LaguerreL

Notations

Traditional name

Laguerre polynomial

Traditional notation

$L_n(z)$

Mathematica StandardForm notation

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LaguerreL[n, z]
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Primary definition

05.02.02.0001.01

$$L_n(z) = \sum_{k=0}^{n} \frac{(-n)_k\, z^k}{k!^2} \;/;\; n \in \mathbb{N}$$

Specific values

Specialized values

For fixed *n*

05.02.03.0001.01

$$L_n(0) = 1$$

For fixed *z*

05.02.03.0002.01

$$L_0(z) = 1$$

05.02.03.0003.01

$$L_1(z) = 1 - z$$

05.02.03.0004.01

$$L_2(z) = \frac{1}{2}\left(2 - 4z + z^2\right)$$

05.02.03.0005.01

$$L_3(z) = \frac{1}{6}\left(6 - 18z + 9z^2 - z^3\right)$$

05.02.03.0006.01

$$L_4(z) = \frac{1}{24}\left(24 - 96 z + 72 z^2 - 16 z^3 + z^4\right)$$

05.02.03.0007.01

$$L_5(z) = \frac{1}{120}\left(120 - 600 z + 600 z^2 - 200 z^3 + 25 z^4 - z^5\right)$$

05.02.03.0008.01

$$L_6(z) = \frac{1}{720}\left(720 - 4320 z + 5400 z^2 - 2400 z^3 + 450 z^4 - 36 z^5 + z^6\right)$$

05.02.03.0009.01

$$L_7(z) = \frac{1}{5040}\left(5040 - 35\,280 z + 52\,920 z^2 - 29\,400 z^3 + 7350 z^4 - 882 z^5 + 49 z^6 - z^7\right)$$

05.02.03.0010.01

$$L_8(z) = \frac{1}{40\,320}\left(40\,320 - 322\,560 z + 564\,480 z^2 - 376\,320 z^3 + 117\,600 z^4 - 18\,816 z^5 + 1568 z^6 - 64 z^7 + z^8\right)$$

05.02.03.0011.01

$$L_9(z) = \frac{1}{362\,880}\left(362\,880 - 3\,265\,920 z + 6\,531\,840 z^2 - 5\,080\,320 z^3 + 1\,905\,120 z^4 - 381\,024 z^5 + 42\,336 z^6 - 2592 z^7 + 81 z^8 - z^9\right)$$

05.02.03.0012.01

$$L_{10}(z) = \frac{1}{3\,628\,800}\left(3\,628\,800 - 36\,288\,000 z + 81\,648\,000 z^2 - 72\,576\,000 z^3 + 31\,752\,000 z^4 - 7\,620\,480 z^5 + 1\,058\,400 z^6 - 86\,400 z^7 + 4050 z^8 - 100 z^9 + z^{10}\right)$$

Values at infinities

05.02.03.0013.01

$$L_n(\infty) = (-1)^n \infty \;/;\; n > 0$$

05.02.03.0014.01

$$L_n(-\infty) = \infty \;/;\; n > 0$$

General characteristics

Domain and analyticity

The function $L_n(z)$is defined over $\mathbb{N} \otimes \mathbb{C}$. For fixed n, the function $L_n(z)$ is a polynomial in z of degree n.

05.02.04.0001.01

$$(n * z) \longrightarrow L_n(z) :: (\mathbb{N} \otimes \mathbb{C}) \longrightarrow \mathbb{C}$$

Symmetries and periodicities

Mirror symmetry

05.02.04.0002.01

$$L_n(\bar{z}) = \overline{L_n(z)}$$

Periodicity

No periodicity

Poles and essential singularities

With respect to z

The function $L_n(z)$ is polynomial and has pole of order n at $z = \tilde{\infty}$.

05.02.04.0003.01

$$\mathcal{Sing}_z(L_n(z)) == \{\{\tilde{\infty}, n\}\}$$

Branch points

With respect to z

The function $L_n(z)$ does not have branch points.

05.02.04.0004.01

$$\mathcal{BP}_z(L_n(z)) == \{\}$$

Branch cuts

With respect to z

The function $L_n(z)$ does not have branch cuts.

05.02.04.0005.01

$$\mathcal{BC}_z(L_n(z)) == \{\}$$

Series representations

Generalized power series

Expansions at generic point $z == z_0$

For the function itself

05.02.06.0011.01

$$L_n(z) \propto L_n(z_0) - L_{n-1}^1(z_0)(z - z_0) + \frac{1}{2} L_{n-2}^2(z_0)(z - z_0)^2 + \ldots /; (z \to z_0)$$

05.02.06.0012.01

$$L_n(z) \propto L_n(z_0) - L_{n-1}^1(z_0)(z - z_0) + \frac{1}{2} L_{n-2}^2(z_0)(z - z_0)^2 + O\left((z - z_0)^3\right)$$

05.02.06.0013.01

$$L_n(z) == \sum_{k=0}^{n} \frac{(-1)^k}{k!} L_{n-k}^k(z_0)(z - z_0)^k$$

05.02.06.0014.01

$$L_n(z) == n! \sum_{k=0}^{n} \frac{(-1)^k}{k!\, \Gamma(n-k+1)} \, {}_1\tilde{F}_1(k-n; k+1; z_0)(z - z_0)^k$$

05.02.06.0015.01

$$L_n(z) == \sum_{k=0}^{n} \frac{z_0^{-k}}{k!} \, {}_1\tilde{F}_1(-n; 1-k; z_0) \, (z-z_0)^k$$

05.02.06.0016.01

$$L_n(z) == \tilde{F}_{1\times 0\times 0}^{1\times 0\times 0}\left(\begin{matrix} -n;;; \\ 1;;; \end{matrix} \; z_0, z-z_0\right)$$

05.02.06.0017.01

$$L_n(z) \propto L_n(z_0) \, (1 + O(z-z_0))$$

Expansions at $z == 0$

For the function itself

05.02.06.0001.02

$$L_n(z) \propto 1 - n\,z - \frac{(1-n)\,n}{4} z^2 - \frac{(2-n)\,(1-n)\,n}{36} z^3 - \ldots \;/;\; (z \to 0)$$

05.02.06.0018.01

$$L_n(z) \propto 1 - n\,z - \frac{(1-n)\,n}{4} z^2 - \frac{(2-n)\,(1-n)\,n}{36} z^3 + O\left(z^4\right)$$

05.02.06.0002.01

$$L_n(z) == \sum_{k=0}^{n} \frac{(-n)_k \, z^k}{k!^2}$$

05.02.06.0005.01

$$L_n(z) == \sum_{k=0}^{n} \frac{(-1)^k}{k!} \binom{n}{n-k} z^k$$

05.02.06.0006.01

$$L_n(z) == \frac{1}{n!} \sum_{k=0}^{n} \frac{(-n)_k \, (k+1)_{n-k} \, z^k}{k!}$$

05.02.06.0003.01

$$L_n(z) == {}_1F_1(-n; 1; z)$$

05.02.06.0004.02

$$L_n(z) \propto 1 + O(z)$$

Expansions at $z == \infty$

For the function itself

05.02.06.0007.02

$$L_n(z) \propto \frac{(-z)^n}{n!} \left(1 - \frac{n^2}{z} + \frac{(1-n)^2\,n^2}{2\,z^2} - \ldots\right) /;\; (|z| \to \infty)$$

05.02.06.0019.01

$$L_n(z) \propto \frac{(-z)^n}{n!} \left(1 - \frac{n^2}{z} + \frac{(1-n)^2\,n^2}{2\,z^2} - O\left(\frac{1}{z^3}\right)\right)$$

05.02.06.0008.01

$$L_n(z) = \frac{(-z)^n}{n!} \sum_{k=0}^{n} \frac{(-1)^k \, (-n)_k^{\,2} \, z^{-k}}{k!}$$

05.02.06.0009.01

$$L_n(z) \propto \frac{(-z)^n}{n!} \, {}_2F_0\left(-n, -n; ; -\frac{1}{z}\right)$$

05.02.06.0010.02

$$L_n(z) \propto \frac{1}{n!} (-z)^n \left(1 + O\left(\frac{1}{z}\right)\right)$$

Expansions at $n == \infty$

05.02.06.0020.01

$$L_n(z) \propto \frac{e^{z/2}}{\sqrt{\pi} \sqrt[4]{z} \sqrt[4]{n}} \left(\cos\left(\frac{\pi}{4} - 2\sqrt{\left(n + \frac{1}{2}\right) z}\right) - \frac{\sin\left(\frac{\pi}{4} - 2\sqrt{\left(n + \frac{1}{2}\right) z}\right)}{16 \sqrt{z} \sqrt{n}} - \frac{\left(64 z^2 + 64 z + 9\right) \cos\left(\frac{\pi}{4} - 2\sqrt{\left(n + \frac{1}{2}\right) z}\right)}{512 \, z \, n} - \frac{1024 (2 n + 1) \cos\left(2\sqrt{\left(n + \frac{1}{2}\right) z} + \frac{\pi}{4}\right) z^3 + 3 (64 z (5 z - 1) - 25) \sin\left(\frac{\pi}{4} - 2\sqrt{\left(n + \frac{1}{2}\right) z}\right)}{8192 \, z^{3/2} \, n^{3/2}} + \ldots \right) /; (n \to \infty)$$

05.02.06.0021.01

$$L_n(z) \propto \frac{e^{z/2} z^{-\frac{1}{4}} n^{-\frac{1}{4}}}{\sqrt{\pi}} \Bigg(\cos\left(\frac{\pi}{4} - 2\sqrt{\left(n+\frac{1}{2}\right)z}\right) + \sum_{k=1}^{\infty} n^{-k} \Bigg(\sum_{j=0}^{k} \sum_{r=0}^{k-j} \sum_{s=0}^{k-j-r} \frac{(-1)^{j+r+s} 2^{2j-2k+s} z^{k-j-2r-s}}{s! \left(\frac{1}{2}\right)_r} A_{2(k-j-r-s)} \cos\left(\pi\left(j-k+r+s-\frac{1}{4}\right) + 2\sqrt{\left(n+\frac{1}{2}\right)z}\right) B_j \delta_j \left(k-j-s+\frac{1}{4}\right)_s \left(k-j-r-s+\frac{1}{4}\right)_r \left(k-j-r-s+\frac{3}{4}\right)_r \left(j-k+r+s+\frac{1}{4}\right)_r \left(j-k+r+s+\frac{3}{4}\right)_r - \frac{2}{z} \sum_{j=0}^{k-1} \sum_{r=0}^{k-j-1} \sum_{s=0}^{k-j-r-1} \frac{(-1)^{j+r+s} 2^{2j-2k+s} z^{k-j-2r-s}}{s! \left(\frac{3}{2}\right)_r} A_{2(k-j-r-s)-1} B_j \delta_j \left(k-j-s+\frac{1}{4}\right)_s \left(k-j-r-s+\frac{1}{4}\right)_r \left(k-j-r-s+\frac{3}{4}\right)_r \left(j-k+r+s+\frac{1}{4}\right)_{r+1} \left(j-k+r+s+\frac{3}{4}\right)_{r+1} \sin\left(\pi\left(j-k+r+s+\frac{1}{4}\right) + 2\sqrt{\left(n+\frac{1}{2}\right)z}\right)\Bigg) + \frac{\sqrt{z}}{2\sqrt{n}} \sum_{k=0}^{\infty} \sum_{j=0}^{k} \sum_{r=0}^{k-j} \sum_{s=0}^{k-j-r} \frac{(-1)^{j+r+s} 2^{2j-2k+s} n^{-k} z^{k-j-2r-s}}{s! \left(\frac{3}{2}\right)_r} B_j \delta_j \left(k-j-s+\frac{3}{4}\right)_s \left(k-j-r-s+\frac{3}{4}\right)_r \left(k-j-r-s+\frac{5}{4}\right)_r \left(j-k+r+s-\frac{1}{4}\right)_r \left(j-k+r+s+\frac{1}{4}\right)_r \Bigg((2r+1) A_{2(k-j-r-s)+1} \cos\left(\pi\left(j-k+r+s-\frac{3}{4}\right) + 2\sqrt{\left(n+\frac{1}{2}\right)z}\right) - \frac{(4j-4k+8r+4s-1)(4j-4k+8r+4s+1)}{8z} A_{2(k-j-r-s)} \sin\left(\pi\left(j-k+r+s-\frac{1}{4}\right) + 2\sqrt{\left(n+\frac{1}{2}\right)z}\right)\Bigg)\Bigg) /;$$

$$(n \to \infty) \bigwedge A_0 = 1 \bigwedge A_1 = 0 \bigwedge A_2 = \frac{1}{2} \bigwedge A_m = \frac{m-1}{m} A_{m-2} - (2n+1) A_{m-3} \bigwedge m \in \mathbb{N}^+$$

05.02.06.0022.01

$$L_n(z) \propto e^{z/2} \sum_{k=0}^{\infty} A_k 2^{-k} z^k \, {}_0\tilde{F}_1\left(; k+1; -\frac{z(2n+1)}{2}\right) /;$$

$$(n \to \infty) \bigwedge A_0 = 1 \bigwedge A_1 = 0 \bigwedge A_2 = \frac{1}{2} \bigwedge A_m = \frac{m-1}{m} A_{m-2} - (2n+1) A_{m-3} \bigwedge m \in \mathbb{N}^+$$

05.02.06.0023.01

$$L_n(z) \propto e^{z/2} \sum_{k=0}^{\infty} A_k \left(\frac{z}{2(2n+1)}\right)^{k/2} J_k\left(\sqrt{2(2n+1)z}\right) /;$$

$$(n \to \infty) \bigwedge A_0 = 1 \bigwedge A_1 = 0 \bigwedge A_2 = \frac{1}{2} \bigwedge A_m = \frac{m-1}{m} A_{m-2} - (2n+1) A_{m-3} \bigwedge m \in \mathbb{N}^+$$

05.02.06.0024.01

$$L_n(z) \propto \frac{1}{\sqrt{\pi}} e^{z/2} z^{-\frac{1}{4}} n^{-\frac{1}{4}} \cos\left(2\sqrt{n z} - \frac{\pi}{4}\right)(1 + \ldots) /; (n \to \infty)$$

Integral representations

On the real axis

Of the direct function

05.02.07.0001.01

$$L_n(z) = \frac{1}{2\pi} \int_0^{2\pi} e^{-z e^{i t}} \left(e^{-i t} + 1\right)^n dt$$

05.02.07.0002.01

$$L_n(z) = \frac{1}{n!} e^z \int_0^{\infty} e^{-t} t^n J_0\left(2\sqrt{t z}\right) dt$$

Integral representations of negative integer order

Rodrigues-type formula.

05.02.07.0003.01

$$L_n(z) = \frac{e^z}{n!} \frac{\partial^n (z^n e^{-z})}{\partial z^n}$$

Limit representations

05.02.09.0001.01

$$L_n(z) = \lim_{b \to \infty} P_n^{(0,b)}\left(1 - \frac{2z}{b}\right)$$

Generating functions

Ordinary generating functions

05.02.11.0001.01

$$L_n(z) = \left([t^n] (1-t)^{-1} \exp\left(\frac{t z}{t-1}\right)\right)$$

Differential generating functions

05.02.11.0002.01

$$L_n(z) = \frac{(-1)^n}{n!} \exp\left(-\frac{\partial}{\partial z} z \frac{\partial .}{\partial z}\right) z^n$$

05.02.11.0003.01

$$L_n(z) = \frac{(-1)^n}{n!} e^z \left(\frac{\partial}{\partial z} z \frac{\partial .}{\partial z}\right) e^{-z}$$

Differential equations

Ordinary linear differential equations and wronskians

For the direct function itself

05.02.13.0001.02

$$z\,w''(z)+(1-z)\,w'(z)+n\,w(z)=0\;/;\;w(z)=c_1\,L_n(z)\;+\;c_2\,G_{1,2}^{2,0}\!\left(-z\;\middle|\;\begin{matrix}n+1\\0,0\end{matrix}\right)$$

05.02.13.0002.02

$$W_z\!\left(L_n(z),\,G_{1,2}^{2,0}\!\left(-z\;\middle|\;\begin{matrix}n+1\\0,0\end{matrix}\right)\right)=-\frac{e^z}{z\,n!}$$

05.02.13.0003.01

$$w''(z)+\left(\frac{g'(z)}{g(z)}-g'(z)-\frac{g''(z)}{g'(z)}\right)w'(z)+\frac{n\,g'(z)^2}{g(z)}\,w(z)=0\;/;\;w(z)=c_1\,L_n(g(z))+c_2\,G_{1,2}^{2,0}\!\left(-g(z)\;\middle|\;\begin{matrix}n+1\\0,0\end{matrix}\right)$$

05.02.13.0004.01

$$W_z\!\left(L_n(g(z)),\,G_{1,2}^{2,0}\!\left(-g(z)\;\middle|\;\begin{matrix}n+1\\0,0\end{matrix}\right)\right)=-\frac{g'(z)\,e^{g(z)}}{g(z)\,n!}$$

05.02.13.0005.01

$$h(z)^2\,w''(z)\;+h(z)^2\left(\frac{g'(z)}{g(z)}-g'(z)-\frac{2\,h'(z)}{h(z)}-\frac{g''(z)}{g'(z)}\right)w'(z)\;+$$
$$\left(2\,h'(z)^2+h(z)\left(g'(z)\,h'(z)+\frac{g''(z)\,h'(z)}{g'(z)}-h''(z)\right)-\frac{h(z)\,g'(z)\,(-n\,h(z)\,g'(z)+h'(z))}{g(z)}\right)w(z)\;=0\;/;$$
$$w(z)=c_1\,h(z)\,L_n(z)\;+c_2\,h(z)\,G_{1,2}^{2,0}\!\left(-g(z)\;\middle|\;\begin{matrix}n+1\\0,0\end{matrix}\right)$$

05.02.13.0006.01

$$W_z\!\left(h(z)\,L_n(g(z)),\,h(z)\,G_{1,2}^{2,0}\!\left(-g(z)\;\middle|\;\begin{matrix}n+1\\0,0\end{matrix}\right)\right)=-\frac{h(z)^2\,g'(z)\,e^{g(z)}}{g(z)\,n!}$$

05.02.13.0007.01

$$z^2\,w''(z)\;+z\,(1-2\,s-a\,r\,z^r)\,w'(z)+\left(a\,r\,(s+r\,n)\,z^r+s^2\right)w(z)=0\;/;\;w(z)=c_1\,z^s\,L_n(a\,z^r)\;+c_2\,z^s\,G_{1,2}^{2,0}\!\left(-a\,z^r\;\middle|\;\begin{matrix}n+1\\0,0\end{matrix}\right)$$

05.02.13.0008.01

$$W_z\!\left(z^s\,L_n(a\,z^r),\,z^s\,G_{1,2}^{2,0}\!\left(-a\,z^r\;\middle|\;\begin{matrix}n+1\\0,0\end{matrix}\right)\right)=-\frac{r\,e^{a\,z^r}\,z^{2\,s-1}}{n!}$$

05.02.13.0009.01

$$w''(z)-(a\,r^z\log(r)+2\log(s))\,w'(z)+\left(a\,n\log^2(r)\,r^z+\log^2(s)+a\,r^z\log(r)\log(s)\right)w(z)=0\;/;$$
$$w(z)=c_1\,s^z\,L_n(a\,r^z)\;+c_2\,s^z\,G_{1,2}^{2,0}\!\left(-a\,r^z\;\middle|\;\begin{matrix}n+1\\0,0\end{matrix}\right)$$

05.02.13.0010.01

$$W_z\!\left(s^z\,L_n(a\,r^z),\,s^z\,G_{1,2}^{2,0}\!\left(-a\,r^z\;\middle|\;\begin{matrix}n+1\\0,0\end{matrix}\right)\right)=-\frac{e^{a\,r^z}\,s^{2\,z}\log(r)}{n!}$$

Transformations

Transformations and argument simplifications

Argument involving basic arithmetic operations

05.02.16.0001.01

$$L_n(-z) = e^{-z} L_{-n-1}(z)$$

Addition formulas

05.02.16.0002.01

$$L_n(z_1 + z_2) = \sum_{k=0}^{n} L_k^{\mu}(z_1)\, L_{n-k}^{-\mu-1}(z_2)$$

05.02.16.0003.01

$$L_n(z_1 + z_2) = e^{z_1} \sum_{k=0}^{\infty} \frac{(-1)^k z_1^k}{k!} L_n^k(z_2)$$

Multiple arguments

05.02.16.0004.01

$$L_n(z_1 z_2) = \sum_{k=0}^{n} \binom{n}{n-k} z_1^k (1 - z_1)^{n-k} L_k(z_2)$$

Identities

Recurrence identities

Consecutive neighbors

05.02.17.0001.01

$$L_n(z) = \frac{2n + 3 - z}{n+1} L_{n+1}(z) - \frac{n+2}{n+1} L_{n+2}(z)$$

05.02.17.0002.01

$$L_n(z) = \frac{2n - 1 - z}{n} L_{n-1}(z) - \frac{n-1}{n} L_{n-2}(z)$$

Distant neighbors

05.02.17.0007.01

$$L_n(z) = C_m(n, z)\, L_{m+n}(z) - \frac{m+n+1}{m+n} C_{m-1}(n, z)\, L_{m+n+1}(z) \;/;$$

$$C_0(n, z) = 1 \bigwedge C_1(n, z) = \frac{2n - z + 3}{n+1} \bigwedge C_m(n, z) = \frac{2m + 2n - z + 1}{m+n} C_{m-1}(n, z) - \frac{m+n}{m+n-1} C_{m-2}(n, z) \bigwedge m \in \mathbb{N}^+$$

05.02.17.0008.01

$$L_n(z) == C_m(n, z) L_{n-m}(z) - \frac{n-m}{n-m+1} C_{m-1}(n, z) L_{n-m-1}(z) /; C_0(n, z) == 1 \bigwedge C_1(n, z) == \frac{2n-z-1}{n} \bigwedge C_m(n, z) == \frac{2n-2m-z+1}{n-m+1} C_{m-1}(n, z) - \frac{n-m+1}{n-m+2} C_{m-2}(n, z) \bigwedge m \in \mathbb{N}^+$$

Functional identities

Relations between contiguous functions

Recurrence relations

05.02.17.0003.01

$$n L_{n-1}(z) + (n+1) L_{n+1}(z) == (2n - z + 1) L_n(z)$$

05.02.17.0004.01

$$L_n(z) == \frac{n L_{n-1}(z) + (n+1) L_{n+1}(z)}{2n - z + 1}$$

Normalized recurrence relation

05.02.17.0005.01

$$z\, p(n, z) == p(n+1, z) + n^2 p(n-1, z) + (2n+1) p(n, z) /; p(n, z) == (-1)^n n! L_n(z) \bigwedge n \in \mathbb{N}$$

Relations of special kind

05.02.17.0009.01

$$L_n(z) = L_{n-1}(z) + L_n^{-1}(z)$$

05.02.17.0006.01

$$L_n(z) == \frac{n L_n^{-1}(z) - (n+1) L_{n+1}^{-1}(z)}{z}$$

Complex characteristics

Real part

05.02.19.0001.01

$$\mathrm{Re}(L_n(x + i y)) == \sum_{j=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{(-1)^j y^{2j}}{(2j)!} L_{n-2j}^{2j}(x) /; x \in \mathbb{R} \bigwedge y \in \mathbb{R}$$

Imaginary part

05.02.19.0002.01

$$\mathrm{Im}(L_n(x + i y)) == \sum_{j=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \frac{(-1)^{j-1} y^{2j+1}}{(2j+1)!} L_{n-2j-1}^{2j+1}(x) /; x \in \mathbb{R} \bigwedge y \in \mathbb{R}$$

Argument

05.02.19.0003.01

$$\arg(L_n(x+i\,y)) = \tan^{-1}\left(\sum_{j=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{(-1)^j}{(2\,j)!}\,L_{n-2\,j}^{2\,j}(x)\,y^{2\,j},\ \sum_{j=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \frac{(-1)^{j-1}}{(2\,j+1)!}\,L_{n-2\,j-1}^{2\,j+1}(x)\,y^{2\,j+1}\right) /;\ x \in \mathbb{R} \bigwedge y \in \mathbb{R}$$

Conjugate value

05.02.19.0004.01

$$\overline{L_n(x+i\,y)} = \sum_{j=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{(-1)^j}{(2\,j)!}\,L_{n-2\,j}^{2\,j}(x)\,y^{2\,j} - i \sum_{j=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \frac{(-1)^{j-1}}{(2\,j+1)!}\,L_{n-2\,j-1}^{2\,j+1}(x)\,y^{2\,j+1} /;\ x \in \mathbb{R} \bigwedge y \in \mathbb{R}$$

Differentiation

Low-order differentiation

With respect to *z*

Forward shift operator:

05.02.20.0001.01

$$\frac{\partial L_n(z)}{\partial z} = -L_{n-1}^1(z)$$

05.02.20.0002.01

$$\frac{\partial^2 L_n(z)}{\partial z^2} = L_{n-2}^2(z)$$

Backward shift operator:

05.02.20.0003.01

$$z\,\frac{\partial L_\nu(z)}{\partial z} - z\,L_\nu(z) = (\nu+1)\,L_{\nu+1}^{-1}(z)$$

05.02.20.0004.01

$$\frac{\partial(e^{-z}\,L_\nu(z))}{\partial z} = (\nu+1)\,e^{-z}\,z^{-1}\,L_{\nu+1}^{-1}(z)$$

Symbolic differentiation

With respect to *z*

05.02.20.0005.02

$$\frac{\partial^m L_n(z)}{\partial z^m} = (-1)^m\,L_{n-m}^m(z) /;\ m \in \mathbb{N}$$

05.02.20.0006.02

$$\frac{\partial^m L_n(z)}{\partial z^m} = z^{-m}\,{}_1\tilde{F}_1(-n;\,1-m;\,z) /;\ m \in \mathbb{N}$$

Fractional integro-differentiation

With respect to z

05.02.20.0007.01

$$\frac{\partial^{\alpha} L_n(z)}{\partial z^{\alpha}} = z^{-\alpha} \, {}_1\tilde{F}_1(-n; 1-\alpha; z)$$

Integration

Indefinite integration

Involving only one direct function

05.02.21.0001.01

$$\int L_n(z) \, dz = L_n(z) - L_{n+1}(z)$$

Involving one direct function and elementary functions

Involving power function

05.02.21.0002.01

$$\int z^{\alpha-1} L_n(c z) \, dz = z^{\alpha} \, \Gamma(\alpha) \, {}_2\tilde{F}_2(-n, \alpha; 1, \alpha+1; c z)$$

05.02.21.0003.01

$$\int z^{\alpha-1} L_n(z) \, dz = \frac{z^{\alpha}}{\alpha} \, {}_2F_2(-n, \alpha; 1, \alpha+1; z)$$

05.02.21.0004.01

$$\int z^{n-2} L_n(z) \, dz = z^{n-1} \, \Gamma(n-1) \, {}_2\tilde{F}_2(-n, n-1; 1, n; z)$$

Involving exponential function

05.02.21.0005.01

$$\int e^{-c z} L_n(c z) \, dz = z \, {}_1\tilde{F}_1(n+1; 2; -c z)$$

Involving exponential function and a power function

05.02.21.0006.01

$$\int z^{\alpha-1} e^{-p z} L_n(z) \, dz = -\frac{z^{\alpha}}{(p z)^{\alpha}} \sum_{k=0}^{n} \frac{(-n)_k \, \Gamma(k+\alpha, \, p z)}{k!^2 \, p^k}$$

05.02.21.0007.01

$$\int z^{\alpha-1} e^{-c z} L_n(c z) \, dz = z^{\alpha} \, \Gamma(\alpha) \, {}_2\tilde{F}_2(n+1, \alpha; 1, \alpha+1; -c z)$$

Definite integration

Involving the direct function

Orthogonality:

05.02.21.0008.01

$$\int_0^\infty e^{-t} L_m(t) L_n(t) \, dt == \delta_{n,m}$$

Summation

Finite summation

05.02.23.0001.01

$$\sum_{k=0}^{n} \frac{(\lambda)_{n-k} L_k(z)}{(n-k)!} == L_n^{\lambda}(z)$$

05.02.23.0002.01

$$\sum_{k=0}^{n} \binom{k+\lambda-1}{k} L_{n-k}(z) == L_n^{\lambda}(z)$$

05.02.23.0003.01

$$\sum_{k=0}^{n} \binom{n}{n-k} L_k(z) w^k (1-w)^{n-k} == L_n(z w)$$

05.02.23.0004.01

$$\sum_{k=0}^{n} L_k(z_1) L_{n-k}(z_2) == L_n^1(z_1 + z_2)$$

Infinite summation

05.02.23.0005.01

$$\sum_{n=0}^{\infty} L_n(z) w^n == \frac{1}{1-w} e^{\frac{w z}{w-1}} /; |w| < 1$$

05.02.23.0006.01

$$\sum_{n=0}^{\infty} L_n(x) L_n(y) == e^{\frac{x+y}{2}} \delta(x-y) /; x > 0 \bigwedge y > 0$$

Operations

Limit operation

05.02.25.0001.01

$$\lim_{n\to\infty} L_n\left(\frac{z^2}{4 n}\right) == J_0(z)$$

05.02.25.0002.01

$$\lim_{n\to\infty} L_n\left(-\frac{z^2}{4 n}\right) == I_0(z)$$

Orthogonality, completeness, and Fourier expansions

The set of functions $L_n(x)$, $n = 0, 1, \ldots$, forms a complete, orthogonal (with weight e^{-x}) system on the interval $(0, \infty)$.

05.02.25.0003.01

$$\sum_{n=0}^{\infty} \left(e^{-\frac{x}{2}} L_n(x) \right) \left(e^{-\frac{y}{2}} L_n(y) \right) = \delta(x - y) /; x > 0 \bigwedge y > 0$$

05.02.25.0004.01

$$\int_0^{\infty} \left(e^{-\frac{t}{2}} L_m(t) \right) \left(e^{-\frac{t}{2}} L_n(t) \right) dt = \delta_{n,m}$$

Any sufficiently smooth function $f(x)$ can be expanded in the system $\{L_n(x)\}_{n=0,1,\ldots}$ as a generalized Fourier series, with its sum converging to $f(x)$ almost everywhere.

05.02.25.0005.01

$$f(x) = \sum_{n=0}^{\infty} c_n \psi_n(x) /; c_n = \int_0^{\infty} \psi_n(t) f(t) dt \bigwedge \psi_n(x) = e^{-\frac{x}{2}} L_n(x) \bigwedge x > 0$$

Representations through more general functions

Through hypergeometric functions

Involving $_1F_1$

05.02.26.0001.01

$$L_n(z) = {}_1F_1(-n; 1; z)$$

Through Meijer G

Classical cases involving exp

05.02.26.0002.01

$$e^{-z} L_n(z) = \frac{1}{\Gamma(n+1)} G_{1,2}^{1,1}\left(z \,\middle|\, \begin{array}{c} -n \\ 0, 0 \end{array}\right)$$

Through other functions

Involving some hypergeometric-type functions

05.02.26.0003.01

$$L_n(z) = L_n^0(z)$$

05.02.26.0004.01

$$L_n(z) = \lim_{b \to \infty} P_n^{(0,b)}\left(1 - \frac{2z}{b}\right)$$

Theorems

Expansions in generalized Fourier series

$$f(x) = \sum_{k=0}^{\infty} c_k \psi_k(x) /; \quad c_k = \int_0^{\infty} f(t) \psi_k(t) \, dt, \ \psi_k(x) = e^{-x/2} L_k(x), \ k \in \mathbb{N}.$$

Excited coherent state

The (normalized) excited coherent state $|\alpha, m\rangle = \frac{(a^\dagger)^m |\alpha\rangle}{\langle\alpha| a^m (a^\dagger)^m |\alpha\rangle}$ can be expressed through the number states $|n\rangle$ as

$$|\alpha, m\rangle = \sum_{n=m}^{\infty} \frac{e^{-|\alpha|^2/2} \alpha^{n-m} \sqrt{n!}}{L_m(-|\alpha|^2 m!) (n-m)!} |n\rangle.$$

History

– J.-L. Lagrange
– R. Murphy (1833)
– E. N. Laguerre (1879)

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