LaguerreLGeneral

Notations

Traditional name

Laguerre function

Traditional notation

$L_\nu(z)$

Mathematica StandardForm notation

LaguerreL[ν, z]

Primary definition

07.02.02.0001.01

$$L_\nu(z) = {}_1F_1(-\nu; 1; z)$$

Specific values

Specialized values

For fixed ν

07.02.03.0001.01

$$L_\nu(0) = 1$$

For fixed z

07.02.03.0002.01

$$L_0(z) = 1$$

07.02.03.0003.01

$$L_1(z) = 1 - z$$

07.02.03.0004.01

$$L_2(z) = \frac{1}{2}\left(2 - 4z + z^2\right)$$

07.02.03.0005.01

$$L_3(z) = \frac{1}{6}\left(6 - 18z + 9z^2 - z^3\right)$$

07.02.03.0006.01

$$L_4(z) = \frac{1}{24}\left(24 - 96 z + 72 z^2 - 16 z^3 + z^4\right)$$

07.02.03.0007.01

$$L_5(z) = \frac{1}{120}\left(120 - 600 z + 600 z^2 - 200 z^3 + 25 z^4 - z^5\right)$$

07.02.03.0008.01

$$L_6(z) = \frac{1}{720}\left(720 - 4320 z + 5400 z^2 - 2400 z^3 + 450 z^4 - 36 z^5 + z^6\right)$$

07.02.03.0009.01

$$L_7(z) = \frac{1}{5040}\left(5040 - 35\,280 z + 52\,920 z^2 - 29\,400 z^3 + 7350 z^4 - 882 z^5 + 49 z^6 - z^7\right)$$

07.02.03.0010.01

$$L_8(z) = \frac{1}{40\,320}\left(40\,320 - 322\,560 z + 564\,480 z^2 - 376\,320 z^3 + 117\,600 z^4 - 18\,816 z^5 + 1568 z^6 - 64 z^7 + z^8\right)$$

07.02.03.0011.01

$$L_9(z) = \frac{1}{362\,880}\left(362\,880 - 3\,265\,920 z + 6\,531\,840 z^2 - 5\,080\,320 z^3 + 1\,905\,120 z^4 - 381\,024 z^5 + 42\,336 z^6 - 2592 z^7 + 81 z^8 - z^9\right)$$

07.02.03.0012.01

$$L_{10}(z) = \frac{1}{3\,628\,800}\left(3\,628\,800 - 36\,288\,000 z + 81\,648\,000 z^2 - 72\,576\,000 z^3 + 31\,752\,000 z^4 - 7\,620\,480 z^5 + 1\,058\,400 z^6 - 86\,400 z^7 + 4050 z^8 - 100 z^9 + z^{10}\right)$$

07.02.03.0013.01

$$L_n(z) = \sum_{k=0}^{n} \frac{(-n)_k z^k}{k!^2} \;/;\; n \in \mathbb{N}$$

07.02.03.0014.01

$$L_{-n}(z) = e^z \sum_{k=0}^{n-1} \frac{(-1)^k (1-n)_k z^k}{k!^2} \;/;\; n \in \mathbb{N}^+$$

General characteristics

Domain and analyticity

$L_\nu(z)$ is an analytical function of ν, zm which is defined in $\mathbb{C}^2$. For fixed ν, it is an entire function of z. For positive integer ν, $L_\nu(z)$ degenerates to a polynomial in z.

07.02.04.0001.01

$$(\nu * z) \longrightarrow L_\nu(z) :: (\mathbb{C} \otimes \mathbb{C}) \longrightarrow \mathbb{C}$$

Symmetries and periodicities

Mirror symmetry

07.02.04.0002.01

$$L_{\bar{\nu}}(\bar{z}) = \overline{L_\nu(z)}$$

Periodicity

No periodicity

Poles and essential singularities

With respect to z

For fixed ν /; $\nu \notin \mathbb{N}$, the function $L_\nu(z)$ has only one singular point at $z = \tilde{\infty}$. It is an essential singular point.

07.02.04.0003.01

$$\mathcal{S}ing_z(L_\nu(z)) = \{\{\tilde{\infty}, \infty\}\} /; \nu \notin \mathbb{N}$$

For positive integer ν, the function $L_\nu(z)$ is polynomial and has pole of order ν at $z = \tilde{\infty}$.

07.02.04.0004.01

$$\mathcal{S}ing_z(L_\nu(z)) = \{\{\tilde{\infty}, \nu\}\} /; \nu \in \mathbb{N}^+$$

With respect to ν

For fixed z, the function $L_\nu(z)$ has only one singular point at $\nu = \tilde{\infty}$. It is an essential singular point. .

07.02.04.0005.01

$$\mathcal{S}ing_\nu(L_\nu(z)) = \{\{\tilde{\infty}, \infty\}\}$$

Branch points

With respect to z

For fixed ν, the function $L_\nu(z)$ does not have branch points.

07.02.04.0006.01

$$\mathcal{BP}_z(L_\nu(z)) = \{\}$$

With respect to ν

For fixed z, the function $L_\nu(z)$ does not have branch points.

07.02.04.0007.01

$$\mathcal{BP}_\nu(L_\nu(z)) = \{\}$$

Branch cuts

With respect to z

For fixed integer ν, the function $L_\nu(z)$ does not have branch cuts.

07.02.04.0008.01

$$\mathcal{BC}_z(L_\nu(z)) = \{\}$$

With respect to ν

For fixed z, the function $L_\nu(z)$ does not have branch cuts.

07.02.04.0009.01

$$\mathcal{BC}_\nu(L_\nu(z)) == \{\}$$

Series representations

Generalized power series

Expansions at generic point $z == z_0$

For the function itself

07.02.06.0009.01

$$L_\nu(z) \propto L_\nu(z_0) - L_{\nu-1}^1(z_0)\,(z - z_0) + \frac{1}{2}\,L_{\nu-2}^2(z_0)\,(z - z_0)^2 + \ldots\,/;\,(z \to z_0)$$

07.02.06.0010.01

$$L_\nu(z) \propto L_\nu(z_0) - L_{\nu-1}^1(z_0)\,(z - z_0) + \frac{1}{2}\,L_{\nu-2}^2(z_0)\,(z - z_0)^2 + O\big((z - z_0)^3\big)$$

07.02.06.0011.01

$$L_\nu(z) == \sum_{k=0}^{\infty} \frac{(-1)^k}{k!}\,L_{\nu-k}^k(z_0)\,(z - z_0)^k$$

07.02.06.0012.01

$$L_\nu(z) == \Gamma(\nu+1) \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(\nu-k+1)}\,{}_1\tilde{F}_1(k-\nu;\,k+1;\,z_0)\,(z - z_0)^k$$

07.02.06.0013.01

$$L_\nu(z) == \sum_{k=0}^{\infty} \frac{z_0^{-k}}{k!}\,{}_1\tilde{F}_1(-\nu;\,1-k;\,z_0)\,(z - z_0)^k$$

07.02.06.0014.01

$$L_\nu(z) == \tilde{F}_{1\times 0\times 0}^{1\times 0\times 0}\left(\begin{matrix} -\nu;;; \\ 1;;; \end{matrix}\, z_0,\, z - z_0\right)$$

07.02.06.0015.01

$$L_\nu(z) \propto L_\nu(z_0)\,(1 + O(z - z_0))$$

Expansions at $z == 0$

For the function itself

General case

07.02.06.0001.02

$$L_\nu(z) \propto 1 - \nu\,z - \frac{(1-\nu)\,\nu}{4}\,z^2 - \frac{(2-\nu)\,(1-\nu)\,\nu}{36}\,z^3 - \ldots\,/;\,(z \to 0)$$

07.02.06.0016.01

$$L_\nu(z) \propto 1 - \nu\,z - \frac{(1-\nu)\,\nu}{4}\,z^2 - \frac{(2-\nu)\,(1-\nu)\,\nu}{36}\,z^3 - O\big(z^4\big)$$

07.02.06.0002.01

$$L_\nu(z) = \sum_{k=0}^{\infty} \frac{(-\nu)_k z^k}{k!^2}$$

07.02.06.0003.01

$$L_\nu(z) = {}_1F_1(-\nu; 1; z)$$

07.02.06.0004.02

$$L_\nu(z) \propto 1 + O(z)$$

07.02.06.0017.01

$$L_\nu(z) = F_\infty(z, \nu) /; \left(\left(F_m(z, \nu) = \sum_{k=0}^{m} \frac{(-\nu)_k z^k}{k!^2} = L_\nu(z) + \frac{\Gamma(m-\nu+1)\,\Gamma(\nu+1)\sin(\pi\nu)}{\pi\,(m+1)!^2} z^{m+1}\, {}_2F_2(1, m-\nu+1; m+2, m+2; z) \right) \bigwedge m \in \mathbb{N} \right)$$

Summed form of the truncated series expansion.

Special cases

07.02.06.0005.01

$$L_n(z) = \sum_{k=0}^{n} \frac{(-1)^k}{k!} \binom{n}{n-k} z^k /; n \in \mathbb{N}$$

07.02.06.0006.01

$$L_n(z) = \frac{1}{n!} \sum_{k=0}^{n} \frac{(-n)_k (k+1)_{n-k} z^k}{k!} /; n \in \mathbb{N}$$

07.02.06.0018.01

$$L_n(z) \propto 1 + O(z) /; n \in \mathbb{N}$$

Asymptotic series expansions

07.02.06.0019.01

$$L_\nu(z) \propto \frac{e^z z^{-\nu-1}}{\Gamma(-\nu)} \left(1 + \frac{(1+\nu)^2}{z} + \frac{(1+\nu)^2 (2+\nu)^2}{2 z^2} + \ldots\right) + \frac{(-z)^\nu}{\nu!} \left(1 - \frac{\nu^2}{z} + \frac{(1-\nu)^2 \nu^2}{2 z^2} - \ldots\right) /; (|z| \to \infty)$$

07.02.06.0020.01

$$L_\nu(z) \propto \frac{e^z z^{-\nu-1}}{\Gamma(-\nu)} \left(\sum_{k=0}^{n} \frac{(\nu+1)_k^{\,2} z^{-k}}{k!} + O\left(z^{-n-1}\right)\right) + \frac{(-z)^\nu}{\nu!} \left(\sum_{k=0}^{n} \frac{(-1)^k (-\nu)_k^{\,2} z^{-k}}{k!} + O\left(z^{-n-1}\right)\right) /; (|z| \to \infty)$$

07.02.06.0007.01

$$L_\nu(z) \propto \frac{(-z)^\nu}{\nu!}\, {}_2F_0\left(-\nu, -\nu; ; -\frac{1}{z}\right) + \frac{1}{\Gamma(-\nu)} z^{-\nu-1} e^z\, {}_2F_0\left(\nu+1, \nu+1; ; \frac{1}{z}\right) /; (|z| \to \infty)$$

07.02.06.0008.01

$$L_\nu(z) \propto \frac{1}{\nu!} (-z)^\nu \left(1 + O\left(\frac{1}{z}\right)\right) + \frac{1}{\Gamma(-\nu)} z^{-\nu-1} e^z \left(1 + O\left(\frac{1}{z}\right)\right) /; (|z| \to \infty)$$

Integral representations

On the real axis

Of the direct function

07.02.07.0001.01

$$L_\nu(z) == -\frac{\sin(\nu\pi)}{\pi}\int_0^1 e^{z t}\, t^{-\nu-1}\,(1-t)^\nu\, dt \,/;\, -1 < \mathrm{Re}(\nu) < 0$$

07.02.07.0002.01

$$L_n(z) == \frac{1}{2\pi}\int_0^{2\pi} e^{-z e^{i t}}\left(e^{-i t}+1\right)^n dt \,/;\, n \in \mathbb{N}$$

07.02.07.0003.01

$$L_\nu(z) == \frac{1}{\nu!}\, e^z \int_0^\infty e^{-t}\, t^\nu\, J_0\!\left(2\sqrt{t z}\right) dt \,/;\, \mathrm{Re}(\nu) > -1$$

Integral representations of negative integer order

Rodrigues-type formula.

07.02.07.0004.01

$$L_n(z) == \frac{e^z}{n!}\,\frac{\partial^n (z^n e^{-z})}{\partial z^n} \,/;\, n \in \mathbb{N}$$

Limit representations

07.02.09.0001.01

$$L_\nu(z) == \lim_{b\to\infty} P_\nu^{(0,b)}\!\left(1-\frac{2z}{b}\right)$$

Generating functions

07.02.11.0001.01

$$L_n(z) == \left([t^n]\,(1-t)^{-1}\exp\!\left(\frac{t z}{t-1}\right)\right) /;\, n \in \mathbb{N}$$

Differential equations

Ordinary linear differential equations and wronskians

For the direct function itself

07.02.13.0003.01

$$z\, w''(z) + (1-z)\, w'(z) + \nu\, w(z) == 0 \,/;\, w(z) == c_1\, L_\nu(z) \;+\; c_2\left(e^z\, G_{1,2}^{2,0}\!\left(z \,\middle|\, \begin{matrix} -\nu \\ 0,\, 0 \end{matrix}\right) + G_{1,2}^{2,0}\!\left(-z \,\middle|\, \begin{matrix} \nu+1 \\ 0,\, 0 \end{matrix}\right)\right)$$

07.02.13.0004.01

$$W_z\!\left(L_\nu(z),\, G_{1,2}^{2,0}\!\left(-z \,\middle|\, \begin{matrix} \nu+1 \\ 0,\, 0 \end{matrix}\right) + e^z\, G_{1,2}^{2,0}\!\left(z \,\middle|\, \begin{matrix} -\nu \\ 0,\, 0 \end{matrix}\right)\right) == \frac{e^z}{z\,\Gamma(-\nu)} - \frac{e^z}{z\,\Gamma(\nu+1)}$$

07.02.13.0005.01

$$z\, w''(z) + (1-z)\, w'(z) + \nu\, w(z) == 0 \,/;\, w(z) == c_1\, L_\nu(z) \;+\; c_2\, G_{1,2}^{2,0}\!\left(-z \,\middle|\, \begin{matrix} \nu+1 \\ 0,\, 0 \end{matrix}\right) \bigwedge -\nu \notin \mathbb{N}^+$$

07.02.13.0006.01

$$W_z\left(L_\nu(z),\, G_{1,2}^{2,0}\left(-z \,\middle|\, \begin{matrix} \nu+1 \\ 0,\, 0 \end{matrix}\right)\right) = -\frac{e^z}{z\,\Gamma(\nu+1)}$$

07.02.13.0001.01

$$z\,w''(z) + (1-z)\,w'(z) + \nu\,w(z) = 0 \;/;\; w(z) = c_1\,L_\nu(z) + c_2\,U(-\nu,\,1,\,z) \;/;\; \nu \notin \mathbb{Z}$$

07.02.13.0002.02

$$W_z(L_\nu(z),\, U(-\nu,\,1,\,z)) = -\frac{e^z}{z\,\Gamma(-\nu)}$$

07.02.13.0007.01

$$w''(z) + \left(\frac{g'(z)}{g(z)} - g'(z) - \frac{g''(z)}{g'(z)}\right) w'(z) + \frac{\nu\,g'(z)^2}{g(z)}\,w(z) = 0 \;/;\; w(z) = c_1\,L_\nu(g(z)) + c_2\,G_{1,2}^{2,0}\left(-g(z) \,\middle|\, \begin{matrix} \nu+1 \\ 0,\, 0 \end{matrix}\right) \bigwedge -\nu \notin \mathbb{N}^+$$

07.02.13.0008.01

$$W_z\left(L_\nu(g(z)),\, G_{1,2}^{2,0}\left(-g(z) \,\middle|\, \begin{matrix} \nu+1 \\ 0,\, 0 \end{matrix}\right)\right) = -\frac{g'(z)\,e^{g(z)}}{g(z)\,\nu!}$$

07.02.13.0009.01

$$h(z)^2\,w''(z) + h(z)^2\left(\frac{g'(z)}{g(z)} - g'(z) - \frac{2\,h'(z)}{h(z)} - \frac{g''(z)}{g'(z)}\right) w'(z) + \left(2\,h'(z)^2 + h(z)\left(g'(z)\,h'(z) + \frac{g''(z)\,h'(z)}{g'(z)} - h''(z)\right) - \frac{h(z)\,g'(z)\,(-\nu\,h(z)\,g'(z) + h'(z))}{g(z)}\right) w(z) = 0 \;/;$$
$$w(z) = c_1\,h(z)\,L_\nu(z) + c_2\,h(z)\,G_{1,2}^{2,0}\left(-g(z) \,\middle|\, \begin{matrix} \nu+1 \\ 0,\, 0 \end{matrix}\right) \bigwedge -\nu \notin \mathbb{N}^+$$

07.02.13.0010.01

$$W_z\left(h(z)\,L_\nu(g(z)),\, h(z)\,G_{1,2}^{2,0}\left(-g(z) \,\middle|\, \begin{matrix} \nu+1 \\ 0,\, 0 \end{matrix}\right)\right) = -\frac{h(z)^2\,g'(z)\,e^{g(z)}}{g(z)\,\nu!}$$

07.02.13.0011.01

$$z^2\,w''(z) + z\,(1 - 2\,s - a\,r\,z^r)\,w'(z) + \left(a\,r\,(s + r\,\nu)\,z^r + s^2\right) w(z) = 0 \;/;\; w(z) = c_1\,z^s\,L_\nu(a\,z^r) + c_2\,z^s\,G_{1,2}^{2,0}\left(-a\,z^r \,\middle|\, \begin{matrix} \nu+1 \\ 0,\, 0 \end{matrix}\right) \bigwedge -\nu \notin \mathbb{N}^+$$

07.02.13.0012.01

$$W_z\left(z^s\,L_\nu(a\,z^r),\, z^s\,G_{1,2}^{2,0}\left(-a\,z^r \,\middle|\, \begin{matrix} \nu+1 \\ 0,\, 0 \end{matrix}\right)\right) = -\frac{r\,e^{a\,z^r}\,z^{2\,s-1}}{\nu!}$$

07.02.13.0013.01

$$w''(z) - (a\,r^z\,\log(r) + 2\log(s))\,w'(z) + \left(a\,\nu\,\log^2(r)\,r^z + \log^2(s) + a\,r^z\,\log(r)\,\log(s)\right) w(z) = 0 \;/;$$
$$w(z) = c_1\,s^z\,L_\nu(a\,r^z) + c_2\,s^z\,G_{1,2}^{2,0}\left(-a\,r^z \,\middle|\, \begin{matrix} \nu+1 \\ 0,\, 0 \end{matrix}\right) \bigwedge -\nu \notin \mathbb{N}^+$$

07.02.13.0014.01

$$W_z\left(s^z\,L_\nu(a\,r^z),\, s^z\,G_{1,2}^{2,0}\left(-a\,r^z \,\middle|\, \begin{matrix} \nu+1 \\ 0,\, 0 \end{matrix}\right)\right) = -\frac{e^{a\,r^z}\,s^{2\,z}\,\log(r)}{\nu!}$$

Transformations

Transformations and argument simplifications

Argument involving basic arithmetic operations

07.02.16.0001.01

$$L_\nu(-z) = e^{-z} L_{-\nu-1}(z)$$

07.02.16.0002.01

$$L_{-\nu}(z) = e^z L_{\nu-1}(-z)$$

Addition formulas

07.02.16.0003.01

$$L_n(z_1 + z_2) = \sum_{k=0}^{n} L_k^{\mu}(z_1) L_{n-k}^{-\mu-1}(z_2) \;/;\; n \in \mathbb{N}$$

07.02.16.0004.01

$$L_n(z_1 + z_2) = e^{z_1} \sum_{k=0}^{\infty} \frac{(-1)^k z_1^k}{k!} L_n^k(z_2) \;/;\; n \in \mathbb{N}$$

Multiple arguments

07.02.16.0005.01

$$L_n(z_1 z_2) = \sum_{k=0}^{n} \binom{n}{n-k} z_1^k (1 - z_1)^{n-k} L_k(z_2) \;/;\; n \in \mathbb{N}$$

Identities

Recurrence identities

Consecutive neighbors

07.02.17.0001.01

$$L_\nu(z) = \frac{2\nu + 3 - z}{\nu + 1} L_{\nu+1}(z) - \frac{\nu + 2}{\nu + 1} L_{\nu+2}(z)$$

07.02.17.0002.01

$$L_\nu(z) = \frac{2\nu - 1 - z}{\nu} L_{\nu-1}(z) - \frac{\nu - 1}{\nu} L_{\nu-2}(z)$$

Distant neighbors

07.02.17.0007.01

$$L_\nu(z) = C_n(\nu, z) L_{\nu+n}(z) - \frac{n + \nu + 1}{n + \nu} C_{n-1}(\nu, z) L_{\nu+n+1}(z) \;/;$$
$$C_0(\nu, z) = 1 \bigwedge C_1(\nu, z) = \frac{-z + 2\nu + 3}{\nu + 1} \bigwedge C_n(\nu, z) = \frac{2n - z + 2\nu + 1}{n + \nu} C_{n-1}(\nu, z) - \frac{n + \nu}{n + \nu - 1} C_{n-2}(\nu, z) \bigwedge n \in \mathbb{N}^+$$

07.02.17.0008.01

$$L_\nu(z) = C_n(\nu, z) L_{\nu-n}(z) - \frac{\nu - n}{\nu - n + 1} C_{n-1}(\nu, z) L_{\nu-n-1}(z) \;/;$$
$$C_0(\nu, z) = 1 \bigwedge C_1(\nu, z) = \frac{-z + 2\nu - 1}{\nu} \bigwedge C_n(\nu, z) = \frac{-2n - z + 2\nu + 1}{-n + \nu + 1} C_{n-1}(\nu, z) - \frac{-n + \nu + 1}{-n + \nu + 2} C_{n-2}(\nu, z) \bigwedge n \in \mathbb{N}^+$$

Functional identities

Relations between contiguous functions

Recurrence relations

07.02.17.0003.01

$$\nu L_{\nu-1}(z) + (\nu+1) L_{\nu+1}(z) = (2\nu - z + 1) L_\nu(z)$$

07.02.17.0004.01

$$L_\nu(z) = \frac{\nu L_{\nu-1}(z) + (\nu+1) L_{\nu+1}(z)}{2\nu - z + 1}$$

Normalized recurrence relation

07.02.17.0005.01

$$z\, p(n, z) = p(n+1, z) + n^2\, p(n-1, z) + (2n+1)\, p(n, z) \;/;\; p(n, z) = (-1)^n\, n!\, L_n(z)$$

Relations of special kind

07.02.17.0009.01

$$L_\nu(z) = L_{\nu-1}(z) + L_\nu^{-1}(z)$$

07.02.17.0006.01

$$L_\nu(z) = \frac{\nu L_\nu^{-1}(z) - (\nu+1) L_{\nu+1}^{-1}(z)}{z}$$

Complex characteristics

Real part

07.02.19.0001.01

$$\operatorname{Re}(L_n(x + i y)) = \sum_{j=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{(-1)^j\, y^{2j}}{(2j)!} L_{n-2j}^{2j}(x) \;/;\; x \in \mathbb{R} \land y \in \mathbb{R} \land n \in \mathbb{N}$$

Imaginary part

07.02.19.0002.01

$$\operatorname{Im}(L_n(x + i y)) = \sum_{j=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \frac{(-1)^{j-1}\, y^{2j+1}}{(2j+1)!} L_{n-2j-1}^{2j+1}(x) \;/;\; x \in \mathbb{R} \land y \in \mathbb{R} \land n \in \mathbb{N}$$

Differentiation

Low-order differentiation

With respect to ν

07.02.20.0001.01

$$\frac{\partial L_\nu(z)}{\partial \nu} = -z\, F_{2\times 0\times 1}^{1\times 1\times 2}\left(\begin{matrix} 1-\nu;\, 1;\, 1,\, -\nu; \\ 2,\, 2;;\, 1-\nu; \end{matrix}\; z,\, z\right)$$

07.02.20.0002.01

$$\frac{\partial L_\nu(z)}{\partial \nu} == (\pi\cot(\nu\pi) + \psi(\nu+1))\, L_\nu(z) - \sum_{k=0}^{\infty} \frac{(-\nu)_k\, \psi(k-\nu)\, z^k}{k!^2}$$

07.02.20.0003.01

$$\frac{\partial^2 L_\nu(z)}{\partial \nu^2} == L_\nu(z)\left(\psi(\nu+1)^2 + 2\pi\cot(\nu\pi)\,\psi(\nu+1) - \pi^2 + \psi^{(1)}(\nu+1)\right) + \frac{1}{\nu!}\sum_{k=0}^{\infty}\frac{(-\nu)_k\, z^k}{k!^2}\left(\Gamma(\nu+1)\,\psi(k-\nu)\,\psi(\nu+1) + \nu!\left(\psi(k-\nu)^2 - \left(2\pi\cot(\nu\pi) + 3\,\psi(\nu) + \frac{3}{\nu}\right)\psi(k-\nu) + \psi^{(1)}(k-\nu)\right)\right)$$

With respect to z

Forward shift operator:

07.02.20.0004.01

$$\frac{\partial L_\nu(z)}{\partial z} == -L_{\nu-1}^{1}(z)$$

07.02.20.0005.01

$$\frac{\partial^2 L_\nu(z)}{\partial z^2} == L_{\nu-2}^{2}(z)$$

Backward shift operator:

07.02.20.0006.01

$$z\,\frac{\partial L_n(z)}{\partial z} - z\, L_n(z) == (n+1)\, L_{n+1}^{-1}(z)$$

07.02.20.0007.01

$$\frac{\partial(e^{-z}\, L_n(z))}{\partial z} == (n+1)\, e^{-z}\, z^{-1}\, L_{n+1}^{-1}(z)$$

Symbolic differentiation

With respect to ν

07.02.20.0008.02

$$\frac{\partial^m L_\nu(z)}{\partial \nu^m} == \sum_{k=0}^{\infty}\frac{(-1)^k z^k}{k!^2}\sum_{j=1}^{k} S_k^{(j)}\,(j-m+1)_m\,\nu^{j-m} \;/;\; m\in\mathbb{N}$$

With respect to z

07.02.20.0009.02

$$\frac{\partial^m L_\nu^\lambda(z)}{\partial z^m} == (-1)^m\, L_{\nu-m}^{m}(z) \;/;\; m\in\mathbb{N}$$

07.02.20.0010.02

$$\frac{\partial^m L_\nu(z)}{\partial z^m} == z^{-m}\; {}_1\tilde{F}_1(-\nu;\, 1-m;\, z) \;/;\; m\in\mathbb{N}$$

Fractional integro-differentiation

With respect to z

07.02.20.0011.01

$$\frac{\partial^\alpha L_\nu(z)}{\partial z^\alpha} = z^{-\alpha}\, {}_1\tilde{F}_1(-\nu;\, 1-\alpha;\, z)$$

Integration

Indefinite integration

Involving only one direct function

07.02.21.0001.01

$$\int L_\nu(z)\, dz = L_\nu(z) - L_{\nu+1}(z)$$

Involving one direct function and elementary functions

Involving power function

07.02.21.0002.01

$$\int z^{\alpha-1} L_\nu(c\, z)\, dz = z^\alpha\, \Gamma(\alpha)\, {}_2\tilde{F}_2(-\nu,\, \alpha;\, 1,\, \alpha+1;\, c\, z)$$

07.02.21.0003.01

$$\int z^{\alpha-1} L_\nu(z)\, dz = \frac{z^\alpha}{\alpha}\, {}_2F_2(-\nu,\, \alpha;\, 1,\, \alpha+1;\, z)$$

07.02.21.0004.01

$$\int z^{\nu-2} L_\nu(z)\, dz = z^{\nu-1}\, \Gamma(\nu-1)\, {}_2\tilde{F}_2(-\nu,\, \nu-1;\, 1,\, \nu;\, z)$$

Involving exponential function

07.02.21.0005.01

$$\int e^{-c\, z} L_\nu(c\, z)\, dz = z\, {}_1\tilde{F}_1(\nu+1;\, 2;\, -c\, z)$$

Involving exponential function and a power function

07.02.21.0006.01

$$\int z^{\alpha-1} e^{-p\, z} L_\nu(z)\, dz = -\frac{z^\alpha}{(p\, z)^\alpha} \sum_{k=0}^{\infty} \frac{(-\nu)_k\, \Gamma(k+\alpha,\, p\, z)}{k!^2\, p^k}$$

07.02.21.0007.01

$$\int z^{\alpha-1} e^{-c\, z} L_\nu(c\, z)\, dz = z^\alpha\, \Gamma(\alpha)\, {}_2\tilde{F}_2(\nu+1,\, \alpha;\, 1,\, \alpha+1;\, -c\, z)$$

07.02.21.0008.01

$$\int z^{\nu-1} e^{-z} L_\nu(z)\, dz = \frac{z^\nu}{\nu}\, {}_1\tilde{F}_1(\nu;\, 1;\, -z)$$

Definite integration

Involving the direct function

Orthogonality:

07.02.21.0009.01

$$\int_0^{\infty} e^{-t} L_m(t) L_n(t) \, dt == \delta_{n,m} \;/;\; n \in \mathbb{N} \wedge m \in \mathbb{N}$$

Summation

Finite summation

07.02.23.0001.01

$$\sum_{k=0}^{n} \frac{(\lambda)_{n-k} L_k(z)}{(n-k)!} == L_n^{\lambda}(z) \;/;\; n \in \mathbb{N}$$

07.02.23.0002.01

$$\sum_{k=0}^{n} \binom{k+\lambda-1}{k} L_{n-k}(z) == L_n^{\lambda}(z) \;/;\; n \in \mathbb{N}$$

07.02.23.0003.01

$$\sum_{k=0}^{n} \binom{n}{n-k} L_k(z) \, w^k \, (1-w)^{n-k} == L_n(z\,w) \;/;\; n \in \mathbb{N}$$

07.02.23.0004.01

$$\sum_{k=0}^{n} L_k(z_1) \, L_{n-k}(z_2) == L_n^1(z_1 + z_2) \;/;\; n \in \mathbb{N}$$

Infinite summation

07.02.23.0005.01

$$\sum_{n=0}^{\infty} L_n(z) \, w^n == \frac{1}{1-w} e^{\frac{w\,z}{w-1}} \;/;\; |w| < 1$$

07.02.23.0006.01

$$\sum_{n=0}^{\infty} L_n(x) \, L_n(y) == e^{\frac{x+y}{2}} \delta(x-y) \;/;\; x > 0 \bigwedge y > 0$$

Operations

Limit operation

07.02.25.0001.01

$$\lim_{\nu \to \infty} L_{\nu}\left(\frac{z^2}{4\,\nu}\right) == J_0(z)$$

07.02.25.0002.01

$$\lim_{\nu \to \infty} L_{\nu}\left(-\frac{z^2}{4\,\nu}\right) == I_0(z)$$

Orthogonality, completeness, and Fourier expansions

The set of functions $L_n(x)$, $n = 0, 1, \ldots$, forms a complete, orthogonal (with weight e^{-x}) system on the interval $(0, \infty)$.

07.02.25.0003.01

$$\sum_{n=0}^{\infty} \left(e^{-\frac{x}{2}} L_n(x)\right)\left(e^{-\frac{y}{2}} L_n(y)\right) = \delta(x-y) \;/;\; x > 0 \land y > 0$$

07.02.25.0004.01

$$\int_0^{\infty} \left(e^{-\frac{t}{2}} L_m(t)\right)\left(e^{-\frac{t}{2}} L_n(t)\right) dt = \delta_{n,m}$$

Any sufficiently smooth function $f(x)$ can be expanded in the system $\{L_n(x)\}_{n=0,1,\ldots}$ as a generalized Fourier series, with its sum converging to $f(x)$ almost everywhere.

07.02.25.0005.01

$$f(x) = \sum_{n=0}^{\infty} c_n \psi_n(x) \;/;\; c_n = \int_0^{\infty} \psi_n(t)\, f(t)\, dt \bigwedge \psi_n(x) = e^{-\frac{x}{2}} L_n(x) \;/;\; x > 0$$

Representations through more general functions

Through hypergeometric functions

Involving $_1F_1$

07.02.26.0001.01

$$L_\nu(z) = {}_1F_1(-\nu; 1; z)$$

Through Meijer G

Classical cases for the direct function itself

07.02.26.0002.01

$$L_\nu(z) = \Gamma(\nu+1)\, G_{1,2}^{1,0}\left(z \,\middle|\, \begin{matrix} \nu+1 \\ 0, 0 \end{matrix}\right) \;/;\; \nu \notin \mathbb{Z}$$

07.02.26.0006.01

$$L_\nu(z) + L_\nu(-z) = -2^{-\nu} \sqrt{\pi}\, \Gamma(\nu+1) \sin(\pi\nu)\, G_{3,5}^{1,2}\left(\frac{z}{2}, \frac{1}{2} \,\middle|\, \begin{matrix} \frac{\nu+1}{2}, \frac{\nu}{2}+1, \frac{1}{2} \\ 0, 0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \end{matrix}\right)$$

07.02.26.0007.01

$$L_\nu(z) - L_\nu(-z) = -2^{-\nu} \sqrt{\pi}\, \Gamma(\nu+1) \sin(\pi\nu)\, G_{3,5}^{1,2}\left(\frac{z}{2}, \frac{1}{2} \,\middle|\, \begin{matrix} \frac{\nu+1}{2}, \frac{\nu+2}{2}, 1 \\ \frac{1}{2}, 0, 0, \frac{1}{2}, 1 \end{matrix}\right)$$

Classical cases involving exp

07.02.26.0003.01

$$e^{-z} L_\nu(z) = \frac{1}{\Gamma(\nu+1)} G_{1,2}^{1,1}\left(z \,\middle|\, \begin{matrix} -\nu \\ 0, 0 \end{matrix}\right)$$

Classical cases for products of Laguerre *L*

07.02.26.0008.01

$$L_\nu(z)\,L_\nu(-z) = -\frac{\sin(\pi\,\nu)}{\sqrt{\pi}}\,G_{2,4}^{1,2}\!\left(-\frac{z^2}{4}\;\middle|\;\begin{matrix}\nu+1,\,-\nu\\ 0,\,0,\,0,\,\frac{1}{2}\end{matrix}\right)$$

Classical cases involving exp and products of Laguerre *L*

07.02.26.0009.01

$$e^{-z}\,L_{-\nu-1}(z)\,L_\nu(z) = -2^{-\lambda}\,\sqrt{\pi}\,\sin(\pi\,\nu)\,G_{3,5}^{1,2}\!\left(\frac{z^2}{4}\;\middle|\;\begin{matrix}\nu+1,\,-\nu,\,\frac{1}{2}\\ 0,\,0,\,0,\,\frac{1}{2},\,\frac{1}{2}\end{matrix}\right)$$

Classical cases involving $_1F_1$

07.02.26.0010.01

$$_1F_1(-\nu;\,1;\,-z)\,L_\nu(z) = -\frac{\sin(\pi\,\nu)}{\sqrt{\pi}}\,G_{2,4}^{1,2}\!\left(-\frac{z^2}{4}\;\middle|\;\begin{matrix}\nu+1,\,-\nu\\ 0,\,0,\,0,\,\frac{1}{2}\end{matrix}\right)$$

07.02.26.0011.01

$$_1F_1(-\nu;\,1;\,-z)\,L_\nu(z) = -\sqrt{\pi}\,\sin(\pi\,\nu)\,G_{3,5}^{1,2}\!\left(\frac{z^2}{4}\;\middle|\;\begin{matrix}\nu+1,\,-\nu,\,\frac{1}{2}\\ 0,\,0,\,0,\,\frac{1}{2},\,\frac{1}{2}\end{matrix}\right)$$

Classical cases involving exp and $_1F_1$

07.02.26.0012.01

$$e^{-z}\,{}_1F_1(\nu+1;\,1;\,z)\,L_\nu(z) = -\sqrt{\pi}\,\sin(\pi\,\nu)\,G_{3,5}^{1,2}\!\left(\frac{z^2}{4}\;\middle|\;\begin{matrix}\nu+1,\,-\nu,\,\frac{1}{2}\\ 0,\,0,\,0,\,\frac{1}{2},\,\frac{1}{2}\end{matrix}\right)$$

Classical cases involving $_1\tilde{F}_1$

07.02.26.0013.01

$$_1\tilde{F}_1(-\nu;\,1;\,-z)\,L_\nu(z) = -\frac{\sin(\pi\,\nu)}{\sqrt{\pi}}\,G_{2,4}^{1,2}\!\left(-\frac{z^2}{4}\;\middle|\;\begin{matrix}\nu+1,\,-\nu\\ 0,\,0,\,0,\,\frac{1}{2}\end{matrix}\right)$$

07.02.26.0014.01

$$_1\tilde{F}_1(-\nu;\,1;\,-z)\,L_\nu(z) = -\sqrt{\pi}\,\sin(\pi\,\nu)\,G_{3,5}^{1,2}\!\left(\frac{z^2}{4}\;\middle|\;\begin{matrix}\nu+1,\,-\nu,\,\frac{1}{2}\\ 0,\,0,\,0,\,\frac{1}{2},\,\frac{1}{2}\end{matrix}\right)$$

Classical cases involving exp and $_1\tilde{F}_1$

07.02.26.0015.01

$$e^{-z}\,{}_1\tilde{F}_1(\nu+1;\,1;\,z)\,L_\nu(z) = -\sqrt{\pi}\,\sin(\pi\,\nu)\,G_{3,5}^{1,2}\!\left(\frac{z^2}{4}\;\middle|\;\begin{matrix}\nu+1,\,-\nu,\,\frac{1}{2}\\ 0,\,0,\,0,\,\frac{1}{2},\,\frac{1}{2}\end{matrix}\right)$$

Classical cases involving hypergeometric *U*

07.02.26.0016.01

$$U(-\nu,\,1,\,-z)\,L_\nu(z) = -\frac{\nu\,\Gamma(\nu)\sin(\pi\,\nu)}{2\,\pi^{3/2}}\,G_{2,4}^{3,1}\!\left(-\frac{z^2}{4}\;\middle|\;\begin{matrix}\nu+1,\,-\nu\\ 0,\,0,\,\frac{1}{2},\,0\end{matrix}\right)\,/;\,\mathrm{Re}(z)<0\,\bigvee\,\arg(z)=-\frac{\pi}{2}$$

Classical cases involving exp and hypergeometric *U*

07.02.26.0017.01

$$e^{-z}\,U(\nu+1,1,z)\,L_\nu(z)=\frac{1}{2\sqrt{\pi}\,\Gamma(\nu+1)}\,G_{2,4}^{3,1}\!\left(\frac{z^2}{4}\,\middle|\,\begin{matrix}-\nu,\,\nu+1\\0,\,0,\,\frac{1}{2},\,0\end{matrix}\right)\,/;\,-\frac{\pi}{2}<\arg(z)\leq\frac{\pi}{2}$$

Generalized cases for products of Laguerre *L*

07.02.26.0018.01

$$L_\nu(z)\,L_\nu(-z)=-\frac{\sin(\pi\,\nu)}{\sqrt{\pi}}\,G_{2,4}^{1,2}\!\left(\frac{i\,z}{2},\,\frac{1}{2}\,\middle|\,\begin{matrix}\nu+1,\,-\nu\\0,\,0,\,0,\,\frac{1}{2}\end{matrix}\right)$$

Generalized cases involving exp and products of Laguerre *L*

07.02.26.0019.01

$$e^{-z}\,L_{-\nu-1}^{\lambda}(z)\,L_\nu(z)=-\sqrt{\pi}\,\sin(\pi\,\nu)\,G_{3,5}^{1,2}\!\left(\frac{z}{2},\,\frac{1}{2}\,\middle|\,\begin{matrix}\nu+1,\,-\nu,\,\frac{1}{2}\\0,\,0,\,0,\,\frac{1}{2},\,\frac{1}{2}\end{matrix}\right)$$

Generalized cases involving ${}_1F_1$

07.02.26.0020.01

$${}_1F_1(-\nu;\,1;\,-z)\,L_\nu(z)=-\frac{\sin(\pi\,\nu)}{\sqrt{\pi}}\,G_{2,4}^{1,2}\!\left(\frac{i\,z}{2},\,\frac{1}{2}\,\middle|\,\begin{matrix}\nu+1,\,-\nu\\0,\,0,\,0,\,\frac{1}{2}\end{matrix}\right)$$

07.02.26.0021.01

$${}_1F_1(-\nu;\,1;\,-z)\,L_\nu(z)=-\sqrt{\pi}\,\sin(\pi\,\nu)\,G_{3,5}^{1,2}\!\left(\frac{z}{2},\,\frac{1}{2}\,\middle|\,\begin{matrix}\nu+1,\,-\nu,\,\frac{1}{2}\\0,\,0,\,0,\,\frac{1}{2},\,\frac{1}{2}\end{matrix}\right)$$

Generalized cases involving exp and ${}_1F_1$

07.02.26.0022.01

$$e^{-z}\,{}_1F_1(\nu+1;\,1;\,z)\,L_\nu(z)=-\sqrt{\pi}\,\sin(\pi\,\nu)\,G_{3,5}^{1,2}\!\left(\frac{z}{2},\,\frac{1}{2}\,\middle|\,\begin{matrix}\nu+1,\,-\nu,\,\frac{1}{2}\\0,\,0,\,0,\,\frac{1}{2},\,\frac{1}{2}\end{matrix}\right)$$

Generalized cases involving ${}_1\tilde{F}_1$

07.02.26.0023.01

$${}_1\tilde{F}_1(-\nu;\,1;\,-z)\,L_\nu(z)=-\frac{\sin(\pi\,\nu)}{\sqrt{\pi}}\,G_{2,4}^{1,2}\!\left(\frac{i\,z}{2},\,\frac{1}{2}\,\middle|\,\begin{matrix}\nu+1,\,-\nu\\0,\,0,\,0,\,\frac{1}{2}\end{matrix}\right)$$

07.02.26.0024.01

$${}_1\tilde{F}_1(-\nu;\,1;\,-z)\,L_\nu(z)=-\sqrt{\pi}\,\sin(\pi\,\nu)\,G_{3,5}^{1,2}\!\left(\frac{z}{2},\,\frac{1}{2}\,\middle|\,\begin{matrix}\nu+1,\,-\nu,\,\frac{1}{2}\\0,\,0,\,0,\,\frac{1}{2},\,\frac{1}{2}\end{matrix}\right)$$

Generalized cases involving exp and ${}_1\tilde{F}_1$

07.02.26.0025.01

$$e^{-z}\,{}_1\tilde{F}_1(\nu+1;\,1;\,z)\,L_\nu(z)=-\sqrt{\pi}\,\sin(\pi\,\nu)\,G_{3,5}^{1,2}\!\left(\frac{z}{2},\,\frac{1}{2}\,\middle|\,\begin{matrix}\nu+1,\,-\nu,\,\frac{1}{2}\\0,\,0,\,0,\,\frac{1}{2},\,\frac{1}{2}\end{matrix}\right)$$

Generalized cases involving hypergeometric *U*

07.02.26.0026.01

$$U(-\nu,\,1,\,-z)\,L_\nu(z) = -\frac{\nu\,\Gamma(\nu)\sin(\pi\,\nu)}{2\,\pi^{3/2}}\,G_{2,4}^{3,1}\!\left(-\frac{z}{2},\,\frac{1}{2}\,\middle|\,\begin{matrix}\nu+1,\,-\nu\\0,\,0,\,\frac{1}{2},\,0\end{matrix}\right)$$

Generalized cases involving exp and hypergeometric *U*

07.02.26.0027.01

$$e^{-z}\,e^{-z}\,U(\nu+1,\,1,\,z)\,L_\nu(z) = \frac{1}{2\sqrt{\pi}\;\Gamma(\nu+1)}\,G_{2,4}^{3,1}\!\left(\frac{z}{2},\,\frac{1}{2}\,\middle|\,\begin{matrix}-\nu,\,\nu+1\\0,\,0,\,\frac{1}{2},\,0\end{matrix}\right)$$

Through other functions

Involving some hypergeometric-type functions

07.02.26.0004.01

$$L_\nu(z) == L_\nu^0(z)$$

07.02.26.0005.01

$$L_\nu(z) == \lim_{b\to\infty} P_\nu^{(0,b)}\!\left(1-\frac{2\,z}{b}\right)$$

Theorems

Expansions in generalized Fourier series

$$f(x) == \sum_{k=0}^{\infty} c_k\,\psi_k(x)\;/;\quad c_k == \int_0^\infty f(t)\,\psi_k(t)\,dt,\;\;\psi_k(x) == e^{-x/2}\,L_k(x),\;\;k\in\mathbb{N}.$$

Excited coherent state

The (normalized) excited coherent state $|\,\alpha,\,m\rangle == \frac{(a^\dagger)^m|\alpha\rangle}{\langle\alpha|a^m(a^\dagger)^m|\alpha\rangle}$ can be expressed through the number states $|\,n\rangle$ as

$$|\,\alpha,\,m\rangle == \sum_{n=m}^{\infty}\frac{e^{-|\alpha|^2/2}\,\alpha^{n-m}\,\sqrt{n!}}{L_m\left(-|\,\alpha\,|^2\,m!\right)(n-m)!}\,|\,n\rangle.$$

History

– J.-L. Lagrange
– R. Murphy (1833)
– E. N. Laguerre (1879)

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