JacobiZeta

Notations

Traditional name

Jacobi zeta function

Traditional notation

$Z(z \mid m)$

Mathematica StandardForm notation

`JacobiZeta[z, m]`

Primary definition

08.07.02.0001.01

$$Z(z \mid m) = E(z \mid m) - \frac{E(m)}{K(m)} F(z \mid m)$$

Specific values

Specialized values

For fixed *z*

08.07.03.0001.01

$$Z(z \mid 0) = 0$$

08.07.03.0002.01

$$Z(z \mid 1) = \sin(z) \;/;\; |\mathrm{Re}(z)| \leq \frac{\pi}{2}$$

For fixed *m*

08.07.03.0003.01

$$Z(0 \mid m) = 0$$

08.07.03.0004.01

$$Z\left(\frac{\pi}{2} \middle| m\right) = 0$$

08.07.03.0005.01

$$Z\left(\frac{k\pi}{2} \middle| m\right) = 0 \;/;\; k \in \mathbb{Z}$$

08.07.03.0008.01

$$Z\left(\csc^{-1}\left(\sqrt{m}\right) \,\middle|\, m\right) = \frac{\pi \sqrt{1-m} \sqrt{-\frac{1}{m}} \sqrt{\frac{m}{1-m}}}{2\, K(m)}$$

Values at infinities

08.07.03.0006.01

$$Z(z \mid \infty) = \tilde{\infty}$$

08.07.03.0007.01

$$Z(z \mid -\infty) = \tilde{\infty}$$

General characteristics

Domain and analyticity

$Z(z \mid m)$ is an analytical function of z and m which is defined over $\mathbb{C}^2$.

08.07.04.0001.01

$$(z * m) \longrightarrow Z(z \mid m) :: (\mathbb{C} \otimes \mathbb{C}) \longrightarrow \mathbb{C}$$

Symmetries and periodicities

Parity

$Z(z \mid m)$ is an odd function with respect to z.

08.07.04.0002.01

$$E(-z \mid m) = -E(z \mid m)$$

Mirror symmetry

08.07.04.0003.01

$$Z(\bar{z} \mid \bar{m}) = \overline{Z(z \mid m)}$$

Periodicity

$Z(z \mid m)$ is a periodic function with respect to z with period π.

08.07.04.0010.01

$$Z(z + \pi \mid m) = Z(z \mid m)$$

08.07.04.0004.01

$$Z(z + \pi\, k \mid m) = Z(z \mid m) \;/;\; k \in \mathbb{Z}$$

08.07.04.0011.01

$$Z(x + i\, y \mid m) = Z\left(\pi \operatorname{frac}\left(\frac{x}{\pi}\right) + i\, y \,\middle|\, m\right) \;/;\; x \in \mathbb{R} \land y \in \mathbb{R}$$

Poles and essential singularities

With respect to z

The function $Z(z \mid m)$ does not have poles and essential singularities with respect to z.

08.07.04.0005.01

$\mathcal{S}ing_z(\mathrm{Z}(z \mid m)) == \{\}$

With respect to *m*

The function $\mathrm{Z}(z \mid m)$ does not have poles and essential singularities with respect to m.

08.07.04.0006.01

$\mathcal{S}ing_m(\mathrm{Z}(z \mid m)) == \{\}$

Branch points

With respect to *z*

For fixed m, the function $\mathrm{Z}(z \mid m)$ has an infinite number of branch points at $z == \pm \sin^{-1}\left(\frac{1}{\sqrt{m}}\right) + 2\pi k /; k \in \mathbb{Z}$ and $z == \tilde{\infty}$.

08.07.04.0007.01

$$\mathcal{BP}_a(\mathrm{Z}(z \mid m)) == \left\{\left\{\sin^{-1}\left(\frac{1}{\sqrt{m}}\right) + 2\pi k /; k \in \mathbb{Z}\right\}, \left\{-\sin^{-1}\left(\frac{1}{\sqrt{m}}\right) + 2\pi k /; k \in \mathbb{Z}\right\}, \tilde{\infty}\right\}$$

08.07.04.0008.01

$$\mathcal{R}_z\left(\mathrm{Z}(z \mid m), \sin^{-1}\left(\frac{1}{\sqrt{m}}\right) + 2\pi k\right) == 2 /; k \in \mathbb{Z}$$

08.07.04.0009.01

$$\mathcal{R}_z\left(\mathrm{Z}(z \mid m), -\sin^{-1}\left(\frac{1}{\sqrt{m}}\right) + 2\pi k\right) == 2 /; k \in \mathbb{Z}$$

Branch cuts

Branch cut locations: complicated

Series representations

Generalized power series

Expansions at generic point $z == z_0$

For the function itself

08.07.06.0008.01

$$Z(z\mid m)\propto Z(z_0\mid m)+\left(\sqrt{1-m\sin^2(z_0)}-\frac{E(m)}{K(m)\sqrt{1-m\sin^2(z_0)}}\right)(z-z_0)+\frac{1}{4}(m\sin(2z_0))\left(-\frac{E(m)}{K(m)\left(1-m\sin^2(z_0)\right)^{3/2}}-\frac{1}{\sqrt{1-m\sin^2(z_0)}}\right)(z-z_0)^2+\left(m\left((4(m-2)\cos(2z_0)+m(\cos(4z_0)-5))E(m)+8K(m)\left(m\sin^2(z_0)-1\right)\left(m\sin^4(z_0)-\sin^2(z_0)+\cos^2(z_0)\right)\right)\right)\Big/\left(48\left(1-m\sin^2(z_0)\right)^{5/2}K(m)\right)(z-z_0)^3-\frac{m\sin(2z_0)}{48\left(1-m\sin^2(z_0)\right)^{7/2}K(m)}\left(K(m)\left(m\sin^2(z_0)-1\right)\left(-3m\cos^2(z_0)+m\sin^2(z_0)\left(m\sin^2(z_0)-5\right)+4\right)+E(m)\left(m\left(5m\sin^4(z_0)+\left(6m\cos^2(z_0)-1\right)\sin^2(z_0)+9\cos^2(z_0)\right)-4\right)\right)(z-z_0)^4-\frac{m}{120\left(1-m\sin^2(z_0)\right)^{9/2}K(m)}\left(K(m)\left(m\sin^2(z_0)-1\right)\left(-3m\left(4m\sin^2(z_0)+1\right)\cos^4(z_0)-2\left(m\sin^2(z_0)-1\right)\left(7m\sin^2(z_0)+2\right)\cos^2(z_0)-4\sin^2(z_0)+m\sin^4(z_0)\left(m\sin^2(z_0)-3\right)^2\right)+E(m)\left(3m\left(8m\left(m\sin^2(z_0)+3\right)\sin^2(z_0)+3\right)\cos^4(z_0)+2\left(m\sin^2(z_0)-1\right)\left(m\sin^2(z_0)+2\right)\left(14m\sin^2(z_0)+1\right)\cos^2(z_0)+\sin^2(z_0)\left(m\sin^2(z_0)-1\right)^2\left(5m\sin^2(z_0)+4\right)\right)\right)(z-z_0)^5+\ldots\ /;\ (z\to z_0)$$

08.07.06.0009.01

$$Z(z\mid m)=Z(z_0\mid m)+\left(\sqrt{1-m\sin^2(z_0)}-\frac{E(m)}{K(m)\sqrt{1-m\sin^2(z_0)}}\right)(z-z_0)+\sum_{k=2}^{\infty}\frac{i^{k-1}\binom{k-\frac{3}{2}}{k-1}}{k!}\sum_{q=1}^{k-1}\frac{(-1)^q\binom{k-1}{q}}{\left(1-m\sin^2(z_0)\right)^q}\left(\frac{E(m)(1-2k)}{K(m)\left((2q+1)\sqrt{1-m\sin^2(z_0)}\right)}+\frac{\sqrt{1-m\sin^2(z_0)}}{1-2q}\right)\sum_{j=0}^{q}\binom{q}{j}m^j(2-m)^{q-j}2^{-j+k-q-1}\sum_{i=0}^{j}\binom{j}{i}(2i-j)^{k-1}e^{2(2i-j)iz_0}(z-z_0)^k$$

08.07.06.0010.01

$$Z(z\mid m)=Z(z_0\mid m)+\sum_{k=1}^{\infty}\frac{1}{k!}\sum_{j=1}^{k}\frac{1}{j!}\sum_{q=0}^{j-1}\binom{j}{q}\sum_{p=0}^{j-q}(-1)^q2^{q-j}\sin^q(z_0)(-j+q+2p)^k e^{-\frac{1}{2}i(\pi(j+k-q-2p)+2(-j+q+2p)z_0)}\binom{j-q}{p}\sum_{i=0}^{j-1}(1-j)_{2(j-i)-2}\sum_{s=0}^{i}\binom{i}{s}\left(\frac{1}{2}\right)_s\frac{m^{i-s}\cos^{-2s-1}(z_0)\left(1-m\sin^2(z_0)\right)^{s-i}}{(j-i-1)!\,(2\sin(z_0))^{-2i+j-1}}\left(\left(-\frac{1}{2}\right)_{i-s}\sqrt{1-m\sin^2(z_0)}-\frac{\left(\frac{1}{2}\right)_{i-s}}{K(m)\sqrt{1-m\sin^2(z_0)}}E(m)\right)(z-z_0)^k$$

08.07.20.0006.01

$$\frac{\partial^n \mathrm{Z}(z \mid m)}{\partial z^n} = \frac{1}{\sqrt{\pi}} \Gamma\left(n - \frac{1}{2}\right) \sum_{k=0}^{n-1} \frac{(-1)^k \left(1 - m \sin^2(z)\right)^{\frac{1}{2}-k}}{k!\,(n-k-1)!} \left(\frac{1}{1-2k} - \frac{E(m)\,(2n-1)}{K(m)\left(1 - m\sin^2(z)\right)(2k+1)}\right) \frac{\partial^{n-1}\left(1 - m\sin^2(z)\right)^k}{\partial z^{n-1}} \;/;\; n \in \mathbb{N}^+$$

08.07.06.0011.01

$$\mathrm{Z}(z \mid m) = \sum_{k=0}^{\infty} \frac{1}{k!} \mathrm{Z}^{(k,0)}(z_0 \mid m)\,(z - z_0)^k$$

08.07.06.0012.01

$$\mathrm{Z}(z \mid m) \propto \mathrm{Z}(z_0 \mid m)\,(1 + O(z - z_0))$$

Expansions on branch cuts

Formulas on real axis for real *m*

For $m > 1,\ \csc^{-1}\left(\sqrt{m}\right) + \pi u < x < \pi\left(u + \frac{1}{2}\right) /;\ u \in \mathbb{Z}$

08.07.06.0013.01

$$\mathrm{Z}(z \mid m) \propto \mathrm{Z}(x \mid m)\, e^{-\pi i \left\lfloor \frac{\arg(x-z)}{2\pi} \right\rfloor} + \left(1 - e^{-\pi i \left\lfloor \frac{\arg(x-z)}{2\pi} \right\rfloor}\right) \mathrm{Z}\left(\csc^{-1}\left(\sqrt{m}\right) \middle| m\right) - e^{-\pi i \left\lfloor \frac{\arg(x-z)}{2\pi} \right\rfloor} \left(\frac{2E(m) + (-\cos(2x)\,m + m - 2)\,K(m)}{2K(m)\sqrt{1 - m\sin^2(x)}}(z - x) + \frac{m\,(2E(m) + (\cos(2x)\,m - m + 2)\,K(m))\sin(2x)}{8K(m)\left(1 - m\sin^2(x)\right)^{3/2}}(z - x)^2 + \ldots\right) /;$$
$$(z \to x) \bigwedge x \in \mathbb{R} \bigwedge m \in \mathbb{R} \bigwedge m > 1 \bigwedge \pi u + \csc^{-1}\left(\sqrt{m}\right) < x < \pi u + \frac{\pi}{2} \bigwedge u \in \mathbb{Z}$$

08.07.06.0014.01

$$\mathrm{Z}(z \mid m) = e^{-\pi i \left\lfloor \frac{\arg(x-z)}{2\pi} \right\rfloor} \mathrm{Z}(x \mid m) + \left(1 - e^{-\pi i \left\lfloor \frac{\arg(x-z)}{2\pi} \right\rfloor}\right) \mathrm{Z}\left(\csc^{-1}\left(\sqrt{m}\right) \middle| m\right) + e^{-\pi i \left\lfloor \frac{\arg(x-z)}{2\pi} \right\rfloor} \sum_{k=1}^{\infty} \frac{1}{k!} \sum_{j=1}^{k} \frac{1}{j!} \sum_{q=0}^{j-1} \binom{j}{q} \sum_{p=0}^{j-q} (-1)^q\, 2^{q-j} \sin^q(x)\,(-j + q + 2p)^k\, e^{-\frac{1}{2} i (\pi (j+k-q-2p) + 2(-j+q+2p)x)}$$
$$\binom{j-q}{p} \sum_{i=0}^{j-1} (1-j)_{2(j-i)-2} \sum_{s=0}^{i} \binom{i}{s} \left(\frac{1}{2}\right)_s \frac{m^{i-s} \cos^{-2s-1}(x)\left(1 - m\sin^2(x)\right)^{s-i}}{(j-i-1)!\,(2\sin(x))^{-2i+j-1}}$$
$$\left(\left(-\frac{1}{2}\right)_{i-s} \sqrt{1 - m\sin^2(x)} - \frac{\left(\frac{1}{2}\right)_{i-s}}{K(m)\sqrt{1 - m\sin^2(x)}} E(m)\right)(z - x)^k /;$$
$$x \in \mathbb{R} \bigwedge m \in \mathbb{R} \bigwedge m > 1 \bigwedge \pi u + \csc^{-1}\left(\sqrt{m}\right) < x < \pi u + \frac{\pi}{2} \bigwedge u \in \mathbb{Z}$$

08.07.06.0015.01

$$Z(z \mid m) \propto \left(Z(x \mid m)\, e^{-\pi i \left\lfloor \frac{\arg(x-z)}{2\pi} \right\rfloor} + \left(1 - e^{-\pi i \left\lfloor \frac{\arg(x-z)}{2\pi} \right\rfloor}\right) Z\left(\csc^{-1}\left(\sqrt{m}\right) \,\middle|\, m\right) \right) (1 + O(z-x)) \;/;$$
$$(z \to x) \bigwedge x \in \mathbb{R} \bigwedge m \in \mathbb{R} \bigwedge m > 1 \bigwedge \pi u + \csc^{-1}\left(\sqrt{m}\right) < x < \pi u + \frac{\pi}{2} \bigwedge u \in \mathbb{Z}$$

For $m > 1,\ \pi\left(u + \frac{1}{2}\right) < x < \pi(u+1) - \csc^{-1}\left(\sqrt{m}\right) \;/;\ u \in \mathbb{Z}$

08.07.06.0016.01

$$Z(z \mid m) = e^{-\pi i \left\lfloor \frac{\arg(z-x)}{2\pi} \right\rfloor} Z(x \mid m) - Z\left(\csc^{-1}\left(\sqrt{m}\right) \,\middle|\, m\right) \left(1 - e^{-\pi i \left\lfloor \frac{\arg(z-x)}{2\pi} \right\rfloor}\right) - e^{-\pi i \left\lfloor \frac{\arg(z-x)}{2\pi} \right\rfloor}$$
$$\left(\frac{2 E(m) + (-\cos(2 z_0)\, m + m - 2) K(m)}{2 K(m) \sqrt{1 - m \sin^2(z_0)}} (z-x) + \frac{m\, (2 E(m) + (\cos(2 z_0)\, m - m + 2) K(m)) \sin(2 z_0)}{8 K(m) \left(1 - m \sin^2(z_0)\right)^{3/2}} (z-x)^2 + \ldots \right) /;$$
$$(z \to x) \bigwedge x \in \mathbb{R} \bigwedge m \in \mathbb{R} \bigwedge m > 1 \bigwedge \pi u + \frac{\pi}{2} < x < \pi (u+1) - \csc^{-1}\left(\sqrt{m}\right) \bigwedge u \in \mathbb{Z}$$

08.07.06.0017.01

$$Z(z \mid m) = e^{-\pi i \left\lfloor \frac{\arg(z-x)}{2\pi} \right\rfloor} Z(x \mid m) - Z\left(\csc^{-1}\left(\sqrt{m}\right) \,\middle|\, m\right) \left(1 - e^{-\pi i \left\lfloor \frac{\arg(z-x)}{2\pi} \right\rfloor}\right) +$$
$$e^{-\pi i \left\lfloor \frac{\arg(z-x)}{2\pi} \right\rfloor} \sum_{k=1}^{\infty} \frac{1}{k!} \sum_{j=1}^{k} \frac{1}{j!} \sum_{q=0}^{j-1} \binom{j}{q} \sum_{p=0}^{j-q} (-1)^q\, 2^{q-j} \sin^q(x)\, (-j+q+2p)^k\, e^{-\frac{1}{2} i (\pi (j+k-q-2p) + 2(-j+q+2p) x)}$$
$$\binom{j-q}{p} \sum_{i=0}^{j-1} (1-j)_{2(j-i)-2} \sum_{s=0}^{i} \binom{i}{s} \left(\frac{1}{2}\right)_s \frac{m^{i-s} \cos^{-2s-1}(x) \left(1 - m \sin^2(x)\right)^{s-i}}{(j-i-1)!\, (2 \sin(x))^{-2i+j-1}}$$
$$\left(\left(-\frac{1}{2}\right)_{i-s} \sqrt{1 - m \sin^2(x)} - \frac{\left(\frac{1}{2}\right)_{i-s}}{K(m) \sqrt{1 - m \sin^2(x)}} E(m) \right) (z-x)^k \;/;$$
$$x \in \mathbb{R} \bigwedge m \in \mathbb{R} \bigwedge m > 1 \bigwedge \pi u + \frac{\pi}{2} < x < \pi (u+1) - \csc^{-1}\left(\sqrt{m}\right) \bigwedge$$
$$u \in \mathbb{Z}$$

08.07.06.0018.01

$$Z(z \mid m) = \left(e^{-\pi i \left\lfloor \frac{\arg(z-x)}{2\pi} \right\rfloor} Z(x \mid m) - Z\left(\csc^{-1}\left(\sqrt{m}\right) \,\middle|\, m\right) \left(1 - e^{-\pi i \left\lfloor \frac{\arg(z-x)}{2\pi} \right\rfloor}\right) \right) (1 + O(z-x)) \;/;$$
$$(z \to x) \bigwedge x \in \mathbb{R} \bigwedge m \in \mathbb{R} \bigwedge m > 1 \bigwedge \pi u + \frac{\pi}{2} < x < \pi (u+1) - \csc^{-1}\left(\sqrt{m}\right) \bigwedge u \in \mathbb{Z}$$

Expansions at $z == 0$

08.07.06.0019.01

$$Z(z\mid m)\propto\left(1-\frac{E(m)}{K(m)}\right)z-\frac{m}{6}\left(\frac{E(m)}{K(m)}+1\right)z^3+\frac{m}{120}\left(4-3m-\frac{(9m-4)E(m)}{K(m)}\right)z^5+\frac{m}{5040}\left(-45m^2+60m-16-\frac{\left(225m^2-180m+16\right)E(m)}{K(m)}\right)z^7+\frac{m}{362\,880}\left(-1575m^3+2520m^2-1008m+64-\left(11\,025m^3-12\,600m^2+3024m-64\right)\frac{E(m)}{K(m)}\right)z^9+\ldots\;/;\;(z\to 0)$$

08.07.06.0001.02

$$Z(z\mid m)\propto\left(1-\frac{E(m)}{K(m)}\right)z-\frac{m}{6}\left(\frac{E(m)}{K(m)}+1\right)z^3+\frac{m}{120}\left(4-3m-\frac{(9m-4)E(m)}{K(m)}\right)z^5+\frac{m}{5040}\left(-45m^2+60m-16-\frac{\left(225m^2-180m+16\right)E(m)}{K(m)}\right)z^7+\frac{m}{362\,880}\left(-1575m^3+2520m^2-1008m+64-\left(11\,025m^3-12\,600m^2+3024m-64\right)\frac{E(m)}{K(m)}\right)z^9+O\left(z^{11}\right)$$

08.07.06.0020.01

$$Z(z\mid m)=z-\frac{E(m)\,z}{K(m)}+\sum_{i=1}^{\infty}\sum_{k=1}^{\infty}\sum_{j=0}^{k-1}\frac{\binom{2k}{j}(-1)^{i-j+k}\,2^{2i-2k+1}\,(j-k)^{2i}\,m^k}{k!\,(2i+1)!}\left(\left(-\frac{1}{2}\right)_k-\frac{E(m)}{K(m)}\left(\frac{1}{2}\right)_k\right)z^{2i+1}$$

08.07.06.0021.01

$$Z(z\mid m)\propto\left(1-\frac{E(m)}{K(m)}\right)z\left(1+O\left(z^2\right)\right)$$

Expansions at $z=\csc^{-1}\left(\sqrt{m}\right)+\pi u\;/;\;u\in\mathbb{Z}$

08.07.06.0022.01

$$Z(z\mid m)\propto Z\left(\csc^{-1}\left(\sqrt{m}\right)\middle|\,m\right)+\frac{\sqrt{2}\,E(m)}{\sqrt{m-1}\,K(m)}\sqrt{-\sqrt{m-1}\,(z-z_0)}+\sqrt{2}\sqrt{-\sqrt{m-1}\,(z-z_0)}\,(z-z_0)\left(\frac{2}{3}-\frac{(m-2)E(m)}{12(m-1)K(m)}+\left(\frac{m-2}{10\sqrt{m-1}}+\frac{\left(9m^2-4m+4\right)E(m)}{480(m-1)^{3/2}K(m)}\right)(z-z_0)-\left(\frac{3m^2+20m-20}{336(m-1)}+\frac{\left(15m^3-26m^2-12m+8\right)E(m)}{2688(m-1)^2K(m)}\right)(z-z_0)^2+\ldots\right)\;/;\;(z\to z_0)\bigwedge z_0=\csc^{-1}\left(\sqrt{m}\right)+\pi u\bigwedge u\in\mathbb{Z}$$

08.07.06.0023.01

$$Z(z \mid m) = Z\left(\csc^{-1}\left(\sqrt{m}\right) \middle| m\right) + \frac{\sqrt{2}\, E(m)}{\sqrt{m-1}\, K(m)} \sqrt{-\sqrt{m-1}\, (z-z_0)} + \sqrt{2}\, (z-z_0) \sqrt{-\sqrt{m-1}\, (z-z_0)}$$
$$\sum_{k=0}^{\infty} \frac{2}{2k+3} \binom{k-\frac{1}{2}}{k} \left(\frac{E(m)\left(4k^2+8k+3\right)}{4\sqrt{m-1}\, K(m)\,(k+1)} \sum_{j=0}^{k+1} \frac{(-1)^j}{2j+1} \binom{k+1}{j} p_{j,k+1} + \sum_{j=0}^{k} \frac{(-1)^j}{1-2j} \binom{k}{j} p_{j,k} \right) (z-z_0)^k /;$$
$$z_0 = \pi u + \csc^{-1}\left(\sqrt{m}\right) \bigwedge u \in \mathbb{Z} \bigwedge a_0 = 1 \bigwedge a_{2k} = \frac{(-1)^k\, 2^{2k}}{(2k+1)!} \bigwedge a_{2k+1} = \frac{(-1)^{k-1}\, 2^{2k}\, (2-m)}{\sqrt{m-1}\, (2k+2)!} \bigwedge$$
$$k \in \mathbb{N} \bigwedge p_{u,0} = 1 \bigwedge p_{u,v} = \frac{1}{v} \sum_{j=1}^{v} (u\, j + j - v)\, a_j\, p_{u,v-j}$$

08.07.06.0024.01

$$Z(z \mid m) \propto Z\left(\csc^{-1}\left(\sqrt{m}\right) \middle| m\right) + \frac{\sqrt{2}\, E(m)}{\sqrt{m-1}\, K(m)} \sqrt{-\sqrt{m-1}\, (z-z_0)}\, (1 + O(z-z_0)) /;$$
$$(z \to z_0) \bigwedge z_0 = \csc^{-1}\left(\sqrt{m}\right) + \pi u \bigwedge u \in \mathbb{Z}$$

Expansions at $z == -\csc^{-1}\left(\sqrt{m}\right) + \pi u /; u \in \mathbb{Z}$

08.07.06.0025.01

$$Z(z \mid m) \propto -Z\left(\csc^{-1}\left(\sqrt{m}\right) \middle| m\right) - \frac{\sqrt{2}\, E(m)}{\sqrt{m-1}\, K(m)} \sqrt{\sqrt{m-1}\, (z-z_0)} +$$
$$\sqrt{2}\, (z-z_0) \sqrt{\sqrt{m-1}\, (z-z_0)} \left(\frac{2}{3} - \frac{(m-2)\, E(m)}{12\,(m-1)\, K(m)} - \left(\frac{m-2}{10\sqrt{m-1}} + \frac{\left(9m^2-4m+4\right) E(m)}{480\,(m-1)^{3/2}\, K(m)} \right) (z-z_0) - \right.$$
$$\left. \left(\frac{3m^2+20m-20}{336\,(m-1)} + \frac{\left(15m^3-26m^2-12m+8\right) E(m)}{2688\,(m-1)^2\, K(m)} \right) (z-z_0)^2 + \ldots \right) /; (z \to z_0) \bigwedge z_0 = \pi u - \csc^{-1}\left(\sqrt{m}\right) \bigwedge u \in \mathbb{Z}$$

08.07.06.0026.01

$$Z(z \mid m) \propto -Z\left(\csc^{-1}\left(\sqrt{m}\right) \middle| m\right) - \frac{\sqrt{2}\, E(m)}{\sqrt{m-1}\, K(m)} \sqrt{\sqrt{m-1}\, (z-z_0)} + \sqrt{2} \sqrt{\sqrt{m-1}\, (z-z_0)}\, (z-z_0)$$
$$\sum_{k=0}^{\infty} \frac{2\,(-1)^k}{2k+3} \binom{k-\frac{1}{2}}{k} \left(\frac{\left(4k^2+8k+3\right) E(m)}{4\sqrt{m-1}\, (k+1)\, K(m)} \sum_{j=0}^{k+1} \frac{(-1)^j}{2j+1} \binom{k+1}{j} p_{j,k+1} + \sum_{j=0}^{k} \frac{(-1)^j}{1-2j} \binom{k}{j} p_{j,k} \right) (z-z_0)^k /;$$
$$z_0 == -\csc^{-1}\left(\sqrt{m}\right) + \pi u \bigwedge u \in \mathbb{Z} \bigwedge a_0 == 1 \bigwedge a_{2k} == \frac{(-1)^k\, 2^{2k}}{(2k+1)!} \bigwedge a_{2k+1} == \frac{(-1)^{k-1}\, 2^{2k}\, (2-m)}{\sqrt{m-1}\, (2k+2)!} \bigwedge$$
$$k \in \mathbb{N} \bigwedge p_{u,0} == 1 \bigwedge p_{u,v} == \frac{1}{v} \sum_{j=1}^{v} (u\, j + j - v)\, a_j\, p_{u,v-j}$$

08.07.06.0027.01

$$Z(z \mid m) \propto -Z\left(\csc^{-1}\left(\sqrt{m}\right) \middle| m\right) - \frac{\sqrt{2}\, E(m)}{\sqrt{m-1}\, K(m)} \sqrt{\sqrt{m-1}\, (z-z_0)}\, (1 + O(z-z_0)) /;$$
$$(z \to z_0) \bigwedge z_0 = \pi u - \csc^{-1}\left(\sqrt{m}\right) \bigwedge u \in \mathbb{Z}$$

Expansions at $z == \pi/2 + \pi u /; u \in \mathbb{Z} \bigwedge m > 1$

08.07.06.0028.01

$$Z(z \mid m) \propto -Z\left(\csc^{-1}\left(\sqrt{m}\right) \middle| m\right)\left(i \sqrt{-\frac{1}{(z-z_0)^2}}\, (z-z_0) + \frac{\sqrt{(z-z_0)^2}}{z-z_0}\right) + \left(\sqrt{1-m} - \frac{E(m)}{\sqrt{1-m}\, K(m)}\right)(z-z_0) + \left(\frac{m}{6\sqrt{1-m}} + \frac{E(m)\, m}{6\,(1-m)^{3/2}\, K(m)}\right)(z-z_0)^3 + \left(\frac{(m-4)\, m}{120\,(1-m)^{3/2}} - \frac{m\,(5\,m+4)\, E(m)}{120\,(1-m)^{5/2}\, K(m)}\right)(z-z_0)^5 + \ldots /; (z \to z_0) \bigwedge z_0 = \frac{\pi}{2} + \pi\, u \bigwedge u \in \mathbb{Z} \bigwedge m \in \mathbb{R} \bigwedge m > 1$$

08.07.06.0029.01

$$Z(z \mid m) = -Z\left(\csc^{-1}\left(\sqrt{m}\right) \middle| m\right)\left(i \sqrt{-\frac{1}{(z-z_0)^2}}\, (z-z_0) + \frac{\sqrt{(z-z_0)^2}}{z-z_0}\right) + \left(\sqrt{1-m} - \frac{E(m)}{\sqrt{1-m}\, K(m)}\right)(z-z_0) + \sum_{k=2}^{\infty} \frac{i^{k-1}}{k!} \binom{k-\frac{3}{2}}{k-1} \sum_{q=1}^{k-1} \left(\frac{\sqrt{1-m}}{1-2\,q} - \frac{(2\,k-1)\, E(m)}{\sqrt{1-m}\, (2\,q+1)\, K(m)}\right) (-1)^q \binom{k-1}{q} (1-m)^{-q} \sum_{j=0}^{q} \binom{q}{j} (-m)^j\, (2-m)^{q-j}\, 2^{-j+k-q-1} \sum_{i=0}^{j} \binom{j}{i} (2\,i-j)^{k-1}\, (z-z_0)^k /; (z \to z_0) \bigwedge z_0 = \pi\, u + \frac{\pi}{2} \bigwedge u \in \mathbb{Z} \bigwedge m \in \mathbb{R} \bigwedge m > 1$$

08.07.06.0030.01

$$Z(z \mid m) \propto -Z\left(\csc^{-1}\left(\sqrt{m}\right) \middle| m\right)\left(i \sqrt{-\frac{1}{(z-z_0)^2}}\, (z-z_0) + \frac{\sqrt{(z-z_0)^2}}{z-z_0}\right) + \left(\sqrt{1-m} - \frac{E(m)}{\sqrt{1-m}\, K(m)}\right)(z-z_0)\left(1 + O\left((z-z_0)^2\right)\right) /; (z \to z_0) \bigwedge z_0 = \frac{\pi}{2} + \pi\, u \bigwedge u \in \mathbb{Z} \bigwedge m \in \mathbb{R} \bigwedge m > 1$$

Expansions at generic point $m == m_0$

For the function itself

08.07.06.0031.01

$$Z(z \mid m) \propto Z(z \mid m_0) + \frac{1}{2\,(m_0-1)\, m_0\, K(m_0)}\left(E(m_0)\left(Z(z \mid m_0) - \frac{m_0 \sin(2\,z)}{2\sqrt{1-m_0 \sin^2(z)}}\right) + K(m_0)\, Z(z \mid m_0)\, (m_0-1)\right)(m-m_0) + \frac{1}{16\,(m_0-1)^2\, m_0^2 \left(1-m_0 \sin^2(z)\right)^{3/2} K(m_0)^2}\left(-(\cos(2\,z)\, m_0 - m_0 + 2)\left(m_0 \sin(2\,z) - 2\,Z(z \mid m_0)\sqrt{1-m_0 \sin^2(z)}\right) E(m_0)^2 - K(m_0)\left(4\,Z(z \mid m_0)\left(1-m_0 \sin^2(z)\right)^{3/2} - m_0\, (\cos(2\,z)\, m_0 - 2\,m_0 + 3) \sin(2\,z)\right) E(m_0) + \frac{1}{2}(m_0-1)\,(\cos(2\,z)\, m_0 - m_0 + 2)\, K(m_0)^2 \left(\sin(2\,z) - 2\,Z(z \mid m_0)\sqrt{1-m_0 \sin^2(z)}\right) m_0\right)(m-m_0)^2 + \ldots /; (m \to m_0)$$

08.07.06.0032.01

$$Z(z \mid m) \propto Z(z \mid m_0) + \frac{1}{2\,(m_0-1)\,m_0\,K(m_0)} \left(E(m_0) \left(Z(z \mid m_0) - \frac{m_0 \sin(2\,z)}{2\sqrt{1-m_0 \sin^2(z)}} \right) + K(m_0)\,Z(z \mid m_0)\,(m_0-1) \right) (m-m_0) +$$

$$\frac{1}{16\,(m_0-1)^2\,m_0^2 \left(1-m_0 \sin^2(z)\right)^{3/2} K(m_0)^2} \left(-(\cos(2\,z)\,m_0 - m_0 + 2) \left(m_0 \sin(2\,z) - 2\,Z(z \mid m_0) \sqrt{1-m_0 \sin^2(z)} \right) E(m_0)^2 - \right.$$

$$K(m_0) \left(4\,Z(z \mid m_0) \left(1-m_0 \sin^2(z)\right)^{3/2} - m_0\,(\cos(2\,z)\,m_0 - 2\,m_0 + 3) \sin(2\,z) \right) E(m_0) +$$

$$\left. \frac{1}{2}\,(m_0-1)\,(\cos(2\,z)\,m_0 - m_0 + 2)\,K(m_0)^2 \left(\sin(2\,z) - 2\,Z(z \mid m_0) \sqrt{1-m_0 \sin^2(z)} \right) m_0 \right) (m-m_0)^2 + O\!\left((m-m_0)^3\right)$$

08.07.06.0033.01

$$Z(z \mid m) = \sum_{k=0}^{\infty} \left(\frac{1}{k!}\,Z^{(0,k)}(z \mid m_0) - \frac{1}{c_0} \sum_{j=0}^{k} (j+1)\,d_{k-j} \sum_{r=0}^{j} \frac{(-1)^r \binom{j}{r}}{r+1}\,p_{r,j} \right) (m-m_0)^k \;/;$$

$$a_k = \frac{\left(-\frac{1}{2}\right)_k \left(\frac{1}{2}\right)_k}{(k!)^2}\,{}_2F_1\!\left(k-\frac{1}{2}, k+\frac{1}{2}; k+1; m_0\right) \bigwedge c_k = \frac{\left(\frac{1}{2}\right)_k^2}{(k!)^2}\,{}_2F_1\!\left(k+\frac{1}{2}, k+\frac{1}{2}; k+1; m_0\right) \bigwedge$$

$$b_k = \frac{1}{k!}\,F^{(0,k)}(z \mid m_0) \bigwedge d_k = \sum_{n=0}^{k} a_n\,b_{k-n} \bigwedge p_{j,0} = 1 \bigwedge p_{j,k} = \frac{1}{c_0\,k} \sum_{m=1}^{k} (j\,m - k + m)\,c_m\,p_{j,k-m} \bigwedge k \in \mathbb{N}^+$$

08.07.06.0034.01

$$Z(z \mid m) = \sum_{k=0}^{\infty} \frac{1}{k!}\,Z^{(0,k)}(z \mid m_0)\,(m-m_0)^k$$

08.07.06.0035.01

$$Z(z \mid m) \propto Z(z \mid m_0)\,(1 + O(m-m_0))$$

Expansions at *m* == 0

08.07.06.0002.02

$$Z(z \mid m) \propto \frac{1}{4} \sin(2\,z)\,m + \left(\frac{1}{16} \sin(2\,z) - \frac{1}{64} \sin(4\,z) \right) m^2 +$$

$$\left(\frac{17}{512} \sin(2\,z) - \frac{3}{256} \sin(4\,z) + \frac{1}{512} \sin(6\,z) \right) m^3 + \left(\frac{45 \sin(2\,z)}{2048} + \frac{5 \sin(6\,z)}{2048} - \frac{37 \sin(4\,z)}{4096} - \frac{5 \sin(8\,z)}{16\,384} \right) m^4 +$$

$$\left(\frac{1059 \sin(2\,z)}{65\,536} + \frac{325 \sin(6\,z)}{131\,072} + \frac{7 \sin(10\,z)}{131\,072} - \frac{119 \sin(4\,z)}{16\,384} - \frac{35 \sin(8\,z)}{65\,536} \right) m^5 +$$

$$\left(\frac{3315 \sin(2\,z)}{262\,144} + \frac{1245 \sin(6\,z)}{524\,288} + \frac{63 \sin(10\,z)}{524\,288} - \frac{707 \sin(8\,z)}{1\,048\,576} - \frac{12\,653 \sin(4\,z)}{2\,097\,152} - \frac{21 \sin(12\,z)}{2\,097\,152} \right) m^6 +$$

$$\left(\frac{172\,989 \sin(2\,z)}{16\,777\,216} + \frac{37\,377 \sin(6\,z)}{16\,777\,216} + \frac{3045 \sin(10\,z)}{16\,777\,216} + \frac{33 \sin(14\,z)}{16\,777\,216} - \frac{3157 \sin(8\,z)}{4\,194\,304} - \frac{43\,087 \sin(4\,z)}{8\,388\,608} - \frac{231 \sin(12\,z)}{8\,388\,608} \right) m^7 +$$

$$O\!\left(m^8\right) \;/;\; (m \to 0)$$

08.07.06.0003.01

$$Z(z \mid m) = \sum_{k=1}^{\infty} \frac{(-1)^k \sin(2kz)}{k} \binom{k-\frac{3}{2}}{k} \left(\frac{(2k-1)E(m)}{K(m)} \, {}_2F_1\left(k+\frac{1}{2}, k+\frac{1}{2}; 2k+1; m\right) + {}_2F_1\left(k-\frac{1}{2}, k+\frac{1}{2}; 2k+1; m\right) \right) \left(\frac{m}{4}\right)^k$$

08.07.06.0004.01

$$Z(z \mid m) = \sum_{k=0}^{\infty} \frac{1}{k!} \left(\left(-\frac{1}{2}\right)_k - \frac{E(m)}{K(m)} \left(\frac{1}{2}\right)_k \right) \sum_{j=1}^{k} \binom{2k}{k-j} \frac{(-1)^j \sin(2jz)}{j} \left(\frac{m}{4}\right)^k \;/;\; |m| < 1 \bigwedge -\frac{\pi}{2} \le z \le \frac{\pi}{2}$$

08.07.06.0005.01

$$Z(z \mid m) = \left(1 - \frac{2z}{\pi}\right) E(m) - \cos(z) \sum_{k=0}^{\infty} \frac{1}{k!} \left(-\frac{1}{2}\right)_k \, {}_2F_1\left(\frac{1}{2}, \frac{1}{2}-k; \frac{3}{2}; \cos^2(z)\right) m^k - \frac{E(m)}{K(m)} \sum_{k=1}^{\infty} \frac{(-1)^k \sin(2kz)}{k} \binom{k-\frac{1}{2}}{k} \, {}_2F_1\left(k+\frac{1}{2}, k+\frac{1}{2}; 2k+1; m\right) \left(\frac{m}{4}\right)^k$$

08.07.06.0036.01

$$Z(z \mid m) = z - \frac{E(m)\, z}{K(m)} + z \sum_{k=1}^{\infty} \left(\frac{\left(\frac{1}{2}\right)_k}{k!^2} \left(\left(-\frac{1}{2}\right)_k - \frac{E(m)}{K(m)} \left(\frac{1}{2}\right)_k \right) + \frac{(-1)^k \sin(2kz)}{4^k k} \left(\binom{k-\frac{3}{2}}{k} \, {}_2F_1\left(k-\frac{1}{2}, k+\frac{1}{2}; 2k+1; m\right) - \frac{E(m)}{K(m)} \binom{k-\frac{1}{2}}{k} \, {}_2F_1\left(k+\frac{1}{2}, k+\frac{1}{2}; 2k+1; m\right) \right) \right) m^k$$

08.07.06.0006.01

$$Z(z \mid m) = E(m) - \frac{\sin(z)\, E(m)}{K(m)} F_{1\times0\times0}^{1\times1\times1}\left(\begin{matrix} \frac{1}{2}; \frac{1}{2}; \frac{1}{2}; \\ \frac{3}{2};;; \end{matrix} \; m \sin^2(z), \sin^2(z) \right) - \cos(z) \left(F_{2\times0\times0}^{1\times2\times2}\left(\begin{matrix} \frac{1}{2}; -\frac{1}{2}, \frac{1}{2}; \frac{1}{2}, 1; \\ 1, \frac{3}{2};;; \end{matrix} \; -m\cos^2(z), \cos^2(z) \right) - \frac{m}{2} F_{1\times1\times1}^{2\times1\times1}\left(\begin{matrix} \frac{1}{2}, \frac{3}{2}; \frac{1}{2}; 1; \\ 2; \frac{3}{2}; \frac{3}{2}; \end{matrix} \; -m\cos^2(z), m \right) \right)$$

08.07.06.0037.01

$$Z(z \mid m) \propto \frac{1}{4} m \sin(2z) (1 + O(m))$$

Other series representations

08.07.06.0007.01

$$Z(z \mid m) = \frac{2\pi}{K(m)} \sum_{k=1}^{\infty} \frac{q(m)^k}{1 - q(m)^{2k}} \sin\left(\frac{(k\pi)\, F(z \mid m)}{K(m)} \right)$$

Integral representations

On the real axis

Of the direct function

08.07.07.0001.01

$$Z(z \mid m) = \int_0^z \left(\sqrt{1 - m \sin^2(t)} - \frac{E(m)}{K(m) \sqrt{1 - m \sin^2(t)}} \right) dt$$

08.07.07.0002.01

$$Z(z \mid m) = \int_0^{\sin(z)} \frac{1}{\sqrt{1-t^2}} \left(\sqrt{1 - m t^2} - \frac{E(m)}{K(m) \sqrt{1 - m t^2}} \right) dt \;/;\; -\frac{\pi}{2} < z < \frac{\pi}{2}$$

Differential equations

Ordinary nonlinear differential equations

08.07.13.0001.01

$$m^2 \cos^2(z) \sin^2(z) w^{(1,0)}(z, m)^2 + \left(m \sin^2(z) - 1\right) \left(4 m^2 \cos^2(z) \sin^2(z) + \left(m \sin^2(z) - 1\right) w^{(2,0)}(z, m)^2 + m \sin(2 z) w^{(1,0)}(z, m) w^{(2,0)}(z, m)\right) = 0 \;/;\; w(z) = Z(z \mid m)$$

08.07.13.0002.01

$$16 E(m)^6 + 8 \left(w'(z)^2 + w''(z)^2 + 4 m - 8\right) K(m) E(m)^5 + \left(w'(z)^4 + 2 \left(w''(z)^2 + 16 (m-2)\right) w'(z)^2 + \left(w''(z)^2 + 4 m\right)^2 - 16 \left(w''(z)^2 + 6 m - 6\right)\right) K(m)^2 E(m)^4 + 2 \left(5 (m-2) w'(z)^4 + \left(3 (m-2) w''(z)^2 + 4 ((m-14) m + 14)\right) w'(z)^2 - 4 (m-1) \left(w''(z)^2 + 4 (m-2)\right)\right) K(m)^3 E(m)^3 + \left((m-2) w'(z)^6 + \left((m-2) w''(z)^2 + (m-54) m + 54\right) w'(z)^4 - 2 (m-1) \left(9 w''(z)^2 + 16 (m-2)\right) w'(z)^2 + 16 (m-1)^2\right) K(m)^4 E(m)^2 - 2 (m-1) K(m)^5 w'(z)^2 \left(6 w'(z)^4 + \left(4 w''(z)^2 + 5 m - 10\right) w'(z)^2 - 4 m + 4\right) E(m) - (m-1) K(m)^6 w'(z)^4 \left(w'(z)^4 + \left(w''(z)^2 + m - 2\right) w'(z)^2 - m + 1\right) = 0 \;/;\; w(z) = Z(z \mid m)$$

Transformations

Transformations and argument simplifications

Argument involving basic arithmetic operations

08.07.16.0001.01

$$E(-z \mid m) = -E(z \mid m)$$

08.07.16.0002.01

$$Z(z + \pi k \mid m) = Z(z \mid m) \;/;\; k \in \mathbb{Z}$$

Products, sums, and powers of the direct function

Sums of the direct function

08.07.16.0003.01

$$Z(z_1 \mid m) + Z(z_2 \mid m) = Z(z \mid m) - m \sin(z_1) \sin(z_2) \sin(z) \;/;\; z = 2 \tan^{-1}\left(\frac{\sin(z_1) \sqrt{1 - m \sin^2(z_2)} - \sin(z_2) \sqrt{1 - m \sin^2(z_1)}}{\cos(z_1) - \cos(z_2)} \right) \bigwedge 0 < m < 1 \bigwedge |z_1| < 1 \bigwedge |z_2| < 1$$

08.07.16.0004.01

$$Z(z_1 \mid m) + Z(z_2 \mid m) == Z\left(\sin^{-1}(w) \,\middle|\, m\right) + m\, w \sin(z_1) \sin(z_2) \,/;$$
$$w == \frac{\cos(z_2)\sqrt{1 - m\sin^2(z_2)}\, \sin(z_1) + \cos(z_1)\sqrt{1 - m\sin^2(z_1)}\, \sin(z_2)}{1 - m\sin^2(z_1)\sin^2(z_2)} \bigwedge 0 \le m < 1 \bigwedge |z_1| < 1 \bigwedge |z_2| < 1$$

Identities

Functional identities

08.07.17.0001.01

$$Z(z \mid m) == i\, \text{dn}(-i\, F(z \mid m) \mid 1 - m)\, \text{sc}(-i\, F(z \mid m) \mid 1 - m) - i\, Z(\text{am}(-i\, F(z \mid m) \mid 1 - m) \mid 1 - m) - \frac{\pi F(z \mid m)}{2 K(m) K(1 - m)}$$

Differentiation

Low-order differentiation

With respect to z

08.07.20.0001.01

$$\frac{\partial Z(z \mid m)}{\partial z} == \sqrt{1 - m\sin^2(z)} - \frac{E(m)}{K(m)\sqrt{1 - m\sin^2(z)}}$$

08.07.20.0002.01

$$\frac{\partial^2 Z(z \mid m)}{\partial z^2} == -\frac{m\sin(2z)}{2\sqrt{1 - m\sin^2(z)}} - \frac{m\, E(m)\sin(2z)}{K(m)\left(1 - m\sin^2(z)\right)^{3/2}}$$

With respect to m

08.07.20.0003.01

$$\frac{\partial Z(z \mid m)}{\partial m} == \frac{1}{2m}\left(1 + \frac{E(m)}{(m-1)K(m)}\right) Z(z \mid m) - \frac{E(m)\sin(2z)}{4(m-1)\sqrt{1 - m\sin^2(z)}\, K(m)}$$

08.07.20.0004.01

$$\frac{\partial^2 Z(z \mid m)}{\partial m^2} == \left(\frac{(m-1)m}{2}(\cos(2z)\, m - m + 2)\left(\sin(2z) - 2 Z(z \mid m)\sqrt{1 - m\sin^2(z)}\right) K(m)^2 - \right.$$
$$(\cos(2z)\, m - m + 2)\left(m\sin(2z) - 2 Z(z \mid m)\sqrt{1 - m\sin^2(z)}\right) E(m)^2 -$$
$$\left.\left(4\left(1 - m\sin^2(z)\right)^{3/2} Z(z \mid m) - m\, \sin(2z)(\cos(2z)\, m - 2m + 3)\right) K(m)\, E(m)\right) \Big/ \left(8(m-1)^2 m^2 \left(1 - m\sin^2(z)\right)^{3/2} K(m)^2\right)$$

Symbolic differentiation

With respect to z

08.07.20.0008.01

$$\frac{\partial^n \operatorname{Z}(z\mid m)}{\partial z^n} = \delta_n \operatorname{Z}(z\mid m) + \delta_{n-1}\left(\sqrt{1-m\sin^2(z)} - \frac{E(m)}{K(m)\sqrt{1-m\sin^2(z)}}\right) - \frac{2\, i^{n-1}\left(-\frac{1}{2}\right)_n}{(n-1)!}\sum_{q=1}^{n-1}\frac{(-1)^q\binom{n-1}{q}}{\left(1-m\sin^2(z)\right)^q}\left(\frac{E(m)\,(1-2n)}{K(m)\left((2q+1)\sqrt{1-m\sin^2(z)}\right)} + \frac{\sqrt{1-m\sin^2(z)}}{1-2q}\right) \sum_{j=0}^{q}\binom{q}{j} m^j\,(2-m)^{q-j}\,2^{-j+n-q-1}\sum_{i=0}^{j}\binom{j}{i}(2i-j)^{n-1}\,e^{2(2i-j)iz} \;/;\; n\in\mathbb{N}$$

08.07.20.0009.01

$$\frac{\partial^n \operatorname{Z}(z\mid m)}{\partial z^n} = \operatorname{Z}(z\mid m)\,\delta_n + \sum_{j=1}^{n}\frac{1}{j!}\sum_{q=0}^{j-1}\binom{j}{q}\sum_{p=0}^{j-q}(-1)^q\,2^{q-j}\sin^q(z)\,(q-j+2p)^n\,e^{-\frac{1}{2}i(\pi(j+n-q-2p)+2(-j+q+2p)z)}\binom{j-q}{p}\sum_{i=0}^{j-1}(1-j)_{2(j-i)-2}\sum_{s=0}^{i}\binom{i}{s}\left(\frac{1}{2}\right)_s \frac{m^{i-s}\cos^{-2s-1}(z)\left(1-m\sin^2(z)\right)^{s-i}}{(-i+j-1)!\,(2\sin(z))^{-2i+j-1}}\left(\left(-\frac{1}{2}\right)_{i-s}\sqrt{1-m\sin^2(z)} - \frac{\left(\frac{1}{2}\right)_{i-s}}{K(m)\sqrt{1-m\sin^2(z)}}E(m)\right) \;/;\; n\in\mathbb{N}$$

08.07.20.0005.01

$$\frac{\partial^n \operatorname{Z}(z\mid m)}{\partial z^n} = -\frac{(2i)^{n-1}\sqrt{\pi}}{4\sqrt{1-m}\;m\sqrt{1-m\sin^2(z)}}\left(\frac{e^{2iz}\left(m+2\sqrt{1-m}-2\right)}{m}\right)^{-1/2}\left(\frac{m\left(1-e^{2iz}\right)+2\sqrt{1-m}-2}{\sqrt{1-m}}\right)^{-1/2}\sum_{k=0}^{n-1}\frac{e^{2ikz}}{\Gamma\left(\frac{3}{2}-k\right)}\left(\frac{\left(e^{2iz}-1\right)m+2\sqrt{1-m}+2}{m}\right)^{-k}\mathcal{S}_{n-1}^{(k)} \left(\frac{E(m)\,(2k-1)\,m}{K(m)}\left(\left(e^{2iz}-1\right)m-2\sqrt{1-m}+2\right)F_1\left(\frac{1}{2};-\frac{1}{2},\frac{1}{2};\frac{1}{2}-k;\frac{\left(e^{2iz}-1\right)m+2\sqrt{1-m}+2}{2\sqrt{1-m}+2-m},\frac{\left(e^{2iz}-1\right)m+2\sqrt{1-m}+2}{4\sqrt{1-m}}\right) - \left(m-2e^{2iz}(m-2)+e^{4iz}m\right)\left(\left(\sqrt{1-m}-2\right)m-2\sqrt{1-m}+2\right) F_1\left(\frac{3}{2};\frac{1}{2},-\frac{1}{2};\frac{3}{2}-k;\frac{\left(e^{2iz}-1\right)m+2\sqrt{1-m}+2}{-m+2\sqrt{1-m}+2},\frac{\left(e^{2iz}-1\right)m+2\sqrt{1-m}+2}{4\sqrt{1-m}}\right)\right) \;/;\; n\in\mathbb{N}^+$$

08.07.20.0006.02

$$\frac{\partial^n \operatorname{Z}(z\mid m)}{\partial z^n} = \operatorname{Z}(z\mid m)\,\delta_n + 2\left(-\frac{1}{2}\right)_n\sum_{k=0}^{n-1}\frac{(-1)^k\left(1-m\sin^2(z)\right)^{-k}}{k!\,(-k+n-1)!}\left(\frac{E(m)\,(2n-1)}{K(m)\left((2k+1)\sqrt{1-m\sin^2(z)}\right)} + \frac{\sqrt{1-m\sin^2(z)}}{2k-1}\right)\frac{\partial^{n-1}\left(1-m\sin^2(z)\right)^k}{\partial z^{n-1}} \;/;\; n\in\mathbb{N}$$

Fractional integro-differentiation

With respect to z

08.07.20.0007.01

$$\frac{\partial^\alpha Z(z\mid m)}{\partial z^\alpha} = z^{1-\alpha}\sqrt{\pi}\, 2^\alpha \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \left(\left(-\frac{1}{2}\right)_k {}_2F_1\left(k-\frac{1}{2}, k+\frac{1}{2}; 2k+1; m\right) - \frac{E(m)}{K(m)}\left(\frac{1}{2}\right)_k {}_2F_1\left(k+\frac{1}{2}, k+\frac{1}{2}; 2k+1; m\right)\right) {}_1\tilde{F}_2\left(1; 1-\frac{\alpha}{2}, \frac{3-\alpha}{2}; -k^2 z^2\right)\left(\frac{m}{4}\right)^k$$

Integration

Indefinite integration

Involving only one direct function

08.07.21.0001.01

$$\int Z(z\mid m)\, dz = \frac{1}{2}\sum_{k=1}^{\infty} \frac{(-1)^k (1-\cos(2kz))}{k!\, k^2}\left(-\frac{1}{2}\right)_k \left({}_2F_1\left(k-\frac{1}{2}, k+\frac{1}{2}; 2k+1; m\right) + \frac{(2k-1)E(m)}{K(m)}\, {}_2F_1\left(k+\frac{1}{2}, k+\frac{1}{2}; 2k+1; m\right)\right)\left(\frac{m}{4}\right)^k$$

Representations through more general functions

Through hypergeometric functions of two variables

08.07.26.0001.01

$$Z(z\mid m) = E(m) - \frac{\sin(z)E(m)}{K(m)} F_{1\times 0\times 0}^{1\times 1\times 1}\left(\begin{matrix}\frac{1}{2}; \frac{1}{2}; \frac{1}{2}; \\ \frac{3}{2};;;\end{matrix}\; m\sin^2(z), \sin^2(z)\right) - \cos(z)\left(F_{2\times 0\times 0}^{1\times 2\times 2}\left(\begin{matrix}\frac{1}{2}; -\frac{1}{2}, \frac{1}{2}; \frac{1}{2}, 1; \\ 1, \frac{3}{2};;;\end{matrix}\; -m\cos^2(z), \cos^2(z)\right) - \frac{m}{2} F_{1\times 1\times 1}^{2\times 1\times 1}\left(\begin{matrix}\frac{1}{2}, \frac{3}{2}; \frac{1}{2}; 1; \\ 2; \frac{3}{2}; \frac{3}{2};\end{matrix}\; -m\cos^2(z), m\right)\right)$$

Through Meijer G

Classical cases involving $\tan^{-1}$ in the arguments

08.07.26.0002.01

$$Z\left(\tan^{-1}\left(\sqrt[4]{z}\right)\,\middle|\, 1-\frac{1}{z}\right) = \frac{\sqrt{z}-1}{2\sqrt{z}} - \frac{1}{4\pi} G_{2,2}^{2,2}\left(z \,\middle|\, \begin{matrix}0, 1 \\ -\frac{1}{2}, \frac{1}{2}\end{matrix}\right) - \frac{1}{4\pi K\left(1-\frac{1}{z}\right)} E\left(1-\frac{1}{z}\right) G_{2,2}^{2,2}\left(z \,\middle|\, \begin{matrix}1, 1 \\ \frac{1}{2}, \frac{1}{2}\end{matrix}\right) /; z \notin (-\infty, 0)$$

Classical cases involving $\cot^{-1}$ in the arguments

08.07.26.0003.01

$$Z\left(\cot^{-1}\left(\sqrt[4]{z}\right)\,\middle|\, 1-z\right) = \frac{1-\sqrt{z}}{2} - \frac{E(1-z)}{4\pi K(1-z)} G_{2,2}^{2,2}\left(z \,\middle|\, \begin{matrix}\frac{1}{2}, \frac{1}{2} \\ 0, 0\end{matrix}\right) - \frac{1}{4\pi} G_{2,2}^{2,2}\left(z \,\middle|\, \begin{matrix}\frac{1}{2}, \frac{3}{2} \\ 0, 1\end{matrix}\right) /; z \notin (-\infty, -1)$$

Through other functions

Involving some elliptic-type functions

08.07.26.0004.01

$$\mathrm{Z}(z \mid m) = -\frac{E(m)\,\Pi(0; z \mid m)}{K(m)} + (1-m)\,\Pi(m; z \mid m) + \frac{m \sin(2z)}{2\sqrt{1-m\sin^2(z)}}$$

08.07.26.0005.01

$$\mathrm{Z}(z \mid m) = \sqrt{1-m\sin^2(z)}\;\frac{\partial \log(\vartheta_4(\pi F(z \mid m)/(2K(m)), q(m)))}{\partial z}$$

08.07.26.0006.01

$$\mathrm{Z}(z \mid m) = \frac{\omega_1}{K(m)^2}\left(K(m)\left(\eta_3 + \zeta\left(\frac{\omega_1 F(z \mid m)}{K(m)} - \omega_3; g_2, g_3\right)\right) - F(z \mid m)\,\eta_1\right) /;$$

$$m = q^{-1}\left(\exp\left(\frac{i\pi\omega_3}{\omega_1}\right)\right) \bigwedge \{\omega_1, \omega_3\} = \{\omega_1(g_2, g_3), \omega_3(g_2, g_3)\} \bigwedge \{\eta_1, \eta_3\} = \{\zeta(\omega_1; g_2, g_3), \zeta(\omega_3; g_2, g_3)\}$$

Involving some hypergeometric-type functions

08.07.26.0007.01

$$\mathrm{Z}(z \mid m) = \sin(z)\left(F_1\left(\frac{1}{2}; \frac{1}{2}, -\frac{1}{2}; \frac{3}{2}; \sin^2(z), m\sin^2(z)\right) - \frac{E(m)}{K(m)} F_1\left(\frac{1}{2}; \frac{1}{2}, \frac{1}{2}; \frac{3}{2}; \sin^2(z), m\sin^2(z)\right)\right) /; |\mathrm{Re}(z)| < \frac{\pi}{2}$$

Representations through equivalent functions

With related functions

08.07.27.0003.01

$$\mathrm{Z}(z \mid m) = \sin(z)\left(F_{1\times 0\times 0}^{1\times 1\times 1}\left(\begin{matrix}\frac{1}{2}; \frac{1}{2}; -\frac{1}{2}; \\ \frac{3}{2};;;\end{matrix}\ \sin^2(z), m\sin^2(z)\right) - \frac{E(m)}{K(m)} F_{1\times 0\times 0}^{1\times 1\times 1}\left(\begin{matrix}\frac{1}{2}; \frac{1}{2}; \frac{1}{2}; \\ \frac{3}{2};;;\end{matrix}\ m\sin^2(z), \sin^2(z)\right)\right)$$

08.07.27.0001.01

$$\mathrm{Z}(\mathrm{am}(z \mid m) \mid m) = E(\mathrm{am}(z \mid m) \mid m) - z\,\frac{E(m)}{K(m)}$$

08.07.27.0002.01

$$\mathrm{Z}(\mathrm{am}(z \mid m) \mid m) = (1-m)\,\Pi(m; \mathrm{am}(z \mid m) \mid m) - z\,\frac{E(m)}{K(m)} + \frac{m\sin(2\,\mathrm{am}(z \mid m))}{2\sqrt{1-m\sin^2(\mathrm{am}(z \mid m))}}$$

History

– C. G. J. Jacobi (1829)

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