(z) = -\frac{63}{8}(z-1)\,z\,(z+1)\left(143z^4-110z^2+15\right)$$

05.07.03.0055.01

$$P_7^3(z) = -\frac{315}{8}\left(1-z^2\right)^{3/2}\left(143 z^4-66 z^2+3\right)$$

05.07.03.0056.01

$$P_7^4(z) = \frac{3465}{2} z\left(z^2-1\right)^2\left(13 z^2-3\right)$$

05.07.03.0057.01

$$P_7^5(z) = -\frac{10\,395}{2}\left(1-z^2\right)^{5/2}\left(13 z^2-1\right)$$

05.07.03.0058.01

$$P_7^6(z) = 135\,135\, z\left(1-z^2\right)^3$$

05.07.03.0059.01

$$P_7^7(z) = -135\,135\left(1-z^2\right)^{7/2}$$

05.07.03.0060.01

$$P_8^0(z) = \frac{1}{128}\left(6435 z^8-12\,012 z^6+6930 z^4-1260 z^2+35\right)$$

05.07.03.0061.01

$$P_8^1(z) = -\frac{9}{16} z \sqrt{1-z^2}\left(715 z^6-1001 z^4+385 z^2-35\right)$$

05.07.03.0062.01

$$P_8^2(z) = -\frac{315}{16}\left(z^2-1\right)\left(143 z^6-143 z^4+33 z^2-1\right)$$

05.07.03.0063.01

$$P_8^3(z) = -\frac{3465}{8} z\left(1-z^2\right)^{3/2}\left(39 z^4-26 z^2+3\right)$$

05.07.03.0064.01

$$P_8^4(z) = \frac{10\,395}{8}\left(z^2-1\right)^2\left(65 z^4-26 z^2+1\right)$$

05.07.03.0065.01

$$P_8^5(z) = -\frac{135\,135}{2} z\left(1-z^2\right)^{5/2}\left(5 z^2-1\right)$$

05.07.03.0066.01

$$P_8^6(z) = -\frac{135\,135}{2}\left(z^2-1\right)^3\left(15 z^2-1\right)$$

05.07.03.0067.01

$$P_8^7(z) = -2\,027\,025\, z\left(1-z^2\right)^{7/2}$$

05.07.03.0068.01

$$P_8^8(z) = 2\,027\,025\left(z^2-1\right)^4$$

05.07.03.0069.01

$$P_9^0(z) = \frac{1}{128} z\left(12\,155 z^8-25\,740 z^6+18\,018 z^4-4620 z^2+315\right)$$

05.07.03.0070.01

$$P_9^1(z) = -\frac{45}{128}\sqrt{1-z^2}\left(2431 z^8 - 4004 z^6 + 2002 z^4 - 308 z^2 + 7\right)$$

05.07.03.0071.01

$$P_9^2(z) = -\frac{495}{16}(z-1)\,z\,(z+1)\left(221 z^6 - 273 z^4 + 91 z^2 - 7\right)$$

05.07.03.0072.01

$$P_9^3(z) = -\frac{3465}{16}\left(1-z^2\right)^{3/2}\left(221 z^6 - 195 z^4 + 39 z^2 - 1\right)$$

05.07.03.0073.01

$$P_9^4(z) = \frac{135\,135}{8}\left(z^2-1\right)^2\left(17 z^5 - 10 z^3 + z\right)$$

05.07.03.0074.01

$$P_9^5(z) = -\frac{135\,135}{8}\left(1-z^2\right)^{5/2}\left(85 z^4 - 30 z^2 + 1\right)$$

05.07.03.0075.01

$$P_9^6(z) = -\frac{675\,675}{2}\,z\left(z^2-1\right)^3\left(17 z^2 - 3\right)$$

05.07.03.0076.01

$$P_9^7(z) = -\frac{2\,027\,025}{2}\left(1-z^2\right)^{7/2}\left(17 z^2 - 1\right)$$

05.07.03.0077.01

$$P_9^8(z) = 34\,459\,425\,z\left(z^2-1\right)^4$$

05.07.03.0078.01

$$P_9^9(z) = -34\,459\,425\left(1-z^2\right)^{9/2}$$

05.07.03.0079.01

$$P_{10}^0(z) = \frac{1}{256}\left(46\,189 z^{10} - 109\,395 z^8 + 90\,090 z^6 - 30\,030 z^4 + 3465 z^2 - 63\right)$$

05.07.03.0080.01

$$P_{10}^1(z) = -\frac{55}{128}\,z\sqrt{1-z^2}\left(4199 z^8 - 7956 z^6 + 4914 z^4 - 1092 z^2 + 63\right)$$

05.07.03.0081.01

$$P_{10}^2(z) = -\frac{495}{128}\left(z^2-1\right)\left(4199 z^8 - 6188 z^6 + 2730 z^4 - 364 z^2 + 7\right)$$

05.07.03.0082.01

$$P_{10}^3(z) = -\frac{6435}{16}\,z\left(1-z^2\right)^{3/2}\left(323 z^6 - 357 z^4 + 105 z^2 - 7\right)$$

05.07.03.0083.01

$$P_{10}^4(z) = \frac{45\,045}{16}\left(z^2-1\right)^2\left(323 z^6 - 255 z^4 + 45 z^2 - 1\right)$$

05.07.03.0084.01

$$P_{10}^{5}(z) = -\frac{135135}{8} z\left(1-z^2\right)^{5/2}\left(323 z^4-170 z^2+15\right)$$

05.07.03.0085.01

$$P_{10}^{6}(z) = -\frac{675675}{8}\left(z^2-1\right)^3\left(323 z^4-102 z^2+3\right)$$

05.07.03.0086.01

$$P_{10}^{7}(z) = -\frac{11486475}{2} z\left(1-z^2\right)^{7/2}\left(19 z^2-3\right)$$

05.07.03.0087.01

$$P_{10}^{8}(z) = \frac{34459425}{2}\left(z^2-1\right)^4\left(19 z^2-1\right)$$

05.07.03.0088.01

$$P_{10}^{9}(z) = -654729075 z\left(1-z^2\right)^{9/2}$$

05.07.03.0089.01

$$P_{10}^{10}(z) = 654729075\left(1-z^2\right)^5$$

General characteristics

Domain and analyticity

The function $P_n^\mu(z)$is defined over $\mathbb{N} \otimes \mathbb{C} \otimes \{2\} \otimes \mathbb{C}$. For fixed n, λ, the function $P_n^\mu(z)$ is a polynomial in z of degree n multiplied on function $\frac{(1+z)^{\mu/2}}{(1-z)^{\mu/2}}$.

05.07.04.0001.01

$$(n * \mu * 2 * z) \longrightarrow P_n^\mu(z) :: (\mathbb{N} \otimes \mathbb{C} \otimes \{2\} \otimes \mathbb{C}) \longrightarrow \mathbb{C}$$

Symmetries and periodicities

Mirror symmetry

05.07.04.0002.01

$$P_n^{\bar{\mu}}(\bar{z}) = \overline{P_n^\mu(z)}$$

Periodicity

No periodicity

Poles and essential singularities

With respect to *z*

For fixed n, μ /; $\frac{\mu}{2} \notin \mathbb{Z}$, the function $P_n^\mu(z)$ does not have poles and essential singularities.

05.07.04.0003.01

$$\mathcal{S}ing_z\left(P_n^\mu(z)\right) = \{\} \;/;\; \frac{\mu}{2} \notin \mathbb{Z}$$

For integer n and integer $\frac{\mu}{2}$, the function $P_n^\mu(z)$ is polynomial and has pole of order n at $z = \tilde{\infty}$.

05.07.04.0004.01

$$\mathcal{S}ing_z\left(P_n^\mu(z)\right) == \{\{\tilde{\infty}, n\}\} /; \frac{\mu}{2} \in \mathbb{Z}$$

With respect to μ

For fixed n, z, the function $P_n^\mu(z)$ has only one singular point at $\mu = \tilde{\infty}$. It is an essential singular point. .

05.07.04.0005.01

$$\mathcal{S}ing_\mu\left(P_n^\mu(z)\right) == \{\{\tilde{\infty}, \infty\}\}$$

Branch points

With respect to z

For fixed n and integers $\frac{\mu}{2}$, the function $P_n^\mu(z)$does not have branch points.

05.07.04.0006.01

$$\mathcal{BP}_z\left(P_n^\mu(z)\right) == \{\} /; \frac{\mu}{2} \in \mathbb{Z}$$

With respect to μ

For fixed n, z, the function $P_n^\mu(z)$ does not have branch points.

05.07.04.0007.01

$$\mathcal{BP}_\mu\left(P_n^\mu(z)\right) == \{\}$$

Branch cuts

With respect to z

For fixed generic n, μ /; $\frac{\mu}{2} \notin \mathbb{Z}$, the function $P_n^\mu(z)$ is a single-valued function on the z-plane cut along the intervals $(-\infty, -1)$ and $(1, \infty)$. The function $P_n^\mu(z)$ is continuous from above on the interval $(-\infty, -1]$ and from below on the interval $[1, \infty)$.

05.07.04.0008.01

$$\mathcal{BC}_z\left(P_n^\mu(z)\right) == \{\{(-\infty, -1), -i\}, \{(1, \infty), i\}\} /; \frac{\mu}{2} \notin \mathbb{Z}$$

05.07.04.0009.01

$$\lim_{\epsilon \to +0} P_n^\mu(x + i\,\epsilon) == P_n^\mu(x) /; x < -1$$

05.07.04.0010.01

$$\lim_{\epsilon \to +0} P_n^\mu(x - i\,\epsilon) == e^{i\,\pi\,\mu}\left(\frac{2\,i\,\pi}{\Gamma(-\mu - n)\,\Gamma(n - \mu + 1)} P_n^{-\mu}(-x) + P_n^\mu(x)\right) /; x < -1$$

05.07.04.0011.01

$$\lim_{\epsilon \to +0} P_n^\mu(x + i\,\epsilon) == e^{i\,\pi\,\mu}\, P_n^\mu(x) /; x > 1$$

05.07.04.0012.01

$$\lim_{\epsilon\to+0} P_n^\mu(x - i\,\epsilon) == P_n^\mu(x) \;/;\; x > 1$$

For fixed integers $\frac{\mu}{2}$, the function $P_n^\mu(z)$ is a polynomial and does not have branch cuts.

05.07.04.0013.01

$$\mathcal{BC}_z\left(P_n^\mu(z)\right) == \{\} \;/;\; \frac{\mu}{2} \in \mathbb{Z}$$

With respect to μ

For fixed n, z, the function $P_n^\mu(z)$ does not have branch cuts.

05.07.04.0014.01

$$\mathcal{BC}_\mu\left(P_n^\mu(z)\right) == \{\}$$

Series representations

Generalized power series

Expansions at $z == 0$

05.07.06.0001.01

$$P_n^\mu(z) == \sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{j=0}^{\infty} \frac{(-n)_k\,(n+1)_k\left(\frac{\mu}{2}-k\right)_m\left(-\frac{\mu}{2}\right)_j\,(-1)^j\,z^{j+m}}{\Gamma(k-\mu+1)\,k!\,m!\,j!\,2^k}$$

05.07.06.0002.01

$$P_n^\mu(z) == \frac{\mu}{2}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{j=0}^{\infty} \frac{(-n)_k\,(n+1)_k\left(\frac{\mu}{2}+1\right)_m\left(1-\frac{\mu}{2}\right)_k\left(-\frac{\mu}{2}\right)_j\,(-1)^{j+k}\,z^{j+k+m+1}}{(2)_{k+m}\,\Gamma(k-\mu+1)\,k!\,j!\,2^k} +$$
$$\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{j=0}^{\infty} \frac{(-n)_{k+m}\,(n+1)_{k+m}\left(m-\frac{\mu}{2}+1\right)_k\left(-\frac{\mu}{2}\right)_j\,(-z)^{j+k}}{\Gamma(k+m-\mu+1)\,(k+m)!\;k!\,j!\,2^{k+m}}$$

05.07.06.0003.01

$$P_n^\mu(z) == \frac{\mu\,z}{2}\sum_{k=0}^{\infty}(k+1)!\left(\frac{\mu}{2}+1\right)_k z^k\;{}_1F_0\left(-\frac{\mu}{2};;-z\right){}_3\tilde{F}_2\left(-n,\,n+1,\,1-\frac{\mu}{2};\,k+2,\,1-\mu;\,-\frac{z}{2}\right)+$$
$$\Gamma\left(1-\frac{\mu}{2}\right)\sum_{k=0}^{\infty}\left(-\frac{\mu}{2}\right)_k(-z)^k\,\tilde{F}_{2\times1\times0}^{3\times1\times0}\left(\begin{matrix}-n,\,n+1,\,1-\frac{\mu}{2};\,1;\,; & \frac{1}{2},\,-\frac{z}{2}\\ 1,\,1-\mu;\,1-\frac{\mu}{2};\,; & \end{matrix}\right)$$

05.07.06.0004.01

$$P_n^\mu(z) \propto \frac{2^\mu\sqrt{\pi}}{\Gamma\left(\frac{1-\mu-n}{2}\right)\Gamma\left(1-\frac{\mu-n}{2}\right)}\,(1+O(z)) \;/;\; (z\to 0)$$

05.07.06.0005.01

$$P_n^m(z) == (-1)^m\left(1-z^2\right)^{m/2}2^{-n}\sum_{k=0}^{\left\lfloor\frac{n}{2}\right\rfloor}(-1)^k\binom{n}{k}\binom{2n-2k}{n}(n-m-2k+1)_m\,z^{n-m-2k} \;/;\; m\in\mathbb{N}$$

05.07.06.0006.01

$$P_n^m(z) \propto (-1)^{m+\left\lfloor \frac{n-m}{2} \right\rfloor} 2^{-n} \binom{n}{\left\lfloor \frac{n-m}{2} \right\rfloor} \binom{2n-2\left\lfloor \frac{n-m}{2} \right\rfloor}{n} \left(n-m-2\left\lfloor \frac{n-m}{2} \right\rfloor+1\right)_m z^{n-m-2\left\lfloor \frac{n-m}{2} \right\rfloor}(1+O(z)) /; (z \to 0) \land m \in \mathbb{N} \land n \geq m$$

Expansions at $z == 1$

05.07.06.0007.01

$$P_n^\mu(z) == \frac{(1+z)^{\mu/2}}{(1-z)^{\mu/2}} \left(\frac{1}{\Gamma(1-\mu)} - \frac{(-n)(n+1)(z-1)}{2\,\Gamma(2-\mu)} + \frac{(-n)(1-n)(n+1)(n+2)(z-1)^2}{8\,\Gamma(3-\mu)} - \ldots \right)$$

05.07.06.0008.01

$$P_n^\mu(z) == \frac{(1+z)^{\mu/2}}{(1-z)^{\mu/2}} \sum_{k=0}^{n} \frac{(-n)_k (n+1)_k}{\Gamma(k-\mu+1)\,k!} \left(\frac{1-z}{2}\right)^k$$

05.07.06.0009.01

$$P_n^\mu(z) == \frac{(1+z)^{\mu/2}}{(1-z)^{\mu/2}} \,_2\tilde{F}_1\left(-n, n+1; 1-\mu; \frac{1-z}{2}\right)$$

05.07.06.0010.01

$$P_n^\mu(z) == \frac{2^{\mu/2}}{(1-z)^{\mu/2}} \sum_{k=0}^{\infty} \sum_{j=0}^{k} \frac{(-1)^k \left(-\frac{\mu}{2}\right)_{k-j} (-n)_j (n+1)_j (z-1)^k}{\Gamma(j-\mu+1)(k-j)!\,j!\,2^k}$$

05.07.06.0011.01

$$P_n^\mu(z) \propto \frac{2^{\mu/2}}{\Gamma(1-\mu)} (1-z)^{-\mu/2} (1+O(z-1)) /; (z \to 1) \bigwedge \mu \notin \mathbb{N}^+$$

05.07.06.0012.01

$$P_n^m(z) == \left(1-z^2\right)^{m/2} 2^{-m} (-n)_m (n+1)_m \sum_{k=0}^{\infty} \frac{(m-n)_k (m+n+1)_k}{\Gamma(k+m+1)\,k!} \left(\frac{1-z}{2}\right)^k /; m \in \mathbb{N}$$

05.07.06.0013.01

$$P_n^\mu(z) == \frac{(1+z)^{\mu/2}}{(1-z)^{\mu/2}} \sum_{k=0}^{n} \frac{(-n)_k (n+1)_k}{\Gamma(k-\mu+1)\,k!} \left(\frac{1-z}{2}\right)^k$$

Expansions at $z == -1$

05.07.06.0014.01

$$P_n^\mu(z) == \frac{2^{-\mu/2}\,\Gamma(-\mu)(z+1)^{\mu/2}}{\Gamma(-\mu-n)\,\Gamma(1-\mu+n)} \left(1 + \frac{\mu(1+\mu) - 2n(1+n)}{4(1+\mu)}(z+1) + \frac{\mu(1+\mu)(2+\mu)^2 - 4(2+\mu(2+\mu))n - 4(1+\mu)^2 n^2 + 8n^3 + 4n^4}{32(1+\mu)(2+\mu)}(z+1)^2 + \ldots\right) /; \mu \notin \mathbb{Z}$$

05.07.06.0015.01

$$P_n^\mu(z) == \frac{\Gamma(-\mu)}{\Gamma(-\mu-n)\,\Gamma(n-\mu+1)} \frac{(1+z)^{\mu/2}}{(1-z)^{\mu/2}} \sum_{k=0}^{n} \frac{(-n)_k (n+1)_k}{(\mu+1)_k\,k!} \left(\frac{z+1}{2}\right)^k /; \mu \notin \mathbb{Z}$$

05.07.06.0016.01

$$P_n^\mu(z) == \frac{\Gamma(-\mu)}{\Gamma(-\mu-n)\,\Gamma(n-\mu+1)} \frac{(1+z)^{\mu/2}}{(1-z)^{\mu/2}} \,_2F_1\left(-n, n+1; \mu+1; \frac{z+1}{2}\right) /; \mu \notin \mathbb{Z}$$

05.07.06.0017.01

$$P_n^\mu(z) = \frac{\Gamma(-\mu)\, 2^{-\frac{\mu}{2}}}{\Gamma(-\mu-n)\,\Gamma(n-\mu+1)} (z+1)^{\mu/2} \sum_{k=0}^{\infty} \sum_{j=0}^{k} \frac{\left(\frac{\mu}{2}\right)_{k-j} (-n)_j\, (n+1)_j\, 2^{-k}\, (z+1)^k}{(\mu+1)_j\, j!\, (k-j)!} \;/;\; \mu \notin \mathbb{Z}$$

05.07.06.0018.01

$$P_n^\mu(z) \propto \frac{2^{-\mu/2}\, \Gamma(-\mu)}{\Gamma(-\mu-n)\,\Gamma(n-\mu+1)} (z+1)^{\mu/2} (1 + O(z+1)) \;/;\; (z \to -1) \bigwedge \mu \notin \mathbb{Z}$$

Expansions at $z = \infty$

05.07.06.0019.01

$$P_n^\mu(z) = \frac{(-1)^n\, 2^{-n}}{n!} \frac{(1+z)^{\mu/2}}{(1-z)^{\mu/2-n}} \sum_{k=0}^{n} \frac{(2n-k)!\, (-n)_k}{k!\, \Gamma(n-k-\mu+1)} \left(\frac{2}{1-z}\right)^k$$

05.07.06.0020.01

$$P_n^\mu(z) \propto \frac{2^n\, \Gamma\left(n+\frac{1}{2}\right) z^{n+\frac{\mu}{2}}}{\sqrt{\pi}\; \Gamma(n-\mu+1)\, (1-z)^{\mu/2}} \left(1 + O\left(\frac{1}{z}\right)\right) \;/;\; (|z| \to \infty)$$

Integral representations

On the real axis

Of the direct function

05.07.07.0001.01

$$P_n^\mu(z) = \frac{(-1)^{-n}\, 2^{\mu-2n}}{\Gamma(-\mu-n)\, n!} \frac{(1+z)^{\mu/2}}{(1-z)^{\mu/2-n}} \int_{-1}^{1} \frac{(t-1)^n}{(t+1)^{\mu+n+1}} \left(t - \frac{z+3}{z-1}\right)^n dt \;/;\; n < -\mathrm{Re}(\mu)$$

05.07.07.0002.01

$$P_n^m(z) = \frac{(-n)_m}{\pi}\, e^{\frac{m\pi i}{2}} \int_0^\pi \frac{\cos(m\,t)}{\left(z + \sqrt{z^2-1}\, \cos(t)\right)^{n+1}} dt \;/;\; m \in \mathbb{N} \land \mathrm{Re}(z) > 0$$

05.07.07.0003.01

$$P_n^\mu(z) = \frac{1}{\Gamma(-\mu)} \left(1-z^2\right)^{\mu/2} \int_z^1 P_n(t)\, (t-z)^{-\mu-1}\, dt \;/;\; \mathrm{Re}(\mu) < 0$$

Integral representations of negative integer order

Rodrigues-type formula.

05.07.07.0004.01

$$P_n^m(z) = (-1)^m \left(1-z^2\right)^{m/2} \frac{\partial^m P_n(z)}{\partial z^m} \;/;\; m \in \mathbb{N}$$

05.07.07.0005.01

$$P_n^m(z) = \frac{(-1)^{m+n} \left(1-z^2\right)^{m/2}}{2^n\, n!} \frac{\partial^{m+n} \left(1-z^2\right)^n}{\partial z^{m+n}} \;/;\; m \in \mathbb{N}$$

Differential equations

Ordinary linear differential equations and wronskians

For the direct function itself

05.07.13.0001.01

$$\left(1-z^2\right) w''(z)-2 z\, w'(z)+\left(n (n+1)-\frac{\mu^2}{1-z^2}\right) w(z)=0\ /;\ w(z)=c_1 P_n^{\mu}(z)+c_2 Q_n^{\mu}(z)$$

05.07.13.0002.02

$$W_z\left(P_n^{\mu}(z), Q_n^{\mu}(z)\right)=\frac{\Gamma(n+\mu+1)}{\left(1-z^2\right) \Gamma(n-\mu+1)}$$

05.07.13.0003.01

$$g'(z)\, w''(z)-\left(\frac{2 g(z)\, g'(z)^2}{1-g(z)^2}+g''(z)\right) w'(z)-\frac{\left(\mu^2-n (n+1) \left(1-g(z)^2\right)\right) g'(z)^3}{\left(1-g(z)^2\right)^2} w(z)=0\ /;\ w(z)=c_1 P_n^{\mu}(g(z))+c_2 Q_n^{\mu}(g(z))$$

05.07.13.0004.01

$$W_z\left(P_n^{\mu}(g(z)), Q_n^{\mu}(g(z))\right)=\frac{\Gamma(\mu+n+1)\, g'(z)}{\left(1-g(z)^2\right) \Gamma(1-\mu+n)}$$

05.07.13.0005.01

$$g'(z)\, h(z)^2 w''(z)-\left(\left(\frac{2 g(z)\, g'(z)^2}{1-g(z)^2}+g''(z)\right) h(z)^2+2 g'(z)\, h'(z)\, h(z)\right) w'(z)+\left(-\frac{\mu^2-n (n+1) \left(1-g(z)^2\right)}{\left(1-g(z)^2\right)^2} h(z)^2 g'(z)^3+2 h'(z)^2 g'(z)+h(z) \left(h'(z) \left(\frac{2 g(z)\, g'(z)^2}{1-g(z)^2}+g''(z)\right)-g'(z)\, h''(z)\right)\right) w(z)=0\ /;\ w(z)=c_1 h(z) P_n^{\mu}(g(z))+c_2 h(z) Q_n^{\mu}(g(z))$$

05.07.13.0006.01

$$W_z\left(h(z) P_n^{\mu}(g(z)), h(z) Q_n^{\mu}(g(z))\right)=\frac{\Gamma(\mu+n+1)\, h(z)^2 g'(z)}{\left(1-g(z)^2\right) \Gamma(1-\mu+n)}$$

05.07.13.0007.01

$$z^2 w''(z)-z\left(2 s+\frac{r\left(a^2 z^{2 r}+1\right)}{1-a^2 z^{2 r}}-1\right) w'(z)+\left(-\frac{a^2 r^2 \left(\mu^2+\left(a^2 z^{2 r}-1\right) n (n+1)\right) z^{2 r}}{\left(1-a^2 z^{2 r}\right)^2}+s^2+\frac{r s\left(a^2 z^{2 r}+1\right)}{1-a^2 z^{2 r}}\right) w(z)=0\ /;\ w(z)=c_1 z^s P_n^{\mu}(a z^r)+c_2 z^s Q_n^{\mu}(a z^r)$$

05.07.13.0008.01

$$W_z\left(z^s P_n^{\mu}(a z^r), z^s Q_n^{\mu}(a z^r)\right)=\frac{a r z^{r+2 s-1}\, \Gamma(\mu+n+1)}{\left(1-a^2 z^{2 r}\right) \Gamma(-\mu+n+1)}$$

05.07.13.0009.01

$$w''(z) - \frac{a^2 (\log(r) - 2\log(s))\, r^{2z} + \log(r) + 2\log(s)}{1 - a^2 r^{2z}}\, w'(z) + \left(-\frac{a^2 \left(\mu^2 - \left(1 - a^2 r^{2z}\right) n\,(n+1)\right) \log^2(r)\, r^{2z}}{\left(1 - a^2 r^{2z}\right)^2} + \log^2(s) + \frac{\left(a^2 r^{2z} + 1\right) \log(r) \log(s)}{1 - a^2 r^{2z}}\right) w(z) = 0 \;/;\; w(z) = c_1\, s^z\, P_n^\mu\left(a\, r^z\right) + c_2\, s^z\, Q_n^\mu(a\, r^z)$$

05.07.13.0010.01

$$W_z\left(s^z\, P_n^\mu(a\, r^z),\, s^z\, Q_n^\mu(a\, r^z)\right) = \frac{a\, r^z\, s^{2z}\, \Gamma(\mu + n + 1) \log(r)}{\left(1 - a^2 r^{2z}\right) \Gamma(-\mu + n + 1)}$$

Transformations

Transformations and argument simplifications

Argument involving basic arithmetic operations

05.07.16.0001.01

$$P_n^{-m}(z) = \frac{(-1)^m\, (n-m)!}{(m+n)!}\, P_n^m(z) \;/;\; m \in \mathbb{Z}$$

05.07.16.0002.01

$$P_n^\mu(-z) = -\frac{\pi \csc(\pi\, \mu)\, P_n^{-\mu}(z)}{\Gamma(-n-\mu)\, \Gamma(n-\mu+1)}$$

Identities

Recurrence identities

Consecutive neighbors

05.07.17.0001.01

$$P_n^\mu(z) = \frac{(2\,n+3)\, z}{\mu+n+1}\, P_{n+1}^\mu(z) + \frac{\mu-n-2}{n+\mu+1}\, P_{n+2}^\mu(z)$$

05.07.17.0002.01

$$P_n^\mu(z) = \frac{(2\,n-1)\, z}{n-\mu}\, P_{n-1}^\mu(z) - \frac{\mu+n-1}{n-\mu}\, P_{n-2}^\mu(z)$$

05.07.17.0003.01

$$P_n^\mu(z) = \frac{2\, z\, (1+\mu)}{(\mu\, (1+\mu) - n\, (1+n)) \sqrt{1-z^2}}\, P_n^{\mu+1}(z) + \frac{1}{\mu\, (\mu+1) - n\, (n+1)}\, P_n^{\mu+2}(z)$$

05.07.17.0004.01

$$P_n^\mu(z) = \frac{2\, (1-\mu)\, z}{\sqrt{1-z^2}}\, P_n^{\mu-1}(z) + ((\mu-1)\, (\mu-2) - n\, (n+1))\, P_n^{\mu-2}(z)$$

Distant neighbors

05.07.17.0011.01

$$P_n^\mu(z) = C_m(n, \mu, z)\, P_{n+m}^\mu(z) + \frac{\mu - n - m - 1}{m + \mu + n} C_{m-1}(n, \mu, z)\, P_{n+m+1}^\mu(z) \;/;\; C_0(n, \mu, z) = 1 \bigwedge C_1(n, \mu, z) = \frac{(2n+3)z}{\mu + n + 1} \bigwedge C_m(n, \mu, z) = \frac{z(2m + 2n + 1)}{m + \mu + n} C_{m-1}(n, \mu, z) + \frac{\mu - n - m}{m + \mu + n - 1} C_{m-2}(n, \mu, z) \bigwedge m \in \mathbb{N}^+$$

05.07.17.0012.01

$$P_n^\mu(z) = C_m(n, \mu, z)\, P_{n-m}^\mu(z) - \frac{\mu + n - m}{n - \mu - m + 1} C_{m-1}(n, \mu, z)\, P_{n-m-1}^\mu(z) \;/;\; C_0(n, \mu, z) = 1 \bigwedge C_1(n, \mu, z) = \frac{(2n-1)z}{n - \mu} \bigwedge C_m(n, \mu, z) = \frac{z(2m - 2n - 1)}{m + \mu - n - 1} C_{m-1}(n, \mu, z) - \frac{\mu + n - m + 1}{n - \mu - m + 2} C_{m-2}(n, \mu, z) \bigwedge m \in \mathbb{N}^+$$

05.07.17.0013.01

$$P_n^\mu(z) = C_m(n, \mu, z)\, P_n^{\mu+m}(z) + \frac{1}{(m + \mu - 1)(m + \mu) - n(n+1)} C_{m-1}(n, \mu, z)\, P_n^{\mu+m+1}(z) \;/;\; C_0(n, \mu, z) = 1 \bigwedge C_1(n, \mu, z) = \frac{2(\mu+1)z}{(\mu(\mu+1) - n(1+n))\sqrt{1 - z^2}} \bigwedge C_m(n, \mu, z) = \frac{2z(m + \mu)}{\sqrt{1 - z^2}\,((m - 1 + \mu)(m + \mu) - n(1+n))} C_{m-1}(n, \mu, z) + \frac{1}{(m - 2 + \mu)(m - 1 + \mu) - n(1+n)} C_{m-2}(n, \mu, z) \bigwedge m \in \mathbb{N}^+$$

05.07.17.0014.01

$$P_n^\mu(z) = C_m(n, \mu, z)\, P_n^{\mu-m}(z) + ((\mu - m - 1)(\mu - m) - n(n+1))\, C_{m-1}(n, \mu, z)\, P_n^{\mu-m-1}(z) \;/;\; C_0(n, \mu, z) = 1 \bigwedge C_1(n, \mu, z) = \frac{2(1-\mu)z}{\sqrt{1 - z^2}} \bigwedge C_m(n, \mu, z) = \frac{2z(m - \mu)}{\sqrt{1 - z^2}} C_{m-1}(n, \mu, z) + ((\mu - m)(\mu - m + 1) - n(n+1))\, C_{m-2}(n, \mu, z) \bigwedge m \in \mathbb{N}^+$$

Functional identities

Relations between contiguous functions

05.07.17.0005.01

$$(\mu + n)\, P_{n-1}^\mu(z) + (n - \mu + 1)\, P_{n+1}^\mu(z) = (2n + 1)\, z\, P_n^\mu(z)$$

05.07.17.0006.01

$$P_n^\mu(z) = \frac{(\mu + n)\, P_{n-1}^\mu(z) + (n - \mu + 1)\, P_{n+1}^\mu(z)}{(2n + 1)\, z}$$

05.07.17.0007.01

$$P_n^{\mu+1}(z) - (\mu(\mu - 1) - n(n+1))\, P_n^{\mu-1}(z) + \frac{2\mu z}{\sqrt{1 - z^2}} P_n^\mu(z) = 0$$

Expansion with respect to parameters

05.07.17.0009.01

$$P_{n_1}^{\mu_1}(z)\, P_{n_2}^{\mu_2}(z) == (-1)^{\mu_1} \sqrt{\frac{(n_1+\mu_1)!\,(n_2+\mu_2)!}{(n_1-\mu_1)!\,(n_2-\mu_2)!}} \sum_{k=|n_1-n_2|}^{n_1+n_2} \mathrm{If}\Big[k \ge |\mu_1-\mu_2| \wedge \mathrm{EvenQ}[k+n_1+n_2],$$
$$(-1)^{\mu_2-\mu_1}\,(2k+1) \begin{pmatrix} n_1 & n_2 & k \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} n_1 & n_2 & k \\ -\mu_1 & \mu_2 & \mu_1-\mu_2 \end{pmatrix} \sqrt{\frac{(k+\mu_1-\mu_2)!}{(k-\mu_1+\mu_2)!}}\; P_k^{\mu_2-\mu_1}(z),\, 0\Big] /;\, \mu_1 \in \mathbb{Z} \wedge \mu_2 \in \mathbb{Z}$$

05.07.17.0010.01

$$P_n^m(\cos(\beta)) == (-1)^m \left(\frac{a}{c}\right)^n \sum_{k=0}^{n-m} \binom{m+n}{k} \left(-\frac{b}{a}\right)^{n-k} P_{n-k}^m(\cos(\gamma)) /;\, n \in \mathbb{N} \bigwedge m \in \mathbb{Z} \bigwedge -n \le m \le n \bigwedge$$
$$0 \le \beta \le \pi \bigwedge 0 \le \gamma \le \pi \bigwedge \alpha == -\beta-\gamma+\pi \bigwedge \beta+\gamma \le \pi \bigwedge a \in \mathbb{R} \bigwedge a > 0 \bigwedge b == a\, \frac{\sin(\beta)}{\sin(\alpha)} \bigwedge c == a\, \frac{\sin(\gamma)}{\sin(\alpha)}$$

The conditions on β, γ, a, b, and c allow an interpretation of α, β, and γ as angles of a triangle and a, b, and c are the length of the opposite sides.

Additional relations between contiguous functions

05.07.17.0008.01

$$P_n^{\mu+1}(z)\, P_{n_1}^{\mu_1+1}(z) - P_{n-1}^{\mu+1}(z)\, P_{n_1-1}^{\mu_1+1}(z) + (n-\mu)\,(n_1-\mu_1)\, P_n^{\mu}(z)\, P_{n_1}^{\mu_1}(z) - (\mu+n)\,(\mu_1+n_1)\, P_{n-1}^{\mu}(z)\, P_{n_1-1}^{\mu_1}(z) == 0$$

Differentiation

Low-order differentiation

With respect to μ

05.07.20.0001.01

$$\frac{\partial P_n^{\mu}(z)}{\partial \mu} == \frac{(1+z)^{\mu/2}}{(1-z)^{\mu/2}} \sum_{k=0}^{n} \frac{(-n)_k\,(n+1)_k}{\Gamma(k-\mu+1)\,k!}\, \psi(k-\mu+1) \left(\frac{1-z}{2}\right)^k + \frac{1}{2}\,(\log(1+z)-\log(1-z))\, P_n^{\mu}(z) /;\, \left|\frac{1-z}{2}\right| < 1 \bigwedge \mu \notin \mathbb{N}^+$$

05.07.20.0002.01

$$\frac{\partial P_n^{\mu}(z)}{\partial \mu} == \left(\psi(1-\mu) - \frac{1}{2}\,(\log(1-z)-\log(1+z))\right) P_n^{\mu}(z) -$$
$$\frac{n\,(n+1)}{2\,(1-\mu)\,\Gamma(2-\mu)}\, \frac{(1+z)^{\mu/2}}{(1-z)^{\mu/2-1}}\, F_{2\times0\times1}^{2\times1\times2}\!\left(\begin{matrix} 1-n,\, n+2;\, 1;\, 1,\, 1-\mu; \\ 2,\, 2-\mu;;\, 2-\mu; \end{matrix}\; \frac{1-z}{2},\, \frac{1-z}{2}\right)$$

05.07.20.0003.01

$$\frac{\partial^2 P_n^{\mu}(z)}{\partial \mu^2} == \frac{(1+z)^{\mu/2}}{(1-z)^{\mu/2}} \sum_{k=0}^{n} \frac{(-n)_k\,(n+1)_k}{k!\,\Gamma(k-\mu+1)} \left(\frac{1-z}{2}\right)^k \left(\psi(k-\mu+1)^2 + (\log(1+z)-\log(1-z))\,\psi(k-\mu+1) - \psi^{(1)}(k-\mu+1)\right) +$$
$$\frac{1}{4}\,(\log(1+z)-\log(1-z))^2\, P_n^{\mu}(z) \;/;\, \left|\frac{1-z}{2}\right| < 1 \bigwedge \mu \notin \mathbb{N}^+$$

With respect to z

05.07.20.0004.01

$$\frac{\partial P_n^{\mu}(z)}{\partial z} == \frac{1}{z^2-1}\left(z\,n\,P_n^{\mu}(z) - (\mu+n)\,P_{n-1}^{\mu}(z)\right)$$

05.07.20.0005.01

$$\frac{\partial^2 P_n^\mu(z)}{\partial z^2} = \frac{2\,z\,(\mu+n)\,P_{n-1}^\mu(z) + \left(\mu^2 + \left((n-1)\,z^2 - n - 1\right)n\right)P_n^\mu(z)}{\left(z^2-1\right)^2}$$

Symbolic differentiation

With respect to z

05.07.20.0006.02

$$\frac{\partial^m P_n^\mu(z)}{\partial z^m} = \frac{(-1)^m\,(1+z)^{\mu/2}}{(1-z)^{\mu/2+m}}\,\Gamma\!\left(\frac{\mu}{2}+1\right) \sum_{j=0}^m \binom{m}{j}\,{}_2\tilde{F}_1\!\left(-j, \frac{\mu}{2}; \frac{\mu}{2}-j+1; \frac{z+1}{z-1}\right){}_3\tilde{F}_2\!\left(1, -n, n+1; j-m+1, 1-\mu; \frac{1-z}{2}\right)\left(\frac{z-1}{z+1}\right)^j \text{ /; } m \in \mathbb{N}$$

05.07.20.0008.01

$$\frac{\partial^m P_n^\mu(z)}{\partial z^m} = \frac{\Gamma\left(1-\frac{\mu}{2}\right)\Gamma(\mu+n+1)}{\Gamma(1-\mu+n)} \sum_{k=0}^m \sum_{j=0}^k \frac{(-1)^j\,2^{2j-k}\,k!\,\binom{m}{k}\,\Gamma(1-k+m-\mu+n)}{(k-j)!\,(2j-k)!\,\Gamma\left(1-j-\frac{\mu}{2}\right)\Gamma(k-m+\mu+n+1)}\,z^{2j-k}\left(1-z^2\right)^{\frac{1}{2}(k-2j-m)}\,P_n^{k-m+\mu}(z) \text{ /; } m \in \mathbb{N}$$

05.07.20.0009.01

$$\frac{\partial^m P_n^\mu(z)}{\partial z^m} = \sqrt{\pi}\sum_{k=0}^m (-1)^{m-k}\,z^{-k}\left(1-z^2\right)^{\frac{k-m}{2}}\binom{m}{k}(-\mu-n)_{m-k}\,{}_3\tilde{F}_2\!\left(1, -k, \frac{\mu}{2}; \frac{1-k}{2}, 1-\frac{k}{2}; \frac{z^2}{z^2-1}\right)(1-\mu+n)_{m-k}\,P_n^{k-m+\mu}(z) \text{ /; } m \in \mathbb{N}$$

Fractional integro-differentiation

With respect to z

05.07.20.0007.01

$$\frac{\partial^\alpha P_n^\mu(z)}{\partial z^\alpha} = \frac{\mu}{2}\,z^{1-\alpha}\sum_{k=0}^\infty (k+1)!\left(\frac{\mu}{2}+1\right)_k z^k\,\tilde{F}_{1\times 0\times 2}^{1\times 1\times 3}\!\left(\begin{matrix} k+2; -\frac{\mu}{2}; -n, n+1, 1-\frac{\mu}{2}; \\ k-\alpha+2;; k+2, 1-\mu; \end{matrix} -z, -\frac{z}{2}\right) + \Gamma\!\left(1-\frac{\mu}{2}\right)z^{-\alpha}\sum_{k=0}^\infty \left(-\frac{\mu}{2}\right)_k (-z)^k\,\tilde{F}_{2\times 1\times 1}^{3\times 1\times 1}\!\left(\begin{matrix} -n, n+1, 1-\frac{\mu}{2}; 1; k+1; \\ 1, 1-\mu; 1-\frac{\mu}{2}; k-\alpha+1; \end{matrix} \frac{1}{2}, -\frac{z}{2}\right)$$

Integration

Indefinite integration

Involving only one direct function

05.07.21.0001.01

$$\int P_n^\mu(z)\,dz = 2\sum_{k=0}^n \frac{(-n)_k\,(n+1)_k}{\Gamma(k-\mu+1)\,k!}\,\mathrm{B}_{\frac{z+1}{2}}\!\left(\frac{\mu+2}{2}, k-\frac{\mu}{2}+1\right)$$

Definite integration

Involving the direct function

05.07.21.0004.01

$$\int_0^1 P_n^m(x)\,dx = \frac{(-1)^m \pi\,(m+n)!\,2^{-2m-1}}{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{m+3}{2}\right)(n-m)!}\,{}_3F_2\left(\frac{1}{2}(m+n+1),\,\frac{m-n}{2},\,\frac{m}{2}+1;\,m+1,\,\frac{m+3}{2};\,1\right)\,/;\,n\in\mathbb{N}\land m\in\mathbb{N}$$

05.07.21.0005.01

$$\int_0^1 P_n^m(x)\,dx = \frac{(-1)^m\,2^{-2m-1}\sqrt{\pi}\;m!\,(m+n)!\,\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{1}{2}(-m+n+1)\right)}{(n-m)!\,\frac{m+n}{2}!\,\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{n+3}{2}\right)}\,/;\,n\in\mathbb{N}\land m\in\mathbb{N}$$

05.07.21.0006.01

$$\int_{-1}^1 P_n^m(x)\,dx = \frac{(-1)^n+(-1)^m}{\Gamma\left(\frac{n+3}{2}\right)\frac{n-m}{2}!}\,2^{m-2}\,m\,\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{1}{2}(m+n+1)\right)\,/;\,n\in\mathbb{N}\land m\in\mathbb{N}$$

05.07.21.0002.01

$$\int_{-1}^1 P_n^m(t)^2\,dt = \frac{2\,(m+n)!}{(2n+1)\,(n-m)!}\,/;\,m\in\mathbb{N}\land m\le n$$

Orthogonality:

05.07.21.0003.01

$$\int_{-1}^1 P_n^m(t)\,P_l^m(t)\,dt = \frac{2\,(m+n)!\,\delta_{n,l}}{(2n+1)\,(n-m)!}\,/;\,l\in\mathbb{N}\bigwedge m\in\mathbb{N}$$

05.07.21.0007.01

$$\int_{-1}^1 P_n^m(t)\,dt - \frac{((-1)^m+(-1)^n)\,2^{m-2}\,m\,\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{1}{2}(m+n+1)\right)}{\frac{n-m}{2}!\,\Gamma\left(\frac{n+3}{2}\right)}\,/;\,m\in\mathbb{Z}$$

05.07.21.0008.01

$$\int_0^\infty P_n^m\left(\frac{1-x}{1+x}\right)\frac{J_m\left(y\sqrt{x}\right)}{(1+x)^{3/2}}\,dx = \frac{(-1)^n\,(2y)^m\,e^{-y}}{n+\frac{1}{2}}\,L_{n-m}^{2m}(2y)\,/;\,n\in\mathbb{N}\bigwedge m\in\mathbb{N}\bigwedge y\in\mathbb{R}\bigwedge y\ge 0$$

05.07.21.0009.01

$$\int_{-1}^1 \frac{P_n^m(t)^2}{\left(1-t^2\right)^{p+1}}\,dt = \frac{(m-p-1)!}{2^{m+p}\,(2m-2p-1)!!}\left(\frac{(m+n)!}{(n-m)!\,m!}\right)^2 {}_4F_3\left(m-n,\,m+n+1,\,m+\frac{1}{2},\,m-p;\,2m+1,\,m+1,\,m-p+\frac{1}{2};\,1\right)/;$$

$$n\in\mathbb{N}\bigwedge m\in\mathbb{N}\bigwedge p\in\mathbb{N}\bigwedge p+1\le m\le n$$

05.07.21.0010.01

$$\int_0^\infty \frac{P_n^m\left(\frac{1-x}{x+1}\right)J_m\left(y\sqrt{x}\right)}{(x+1)^{3/2}}\,dx = \frac{(-1)^n}{n+1/2}\,(2y)^m\,e^{-y}\,L_{n-m}^{2m}(2y)\,/;\,n\in\mathbb{N}\bigwedge m\in\mathbb{N}\bigwedge 0\le m\le n\bigwedge y\in\mathbb{R}\bigwedge y\ge 0$$

Summation

Finite summation

05.07.23.0001.01

$$\sum_{m=-n}^{n} (-1)^m P_n^m(z) P_n^{-m}(z) == 1$$

05.07.23.0002.01

$$\sum_{m=-n}^{n} \frac{(n-m)! \cos(m(\phi-\phi_1))}{(m+n)!} P_n^m(\cos(\theta)) P_n^m(\cos(\theta_1)) == P_n(\cos(\theta)\cos(\theta_1) + \cos(\phi-\phi_1)\sin(\theta)\sin(\theta_1)) /; 0 < \theta < \frac{\pi}{2} \bigwedge 0 < \theta_1 < \frac{\pi}{2} \bigwedge 0 < \phi < \frac{\pi}{2} \bigwedge 0 < \phi_1 < \frac{\pi}{2}$$

Operations

Limit operation

05.07.25.0001.01

$$\lim_{n\to\infty} n^{-\mu} P_n^\mu\left(\cos\left(\frac{z}{n}\right)\right) == J_{-\mu}(z)$$

Representations through more general functions

Through hypergeometric functions

Involving ${}_2\tilde{F}_1$

05.07.26.0001.01

$$P_n^\mu(z) == \frac{(1+z)^{\mu/2}}{(1-z)^{\mu/2}} \, {}_2\tilde{F}_1\left(-n, n+1; 1-\mu; \frac{1-z}{2}\right)$$

05.07.26.0003.01

$$P_n^m(z) == \frac{(1+z)^{m/2}}{(1-z)^{m/2}} 2^{-m} (-n)_m (n+1)_m \, {}_2\tilde{F}_1\left(m-n, m+n+1; m+1; \frac{1-z}{2}\right) /; m \in \mathbb{N}^+$$

05.07.26.0024.01

$$P_n^\mu(z) == -\frac{\pi (z+1)^{\mu/2}}{(1-z)^{\mu/2} \sin(\pi\mu) \Gamma(-\mu-n) \Gamma(n-\mu+1)} \, {}_2\tilde{F}_1\left(-n, n+1; \mu+1; \frac{z+1}{2}\right) /; \mu \notin \mathbb{Z}$$

05.07.26.0004.01

$$P_n^\mu(z) == 2^{-n} \frac{(1+z)^{\mu/2+n}}{(1-z)^{\mu/2}} \, {}_2\tilde{F}_1\left(-n, -\mu-n; 1-\mu; \frac{z-1}{z+1}\right)$$

05.07.26.0025.01

$$P_n^\mu(z) = 2^\mu \pi \left(1-z^2\right)^{-\frac{\mu}{2}} \left(\frac{1}{\Gamma\left(\frac{1}{2}(-\mu-n+1)\right)\Gamma\left(\frac{1}{2}(-\mu+n+2)\right)} \, {}_2\tilde{F}_1\left(\frac{1}{2}(-\mu+n+1), -\frac{1}{2}(\mu+n); \frac{1}{2}; z^2\right) - \frac{z}{\Gamma\left(\frac{1}{2}(-\mu-n)\right)\Gamma\left(\frac{1}{2}(-\mu+n+1)\right)} \, {}_2\tilde{F}_1\left(\frac{1}{2}(-\mu-n+1), \frac{1}{2}(-\mu+n+2); \frac{3}{2}; z^2\right)\right)$$

05.07.26.0026.01

$$P_n^\mu(z) = -\frac{2^{-n-3}\,(-1)^n\,(\cos(2\,\pi\,\mu)-1)}{\sqrt{\pi}}\,\left(1-z^2\right)^{-\frac{\mu}{2}}$$
$$\left(\frac{\Gamma(\mu+n+1)}{z}\,\left(-z^2\right)^{\frac{\mu-n}{2}}\left(\csc\left(\frac{1}{2}\,\pi\,(\mu-n)\right)-\frac{\sqrt{-z^2}}{z}\,\sec\left(\frac{1}{2}\,\pi\,(\mu-n)\right)\right){}_2\tilde{F}_1\left(\frac{1}{2}\,(-\mu+n+1),\ \frac{1}{2}\,(-\mu+n+2);\ n+\frac{3}{2};\ \frac{1}{z^2}\right)+\right.$$
$$2^{2\,n+1}\,\Gamma(\mu-n)\left(\csc\left(\frac{1}{2}\,\pi\,(\mu+n)\right)-\frac{\sqrt{-z^2}}{z}\,\sec\left(\frac{1}{2}\,\pi\,(\mu+n)\right)\right)$$
$$\left.\left(-z^2\right)^{\frac{\mu+n}{2}}\,{}_2\tilde{F}_1\left(\frac{1}{2}\,(-\mu-n),\ \frac{1}{2}\,(-\mu-n+1);\ \frac{1}{2}-n;\ \frac{1}{z^2}\right)\right)\ /;\ z\notin(-1,\,0)$$

Involving ${}_2F_1$

05.07.26.0005.01

$$P_n^\mu(z) = \frac{1}{\Gamma(1-\mu)}\,\frac{(1+z)^{\mu/2}}{(1-z)^{\mu/2}}\,{}_2F_1\left(-n,\ n+1;\ 1-\mu;\ \frac{1-z}{2}\right)\ /;\ \mu\notin\mathbb{N}^+$$

05.07.26.0006.01

$$P_n^\mu(z) = \frac{\Gamma(-\mu)}{\Gamma(-\mu-n)\,\Gamma(n-\mu+1)}\,\frac{(1+z)^{\mu/2}}{(1-z)^{\mu/2}}\,{}_2F_1\left(-n,\ n+1;\ \mu+1;\ \frac{z+1}{2}\right)\ /;\ \mu\notin\mathbb{Z}$$

05.07.26.0027.01

$$P_n^\mu(z) = \frac{2^{-n}\,(z+1)^{\frac{\mu}{2}+n}}{\Gamma(\mu)\,(1-z)^{\mu/2}}\,{}_2F_1\left(-n,\ -\mu-n;\ 1-\mu;\ \frac{z-1}{z+1}\right)\ /;\ \mu\notin\mathbb{Z}$$

05.07.26.0028.01

$$P_n^\mu(z) = \frac{2^{-n}\,(-1)^n\,(2\,n)!}{n!\,\Gamma(n-\mu+1)}\,(1-z)^{n-\frac{\mu}{2}}\,(z+1)^{\mu/2}\,{}_2F_1\left(-n,\ \mu-n;\ -2\,n;\ \frac{2}{1-z}\right)$$

05.07.26.0029.01

$$P_n^\mu(z) = 2^\mu\,\sqrt{\pi}\,\left(1-z^2\right)^{-\frac{\mu}{2}}\left(\frac{1}{\Gamma\left(\frac{1}{2}\,(-\mu-n+1)\right)\Gamma\left(\frac{1}{2}\,(-\mu+n+2)\right)}\,{}_2F_1\left(\frac{1}{2}\,(-\mu+n+1),\ -\frac{1}{2}\,(\mu+n);\ \frac{1}{2};\ z^2\right)-\right.$$
$$\left.\frac{2\,z}{\Gamma\left(\frac{1}{2}\,(-\mu-n)\right)\Gamma\left(\frac{1}{2}\,(-\mu+n+1)\right)}\,{}_2F_1\left(\frac{1}{2}\,(-\mu-n+1),\ \frac{1}{2}\,(-\mu+n+2);\ \frac{3}{2};\ z^2\right)\right)$$

05.07.26.0030.01

$$P_n^\mu(z) = \frac{2^{-n-2}(\cos(2\pi\mu)-1)}{\pi\sqrt{\pi}}\left(1-z^2\right)^{-\frac{\mu}{2}}\left(\frac{\Gamma(\mu+n+1)\,\Gamma\left(\frac{1}{2}-n\right)}{z\,(2n+1)}\left(\frac{\sqrt{-z^2}}{z}\sec\left(\frac{1}{2}\pi(\mu-n)\right)-\csc\left(\frac{1}{2}\pi(\mu-n)\right)\right)\right.$$
$$\left(-z^2\right)^{\frac{\mu-n}{2}}{}_2F_1\left(\frac{1}{2}(-\mu+n+1),\,\frac{1}{2}(-\mu+n+2);\,n+\frac{3}{2};\,\frac{1}{z^2}\right)+2^{2n}\,\Gamma(\mu-n)\,\Gamma\left(n+\frac{1}{2}\right)$$
$$\left.\left(\frac{\sqrt{-z^2}\,\sec\left(\frac{1}{2}\pi(\mu+n)\right)}{z}-\csc\left(\frac{1}{2}\pi(\mu+n)\right)\right)\left(-z^2\right)^{\frac{\mu+n}{2}}{}_2F_1\left(\frac{1}{2}(-\mu-n),\,\frac{1}{2}(-\mu-n+1);\,\frac{1}{2}-\nu;\,\frac{1}{z^2}\right)\right)/;\, z\notin(-1,0)$$

Through Meijer *G*

Classical cases for the direct function itself

05.07.26.0007.01

$$P_n^\mu(z) = -\frac{1}{\pi}\frac{(1+z)^{\mu/2}}{(1-z)^{\mu/2}}\lim_{\nu\to n}\sin(\pi\nu)\,G_{2,2}^{1,2}\left(\frac{z-1}{2}\,\middle|\,\begin{matrix}\nu+1,\,-\nu\\0,\,\mu\end{matrix}\right)/;\,n\in\mathbb{Z}$$

Classical cases involving algebraic functions

05.07.26.0008.01

$$(z+1)^n\,P_n^\mu\left(\frac{1-z}{1+z}\right) = \frac{1}{\Gamma(-\mu-n)\,\Gamma(-n)}\,G_{2,2}^{1,2}\left(z\,\middle|\,\begin{matrix}1-\frac{\mu}{2}+n,\,\frac{\mu}{2}+n+1\\-\frac{\mu}{2},\,\frac{\mu}{2}\end{matrix}\right)$$

05.07.26.0009.01

$$(z+1)^n\,P_n^\mu\left(\frac{z-1}{z+1}\right) = \frac{1}{\Gamma(-\mu-n)\,\Gamma(-n)}\,G_{2,2}^{2,1}\left(z\,\middle|\,\begin{matrix}\frac{\mu}{2}+n+1,\,1-\frac{\mu}{2}+n\\\frac{\mu}{2},\,-\frac{\mu}{2}\end{matrix}\right)$$

05.07.26.0010.01

$$(z+1)^{n/2}\,P_n^\mu\left(\frac{1}{\sqrt{z+1}}\right) = \frac{2^{-n-1}}{\sqrt{\pi}\,\Gamma(-\mu-n)}\,G_{2,2}^{1,2}\left(z\,\middle|\,\begin{matrix}\frac{n+1}{2},\,\frac{n}{2}+1\\-\frac{\mu}{2},\,\frac{\mu}{2}\end{matrix}\right)$$

05.07.26.0011.01

$$(z+1)^{n/2}\,P_n^\mu\left(\sqrt{\frac{z}{z+1}}\right) = \frac{2^{-n-1}}{\sqrt{\pi}\,\Gamma(-\mu-n)}\,G_{2,2}^{2,1}\left(z\,\middle|\,\begin{matrix}\frac{\mu+n}{2}+1,\,\frac{n-\mu}{2}+1\\0,\,\frac{1}{2}\end{matrix}\right)/;\,z\notin(-\infty,0)$$

Classical cases involving unit step θ

05.07.26.0012.01

$$\theta(1-|z|)\,(1-z)^{-\frac{\mu}{2}}\,P_n^\mu(2z-1) = G_{2,2}^{2,0}\left(z\,\middle|\,\begin{matrix}-\frac{\mu}{2}-n,\,1-\frac{\mu}{2}+n\\-\frac{\mu}{2},\,\frac{\mu}{2}\end{matrix}\right)/;\,z\notin(-1,0)$$

05.07.26.0013.01

$$\theta(|z|-1)\,(z-1)^{-\frac{\mu}{2}}\,P_n^\mu\left(\frac{2}{z}-1\right) = G_{2,2}^{0,2}\left(z\,\middle|\,\begin{matrix}1,\,1-\mu\\-n,\,n+1\end{matrix}\right)/;\,z\notin(-\infty,-1)$$

05.07.26.0014.01

$$\theta(1-|z|)\,(1-z)^{-\frac{\mu}{2}}\,P_n^\mu\left(\sqrt{z}\right) = 2^\mu\,G_{2,2}^{2,0}\left(z\,\middle|\,\begin{array}{c}\frac{n-\mu}{2}+1,\ \frac{1-\mu-n}{2}\\ 0,\ \frac{1}{2}\end{array}\right)\,/;\, z\notin(-1,0)$$

05.07.26.0015.01

$$\theta(|z|-1)\,(z-1)^{-\frac{\mu}{2}}\,P_n^\mu\left(\frac{1}{\sqrt{z}}\right) = 2^\mu\,G_{2,2}^{0,2}\left(z\,\middle|\,\begin{array}{c}\frac{1-\mu}{2},\ 1-\frac{\mu}{2}\\ -\frac{n}{2},\ \frac{n+1}{2}\end{array}\right)$$

05.07.26.0016.01

$$\frac{\theta(1-|z|)}{\sqrt{1-z}}\left(P_n^\mu\left(-\sqrt{1-z}\right)+P_n^\mu\left(\sqrt{1-z}\right)\right) = \frac{2^{\mu+1}\,\pi}{\Gamma\left(\frac{1}{2}\,(1-\mu-n)\right)\Gamma\left(\frac{1}{2}\,(2-\mu+n)\right)}\,G_{2,2}^{2,0}\left(z\,\middle|\,\begin{array}{c}-\frac{n}{2},\ \frac{n+1}{2}\\ -\frac{\mu}{2},\ \frac{\mu}{2}\end{array}\right)\,/;\, z\notin(-1,0)$$

05.07.26.0017.01

$$\frac{\theta(|z|-1)}{\sqrt{z-1}}\left(P_n^\mu\left(-\sqrt{\frac{z-1}{z}}\right)+P_n^\mu\left(\sqrt{\frac{z-1}{z}}\right)\right) = \frac{2^{\mu+1}\,\pi}{\Gamma\left(\frac{1}{2}\,(-\mu-n+1)\right)\Gamma\left(\frac{1}{2}\,(-\mu+n+2)\right)}\,G_{2,2}^{0,2}\left(z\,\middle|\,\begin{array}{c}\frac{1-\mu}{2},\ \frac{\mu+1}{2}\\ -\frac{n}{2},\ \frac{n+1}{2}\end{array}\right)\,/;\, z\notin(-\infty,-1)$$

Generalized cases involving algebraic functions

05.07.26.0018.01

$$\left(z^2+1\right)^{n/2}\,P_n^\mu\left(\frac{z}{\sqrt{z^2+1}}\right) = \frac{2^{-n-1}}{\sqrt{\pi}\,\Gamma(-\mu-n)}\,G_{2,2}^{2,1}\left(z,\ \frac{1}{2}\,\middle|\,\begin{array}{c}\frac{\mu+n}{2}+1,\ \frac{n-\mu}{2}+1\\ 0,\ \frac{1}{2}\end{array}\right)\,/;\, \mathrm{Re}(z)>0$$

Generalized cases involving unit step θ

05.07.26.0019.01

$$\theta(1-|z|)\left(1-z^2\right)^{-\frac{\mu}{2}}\,P_n^\mu(z) = 2^\mu\,G_{2,2}^{2,0}\left(z,\ \frac{1}{2}\,\middle|\,\begin{array}{c}\frac{n-\mu}{2}+1,\ \frac{1-\mu-n}{2}\\ 0,\ \frac{1}{2}\end{array}\right)$$

05.07.26.0020.01

$$\theta(|z|-1)\left(z^2-1\right)^{-\frac{\mu}{2}}\,P_n^\mu\left(\frac{1}{z}\right) = 2^\mu\,G_{2,2}^{0,2}\left(z,\ \frac{1}{2}\,\middle|\,\begin{array}{c}\frac{1-\mu}{2},\ 1-\frac{\mu}{2}\\ -\frac{n}{2},\ \frac{n+1}{2}\end{array}\right)\,/;\, |z|<0\lor\mathrm{Re}(z)>0$$

05.07.26.0021.01

$$\frac{\theta(|z|-1)}{\sqrt{z^2-1}}\left(P_n^\mu\left(\frac{\sqrt{z^2-1}}{z}\right)+P_n^\mu\left(-\frac{\sqrt{z^2-1}}{z}\right)\right) = \frac{2^{\mu+1}\,\pi}{\Gamma\left(\frac{1-\mu-n}{2}\right)\Gamma\left(\frac{n-\mu}{2}+1\right)}\,G_{2,2}^{0,2}\left(z,\ \frac{1}{2}\,\middle|\,\begin{array}{c}\frac{1-\mu}{2},\ \frac{\mu+1}{2}\\ -\frac{n}{2},\ \frac{n+1}{2}\end{array}\right)\,/;\, |z|<0\lor\mathrm{Re}(z)>0$$

Generalized cases involving sgn

05.07.26.0022.01

$$\frac{\left(\mathrm{sgn}\,(|z|-1)\left(z^2-1\right)\right)^{n+\frac{1}{2}}}{\sqrt{z+1}}\,P_n^{-n-\frac{1}{2}}\left(\frac{2\sqrt{z}}{z+1}\right) =$$

$$-\frac{\Gamma(n+1)}{2^{n+\frac{1}{2}}\sqrt{\pi}}\left(\cos\left(\pi\left(n+\frac{1}{4}\right)\right)G_{2,2}^{1,1}\left(z,\ \frac{1}{2}\,\middle|\,\begin{array}{c}n+1,\ n+\frac{5}{4}\\ \frac{1}{4},\ 0\end{array}\right)+\sin\left(\pi\left(n+\frac{1}{4}\right)\right)G_{2,2}^{1,1}\left(z,\ \frac{1}{2}\,\middle|\,\begin{array}{c}n+\frac{5}{4},\ n+1\\ 0,\ \frac{1}{4}\end{array}\right)\right)\,/;\, \mathrm{Re}\,(z)>0$$

Through other functions

Involving some hypergeometric-type functions

05.07.26.0023.01

$$P_n^\mu(z) = \frac{\Gamma(n+1)}{\Gamma(n-\mu+1)} \frac{(1+z)^{\mu/2}}{(1-z)^{\mu/2}} P_n^{(-\mu,\mu)}(z)$$

Involving spheroidal functions

05.07.26.0031.01

$$P_n^\mu(z) = PS_{n,\mu}(0, z)$$

Representations through equivalent functions

With related functions

Involving Gegenbauer functions

05.07.27.0001.01

$$P_n^\mu(z) = \frac{2^{-\mu}\ \Gamma\left(\frac{1}{2}-\mu\right)\Gamma(\mu+n+1)}{\sqrt{\pi}\ \Gamma(1-\mu+n)} \left(1-z^2\right)^{-\mu/2} C_{\mu+n}^{\frac{1}{2}-\mu}(z)$$

Involving Legendre functions

05.07.27.0002.01

$$P_n^\mu(z) = \frac{(z-1)^{\mu/2}}{(1-z)^{\mu/2}} \mathfrak{P}_n^\mu(z)$$

05.07.27.0003.01

$$P_n^\mu(z) = -\frac{2\csc(\mu\,\pi)}{\pi} \left(\cos(\mu\,\pi)\, Q_n^\mu(z) - (n-\mu+1)_{2\,\mu}\, Q_n^{-\mu}(z)\right) /;\ \mu \notin \mathbb{Z}$$

05.07.27.0004.01

$$P_n^\mu(z) = \frac{\csc(\pi\,\mu)}{\pi^2} \left(\sin(2\,\pi\,\mu)\,\Gamma(\mu-n)\,\Gamma(\mu+n+1)\, Q_n^{-\mu}(-z) - (-1)^n\, 2\,\pi\ Q_n^\mu(-z)\right) /;\ \mu \notin \mathbb{Z}$$

05.07.27.0005.01

$$P_n^\mu(x) = e^{\frac{\pi\,i\,\mu}{2}} \lim_{z\to x+i\,0} \mathfrak{P}_n^\mu(z) /;\ x < -1$$

05.07.27.0006.01

$$P_n^\mu(x) = e^{-\frac{\pi\,i\,\mu}{2}} \lim_{z\to x+i\,0} \mathfrak{P}_n^\mu(z) /;\ x > 1$$

05.07.27.0007.01

$$P_n^\mu(x) = e^{\frac{\pi\,i\,\mu}{2}}\, \mathfrak{P}_n^\mu(x) /;\ x < -1$$

05.07.27.0008.01

$$P_n^\mu(x) = e^{-\frac{\pi\,i\,\mu}{2}}\, \mathfrak{P}_n^\mu(x) /;\ x > 1$$

05.07.27.0009.01

$$P_n^\mu(x) = e^{-\frac{\pi\,i\,\mu}{2}} \lim_{z\to x-i\,0} \mathfrak{P}_n^\mu(z) /;\ -1 < x < 1$$

05.07.27.0010.01

$$P_n^\mu(x) = e^{\frac{\pi i \mu}{2}} \mathfrak{P}_n^\mu(x) /; -1 < x < 1$$

05.07.27.0011.01

$$P_n^\mu(x) = \frac{1}{\pi i} e^{-\pi i \mu} \left(e^{\frac{\pi i \mu}{2}} \lim_{z \to x - i 0} \mathfrak{Q}_n^\mu(z) - e^{-\frac{\pi i \mu}{2}} \lim_{z \to x + i 0} \mathfrak{Q}_n^\mu(z) \right) /; -1 < x < 1$$

Involving spherical harmonic functions

05.07.27.0012.01

$$P_n^\mu(z) = \frac{2\sqrt{\pi}\sqrt{\Gamma(\mu+n+1)}}{\sqrt{2n+1}\sqrt{\Gamma(1-\mu+n)}} Y_n^\mu\left(\cos^{-1}(z), 0\right)$$

Theorems

One infinite sum

$$\sum_{n=0}^{\infty} (2n+1) i^n \sum_{m=0}^{n} (2-\delta_m) \frac{(n-m)!}{(n+m)!} \cos(\varphi-\alpha) P_n^m(\cos(\beta)) P_n^m(\cos(\vartheta)) J_{n+\frac{1}{2}}(|\boldsymbol{k}||\boldsymbol{r}|) = e^{i \boldsymbol{k}.\boldsymbol{r}} /;$$

$$r = |\boldsymbol{r}| \{\cos(\varphi)\sin(\vartheta), \sin(\varphi)\sin(\vartheta), \cos(\vartheta)\} \land \boldsymbol{k} = |\boldsymbol{k}| \{\cos(\alpha)\sin(\beta), \sin(\alpha)\sin(\beta), \cos(\beta)\}$$

Eigenfunctions of the Schrödinger equation

Legendre functions P_n^m are eigenfunctions of the Schrödinger equation with reflectionless, shape invariant, supersymmetric potential: $-\frac{\partial^2 \psi(x)}{\partial x^2} - \left(n(n+1)\operatorname{sech}^2(x)\right)\psi(x) = -m^2 \psi(x)$.

The solution of Dirichlet problem for Laplace equation in spherical coordinates

The solution of the Dirichlet problem $\psi(1, \phi, \vartheta) = \psi_0(\phi, \vartheta)$ for the $\Delta\psi(\phi, \vartheta) = 0$ on unit sphere in spherical coordinates is given by:

$$\psi(r, \phi, \vartheta) = \sum_{n=0}^{\infty}\sum_{m=0}^{n} \frac{2n+1}{2\pi(1+\delta_m)} \frac{(n-m)!}{(n+m)!} r^n P_n^m(\cos(\vartheta)) \int_0^\pi \left(\int_{-\pi}^{\pi} \psi_0\left(\tilde{\phi}, \tilde{\vartheta}\right) \cos\left(m\left(\phi-\tilde{\phi}\right)\right) d\tilde{\phi} \right) P_n^m\left(\cos\left(\tilde{\vartheta}\right)\right) \sin\left(\tilde{\vartheta}\right) d\tilde{\vartheta}$$

History

– D. Bernoulli (1748)
– A. M. Legendre (1782)
– E. Heine (1842)
– F. Neumann (1848)
– L. Schläfli (1881) used complex μ, ν
– E. Hobson (1896)

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