JacobiPGeneral

View the online version at
functions.wolfram.com
Download the
PDF File

Notations

Traditional name

Jacobi function

Traditional notation

$P_{\nu}^{(a,b)}(z)$

Mathematica StandardForm notation

`JacobiP[ν, a, b, z]`

Primary definition

07.15.02.0001.01

$$P_{\nu}^{(a,b)}(z) = \frac{\Gamma(a+\nu+1)}{\Gamma(\nu+1)}\, {}_2\tilde{F}_1\left(-\nu, a+b+\nu+1; a+1; \frac{1-z}{2}\right)$$

Specific values

Specialized values

For fixed ν, a, b

07.15.03.0001.01

$$P_{\nu}^{(a,b)}(0) = \frac{2^{-\nu}\,\Gamma(a+\nu+1)}{\Gamma(a+1)\,\Gamma(\nu+1)}\, {}_2F_1(-b-\nu, -\nu; a+1; -1)$$

07.15.03.0002.01

$$P_{\nu}^{(a,b)}(1) = \frac{\Gamma(a+\nu+1)}{\Gamma(\nu+1)\,\Gamma(a+1)}$$

07.15.03.0003.01

$$P_{\nu}^{(a,b)}(-1) = \frac{\Gamma(-b)}{\Gamma(-b-\nu)\,\Gamma(\nu+1)} \;/;\; \mathrm{Re}(b) < 0$$

07.15.03.0004.01

$$P_{\nu}^{(a,b)}(-1) = \tilde{\infty} \;/;\; \mathrm{Re}(b) > 0$$

For fixed ν, a, z

07.15.03.0005.01

$$P_\nu^{(a,a)}(z) = \frac{(a+1)_\nu}{(2a+1)_\nu} C_\nu^{a+\frac{1}{2}}(z)$$

07.15.03.0006.01

$$P_\nu^{(a,-a)}(z) = \frac{\Gamma(a+\nu+1)}{\Gamma(\nu+1)} \frac{(1+z)^{a/2}}{(1-z)^{a/2}} P_\nu^{-a}(z)$$

07.15.03.0007.01

$$P_\nu^{(a,-a)}(z) = \frac{\Gamma(a+\nu+1)}{\Gamma(\nu+1)} \frac{(z+1)^{a/2}}{(z-1)^{a/2}} \mathcal{P}_\nu^{-a}(z)$$

07.15.03.0008.01

$$P_\nu^{(a,-\nu)}(z) = \frac{2^{-\nu}\,\Gamma(a+\nu+1)}{\Gamma(a+1)\,\Gamma(\nu+1)} (z+1)^\nu$$

07.15.03.0009.01

$$P_\nu^{\left(a,-\frac{1}{2}\right)}(z) = \frac{\Gamma\left(a+\frac{1}{2}\right)\Gamma\left(\nu+\frac{1}{2}\right)}{\sqrt{\pi}\;\Gamma\left(a+\nu+\frac{1}{2}\right)} C_{2\nu}^{a+\frac{1}{2}}\left(\frac{\sqrt{z+1}}{\sqrt{2}}\right)$$

07.15.03.0010.01

$$P_\nu^{\left(a,\frac{1}{2}\right)}(z) = \sqrt{\frac{2}{\pi}}\; \frac{\Gamma\left(a+\frac{1}{2}\right)\Gamma\left(\nu+\frac{3}{2}\right)}{\sqrt{z+1}\;\Gamma\left(a+\nu+\frac{3}{2}\right)} C_{2\nu+1}^{a+\frac{1}{2}}\left(\frac{\sqrt{z+1}}{\sqrt{2}}\right)$$

For fixed *ν*, *b*, *z*

07.15.03.0011.01

$$P_\nu^{(-b,b)}(z) = \frac{\Gamma(\nu-b+1)}{\Gamma(\nu+1)} \frac{(1-z)^{b/2}}{(1+z)^{b/2}} P_\nu^b(z)$$

07.15.03.0012.01

$$P_\nu^{(-b,b)}(z) = \frac{\Gamma(\nu-b+1)}{\Gamma(\nu+1)} \frac{(z-1)^{b/2}}{(z+1)^{b/2}} \mathcal{P}_\nu^b(z)$$

07.15.03.0013.01

$$P_\nu^{(-m-\nu,b)}(z) = \tilde{\infty} \;/;\; m \in \mathbb{N}^+$$

For fixed *a*, *b*, *z*

07.15.03.0014.01

$$P_0^{(a,b)}(z) = 1$$

07.15.03.0015.01

$$P_1^{(a,b)}(z) = \frac{1}{2}((a+b+2)\,z+a-b)$$

07.15.03.0016.01

$$P_2^{(a,b)}(z) = \frac{1}{8}\left((3+a+b)\,(4+a+b)\,z^2+2\left(3\,a+a^2-b\,(3+b)\right)z-4+a^2-b+b^2-a\,(1+2\,b)\right)$$

07.15.03.0017.01

$$P_3^{(a,b)}(z) = \frac{1}{48}\left((a+b+4)(a+b+5)(a+b+6)z^3 + 3(a-b)(a+b+4)(a+b+5)z^2 + 3(a+b+4)\left(a^2-(2b+1)a+b^2-b-6\right)z + (a-b)\left(-16+a^2+(-3+b)b-a(3+2b)\right)\right)$$

07.15.03.0018.01

$$P_4^{(a,b)}(z) = \frac{1}{384}\left((a+b+5)(a+b+6)(a+b+7)(a+b+8)z^4 + 4(a-b)(a+b+5)(a+b+6)(a+b+7)z^3 + 6(a+b+5)(a+b+6)\left(a^2-(2b+1)a+b^2-b-8\right)z^2 + \left(4(a+b+5)\left(a^3-3(b+1)a^2+\left(3b^2-22\right)a+b\left(-b^2+3b+22\right)\right)\right)z + 144+42b-6b^3-37b^2+a^4-2(2b+3)a^3+\left(6b^2+6b-37\right)a^2+2\left(-2b^3+3b^2+43b+21\right)a+b^4\right)$$

07.15.03.0019.01

$$P_5^{(a,b)}(z) = \frac{1}{3840}\left((a+b+6)(a+b+7)(a+b+8)(a+b+9)(a+b+10)z^5 + 5(a-b)(a+b+6)(a+b+7)(a+b+8)(a+b+9)z^4 + 10(a+b+6)(a+b+7)(a+b+8)\left(a^2-(2b+1)a+b^2-b-10\right)z^3 + 10(a+b+6)(a+b+7)\left(a^3-3(b+1)a^2+\left(3b^2-28\right)a+b\left(-b^2+3b+28\right)\right)z^2 + 5(a+b+6)\left(a^4-2(2b+3)a^3+\left(6b^2+6b-49\right)a^2+\left(-4b^3+6b^2+110b+54\right)a+b^4-6b^3-49b^2+54b+240\right)z + a^5-5(b+2)a^4+5\left(2b^2+4b-13\right)a^3-5\left(2b^3-51b-50\right)a^2 + \left(5b^4-20b^3-255b^2+1024\right)a-b\left(b^4-10b^3-65b^2+250b+1024\right)\right)$$

07.15.03.0020.01

$$P_6^{(a,b)}(z) = \frac{1}{46\,080}\left((a+b+7)(a+b+8)(a+b+9)(a+b+10)(a+b+11)(a+b+12)z^6 + 6(a-b)(a+b+7)(a+b+8)(a+b+9)(a+b+10)(a+b+11)z^5 + 15(a+b+7)(a+b+8)(a+b+9)(a+b+10)\left(a^2-(2b+1)a+b^2-b-12\right)z^4 + 20(a+b+7)(a+b+8)(a+b+9)\left(a^3-3(b+1)a^2+\left(3b^2-34\right)a+b\left(-b^2+3b+34\right)\right)z^3 + 15(a+b+7)(a+b+8)\left(a^4-2(2b+3)a^3+\left(6b^2+6b-61\right)a^2+\left(-4b^3+6b^2+134b+66\right)a+b^4-6b^3-61b^2+66b+360\right)z^2 + 6(a+b+7)\left(a^5-5(b+2)a^4+5\left(2b^2+4b-17\right)a^3-5\left(2b^3-63b-62\right)a^2 + \left(5b^4-20b^3-315b^2+1584\right)a-b\left(b^4-10b^3-85b^2+310b+1584\right)\right)z + 64(a+1)(a+2)(a+3)(a+4)(a+5)(a+6) - 192(a+2)(a+3)(a+4)(a+5)(a+6)(a+b+7) + 240(a+3)(a+4)(a+5)(a+6)(a+b+7)(a+b+8) - 160(a+4)(a+5)(a+6)(a+b+7)(a+b+8)(a+b+9) + 60(a+5)(a+6)(a+b+7)(a+b+8)(a+b+9)(a+b+10) - 12(a+6)(a+b+7)(a+b+8)(a+b+9)(a+b+10)(a+b+11) + (a+b+7)(a+b+8)(a+b+9)(a+b+10)(a+b+11)(a+b+12)\right)$$

07.15.03.0021.01

$$P_n^{(a,b)}(z) = \frac{1}{2^n}\sum_{k=0}^{n}\binom{a+n}{k}\binom{b+n}{n-k}(z+1)^k(z-1)^{n-k} \;/;\; n \in \mathbb{N}$$

07.15.03.0022.01

$$P_n^{(a,b)}(z) = \frac{1}{n!}\sum_{k=0}^{n}\frac{(-n)_k\,(a+b+n+1)_k\,(a+k+1)_{n-k}}{k!}\left(\frac{1-z}{2}\right)^k \;/;\; n \in \mathbb{N}$$

07.15.03.0023.01

$$P_{-n}^{(a,b)}(z) = 0 \;/;\; n \in \mathbb{N}^+$$

07.15.03.0024.01

$$P_{-a-b-n-1}^{(a,b)}(z) == \frac{\Gamma(-b-n)}{\Gamma(-a-b-n)\,\Gamma(a+n+1)} \sum_{k=0}^{n} \frac{(a+b+n+1)_k\,(-n)_k\,(a+k+1)_{n-k}}{k!} \left(\frac{1-z}{2}\right)^k /;\, n \in \mathbb{N}$$

07.15.03.0025.01

$$P_{-a-n}^{(a,b)}(z) == \tilde{\infty} /;\, n \in \mathbb{N}^+$$

For fixed ν, z

07.15.03.0026.01

$$P_{\nu}^{\left(\frac{1}{2},\frac{1}{2}\right)}(z) == \frac{1}{\Gamma(\nu+2)} \left(\frac{3}{2}\right)_{\nu} U_{\nu}(z)$$

07.15.03.0027.01

$$P_{\nu}^{\left(-\frac{1}{2},-\frac{1}{2}\right)}(z) == \frac{1}{\Gamma(\nu+1)} \left(\frac{1}{2}\right)_{\nu} T_{\nu}(z)$$

07.15.03.0028.01

$$P_{\nu}^{(0,0)}(z) == P_{\nu}(z)$$

General characteristics

Some abbreviations

07.15.04.0001.01

$$\mathcal{NT}(\{a_1, a_2\}) == \neg\,(-a_1 \in \mathbb{N} \bigvee -a_2 \in \mathbb{N})$$

Domain and analyticity

$P_{\nu}^{(a,b)}(z)$ is an analytical function of ν, a, b, z which is defined in $\mathbb{C}^4$. For fixed ν, a, z, it is an entire function of b. For positive integer ν, the function $P_{\nu}^{(a,b)}(z)$ degenerates to a polynomial in z of order ν.

07.15.04.0002.01

$$(\nu * a * b * z) \longrightarrow P_{\nu}^{(a,b)}(z) :: \mathbb{C}^4 \longrightarrow \mathbb{C}$$

Symmetries and periodicities

Parity

07.15.04.0003.01

$$P_n^{(a,b)}(-z) == (-1)^n\, P_n^{(b,a)}(z) /;\, n \in \mathbb{N}$$

Mirror symmetry

07.15.04.0004.02

$$P_{\bar{\nu}}^{(\bar{a},\bar{b})}(\bar{z}) == \overline{P_{\nu}^{(a,b)}(z)} /;\, z \notin (-\infty, -1)$$

Periodicity

No periodicity

Poles and essential singularities

With respect to z

For fixed ν, a, b in nonpolynomial cases (when $\neg\,(\nu \in \mathbb{N} \bigvee -a-b-\nu-1 \in \mathbb{N})$), the function $P_\nu^{(a,b)}(z)$ does not have poles and essential singularities.

07.15.04.0005.01

$$\mathcal{S}ing_z\left(P_\nu^{(a,b)}(z)\right) = \{\} \;/;\; \mathcal{NT}(\{-\nu,\ a+b+\nu+1\})$$

For positive integer ν or negative integer $a+b+\nu+1$and fixed a, the function $P_\nu^{(a,b)}(z)$ is a polynomial and has pole of order νor $-a-b-\nu-1$at $z = \tilde{\infty}$.

07.15.04.0006.01

$$\mathcal{S}ing_z\left(P_\nu^{(a,b)}(z)\right) = \{\{\tilde{\infty}, -\alpha\}\} \;/;\; \left(\nu \in \mathbb{N}^+ \bigwedge \alpha == -\nu\right) \bigvee \left(-a-b-\nu-1 \in \mathbb{N}^+ \bigwedge \alpha == a+b+\nu+1\right) \bigvee \left(\nu \in \mathbb{N}^+ \bigwedge -a-b-\nu-1 \in \mathbb{N}^+ \bigwedge \alpha == \min(\nu, -a-b-\nu-1)\right)$$

With respect to b

For fixed ν, a, z, the function $P_\nu^{(a,b)}(z)$ has only one singular point at $b = \tilde{\infty}$. It is an essential singular point.

07.15.04.0007.01

$$\mathcal{S}ing_b\left(P_\nu^{(a,b)}(z)\right) = \{\{\tilde{\infty}, \infty\}\}$$

With respect to a

For fixed ν, b, z, the function $P_\nu^{(a,b)}(z)$ has an infinite set of singular points:

a) $a == -\nu - k \;/;\; k \in \mathbb{N}^+$, are the simple poles with residues $\frac{(-1)^{k-1}\,\Gamma(k-b)}{(k-1)!\,\Gamma(-b-\nu)\,\Gamma(\nu+1)}\, P_{-b+k-1}^{(-k-\nu,b)}(z)$;

b) $a == \tilde{\infty}$ is the point of convergence of poles, which is an essential singular point.

07.15.04.0008.01

$$\mathcal{S}ing_a\left(P_\nu^{(a,b)}(z)\right) = \left\{\left\{\{-\nu-k, 1\} \;/;\; k \in \mathbb{N}^+\right\}, \{\tilde{\infty}, \infty\}\right\}$$

07.15.04.0009.01

$$\operatorname{res}_a\left(P_\nu^{(a,b)}(z)\right)(-\nu-k) == \frac{(-1)^{k-1}\,\Gamma(k-b)}{(k-1)!\,\Gamma(-b-\nu)\,\Gamma(\nu+1)}\, P_{-b+k-1}^{(-k-\nu,b)}(z) \;/;\; k \in \mathbb{N}^+$$

With respect to ν

For fixed a, b, z, the function $P_\nu^{(a,b)}(z)$ has an infinite set of singular points:

a) $\nu == -a - k \;/;\; k \in \mathbb{N}^+$, are the simple poles with residues $\frac{(-1)^{k-1}\,\Gamma(k-b)}{(k-1)!\,\Gamma(-b-\nu)\,\Gamma(\nu+1)}\, P_{-b+k-1}^{(-k-\nu,b)}(z)$;

b) $\nu == \tilde{\infty}$ is the point of convergence of poles, which is an essential singular point.

07.15.04.0010.01

$$\mathcal{S}ing_\nu\left(P_\nu^{(a,b)}(z)\right) = \left\{\left\{\{-a-k, 1\} \;/;\; k \in \mathbb{N}^+\right\}, \{\tilde{\infty}, \infty\}\right\}$$

07.15.04.0011.01

$$\operatorname{res}_\nu\left(P_\nu^{(a,b)}(z)\right)(-a-k) == \frac{(-1)^{k-1}\,\Gamma(k-b)}{(k-1)!\,\Gamma(1-a-k)\,\Gamma(a-b+k)}\, P_{-b+k-1}^{(a,b)}(z) \;/;\; k \in \mathbb{N}^+$$

Branch points

With respect to z

For fixed ν, a, b in nonpolynomial cases (when $\neg(\nu \in \mathbb{N} \bigvee -a-b-\nu-1 \in \mathbb{N})$), the function $P_\nu^{(a,b)}(z)$ has two branch points: $z == -1$, $z == \tilde{\infty}$.

For fixed a, b and integer ν, the function $P_\nu^{(a,b)}(z)$ does not have branch points.

07.15.04.0012.01

$$\mathcal{BP}_z\left(P_\nu^{(a,b)}(z)\right) == \{-1, \tilde{\infty}\} /; \mathcal{NT}(\{-\nu, a+b+\nu+1\})$$

07.15.04.0013.01

$$\mathcal{BP}_z\left(P_\nu^{(a,b)}(z)\right) == \{\} /; \nu \in \mathbb{Z}$$

07.15.04.0014.01

$$\mathcal{R}_z\left(P_\nu^{(a,b)}(z), -1\right) == \log /; b \in \mathbb{Z} \bigvee b \notin \mathbb{Q} \bigwedge \mathcal{NT}(\{-\nu, a+b+\nu+1\})$$

07.15.04.0015.01

$$\mathcal{R}_z\left(P_\nu^{(a,b)}(z), -1\right) == s /; b == \frac{r}{s} \bigwedge r \in \mathbb{Z} \bigwedge s-1 \in \mathbb{N}^+ \bigwedge \gcd(r, s) == 1 \bigwedge \mathcal{NT}(\{-\nu, a+b+\nu+1\})$$

07.15.04.0016.01

$$\mathcal{R}_z\left(P_\nu^{(a,b)}(z), \tilde{\infty}\right) == \log /; a+b+2\nu \in \mathbb{Z} \bigvee \neg(\nu \in \mathbb{Q} \bigwedge a+b+\nu \in \mathbb{Q})$$

07.15.04.0017.01

$$\mathcal{R}_z\left(P_\nu^{(a,b)}(z), \tilde{\infty}\right) == \operatorname{lcm}(s, u) /;$$
$$\nu == \frac{r}{s} \bigwedge a+b+\nu == \frac{t}{u} \bigwedge \{r, s, t, u\} \in \mathbb{Z} \bigwedge s > 1 \bigwedge u > 1 \bigwedge \gcd(r, s) == 1 \bigwedge \gcd(t, u) == 1 \bigwedge \mathcal{NT}(\{-\nu, a+b+\nu+1\})$$

With respect to b

For fixed ν, a, z, the function $P_\nu^{(a,b)}(z)$ does not have branch points.

07.15.04.0018.01

$$\mathcal{BP}_b\left(P_\nu^{(a,b)}(z)\right) == \{\}$$

With respect to a

For fixed ν, b, z, the function $P_\nu^{(a,b)}(z)$ does not have branch points.

07.15.04.0019.01

$$\mathcal{BP}_a\left(P_\nu^{(a,b)}(z)\right) == \{\}$$

With respect to ν

For fixed a, b, z, the function $P_\nu^{(a,b)}(z)$ does not have branch points.

07.15.04.0020.01

$$\mathcal{BP}_\nu\left(P_\nu^{(a,b)}(z)\right) == \{\}$$

Branch cuts

With respect to z

For fixed ν, a, b in nonpolynomial cases (when $\neg(\nu \in \mathbb{N} \bigvee -a-b-\nu-1 \in \mathbb{N})$), the function $P_\nu^{(a,b)}(z)$ is a single-valued function on the z-plane cut along the interval $(-\infty, -1)$, where it is continuous from above.

For fixed a, b and integer ν, the function $P_\nu^{(a,b)}(z)$ does not have branch cuts.

07.15.04.0021.01

$$\mathcal{BC}_z\left(P_\nu^{(a,b)}(z)\right) = \{\{(-\infty, -1), -i\}\} /; \mathcal{NT}(\{-\nu, a+b+\nu+1\})$$

07.15.04.0022.01

$$\mathcal{BC}_z\left(P_\nu^{(a,b)}(z)\right) = \{\} /; \nu \in \mathbb{Z}$$

07.15.04.0023.01

$$\lim_{\epsilon \to +0} P_\nu^{(a,b)}(x + i\,\epsilon) = P_\nu^{(a,b)}(x) /; x < -1$$

07.15.04.0024.01

$$\lim_{\epsilon \to +0} P_\nu^{(a,b)}(x - i\,\epsilon) = e^{i\,b\,\pi}\left(e^{i\,b\,\pi} P_\nu^{(a,b)}(x) - 2\,i\sin((b+\nu)\,\pi)\,P_\nu^{(b,a)}(-x)\right) /; x < -1$$

With respect to b

For fixed ν, a, z, the function $P_\nu^{(a,b)}(z)$ does not have branch cuts.

07.15.04.0025.01

$$\mathcal{BC}_b\left(P_\nu^{(a,b)}(z)\right) = \{\}$$

With respect to a

For fixed ν, b, z, the function $P_\nu^{(a,b)}(z)$ does not have branch cuts.

07.15.04.0026.01

$$\mathcal{BC}_a\left(P_\nu^{(a,b)}(z)\right) = \{\}$$

With respect to ν

For fixed a, b, z, the function $P_\nu^{(a,b)}(z)$ does not have branch cuts.

07.15.04.0027.01

$$\mathcal{BC}_\nu\left(P_\nu^{(a,b)}(z)\right) = \{\}$$

Series representations

Generalized power series

Expansions at $z = 0$

07.15.06.0001.01

$$P_\nu^{(a,b)}(z) = 2^{-\nu}\left({}_2F_1(-\nu, -b-\nu; a+1; -1) + \frac{\nu\,(a+b+\nu+1)}{a+1}\,{}_2F_1(1-\nu, -b-\nu; a+2; -1)\,z + \frac{(\nu-1)\,\nu\,(a+b+\nu+1)\,(a+b+\nu+2)}{2\,(a+1)\,(a+2)}\,{}_2F_1(2-\nu, -b-\nu; a+3; -1)\,z^2 + \ldots\right) /; |z| < 1$$

07.15.06.0002.01

$$P_\nu^{(a,b)}(z) = \frac{(a+1)_\nu}{\Gamma(\nu+1)} \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-\nu)_{j+k}\,(a+b+\nu+1)_{j+k}\,(-z)^j}{(a+1)_{j+k}\,j!\,k!\,2^{j+k}} /; |z| < 1$$

07.15.06.0003.01

$$P_\nu^{(a,b)}(z) = \frac{(a+1)_\nu}{\Gamma(\nu+1)}\, \tilde{F}_{1\times0\times0}^{2\times0\times0}\left(\begin{matrix} -\nu,\ a+b+\nu+1;;; \\ a+1;;; \end{matrix} -\frac{z}{2}, \frac{1}{2}\right)$$

07.15.06.0004.01

$$P_\nu^{(a,b)}(z) = \frac{\Gamma(a+\nu+1)}{\Gamma(\nu+1)} \sum_{k=0}^{\infty} \sum_{j=0}^{k} \frac{(-\nu)_k\, (a+b+\nu+1)_k\, (-z)^j}{\Gamma(a+k+1)\, j!\, (k-j)!\, 2^k} \;/;\; |z| < 1$$

07.15.06.0005.01

$$P_\nu^{(a,b)}(z) = \frac{(a+1)_\nu}{\Gamma(\nu+1)}\, 2^{-\nu} \sum_{j=0}^{\infty} \frac{(-\nu)_j\, (a+b+\nu+1)_j}{(a+1)_j\, j!}\, {}_2F_1(-b-\nu,\ j-\nu;\ a+j+1;\ -1)\, (-z)^j$$

07.15.06.0006.01

$$P_n^{(a,b)}(z) = \frac{1}{2^n} \sum_{k=0}^{n} \binom{a+n}{k} \binom{b+n}{n-k} (z+1)^k\, (z-1)^{n-k} \;/;\; n \in \mathbb{N}$$

07.15.06.0007.01

$$P_\nu^{(a,b)}(z) \propto \frac{2^{-\nu}\, \Gamma(a+\nu+1)}{\Gamma(a+1)\, \Gamma(\nu+1)}\, {}_2F_1(-\nu,\ -b-\nu;\ a+1;\ -1)\, (1 + O(z)) \;/;\; (z \to 0)$$

Expansions at $z == 1$

07.15.06.0008.01

$$P_\nu^{(a,b)}(z) = \frac{\Gamma(a+\nu+1)}{\Gamma(\nu+1)} \left(\frac{1}{\Gamma(a+1)} + \frac{\nu\, (a+b+\nu+1)\, (z-1)}{2\, \Gamma(a+2)} - \frac{(1-\nu)\, \nu\, (a+b+\nu+1)\, (a+b+\nu+2)\, (z-1)^2}{8\, \Gamma(a+3)} + \ldots \right) /;\; \left| \frac{1-z}{2} \right| < 1$$

07.15.06.0009.01

$$P_\nu^{(a,b)}(z) = \frac{\Gamma(a+\nu+1)}{\Gamma(\nu+1)} \sum_{k=0}^{\infty} \frac{(-\nu)_k\, (a+b+\nu+1)_k}{\Gamma(a+k+1)\, k!} \left(\frac{1-z}{2} \right)^k /;\; \left| \frac{1-z}{2} \right| < 1$$

07.15.06.0010.01

$$P_\nu^{(a,b)}(z) = \frac{\Gamma(a+\nu+1)}{\Gamma(\nu+1)}\, {}_2\tilde{F}_1\left(-\nu,\ a+b+\nu+1;\ a+1;\ \frac{1-z}{2}\right)$$

07.15.06.0011.01

$$P_n^{(a,b)}(z) = \frac{1}{n!} \sum_{k=0}^{n} \frac{(-n)_k\, (a+b+n+1)_k\, (a+k+1)_{n-k}}{k!} \left(\frac{1-z}{2} \right)^k /;\; n \in \mathbb{N}$$

07.15.06.0012.01

$$P_\nu^{(a,b)}(z) \propto \frac{\Gamma(a+\nu+1)}{\Gamma(a+1)\, \Gamma(\nu+1)}\, (1 + O(z-1)) \;/;\; (z \to 1) \bigwedge a \notin \mathbb{N}^+$$

Expansions at $z == -1$

07.15.06.0013.01

$$P_\nu^{(a,b)}(z) = \frac{\Gamma(-b)}{\Gamma(\nu+1)\, \Gamma(-b-\nu)} \left(1 - \frac{\nu\, (a+b+\nu+1)}{2\, (b+1)} (z+1) - \frac{(1-\nu)\, \nu\, (a+b+\nu+1)\, (a+b+\nu+2)}{8\, (b+1)\, (b+2)} (z+1)^2 - \ldots \right) - \frac{\sin(\nu\, \pi)\, \Gamma(b)\, \Gamma(a+\nu+1)}{\pi\, \Gamma(a+b+\nu+1)} \left(\frac{z+1}{2} \right)^{-b} \left(1 + \frac{(-b-\nu)\, (a+\nu+1)}{2\, (1-b)} (z+1) + \frac{(-b-\nu)\, (-b-\nu+1)\, (a+\nu+1)\, (a+\nu+2)}{8\, (1-b)\, (2-b)} (z+1)^2 + \ldots \right) /;\; \left| \frac{z+1}{2} \right| \bigwedge b \notin \mathbb{Z}$$

07.15.06.0014.01

$$P_\nu^{(a,b)}(z) = \frac{\Gamma(-b)}{\Gamma(\nu+1)\,\Gamma(-b-\nu)} \sum_{k=0}^{\infty} \frac{(-\nu)_k\,(a+b+\nu+1)_k}{(b+1)_k\,k!} \left(\frac{z+1}{2}\right)^k - \frac{\sin(\nu\pi)\,\Gamma(b)\,\Gamma(a+\nu+1)}{\pi\,\Gamma(a+b+\nu+1)} \left(\frac{z+1}{2}\right)^{-b} \sum_{k=0}^{\infty} \frac{(a+\nu+1)_k\,(-b-\nu)_k}{(1-b)_k\,k!} \left(\frac{z+1}{2}\right)^k \;/;\; \left|\frac{z+1}{2}\right| \bigwedge b \notin \mathbb{Z}$$

07.15.06.0015.01

$$P_\nu^{(a,b)}(z) = \frac{\Gamma(-b)}{\Gamma(\nu+1)\,\Gamma(-b-\nu)}\, {}_2F_1\!\left(-\nu, a+b+\nu+1; b+1; \frac{z+1}{2}\right) - \frac{\sin(\nu\pi)\,\Gamma(b)\,\Gamma(a+\nu+1)}{\pi\,\Gamma(a+b+\nu+1)} \left(\frac{z+1}{2}\right)^{-b} {}_2F_1\!\left(a+\nu+1, -b-\nu; 1-b; \frac{z+1}{2}\right) /;\; b \notin \mathbb{Z}$$

07.15.06.0016.01

$$P_\nu^{(a,b)}(z) \propto \frac{\Gamma(-b)}{\Gamma(\nu+1)\,\Gamma(-b-\nu)} (1+O(z+1)) - \frac{2^b \sin(\nu\pi)\,\Gamma(b)\,\Gamma(a+\nu+1)}{\pi\,\Gamma(a+b+\nu+1)} (z+1)^{-b} (1+O(z+1)) \;/;\; (z\to -1) \bigwedge b \notin \mathbb{Z}$$

07.15.06.0017.01

$$P_\nu^{(a,b)}(z) = -\frac{(b-1)!\,\sin(\pi\nu)\,\Gamma(a+\nu+1)}{\pi\,\Gamma(a+b+\nu+1)} \left(\frac{z+1}{2}\right)^{-b} \sum_{k=0}^{b-1} \frac{(-b-\nu)_k\,(a+\nu+1)_k}{k!\,(1-b)_k} \left(\frac{z+1}{2}\right)^k + \frac{(-1)^{b-1}}{b!\,\Gamma(\nu+1)\,\Gamma(-b-\nu)} \log\!\left(\frac{z+1}{2}\right) {}_2F_1\!\left(-\nu, a+b+\nu+1; b+1; \frac{z+1}{2}\right) + \frac{(-1)^b}{\Gamma(\nu+1)\,\Gamma(-b-\nu)} \sum_{k=0}^{\infty} \frac{(-\nu)_k\,(a+b+\nu+1)_k}{k!\,(b+k)!} (\psi(k+1)+\psi(b+k+1)-\psi(a+b+k+\nu+1)-\psi(k-\nu)) \left(\frac{z+1}{2}\right)^k /;\; b \in \mathbb{N}^+$$

07.15.06.0018.01

$$P_\nu^{(a,b)}(z) \propto \frac{(-1)^{b-1}}{b!\,\Gamma(\nu+1)\,\Gamma(-b-\nu)} \left(\log\!\left(\frac{z+1}{2}\right) - \psi(b+1) + \psi(-\nu) + \psi(a+b+\nu+1) + \gamma\right) (1+O(z+1)) - \frac{(b-1)!\,\sin(\pi\nu)\,\Gamma(a+\nu+1)}{\pi\,\Gamma(a+b+\nu+1)} \left(\frac{z+1}{2}\right)^{-b} (1+O(z+1)) \;/;\; (z\to -1) \bigwedge b \in \mathbb{N}^+$$

07.15.06.0019.01

$$P_\nu^{(a,0)}(z) = \frac{\sin(\nu\pi)}{\pi} \log\left(\frac{z+1}{2}\right) {}_2F_1\!\left(-\nu, a+\nu+1; 1; \frac{z+1}{2}\right) - \frac{\sin(\nu\pi)}{\pi} \sum_{k=0}^{\infty} \frac{(-\nu)_k\,(a+\nu+1)_k}{k!^2} (2\,\psi(k+1) - \psi(a+k+\nu+1) - \psi(k-\nu)) \left(\frac{z+1}{2}\right)^k$$

07.15.06.0020.01

$$P_\nu^{(a,0)}(z) \propto \frac{\sin(\nu\pi)}{\pi} \left(\log\left(\frac{z+1}{2}\right) + \psi(-\nu) + \psi(a+\nu+1) + 2\,\gamma\right) (1+O(z+1)) \;/;\; (z\to -1)$$

07.15.06.0021.01

$$P_\nu^{(a,b)}(z) = \frac{\sin(\pi\,\nu)\,\Gamma(a+\nu+1)}{(-b)!\,\pi\,\Gamma(a+b+\nu+1)}\log\left(\frac{z+1}{2}\right)\left(\frac{z+1}{2}\right)^{-b}{}_2F_1\left(a+\nu+1,\,-b-\nu;\,1-b;\,\frac{z+1}{2}\right)-\frac{\sin(\pi\,\nu)\,\Gamma(a+\nu+1)}{\pi\,\Gamma(a+b+\nu+1)}\left(\frac{z+1}{2}\right)^{-b}\sum_{k=0}^{\infty}\frac{(-b-\nu)_k\,(a+\nu+1)_k}{k!\,(k-b)!}\left(\psi(1-b+k)+\psi(k+1)-\psi(a+k+\nu+1)-\psi(k-\nu-b)\right)\left(\frac{z+1}{2}\right)^k+\frac{(-b-1)!}{\Gamma(\nu+1)\,\Gamma(-b-\nu)}\sum_{k=0}^{-b-1}\frac{(-\nu)_k\,(a+b+\nu+1)_k}{k!\,(b+1)_k}\left(\frac{z+1}{2}\right)^k\ /;\ -b\in\mathbb{N}^+$$

07.15.06.0022.01

$$P_\nu^{(a,b)}(z)\propto\frac{\sin(\pi\,\nu)\,\Gamma(a+\nu+1)}{\pi\,(-b)!\,\Gamma(a+b+\nu+1)}\left(\frac{z+1}{2}\right)^{-b}\left(\log\left(\frac{z+1}{2}\right)-\psi(1-b)+\psi(-b-\nu)+\psi(a+\nu+1)+\gamma\right)(1+O(z+1))+\frac{(-b-1)!}{\Gamma(\nu+1)\,\Gamma(-b-\nu)}(1+O(z+1))\ /;\ (z\to-1)\bigwedge -b\in\mathbb{N}^+$$

Expansions at $z = \infty$

07.15.06.0023.01

$$P_\nu^{(a,b)}(z) = \frac{2^{-\nu}\,(a+b+\nu+1)_\nu\,(z-1)^\nu}{\Gamma(\nu+1)}\left(1-\frac{2\,(-a-\nu)\,\nu}{(-a-b-2\,\nu)\,(1-z)}-\frac{2\,(1-\nu)\,(-a-\nu)\,(-a-\nu+1)\,\nu}{(-a-b-2\,\nu)\,(-a-b-2\,\nu+1)\,(1-z)^2}-\ldots\right)-\frac{2^{a+b+\nu+1}\sin(\nu\,\pi)\,\Gamma(-a-b-2\,\nu-1)\,\Gamma(a+\nu+1)}{\pi\,\Gamma(-b-\nu)}(z-1)^{-a-b-\nu-1}\left(1+\frac{2\,(b+\nu+1)\,(a+b+\nu+1)}{(a+b+2\,\nu+2)\,(1-z)}+\frac{2\,(b+\nu+1)\,(b+\nu+2)\,(a+b+\nu+2)\,(a+b+\nu+1)}{(a+b+2\,\nu+2)\,(a+b+2\,\nu+3)\,(1-z)^2}+\ldots\right)/;\ \left|\frac{1-z}{2}\right|>1\bigwedge a+b+2\,\nu\notin\mathbb{Z}$$

07.15.06.0024.01

$$P_\nu^{(a,b)}(z) = \frac{2^{-\nu}\,(a+b+\nu+1)_\nu\,(z-1)^\nu}{\Gamma(\nu+1)}\sum_{k=0}^{\infty}\frac{(-\nu)_k\,(-a-\nu)_k}{(-a-b-2\,\nu)_k\,k!}\left(\frac{2}{1-z}\right)^k-\frac{2^{a+b+\nu+1}\sin(\nu\,\pi)\,\Gamma(-a-b-2\,\nu-1)\,\Gamma(a+\nu+1)}{\pi\,\Gamma(-b-\nu)}(z-1)^{-a-b-\nu-1}\sum_{k=0}^{\infty}\frac{(a+b+\nu+1)_k\,(b+\nu+1)_k}{(a+b+2\,\nu+2)_k\,k!}\left(\frac{2}{1-z}\right)^k\ /;\ \left|\frac{1-z}{2}\right|>1\bigwedge a+b+2\,\nu\notin\mathbb{Z}$$

07.15.06.0025.01

$$P_\nu^{(a,b)}(z) = \frac{2^{-\nu}\,(a+b+\nu+1)_\nu}{\Gamma(\nu+1)}(z-1)^\nu\,{}_2F_1\left(-\nu,\,-a-\nu;\,-a-b-2\,\nu;\,\frac{2}{1-z}\right)-\frac{2^{a+b+\nu+1}\sin(\nu\,\pi)\,\Gamma(-a-b-2\,\nu-1)\,\Gamma(a+\nu+1)}{\pi\,\Gamma(-b-\nu)}(z-1)^{-a-b-\nu-1}\,{}_2F_1\left(a+b+\nu+1,\,b+\nu+1;\,a+b+2\,\nu+2;\,\frac{2}{1-z}\right)/;\ z\notin(-1,\,1)\bigwedge a+b+2\,\nu\notin\mathbb{Z}$$

07.15.06.0026.01

$$P_\nu^{(a,b)}(z)\propto\frac{2^{-\nu}\,(a+b+\nu+1)_\nu\,z^\nu}{\Gamma(\nu+1)}\left(1+O\left(\frac{1}{z}\right)\right)-\frac{2^{a+b+\nu+1}\sin(\nu\,\pi)\,\Gamma(-a-b-2\,\nu-1)\,\Gamma(a+\nu+1)\,z^{-a-b-\nu-1}}{\pi\,\Gamma(-b-\nu)}\left(1+O\left(\frac{1}{z}\right)\right)/;\ (|z|\to\infty)\bigwedge a+b+2\,\nu\notin\mathbb{Z}$$

07.15.06.0027.01

$$P_\nu^{(a,b)}(z) = \frac{2^{a+b+\nu+1}\,\Gamma(a+\nu+1)\,\Gamma(b+\nu+1)\sin(\nu\pi)\sin((a+\nu)\pi)}{\pi^2\,\Gamma(a+b+2\nu+2)}\log\left(\frac{z-1}{2}\right)(z-1)^{-a-b-\nu-1}\,{}_2F_1\left(a+b+\nu+1,\,b+\nu+1;\,a+b+2\nu+2;\,\frac{2}{1-z}\right) - \frac{2^{a+b+\nu+1}\sin(\nu\pi)\,\Gamma(b+\nu+1)}{\pi\,\Gamma(-a-\nu)}(z-1)^{-a-b-\nu-1}\sum_{k=0}^{\infty}\frac{(b+\nu+1)_k\,(a+b+\nu+1)_k}{k!\,(a+b+k+2\nu+1)!}(\psi(k+1)-\psi(a+b+k+\nu+1)-\psi(-b-k-\nu)+\psi(a+b+k+2\nu+2))\left(\frac{2}{1-z}\right)^k + \frac{2^{-\nu}\,(a+b+\nu+1)_\nu\,(z-1)^\nu}{\Gamma(\nu+1)}\sum_{k=0}^{a+b+2\nu}\frac{(-a-\nu)_k\,(-\nu)_k}{k!\,(-a-b-2\nu)_k}\left(\frac{2}{1-z}\right)^k\;/;\;a+b+2\nu+1\in\mathbb{N}\land a+\nu\notin\mathbb{Z}\land|1-z|>2$$

07.15.06.0028.01

$$P_\nu^{(a,b)}(z) \propto \frac{2^{-\nu}\,(a+b+\nu+1)_\nu\,z^\nu}{\Gamma(\nu+1)}\left(1+O\left(\frac{1}{z}\right)\right) - \frac{2^{a+b+\nu+1}\sin(\nu\pi)\,\Gamma(b+\nu+1)}{\pi\,(a+b+2\nu+1)!\,\Gamma(-a-\nu)}z^{-a-b-\nu-1}\left(\log\left(\frac{z-1}{2}\right)-\psi(-b-\nu)-\psi(a+b+\nu+1)+\psi(a+b+2\nu+2)-\gamma\right)\left(1+O\left(\frac{1}{z}\right)\right)\;/;\;(|z|\to\infty)\land a+b+2\nu\in\mathbb{N}\land a+\nu\notin\mathbb{Z}$$

07.15.06.0029.01

$$P_\nu^{(a,b)}(z) \propto -\frac{2^{-\nu}\sin(\nu\pi)\,\Gamma(b+\nu+1)\,z^\nu}{\pi\,\Gamma(-a-\nu)}\left(\log\left(\frac{z-1}{2}\right)-\psi(-b-\nu)-\psi(-\nu)-2\gamma\right)\left(1+O\left(\frac{1}{z}\right)\right)\;/;\;(|z|\to\infty)\land a+b+2\nu=-1\land a+\nu\notin\mathbb{Z}$$

07.15.06.0030.01

$$P_\nu^{(a,b)}(z) = \frac{(-1)^{a+\nu}\,2^{a+b+\nu+1}\sin(\pi\nu)\,(b+\nu)!\,(a+\nu)!}{\pi\,(a+b+2\nu+1)!}(z-1)^{-a-b-\nu-1}\,{}_2F_1\left(a+b+\nu+1,\,b+\nu+1;\,a+b+2\nu+2;\,\frac{2}{1-z}\right) + \frac{2^{-\nu}\,(a+b+2\nu)!\,(z-1)^\nu}{\Gamma(\nu+1)\,\Gamma(a+b+\nu+1)}\sum_{k=0}^{a+\nu}\frac{((-a-\nu)_k\,(-\nu)_k}{k!\,(-a-b-2\nu)_k}\left(\frac{2}{1-z}\right)^k\;/;\;a+b+2\nu+1\in\mathbb{N}\land a+\nu\in\mathbb{N}\land b+\nu\geq 0\land z\notin\{-1,1\}$$

07.15.06.0031.01

$$P_\nu^{(a,b)}(z) \propto \frac{(-1)^{a+\nu}\,2^{a+b+\nu+1}\sin(\pi\nu)\,(b+\nu)!\,(a+\nu)!\,z^{-a-b-\nu-1}}{\pi\,(a+b+2\nu+1)!}\left(1+O\left(\frac{1}{z}\right)\right) + \frac{2^{-\nu}\,(a+b+\nu+1)_\nu\,z^\nu}{\Gamma(\nu+1)}\left(1+O\left(\frac{1}{z}\right)\right)\;/;\;(|z|\to\infty)\land a+b+2\nu+1\in\mathbb{N}\land a+\nu\in\mathbb{N}\land b+\nu\geq 0$$

07.15.06.0032.01

$$P_\nu^{(a,b)}(z) = \frac{(-1)^{a+b+2\nu}\,2^{a+1}\sin(\pi\nu)\,\Gamma(a+1)}{\pi\,(-b-\nu)!\,(a+\nu+1)\,\Gamma(a+b+\nu+1)}(z-1)^{-a-1}\,{}_3F_2\left(1,\,1,\,a+1;\,-b-\nu+1,\,a+\nu+2;\,\frac{2}{1-z}\right) + \frac{(-1)^{a+b+2\nu}\,2^{a+b+\nu+1}\sin(\pi\nu)\,\Gamma(a+\nu+1)}{\pi\,(-b-\nu-1)!}(z-1)^{-a-b-\nu-1}\sum_{k=0}^{-b-\nu-1}\frac{(b+\nu+1)_k\,(a+b+\nu+1)_k}{k!\,(a+b+k+2\nu+1)!}\left(\log\left(\frac{z-1}{2}\right)+\psi(k+1)-\psi(-b-k-\nu)-\psi(a+b+k+\nu+1)+\psi(a+b+k+2\nu+2)\right)\left(\frac{2}{1-z}\right)^k + \frac{2^{-\nu}\,\Gamma(a+b+2\nu+1)}{\Gamma(\nu+1)\,\Gamma(a+b+\nu+1)}(z-1)^\nu\sum_{k=0}^{a+b+2\nu}\frac{(-a-\nu)_k\,(-\nu)_k}{k!\,(-a-b-2\nu)_k}\left(\frac{2}{1-z}\right)^k\;/;\;a+b+2\nu+1\in\mathbb{N}\land a+\nu+1\in\mathbb{N}\land b+\nu\leq 0\land z\notin\{-1,1\}$$

07.15.06.0033.01

$$P_\nu^{(a,b)}(z) \propto \frac{(-1)^{a+b+2\nu}\, 2^{a+1} \sin(\pi\nu)\, \Gamma(a+1)\, z^{-a-1}}{\pi\, (-b-\nu)!\, (a+\nu+1)\, \Gamma(a+b+\nu+1)} \left(1 + O\left(\frac{1}{z}\right)\right) + \frac{(-1)^{a+b+2\nu}\, 2^{a+b+\nu+1} \sin(\pi\nu)\, \Gamma(a+\nu+1)}{\pi\, (-b-\nu-1)!\, (a+b+2\nu+1)!}\; z^{-a-b-\nu-1} \left(1 + O\left(\frac{1}{z}\right)\right) \left(\log\left(\frac{z-1}{2}\right) - \psi(-b-\nu) - \psi(a+b+\nu+1) + \psi(a+b+2\nu+2) - \gamma\right) + \frac{2^{-\nu}\, \Gamma(a+b+2\nu+1)\, z^\nu}{\Gamma(\nu+1)\, \Gamma(a+b+\nu+1)} \left(1 + O\left(\frac{1}{z}\right)\right) /;$$

$$(|z| \to \infty) \bigwedge a+b+2\nu \in \mathbb{N} \bigwedge a+\nu+1 \in \mathbb{N} \bigwedge b+\nu < 0 \bigwedge z \notin \{-1, 1\}$$

07.15.06.0034.01

$$P_\nu^{(a,b)}(z) \propto -\frac{2^{a+1} \sin(\pi\nu)\, \Gamma(a+1)\, z^{-a-1}}{\pi\, (-b-\nu)!\, (a+\nu+1)\, \Gamma(-\nu)} \left(1 + O\left(\frac{1}{z}\right)\right) - \frac{2^{-\nu} \sin(\pi\nu)\, z^\nu}{\pi} \left(\log\left(\frac{z-1}{2}\right) - \psi(-b-\nu) - \psi(-\nu) - 2\gamma\right) \left(1 + O\left(\frac{1}{z}\right)\right) /;$$

$$(|z| \to \infty) \bigwedge a+b+2\nu == -1 \bigwedge a+\nu+1 \in \mathbb{N} \bigwedge b+\nu < 0 \bigwedge z \notin \{-1, 1\}$$

07.15.06.0035.01

$$P_\nu^{(a,b)}(z) == \frac{2^{-\nu} \sin(\pi(b+\nu))\, (z-1)^\nu}{(-a-b-2\nu-1)!\, \sin(\pi(a+\nu))\, \Gamma(\nu+1)\, \Gamma(a+b+\nu+1)} \log\left(\frac{z-1}{2}\right) {}_2F_1\left(-\nu, -a-\nu; -a-b-2\nu; \frac{2}{1-z}\right) - \frac{2^{a+b+\nu+1} \sin(\pi\nu)\, \Gamma(a+\nu+1)}{\pi} (z-1)^{-a-b-\nu-1} \sum_{k=0}^{-a-b-2\nu-2} \frac{(a+b+\nu+1)_k\, \Gamma(-a-b-2\nu-k-1)}{k!\, \Gamma(-b-\nu-k)} \left(\frac{2}{1-z}\right)^k + \frac{2^{-\nu} \sin(\pi(b+\nu))}{\sin(\pi(a+\nu))\, \Gamma(a+b+\nu+1)\, \Gamma(\nu+1)} (z-1)^\nu \sum_{k=0}^{\infty} \frac{(-a-\nu)_k\, (-\nu)_k}{k!\, (k-a-b-2\nu-1)!} (\psi(k+1) - \psi(a-k+\nu+1) - \psi(k-\nu) + \psi(k-a-b-2\nu)) \left(\frac{2}{1-z}\right)^k /;$$

$$-a-b-2\nu-1 \in \mathbb{N}^+ \bigwedge b+\nu \notin \mathbb{Z} \bigwedge |1-z| > 2$$

07.15.06.0036.01

$$P_\nu^{(a,b)}(z) \propto \frac{2^{-\nu} \sin(\pi(b+\nu))\, z^\nu}{(-a-b-2\nu-1)!\, \sin(\pi(a+\nu))\, \Gamma(a+b+\nu+1)\, \Gamma(\nu+1)} \left(\log\left(\frac{z-1}{2}\right) - \psi(-\nu) - \psi(a+\nu+1) + \psi(-a-b-2\nu) - \gamma\right) \left(1 + O\left(\frac{1}{z}\right)\right) - \frac{2^{a+b+\nu+1} \sin(\pi\nu)\, \Gamma(a+\nu+1)\, \Gamma(-a-b-2\nu-1)}{\pi\, \Gamma(-b-\nu)} z^{-a-b-\nu-1} \left(1 + O\left(\frac{1}{z}\right)\right) /; (|z| \to \infty) \bigwedge -a-b-2\nu-1 \in \mathbb{N}^+ \bigwedge b+\nu \notin \mathbb{Z}$$

07.15.06.0037.01

$$P_\nu^{(a,b)}(z) == \frac{(-1)^{a+b+2\nu}\, 2^{a+1} \sin(\pi\nu)\, \Gamma(a+1)}{\pi\, (a+\nu+1)\, \Gamma(1-b-\nu)\, \Gamma(a+b+\nu+1)} (z-1)^{-a-1}\, {}_3F_2\left(1, 1, a+1; -b-\nu+1, a+\nu+2; \frac{2}{1-z}\right) - \frac{2^{a+b+\nu+1} \sin(\pi\nu)\, \Gamma(-a-b-2\nu-1)\, \Gamma(a+\nu+1)}{\pi\, \Gamma(-b-\nu)} (z-1)^{-a-b-\nu-1} \sum_{k=0}^{-a-b-2\nu-2} \frac{(b+\nu+1)_k\, (a+b+\nu+1)_k}{k!\, (a+b+2\nu+2)_k} \left(\frac{2}{1-z}\right)^k + \frac{(-1)^{a+b+2\nu-1}\, 2^{-\nu}\, (z-1)^\nu}{\Gamma(\nu+1)\, \Gamma(a+b+\nu+1)} \sum_{k=0}^{a+\nu} \frac{(-a-\nu)_k\, (-\nu)_k}{k!\, (-a-b+k-2\nu-1)!} \left(\log\left(\frac{z-1}{2}\right) + \psi(k+1) - \psi(k-\nu) - \psi(a-k+\nu+1) + \psi(k-a-b-2\nu)\right) \left(\frac{2}{1-z}\right)^k /;$$

$$-a-b-2\nu-1 \in \mathbb{N}^+ \bigwedge -b-\nu \in \mathbb{N}^+ \bigwedge a+\nu \geq -1 \bigwedge z \notin \{-1, 1\}$$

07.15.06.0038.01

$$P_\nu^{(a,b)}(z) \propto \frac{(-1)^{a+b+2\nu}\, 2^{a+1} \sin(\pi\nu)\, \Gamma(a+1)\, z^{-a-1}}{\pi\,(a+\nu+1)\,\Gamma(1-b-\nu)\,\Gamma(a+b+\nu+1)} \left(1+O\left(\frac{1}{z}\right)\right) - \frac{2^{a+b+\nu+1} \sin(\pi\nu)\, \Gamma(-a-b-2\nu-1)\, \Gamma(a+\nu+1)\, z^{-a-b-\nu-1}}{\pi\,\Gamma(-b-\nu)} \left(1+O\left(\frac{1}{z}\right)\right) + \frac{(-1)^{a+b+2\nu-1}\, 2^{-\nu}}{(-a-b-2\nu-1)!\,\Gamma(\nu+1)\,\Gamma(a+b+\nu+1)} z^\nu \left(\log\left(\frac{z-1}{2}\right) - \psi(-\nu) - \psi(a+\nu+1) + \psi(-a-b-2\nu) - \gamma\right) \left(1+O\left(\frac{1}{z}\right)\right) /; (|z| \to \infty) \bigwedge -a-b-2\nu-1 \in \mathbb{N}^+ \bigwedge -b-\nu \in \mathbb{N}^+ \bigwedge a+\nu \geq 0 \bigwedge z \notin \{-1, 1\}$$

Integral representations

On the real axis

Of the direct function

07.15.07.0001.01

$$P_\nu^{(a,b)}(z) = \frac{\Gamma(a+\nu+1)}{2^\nu\,\Gamma(\nu+1)\,\Gamma(a+b+\nu+1)\,\Gamma(-b-\nu)} \int_0^1 t^{a+b+\nu}\,(1-t)^{-b-\nu-1}\,(z\,t-t+2)^\nu\, dt /; \operatorname{Re}(b+\nu) < 0 \bigwedge \operatorname{Re}(a+b+\nu+1) > 0 \bigwedge |\arg(z+1)| < \pi$$

Integral representations of negative integer order

Rodrigues-type formula.

07.15.07.0002.01

$$P_n^{(a,b)}(z) = \frac{(-1)^n}{n!\,2^n\,(1-z)^a\,(z+1)^b} \frac{\partial^n\left((1-z)^{a+n}\,(z+1)^{b+n}\right)}{\partial z^n} /; n \in \mathbb{N}$$

Generating functions

07.15.11.0001.01

$$P_n^{(a,b)}(z) = \left([t^n]\, \frac{\left(\sqrt{t^2-2\,t\,z+1}+1-t\right)^{-a} \left(\sqrt{t^2-2\,t\,z+1}+1+t\right)^{-b}}{2^{-a-b}\sqrt{t^2-2\,t\,z+1}}\right) /; n \in \mathbb{N} \land -1 < z < 1$$

Differential equations

Ordinary linear differential equations and wronskians

For the direct function itself

07.15.13.0003.01

$$\left(1-z^2\right) w''(z) + (b-a-(a+b+2)\,z)\,w'(z) + \nu\,(\nu+2\,\lambda)\,w(z) = 0 /; w(z) = c_1\,P_\nu^{(a,b)}(z) + w\,(z) = P_n^{(a,b)}(z)\,c_1 + c_2\,G_{2,2}^{2,2}\left(\frac{1-z}{2} \,\middle|\, \begin{matrix} \nu+1, -a-b-\nu \\ 0, -a \end{matrix}\right)$$

07.15.13.0004.01

$$W_z\left(P_\nu^{(a,b)}(z),\, G_{2,2}^{2,2}\left(\frac{1-z}{2}\,\middle|\, \begin{matrix}\nu+1,\, -a-b-\nu \\ 0,\, -a\end{matrix}\right)\right) = -\frac{2^{a+b+1}\,\pi\csc(\pi\,(a+\nu))\,\Gamma(b+\nu+1)}{\Gamma(\nu+1)}\,(1-z)^{-a-1}\,(z+1)^{-b-1}$$

07.15.13.0001.01

$$\left(1-z^2\right)w''(z)+(b-a-(a+b+2)\,z)\,w'(z)+\nu\,(\nu+2\,\lambda)\,w(z)=0\;/;\;w(z)=c_2(1-z)^{-a}\,P_{a+\nu}^{(-a,b)}(z)+c_1\,P_\nu^{(a,b)}(z)\bigwedge a\notin\mathbb{Z}$$

07.15.13.0002.01

$$W_z\left(P_\nu^{(a,b)}(z),\,(1-z)^{-a}\,P_{a+\nu}^{(-a,b)}(z)\right) = \frac{2^{b+1}\sin(a\,\pi)}{\pi}\,(1-z)^{-a-1}\,(z+1)^{-b-1}$$

07.15.13.0005.01

$$w''(z)+\left(\frac{(a-b+(a+b+2)\,g(z))\,g'(z)}{g(z)^2-1}-\frac{g''(z)}{g'(z)}\right)w'(z)-\frac{\nu\,(a+b+\nu+1)\,g'(z)^2}{g(z)^2-1}\,w(z)=0\;/;$$
$$w(z)=c_1\,P_\nu^{(a,b)}(g(z))\;+c_2\,G_{2,2}^{2,2}\left(\frac{1-g(z)}{2}\,\middle|\,\begin{matrix}\nu+1,\,-a-b-\nu\\0,\,-a\end{matrix}\right)$$

07.15.13.0006.01

$$W_z\left(P_\nu^{(a,b)}(g(z)),\,G_{2,2}^{2,2}\left(\frac{1-g(z)}{2}\,\middle|\,\begin{matrix}\nu+1,\,-a-b-\nu\\0,\,-a\end{matrix}\right)\right) = -\frac{2^{a+b+1}\,\pi\csc(\pi\,(a+\nu))\,\Gamma(b+\nu+1)}{\Gamma(\nu+1)}\,(1-g(z))^{-a-1}\,(g(z)+1)^{-b-1}\,g'(z)$$

07.15.13.0007.01

$$w''(z)\,h(z)^2+\left(h(z)\left(\frac{(a-b+(a+b+2)\,g(z))\,h(z)\,g'(z)}{g(z)^2-1}-2\,h'(z)\right)-\frac{h(z)^2\,g''(z)}{g'(z)}\right)w'(z)-$$
$$\left(-2\,h'(z)^2-\frac{h(z)\,g''(z)\,h'(z)}{g'(z)}+\frac{1}{g(z)^2-1}\,h(z)\,g'(z)\,(\nu\,(a+b+\nu+1)\,h(z)\,g'(z)+(a-b+(a+b+2)\,g(z))\,h'(z))+h(z)\,h''(z)\right)$$
$$w(z)=0\;/;\;w(z)=c_1\,h(z)\,P_\nu^{(a,b)}(g(z))\;+c_2\,h(z)\,G_{2,2}^{2,2}\left(\frac{1-g(z)}{2}\,\middle|\,\begin{matrix}\nu+1,\,-a-b-\nu\\0,\,-a\end{matrix}\right)$$

07.15.13.0008.01

$$W_z\left(h(z)\,P_\nu^{(a,b)}(g(z)),\,h(z)\,G_{2,2}^{2,2}\left(\frac{1-g(z)}{2}\,\middle|\,\begin{matrix}\nu+1,\,-a-b-\nu\\0,\,-a\end{matrix}\right)\right) =$$
$$-\frac{2^{a+b+1}\,\pi\csc(\pi\,(a+\nu))\,\Gamma(b+\nu+1)}{\Gamma(\nu+1)}\,(1-g(z))^{-a-1}\,(g(z)+1)^{-b-1}\,h(z)^2\,g'(z)$$

07.15.13.0009.01

$$z^2\,w''(z)+\left(\frac{d\,r\,((a+b+2)\,d\,z^r+a-b)\,z^r}{d^2\,z^{2\,r}-1}-r-2\,s+1\right)z\,w'(z)+$$
$$\frac{-(a-b)\,d\,r\,s\,z^r-d^2\,(s+r\,\nu)\,(r\,(a+b+\nu+1)-s)\,z^{2\,r}-s\,(r+s)}{d^2\,z^{2\,r}-1}\,w(z)=0\;/;$$
$$w(z)=c_1\,z^s\,P_\nu^{(a,b)}(d\,z^r)\;+c_2\,z^s\,G_{2,2}^{2,2}\left(\frac{1-d\,z^r}{2}\,\middle|\,\begin{matrix}\nu+1,\,-a-b-\nu\\0,\,-a\end{matrix}\right)$$

07.15.13.0010.01

$$W_z\left(z^s\,P_\nu^{(a,b)}(d\,z^r),\,z^s\,G_{2,2}^{2,2}\left(\frac{1-d\,z^r}{2}\,\middle|\,\begin{matrix}\nu+1,\,-a-b-\nu\\0,\,-a\end{matrix}\right)\right) =$$
$$-\frac{2^{a+b+1}\,d\,\pi\,r\,z^{r+2\,s-1}}{\Gamma(\nu+1)}\,(1-d\,z^r)^{-a-1}\,(d\,z^r+1)^{-b-1}\csc(\pi\,(a+\nu))\,\Gamma(b+\nu+1)$$

07.15.13.0011.01

$$w''(z) + \left(\frac{d\,((a+b+1)\,d\,r^z + a - b)\,r^z + 1}{d^2\,r^{2z} - 1} \log(r) - 2\log(s) \right) w'(z) + \left(\log^2(s) + \log(r)\log(s) - \frac{d\,r^z \log(r)}{d^2\,r^{2z} - 1} (d\,\nu\,(a+b+\nu+1)\log(r)\,r^z + ((a+b+2)\,d\,r^z + a - b)\log(s)) \right) w(z) = 0 \;/;$$

$$w(z) = c_1\,s^z\,P_\nu^{(a,b)}(d\,r^z) + c_2\,s^z\,G_{2,2}^{2,2}\!\left(\frac{1 - d\,r^z}{2} \,\middle|\, \begin{matrix} \nu+1, -a-b-\nu \\ 0, -a \end{matrix} \right)$$

07.15.13.0012.01

$$W_z\!\left(s^z\,P_\nu^{(a,b)}(d\,r^z),\; s^z\,G_{2,2}^{2,2}\!\left(\frac{1 - d\,r^z}{2} \,\middle|\, \begin{matrix} \nu+1, -a-b-\nu \\ 0, -a \end{matrix} \right) \right) = -\frac{1}{\Gamma(\nu+1)}\,2^{a+b+1}\,d\,\pi\,r^z\,(1 - d\,r^z)^{-a-1}\,(d\,r^z + 1)^{-b-1}\,s^{2z}\csc(\pi\,(a+\nu))\,\Gamma(b+\nu+1)\log(r)$$

Transformations

Transformations and argument simplifications

Argument involving basic arithmetic operations

07.15.16.0001.01

$$P_{-a-b-\nu-1}^{(a,b)}(z) = \frac{\Gamma(-b-\nu)\,\Gamma(\nu+1)}{\Gamma(-a-b-\nu)\,\Gamma(a+\nu+1)}\,P_\nu^{(a,b)}(z)$$

07.15.16.0002.01

$$P_n^{(a,b)}(-z) = (-1)^n\,P_n^{(b,a)}(z) \;/;\; n \in \mathbb{N}$$

Identities

Recurrence identities

Consecutive neighbors

With respect to ν

07.15.17.0001.01

$$P_\nu^{(a,b)}(z) = \frac{(a+b+2\nu+3)\left(a^2 - b^2 + z\,(a+b+2\nu+2)\,(a+b+2\nu+4)\right)}{2\,(a+\nu+1)\,(b+\nu+1)\,(a+b+2\nu+4)}\,P_{\nu+1}^{(a,b)}(z) - \frac{(\nu+2)\,(a+b+\nu+2)\,(a+b+2\nu+2)}{(a+\nu+1)\,(b+\nu+1)\,(a+b+2\nu+4)}\,P_{\nu+2}^{(a,b)}(z)$$

07.15.17.0002.01

$$P_\nu^{(a,b)}(z) = \frac{(a+b+2\nu-1)\left(a^2 - b^2 + z\,(a+b+2\nu-2)\,(a+b+2\nu)\right)}{2\,\nu\,(a+b+\nu)\,(a+b+2\nu-2)}\,P_{\nu-1}^{(a,b)}(z) - \frac{(a+\nu-1)\,(b+\nu-1)\,(a+b+2\nu)}{\nu\,(a+b+\nu)\,(a+b+2\nu-2)}\,P_{\nu-2}^{(a,b)}(z)$$

With respect to a

07.15.17.0014.01

$$P_\nu^{(a,b)}(z) = \frac{-(z-1)\,b + a\,(3-z) - 2\,(z+(z-1)\,\nu-2)}{2\,(a+\nu+1)}\,P_\nu^{(a+1,b)}(z) + \frac{(a+b+\nu+2)\,(z-1)}{2\,(a+\nu+1)}\,P_\nu^{(a+2,b)}(z)$$

07.15.17.0015.01

$$P_\nu^{(a,b)}(z) = \frac{a\,(z-3) + b\,(z-1) + 2\,z\,\nu - 2\,\nu + 2}{(a+b+\nu)\,(z-1)}\,P_\nu^{(a-1,b)}(z) + \frac{2\,(a+\nu-1)}{(a+b+\nu)\,(z-1)}\,P_\nu^{(a-2,b)}(z)$$

With respect to *b*

07.15.17.0016.01

$$P_\nu^{(a,b)}(z) = \frac{(a\,(z+1) + b\,(z+3) + 2\,(\nu\,z + z + \nu + 2))\,P_\nu^{(a,b+1)}(z) - (z+1)\,(a+b+\nu+2)\,P_\nu^{(a,b+2)}(z)}{2\,(b+\nu+1)}$$

07.15.17.0017.01

$$P_\nu^{(a,b)}(z) = \frac{(a\,(z+1) + b\,(z+3) + 2\,(z\,\nu + \nu - 1))\,P_\nu^{(a,b-1)}(z) - 2\,(b+\nu-1)\,P_\nu^{(a,b-2)}(z)}{(z+1)\,(a+b+\nu)}$$

Distant neighbors

07.15.17.0018.01

$$P_\nu^{(a,b)}(z) = C_n(\nu,\,a,\,b,\,z)\,P_{\nu+n}^{(a,b)}(z) - \frac{(n+\nu+1)\,(a+b+n+\nu+1)\,(a+b+2\,n+2\,\nu)}{(a+n+\nu)\,(b+n+\nu)\,(a+b+2\,n+2\,\nu+2)}\,C_{n-1}(\nu,\,a,\,b,\,z)\,P_{\nu+n+1}^{(a,b)}(z)\;/;$$
$$C_0(\nu,\,a,\,b,\,z) = 1 \bigwedge C_1(\nu,\,a,\,b,\,z) = \frac{(a+b+2\,\nu+3)\left(a^2 - b^2 + z\,(a+b+2\,\nu+2)\,(a+b+2\,\nu+4)\right)}{2\,(a+\nu+1)\,(b+\nu+1)\,(a+b+2\,\nu+4)} \bigwedge$$
$$C_n(\nu,\,a,\,b,\,z) = \frac{(a+b+2\,n+2\,\nu+1)\left(a^2 - b^2 + (a+b+2\,n+2\,\nu+2)\,z\,(a+b+2\,n+2\,\nu)\right)}{2\,(a+n+\nu)\,(b+n+\nu)\,(a+b+2\,n+2\,\nu+2)}\,C_{n-1}(\nu,\,a,\,b,\,z) -$$
$$\frac{(n+\nu)\,(a+b+n+\nu)\,(a+b+2\,n+2\,\nu-2)}{(a+n+\nu-1)\,(b+n+\nu-1)\,(a+b+2\,n+2\,\nu)}\,C_{n-2}(\nu,\,a,\,b,\,z) \bigwedge n \in \mathbb{N}^+$$

07.15.17.0019.01

$$P_\nu^{(a,b)}(z) = \frac{(a-n+\nu)\,(b-n+\nu)\,(a+b-2\,n+2\,\nu+2)}{(n-\nu-1)\,(a+b-n+\nu+1)\,(a+b-2\,n+2\,\nu)}\,C_{n-1}(\nu,\,a,\,b,\,z)\,P_{\nu-n-1}^{(a,b)}(z) + C_n(\nu,\,a,\,b,\,z)\,P_{\nu-n}^{(a,b)}(z)\;/;$$
$$C_0(\nu,\,a,\,b,\,z) = 1 \bigwedge C_1(\nu,\,a,\,b,\,z) = \frac{(a+b+2\,\nu-1)\left(a^2 - b^2 + z\,(a+b+2\,\nu-2)\,(a+b+2\,\nu)\right)}{2\,\nu\,(a+b+\nu)\,(a+b+2\,\nu-2)} \bigwedge$$
$$C_n(\nu,\,a,\,b,\,z) = \frac{(a+b-2\,n+2\,\nu+1)\left(a^2 - b^2 + z\,(a+b-2\,n+2\,\nu)\,(a+b-2\,n+2\,\nu+2)\right)}{2\,(-n+\nu+1)\,(a+b-n+\nu+1)\,(a+b-2\,n+2\,\nu)}\,C_{n-1}(\nu,\,a,\,b,\,z) +$$
$$\frac{(a-n+\nu+1)\,(b-n+\nu+1)\,(a+b-2\,n+2\,\nu+4)}{(n-\nu-2)\,(a+b-n+\nu+2)\,(a+b-2\,n+2\,\nu+2)}\,C_{n-2}(\nu,\,a,\,b,\,z) \bigwedge n \in \mathbb{N}^+$$

Functional identities

Relations between contiguous functions

Recurrence relations

07.15.17.0003.01

$$2\,(a+b+2\,\nu+2)\,(a+\nu)\,(b+\nu)\,P_{\nu-1}^{(a,b)}(z)+2\,(a+b+\nu+1)\,(\nu+1)\,(a+b+2\,\nu)\,P_{\nu+1}^{(a,b)}(z)=\left((a+b+2\,\nu+1)\left(a^2-b^2\right)+z\,(a+b+2\,\nu)_3\right)P_\nu^{(a,b)}(z)$$

07.15.17.0004.01

$$P_\nu^{(a,b)}(z)=\left(2\left((a+\nu)\,(b+\nu)\,(a+b+2\,\nu+2)\,P_{\nu-1}^{(a,b)}(z)+(\nu+1)\,(a+b+\nu+1)\,(a+b+2\,\nu)\,P_{\nu+1}^{(a,b)}(z)\right)\right)\Big/\left((a+b+2\,\nu+1)\left(a^2-b^2+z\,(a+b+2\,\nu)\,(a+b+2\,\nu+2)\right)\right)$$

07.15.17.0005.01

$$P_\nu^{(a,b-1)}(z)-P_\nu^{(a-1,b)}(z)=P_{\nu-1}^{(a,b)}(z)$$

07.15.17.0006.01

$$z\,P_\nu^{(a,b)}(z)=\frac{2\,(a+\nu)\,(b+\nu)}{(a+b+2\,\nu)\,(a+b+2\,\nu+1)}\,P_{\nu-1}^{(a,b)}(z)+\frac{2\,(\nu+1)\,(a+b+\nu+1)}{(a+b+2\,\nu+1)\,(a+b+2\,\nu+2)}\,P_{\nu+1}^{(a,b)}(z)+\frac{b^2-a^2}{(a+b+2\,\nu)\,(a+b+2\,\nu+2)}\,P_\nu^{(a,b)}(z)$$

07.15.17.0020.01

$$P_\nu^{(a,b)}(z)=\frac{(a+b+(a-b+2)\,z)\,P_\nu^{(a+1,b-1)}(z)+(z-1)\,(b+\nu-1)\,P_\nu^{(a+2,b-2)}(z)}{(z+1)\,(a+\nu+1)}$$

07.15.17.0021.01

$$P_\nu^{(a,b)}(z)=\frac{(z+1)\,(a+\nu-1)\,P_\nu^{(a-2,b+2)}(z)-(z\,a+a+b-b\,z-2\,z)\,P_\nu^{(a-1,b+1)}(z)}{(z-1)\,(b+\nu+1)}$$

07.15.17.0022.01

$$P_\nu^{(a,b)}(z)=\frac{\nu+1}{a+\nu+1}\,P_{\nu+1}^{(a,b)}(z)+\frac{(a+b+2\,\nu+2)\,(1-z)}{2\,(a+\nu+1)}\,P_\nu^{(a+1,b)}(z)$$

07.15.17.0023.01

$$P_\nu^{(a,b)}(z)=\frac{a+b+\nu+1}{b+\nu+1}\,P_{\nu+1}^{(a,b)}(z)-\frac{a+b+2\,\nu+2}{b+\nu+1}\,P_{\nu+1}^{(a-1,b)}(z)$$

07.15.17.0024.01

$$P_\nu^{(a,b)}(z)=\frac{1}{2}\,(z+1)\,P_\nu^{(a,b+1)}(z)-\frac{1}{2}\,(z-1)\,P_\nu^{(a+1,b)}(z)$$

07.15.17.0025.01

$$P_\nu^{(a,b)}(z)=\frac{a+b+\nu+1}{a+\nu+1}\,P_\nu^{(a+1,b)}(z)-\frac{b+\nu}{a+\nu+1}\,P_\nu^{(a+1,b-1)}(z)$$

Normalized recurrence relation

07.15.17.0007.01

$$z\,p(\nu,z)=p(\nu+1,z)+\frac{\left(b^2-a^2\right)p(\nu,z)}{(a+b+2\,\nu)\,(a+b+2\,\nu+2)}+\frac{4\,\nu\,(a+\nu)\,(b+\nu)\,(a+b+\nu)}{(a+b+2\,\nu-1)\,(a+b+2\,\nu)^2\,(a+b+2\,\nu+1)}\,p(\nu-1,z)\,/;$$

$$p(\nu,z)=\frac{2^\nu\,\Gamma(\nu+1)}{(a+b+\nu+1)_\nu}\,P_\nu^{(a,b)}(z)$$

Additional relations between contiguous functions

07.15.17.0008.01

$$P_\nu^{(a,b)}(z) - P_\nu^{(a-1,b)}(z) = \frac{z+1}{2} P_{\nu-1}^{(a,b+1)}(z)$$

Relations of special kind

07.15.17.0009.01

$$P_\nu^{(a,b)}(z) = \frac{\Gamma(-a-b-\nu)\,\Gamma(a+\nu+1)}{\Gamma(-b-\nu)\,\Gamma(\nu+1)} P_{-a-b-\nu-1}^{(a,b)}(z)$$

07.15.17.0010.01

$$P_\nu^{(a,b)}(z) = \frac{\Gamma(a+\nu+1)\,\Gamma(b+\nu+1)}{\Gamma(\nu+1)\,\Gamma(a+b+\nu+1)} \left(\frac{z+1}{2}\right)^{-b} P_{b+\nu}^{(a,-b)}(z)$$

07.15.17.0011.01

$$P_\nu^{(a,b)}(z) = \frac{\Gamma(a+\nu+1)\,\Gamma(b+\nu+1)}{\Gamma(\nu+1)\,\Gamma(a+b+\nu+1)} \left(\frac{z-1}{2}\right)^{-a} P_{a+\nu}^{(-a,b)}(z) \;/;\; a \in \mathbb{Z}$$

07.15.17.0012.01

$$P_\nu^{(a,b)}(z) = \left(\frac{z-1}{2}\right)^{-a} \left(\frac{z+1}{2}\right)^{-b} P_{a+b+\nu}^{(-a,-b)}(z) \;/;\; a \in \mathbb{Z}$$

07.15.17.0013.01

$$P_\nu^{(a,b)}(-z) = \csc(b\,\pi)\sin((b+\nu)\,\pi)\,P_\nu^{(b,a)}(z) - 2^b\,(1-z)^{-b}\csc(b\,\pi)\csc((a+\nu)\,\pi)\sin(\nu\,\pi)\sin((a+b+\nu)\,\pi)\,P_{-a-\nu-1}^{(-b,a)}(z)$$

Complex characteristics

Real part

07.15.19.0001.01

$$\operatorname{Re}\left(P_n^{(a,b)}(x+i\,y)\right) = \sum_{j=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{(-1)^j\,(a+b+n+1)_{2\,j}\,y^{2\,j}}{2^{2\,j}\,(2\,j)!} P_{n-2\,j}^{(a+2\,j,b+2\,j)}(x) \;/;\; x \in \mathbb{R} \land y \in \mathbb{R} \land a \in \mathbb{R} \land b \in \mathbb{R} \land n \in \mathbb{N}$$

Imaginary part

07.15.19.0002.01

$$\operatorname{Im}\left(P_n^{(a,b)}(x+i\,y)\right) = \sum_{j=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \frac{(-1)^j\,(a+b+n+1)_{2\,j+1}\,y^{2\,j+1}}{2^{2\,j+1}\,(2\,j+1)!} P_{-2\,j+n-1}^{(a+2\,j+1,b+2\,j+1)}(x) \;/;\; x \in \mathbb{R} \land y \in \mathbb{R} \land a \in \mathbb{R} \land b \in \mathbb{R} \land n \in \mathbb{N}$$

Differentiation

Low-order differentiation

With respect to ν

07.15.20.0001.01

$$\frac{\partial P_\nu^{(a,b)}(z)}{\partial \nu} = \frac{(z-1)\,\Gamma(a+\nu+1)}{2\,\Gamma(a+2)\,\Gamma(\nu+1)}\left((a+b+\nu+1)\,\tilde{F}_{2\times 0\times 1}^{2\times 1\times 2}\left(\begin{matrix}1-\nu,\, a+b+\nu+2;\, 1;\, 1,\, -\nu; \\ 2,\, a+2;;\, 1-\nu;\end{matrix}\, \frac{1-z}{2},\, \frac{1-z}{2}\right)+\right.$$
$$\left.\nu\,\tilde{F}_{2\times 0\times 1}^{2\times 1\times 2}\left(\begin{matrix}1-\nu,\, a+b+\nu+2;\, 1;\, 1,\, a+b+\nu+1; \\ 2,\, a+2;;\, a+b+\nu+2;\end{matrix}\, \frac{1-z}{2},\, \frac{1-z}{2}\right)\right)+$$
$$\frac{\Gamma(a+\nu+1)}{\Gamma(\nu+1)}\,(\psi(a+\nu+1)-\psi(\nu+1))\,{}_2\tilde{F}_1\left(-\nu,\, a+b+\nu+1;\, a+1;\, \frac{1-z}{2}\right)$$

With respect to a

07.15.20.0002.01

$$\frac{\partial P_\nu^{(a,b)}(z)}{\partial a} = \frac{\Gamma(a+\nu+1)\,(\psi(a+\nu+1)-\psi(a+1))}{\Gamma(\nu+1)}\,{}_2\tilde{F}_1\left(-\nu,\, a+b+\nu+1;\, a+1;\, \frac{1-z}{2}\right)-$$
$$\frac{(1-z)\,\Gamma(a+\nu+1)}{2\,(a+1)\,\Gamma(\nu)\,\Gamma(a+2)}\left((a+1)\,F_{2\times 0\times 1}^{2\times 1\times 2}\left(\begin{matrix}1-\nu,\, a+b+\nu+2;\, 1;\, 1,\, a+b+\nu+1; \\ 2,\, a+2;;\, a+b+\nu+2;\end{matrix}\, \frac{1-z}{2},\, \frac{1-z}{2}\right)-\right.$$
$$\left.(a+b+\nu+1)\,F_{2\times 0\times 1}^{2\times 1\times 2}\left(\begin{matrix}1-\nu,\, a+b+\nu+2;\, 1;\, 1,\, a+1; \\ 2,\, a+2;;\, a+2;\end{matrix}\, \frac{1-z}{2},\, \frac{1-z}{2}\right)\right)$$

With respect to b

07.15.20.0003.01

$$\frac{\partial P_\nu^{(a,b)}(z)}{\partial b} = -\frac{\Gamma(a+\nu+1)\,(1-z)}{2\,\Gamma(\nu)\,\Gamma(a+2)}\,F_{2\times 0\times 1}^{2\times 1\times 2}\left(\begin{matrix}a+b+\nu+2,\, 1-\nu;\, 1;\, 1,\, a+b+\nu+1; \\ 2,\, a+2;;\, a+b+\nu+2;\end{matrix}\, \frac{1-z}{2},\, \frac{1-z}{2}\right)$$

With respect to z

Forward shift operator:

07.15.20.0004.01

$$\frac{\partial P_\nu^{(a,b)}(z)}{\partial z} = \frac{a+b+\nu+1}{2}\,P_{\nu-1}^{(a+1,b+1)}(z)$$

07.15.20.0005.01

$$\frac{\partial^2 P_\nu^{(a,b)}(z)}{\partial z^2} = \frac{1}{4}\,(a+b+\nu+1)\,(a+b+\nu+2)\,P_{\nu-2}^{(a+2,b+2)}(z)$$

Backward shift operator:

07.15.20.0006.01

$$\left(1-z^2\right)\frac{\partial P_\nu^{(a,b)}(z)}{\partial z}+(b-a-(a+b)\,z)\,P_\nu^{(a,b)}(z) = -2\,(\nu+1)\,P_{\nu+1}^{(a-1,b-1)}(z)$$

07.15.20.0007.01

$$\frac{\partial\left((1-z)^a\,(z+1)^b\,P_\nu^{(a,b)}(z)\right)}{\partial z} = -2\,(\nu+1)\,(1-z)^{a-1}\,(z+1)^{b-1}\,P_{\nu+1}^{(a-1,b-1)}(z)$$

Symbolic differentiation

With respect to z

07.15.20.0008.01

$$\frac{\partial^m P_\nu^{(a,b)}(z)}{\partial z^m} = 2^{-m} (a+b+\nu+1)_m P_{\nu-m}^{(a+m,b+m)}(z) /; m \in \mathbb{N}^+$$

07.15.20.0009.01

$$\frac{\partial^m P_\nu^{(a,b)}(z)}{\partial z^m} = \frac{\Gamma(a+\nu+1)(z-1)^{-m}}{\Gamma(\nu+1)} \, {}_3\tilde{F}_2\left(1, -\nu, a+b+\nu+1; a+1, 1-m; \frac{1-z}{2}\right) /; m \in \mathbb{N}^+$$

Fractional integro-differentiation

With respect to *z*

07.15.20.0010.01

$$\frac{\partial^\alpha P_\nu^{(a,b)}(z)}{\partial z^\alpha} = \frac{\Gamma(a+\nu+1)}{\Gamma(\nu+1)} z^{-\alpha} \, \tilde{F}_{1\times1\times0}^{2\times1\times0}\left(\begin{matrix} -\nu, a+b+\nu+1; 1; ; \\ a+1; 1-\alpha; ; \end{matrix} -\frac{z}{2}, \frac{1}{2}\right)$$

Integration

Indefinite integration

Involving only one direct function

07.15.21.0001.01

$$\int P_\nu^{(a,b)}(z) \, dz = \frac{2}{a+b+\nu} P_{\nu+1}^{(a-1,b-1)}(z)$$

Involving one direct function and elementary functions

Involving power function

07.15.21.0002.01

$$\int z^{\alpha-1} P_\nu^{(a,b)}(z) \, dz = \frac{(a+1)_\nu z^\alpha}{\alpha \, \Gamma(\nu+1)} \tilde{F}_{1\times1\times0}^{2\times1\times0}\left(\begin{matrix} -\nu, a+b+\nu+1; \alpha; ; \\ a+1; \alpha+1; ; \end{matrix} -\frac{z}{2}, \frac{1}{2}\right)$$

Involving algebraic functions

07.15.21.0003.01

$$\int (z-1)^c P_\nu^{(a,b)}(z) \, dz = \frac{(z-1)^{c+1} \Gamma(c+1) \Gamma(a+\nu+1)}{\Gamma(\nu+1)} \, {}_3\tilde{F}_2\left(-\nu, a+b+\nu+1, c+1; a+1, c+2; \frac{1-z}{2}\right)$$

07.15.21.0004.01

$$\int (1-z)^a (z+1)^b P_\nu^{(a,b)}(z) \, dz = -\frac{(1-z)^{a+1} (z+1)^{b+1}}{2\nu} P_{\nu-1}^{(a+1,b+1)}(z)$$

Definite integration

Involving the direct function

07.15.21.0005.01

$$\int_{-1}^{1} (1-t)^a (t+1)^b P_m^{(a,b)}(t) P_n^{(a,b)}(t) \, dt = \frac{2^{a+b+1} \Gamma(a+n+1) \Gamma(b+n+1)}{n! \, (a+b+2n+1) \Gamma(a+b+n+1)} \delta_{m,n} /; m \in \mathbb{N} \wedge n \in \mathbb{N} \wedge \operatorname{Re}(a) > -1 \wedge \operatorname{Re}(b) > -1$$

Summation

Infinite summation

07.15.23.0001.01

$$\sum_{n=0}^{\infty} P_n^{(a,b)}(z)\, w^n = \frac{\left(\sqrt{w^2-2\,z\,w+1}+1-w\right)^{-a}\left(\sqrt{w^2-2\,z\,w+1}+1+w\right)^{-b}}{2^{-a-b}\sqrt{w^2-2\,z\,w+1}} \;/;\; -1<z<1\land |w|<1$$

07.15.23.0002.01

$$\sum_{n=0}^{\infty} \frac{P_n^{(a,b)}(z)\, w^n}{(a+1)_n\,(b+1)_n} = {}_0F_1\!\left(;a+1;\frac{1}{2}(z-1)\,w\right){}_0F_1\!\left(;b+1;\frac{1}{2}(z+1)\,w\right) /;\; -1<z<1\land |w|<1$$

07.15.23.0003.01

$$\sum_{n=0}^{\infty} \frac{(a+b+1)_n}{(a+1)_n} P_n^{(a,b)}(z)\, w^n = (1-w)^{-a-b-1}\,{}_2F_1\!\left(\frac{a+b+1}{2},\frac{a+b}{2}+1;a+1;\frac{2\,(z-1)\,w}{(1-w)^2}\right) /;\; -1<z<1\land |w|<1$$

07.15.23.0004.01

$$\sum_{n=0}^{\infty} \frac{(a+b+1)_n}{(b+1)_n} P_n^{(a,b)}(z)\, w^n = (1+w)^{-a-b-1}\,{}_2F_1\!\left(\frac{a+b+1}{2},\frac{a+b}{2}+1;b+1;\frac{2\,(z+1)\,w}{(1+w)^2}\right) /;\; -1<z<1\land |w|<1$$

07.15.23.0005.01

$$\sum_{n=0}^{\infty} \frac{(c)_n\,(a+b-c+1)_n}{(a+1)_n\,(b+1)_n} P_n^{(a,b)}(z)\, w^n = {}_2F_1\!\left(c,a+b-c+1;a+1;\frac{1}{2}\left(1-\sqrt{w^2-2\,z\,w+1}-w\right)\right)$$
$$ {}_2F_1\!\left(c,a+b-c+1;b+1;\frac{1}{2}\left(1-\sqrt{w^2-2\,z\,w+1}+w\right)\right) /;\; -1<z<1\land |w|<1$$

07.15.23.0006.01

$$\sum_{n=0}^{\infty} \frac{n!\,(a+b+2\,n+1)\,\Gamma(a+b+n+1)}{\Gamma(a+n+1)\,\Gamma(b+n+1)} P_n^{(a,b)}(x)\,P_n^{(a,b)}(y) = 2^{a+b+1}\,(1-x)^{-\frac{a}{2}}\,(x+1)^{-\frac{b}{2}}\,(1-y)^{-\frac{a}{2}}\,(y+1)^{-\frac{b}{2}}\,\delta(x-y) /;$$
$$-1<x<1\bigwedge -1<y<1\bigwedge \mathrm{Re}(a)>-1\bigwedge \mathrm{Re}(b)>-1$$

Operations

Limit operation

07.15.25.0001.01

$$\lim_{\nu\to\infty} \frac{1}{\nu^a}\left(\frac{2}{z}\right)^{-a} P_\nu^{(a,b)}\!\left(\cos\!\left(\frac{z}{\nu}\right)\right) = J_a(z)$$

07.15.25.0002.01

$$\lim_{a\to\infty} a^{-\frac{\nu}{2}}\, P_\nu^{(a,a)}\!\left(\frac{z}{\sqrt{a}}\right) = \frac{H_\nu(z)}{2^\nu\,\Gamma(\nu+1)}$$

07.15.25.0003.01

$$\lim_{b\to\infty} P_\nu^{(0,b)}\!\left(1-\frac{2\,z}{b}\right) = L_\nu(z)$$

07.15.25.0004.01

$$\lim_{b\to\infty} P_\nu^{(a,b)}\left(1-\frac{2z}{b}\right) == L_\nu^a(z)$$

Orthogonality, completeness, and Fourier expansions

The set of functions $P_n^{(a,b)}(x)$, $n = 0, 1, \ldots$, forms a complete, orthogonal (with weight $\frac{n!\,(a+b+2n+1)\,\Gamma(a+b+n+1)}{2^{a+b+1}\,\Gamma(a+n+1)\,\Gamma(b+n+1)}\,(1-x)^a\,(1+x)^b$) system on the interval $(-1, 1)$.

07.15.25.0005.01

$$\sum_{n=0}^{\infty}\left(\sqrt{\frac{n!\,(a+b+2n+1)\,\Gamma(a+b+n+1)}{2^{a+b+1}\,\Gamma(a+n+1)\,\Gamma(b+n+1)}}\,(1-x)^{a/2}\,(1+x)^{b/2}\,P_n^{(a,b)}(x)\right)\left(\sqrt{\frac{n!\,(a+b+2n+1)\,\Gamma(a+b+n+1)}{2^{a+b+1}\,\Gamma(a+n+1)\,\Gamma(b+n+1)}}\,(1-y)^{a/2}\,(1+y)^{b/2}\,P_n^{(a,b)}(y)\right) == \delta(x-y)\ /;\ -1<x<1\land -1<y<1\land \mathrm{Re}(a)>-1\land \mathrm{Re}(b)>-1$$

07.15.25.0006.01

$$\int_{-1}^{1}\left(\sqrt{\frac{m!\,(a+b+2m+1)\,\Gamma(a+b+m+1)}{2^{a+b+1}\,\Gamma(a+m+1)\,\Gamma(b+m+1)}}\,(1-t)^{a/2}\,(1+t)^{b/2}\,P_m^{(a,b)}(t)\right)\left(\sqrt{\frac{n!\,(a+b+2n+1)\,\Gamma(a+b+n+1)}{2^{a+b+1}\,\Gamma(a+n+1)\,\Gamma(b+n+1)}}\,(1-t)^{a/2}\,(1+t)^{b/2}\,P_n^{(a,b)}(t)\right)dt == \delta_{m,n}\ /;\ \mathrm{Re}(a)>-1\land \mathrm{Re}(b)>-1$$

Any sufficiently smooth function $f(x)$ can be expanded in the system $\left\{P_n^{(a,b)}(x)\right\}_{n=0,1,\ldots}$ as a generalized Fourier series, with its sum converging to $f(x)$ almost everywhere.

07.15.25.0007.01

$$f(x) == \sum_{n=0}^{\infty} c_n\,\psi_n(x)\ /;$$

$$c_n == \int_{-1}^{1}\psi_n(t)\,f(t)\,dt \land \psi_n(x) == \sqrt{\frac{n!\,(a+b+2n+1)\,\Gamma(a+b+n+1)}{2^{a+b+1}\,\Gamma(a+n+1)\,\Gamma(b+n+1)}}\,(1-x)^{a/2}\,(1+x)^{b/2}\,P_n^{(a,b)}(x) \land -1<x<1$$

Representations through more general functions

Through hypergeometric functions

Involving ${}_2\tilde{F}_1$

07.15.26.0001.01

$$P_\nu^{(a,b)}(z) == \frac{\Gamma(a+\nu+1)}{\Gamma(\nu+1)}\,{}_2\tilde{F}_1\left(-\nu, a+b+\nu+1; a+1; \frac{1-z}{2}\right)$$

Involving ${}_2F_1$

07.15.26.0002.01

$$P_\nu^{(a,b)}(z) = \frac{\Gamma(a+\nu+1)}{\Gamma(\nu+1)\,\Gamma(a+1)}\,{}_2F_1\!\left(-\nu,\, a+b+\nu+1;\, a+1;\, \frac{1-z}{2}\right) /;\, -a \notin \mathbb{N}^+$$

07.15.26.0003.01

$$P_\nu^{(a,b)}(z) = \frac{\Gamma(-b)}{\Gamma(\nu+1)\,\Gamma(-b-\nu)}\,{}_2F_1\!\left(-\nu,\, a+b+\nu+1;\, b+1;\, \frac{z+1}{2}\right) - \frac{\sin(\nu\pi)\,\Gamma(b)\,\Gamma(a+\nu+1)}{\pi\,\Gamma(a+b+\nu+1)}\left(\frac{z+1}{2}\right)^{-b}{}_2F_1\!\left(a+\nu+1,\, -b-\nu;\, 1-b;\, \frac{z+1}{2}\right) /;\, b \notin \mathbb{Z}$$

07.15.26.0004.01

$$P_\nu^{(a,b)}(z) = \frac{2^{-\nu}\,(a+b+\nu+1)_\nu}{\Gamma(\nu+1)}\,(z-1)^\nu\,{}_2F_1\!\left(-\nu,\, -a-\nu;\, -a-b-2\nu;\, \frac{2}{1-z}\right) - \frac{2^{a+b+\nu+1}\sin(\nu\pi)\,\Gamma(-a-b-2\nu-1)\,\Gamma(a+\nu+1)}{\pi\,\Gamma(-b-\nu)}\,(z-1)^{-a-b-\nu-1}\,{}_2F_1\!\left(a+b+\nu+1,\, b+\nu+1;\, a+b+2\nu+2;\, \frac{2}{1-z}\right) /;\, z \notin (-1,\,1) \bigwedge a+b+2\nu \notin \mathbb{Z}$$

Through hypergeometric functions of two variables

07.15.26.0005.01

$$P_\nu^{(a,b)}(z) = \frac{(a+1)_\nu}{\Gamma(\nu+1)}\,\tilde{F}_{1\times 0\times 0}^{2\times 0\times 0}\!\left(\begin{matrix} -\nu,\, a+b+\nu+1;;; \\ a+1;;; \end{matrix}\; -\frac{z}{2},\, \frac{1}{2}\right)$$

Through Meijer G

Classical cases for the direct function itself

07.15.26.0006.01

$$P_\nu^{(a,b)}(z) = -\frac{\sin(\pi\nu)\,\Gamma(a+\nu+1)}{\pi\,\Gamma(a+b+\nu+1)}\,G_{2,2}^{1,2}\!\left(\frac{z-1}{2}\;\middle|\; \begin{matrix} \nu+1,\, -a-b-\nu \\ 0,\, -a \end{matrix}\right) /;\, \nu \notin \mathbb{Z}$$

07.15.26.0007.01

$$P_n^{(a,b)}(z) = -\frac{1}{\pi}\lim_{m\to n}\frac{\sin(\pi m)\,\Gamma(a+m+1)}{\Gamma(a+b+m+1)}\,G_{2,2}^{1,2}\!\left(\frac{z-1}{2}\;\middle|\; \begin{matrix} m+1,\, -a-b-m \\ 0,\, -a \end{matrix}\right) /;\, n \in \mathbb{Z}$$

07.15.26.0008.01

$$P_\nu^{(a,b)}(2z+1) = -\frac{\sin(\pi\nu)\,\Gamma(a+\nu+1)}{\pi\,\Gamma(a+b+\nu+1)}\,G_{2,2}^{1,2}\!\left(z\;\middle|\; \begin{matrix} \nu+1,\, -a-b-\nu \\ 0,\, -a \end{matrix}\right) /;\, \nu \notin \mathbb{Z}$$

07.15.26.0009.01

$$P_\nu^{(a,b)}(2z-1) = -\frac{\sin(\pi\nu)}{\pi\,\Gamma(-b-\nu)\,\Gamma(a+b+\nu+1)}\,G_{2,2}^{2,2}\!\left(z\;\middle|\; \begin{matrix} \nu+1,\, -a-b-\nu \\ 0,\, -b \end{matrix}\right)$$

Classical cases involving algebraic functions

07.15.26.0010.01

$$(z+1)^b\,P_\nu^{(a,b)}(2z+1) = \frac{1}{\Gamma(\nu+1)\,\Gamma(-b-\nu)}\,G_{2,2}^{1,2}\!\left(z\;\middle|\; \begin{matrix} -a-\nu,\, b+\nu+1 \\ 0,\, -a \end{matrix}\right)$$

07.15.26.0011.01

$$(z+1)^b\,P_\nu^{(a,b)}\!\left(1+\frac{2}{z}\right) = \frac{1}{\Gamma(\nu+1)\,\Gamma(-b-\nu)}\,G_{2,2}^{2,1}\!\left(z\;\middle|\; \begin{matrix} b+1,\, a+b+1 \\ a+b+\nu+1,\, -\nu \end{matrix}\right) /;\, z \notin (-1,\,0)$$

07.15.26.0012.01

$$(z+1)^{-a-b-\nu-1}\,P_{\nu}^{(a,b)}\!\left(\frac{1-z}{1+z}\right)=\frac{1}{\Gamma(\nu+1)\,\Gamma(a+b+\nu+1)}\,G_{2,2}^{1,2}\!\left(z\,\middle|\,\begin{matrix}-a-b-\nu,\,-a-\nu\\0,\,-a\end{matrix}\right)/;\;z\notin(-\infty,-1)$$

07.15.26.0013.01

$$(z+1)^{-a-b-\nu-1}\,P_{\nu}^{(a,b)}\!\left(\frac{z-1}{z+1}\right)=\frac{1}{\Gamma(\nu+1)\,\Gamma(a+b+\nu+1)}\,G_{2,2}^{2,1}\!\left(z\,\middle|\,\begin{matrix}-a-b-\nu,\,-b-\nu\\0,\,-b\end{matrix}\right)/;\;z\notin(-1,0)$$

Classical cases involving unit step θ

07.15.26.0014.01

$$\theta(1-|z|)\,(1-z)^{a}\,P_{\nu}^{(a,b)}(2z-1)=\frac{\Gamma(a+\nu+1)}{\Gamma(\nu+1)}\,G_{2,2}^{2,0}\!\left(z\,\middle|\,\begin{matrix}a+\nu+1,\,-b-\nu\\0,\,-b\end{matrix}\right)/;\;z\notin(-1,0)$$

07.15.26.0015.01

$$\theta(|z|-1)\,(z-1)^{a}\,P_{\nu}^{(a,b)}(2z-1)=\frac{\Gamma(a+\nu+1)}{\Gamma(\nu+1)}\,G_{2,2}^{0,2}\!\left(z\,\middle|\,\begin{matrix}-b-\nu,\,a+\nu+1\\0,\,-b\end{matrix}\right)$$

07.15.26.0016.01

$$\theta(1-|z|)\,(1-z)^{a}\,P_{\nu}^{(a,b)}\!\left(\frac{2}{z}-1\right)=\frac{\Gamma(a+\nu+1)}{\Gamma(\nu+1)}\,G_{2,2}^{2,0}\!\left(z\,\middle|\,\begin{matrix}a+1,\,a+b+1\\-\nu,\,a+b+\nu+1\end{matrix}\right)$$

07.15.26.0017.01

$$\theta(|z|-1)\,(z-1)^{b}\,P_{\nu}^{(b,b)}\!\left(\frac{2}{z}-1\right)=\frac{\Gamma(b+\nu+1)}{\Gamma(\nu+1)}\,G_{2,2}^{0,2}\!\left(z\,\middle|\,\begin{matrix}b+1,\,2b+1\\-\nu,\,2b+\nu+1\end{matrix}\right)/;\;z\notin(-\infty,-1)$$

07.15.26.0018.01

$$\theta(1-|z|)\,(1-z)^{-a-b-n-1}\,P_{n}^{(a,b)}\!\left(\frac{1+z}{1-z}\right)=\frac{(-1)^{n}\,\Gamma(-a-b-n)}{n!}\,G_{2,2}^{2,0}\!\left(z\,\middle|\,\begin{matrix}-a-n,\,-a-b-n\\0,\,-a\end{matrix}\right)/;\;n\in\mathbb{N}$$

07.15.26.0019.01

$$\theta(|z|-1)\,(z-1)^{-a-b-n-1}\,P_{n}^{(a,b)}\!\left(\frac{z+1}{z-1}\right)=\frac{(-1)^{n}\,\Gamma(-a-b-n)}{n!}\,G_{2,2}^{0,2}\!\left(z\,\middle|\,\begin{matrix}-b-n,\,-a-b-n\\0,\,-b\end{matrix}\right)/;\;n\in\mathbb{N}$$

Theorems

Expansions in generalized Fourier series

$$f(x)=\sum_{k=0}^{\infty}c_k\,\psi_k(x)\;/;\quad c_k=\int_{-1}^{1}f(t)\,\psi_k(t)\,dt,$$

$$\psi_k(x)=\sqrt{\frac{n!\,(a+b+2n+1)\,\Gamma(a+b+n+1)}{2^{a+b+1}\,\Gamma(a+n+1)\,\Gamma(b+n+1)}}\,(1-x)^{a/2}\,(1+x)^{b/2}\,P_{k}^{(a,b)}(x),\;\;k\in\mathbb{N}.$$

The quantum mechanical representation matrices of angular momentum

The quantum mechanical representation matrices $D_{m\,m'}^{L}(\alpha,\,\beta,\,\gamma)$ of angular momentum L are given by

$$D_{m\,m'}^{L}(\alpha,\,\beta,\,\gamma)=e^{i\,(m\,\alpha+m'\,\gamma)}\,\sqrt{\frac{(L+m')!\,(L-m')!}{(L+m)!\,(L-m)!}}\,\left(\cos\left(\frac{\beta}{2}\right)\right)^{m+m'}\left(\sin\left(\frac{\beta}{2}\right)\right)^{m-m'}\,P_{L-m'}^{(m-m',m+m')}(\cos(\beta))$$

where α, β, γ are the Euler angles and L, m, $m' \in \mathbb{N}$, $-L \leq m$, $m' \leq L$.

The expected value of the number of real eigenvalues of a one matrix

The expected value r_n of the number of real eigenvalues of a $n \times n$ matrix whose matrix elements are random variables with Gaussian distribution (mean = 0, variance = 1) is

$$r_n == \left(1 - (-1)^n\right) \big/ 2 + \sqrt{2}\ P_{n-2}^{(1-n,3/2)}(3).$$

The equilibrium positions of *n* unit charges

The equilibrium positions of n unit charges, a charge q at -1, and a charge p at $+1$ interacting with potential $-\log(x)$ are the zeros of $P_n^{(2\,p-1,2\,q-1)}(x)$.

History

– C. J. Jacobi (1859)

– P. L. Chebyshev (1870).

Copyright