InverseWeierstrassP

Notations

Traditional name

Inverse Weierstrass elliptic function

Traditional notation

$\wp^{-1}(z; g_2, g_3)$

Mathematica StandardForm notation

InverseWeierstrassP[z, {g2, g3}]

Primary definition

09.22.02.0001.01

$z \equiv \wp(w; g_2, g_3) /; w \equiv \wp^{-1}(z; g_2, g_3)$

09.22.02.0002.01

$$\wp^{-1}(z; g_2, g_3) \equiv \int_{\infty}^{z} \frac{1}{\sqrt{4 t^3 - g_2 t - g_3}} \, dt /; z \in \mathbb{R} \bigwedge \operatorname{Re}(4 z^3 - g_2 z - g_3) > 0$$

General characteristics

Domain and analyticity

$\wp^{-1}(z; g_2, g_3)$ is an analytical function of z, g_2 and g_3 which is defined in $\mathbb{C}^3$.

09.22.04.0001.01

$\{z * \{g_2 * g_3\}\} \longrightarrow \wp^{-1}(z; g_2, g_3) :: (\mathbb{C} \otimes \{\mathbb{C} \otimes \mathbb{C}\}) \longrightarrow \mathbb{C}$

Symmetries and periodicities

Mirror symmetry

09.22.04.0002.01

$\wp^{-1}(\bar{z}; \overline{g_2}, \overline{g_3}) \equiv \overline{\wp^{-1}(z; g_2, g_3)}$

Periodicity

No periodicity

Poles and essential singularities

With respect to g_3

The function $\wp^{-1}(z; g_2, g_3)$ does not have poles and essential singularities with respect to g_3.

09.22.04.0003.01

$$\mathcal{S}ing_{g_3}\left(\wp^{-1}(z; g_2, g_3)\right) = \{\}$$

With respect to g_2

The function $\wp^{-1}(z; g_2, g_3)$ does not have poles and essential singularities with respect to g_2.

09.22.04.0004.01

$$\mathcal{S}ing_{g_2}\left(\wp^{-1}(z; g_2, g_3)\right) = \{\}$$

With respect to z

The function $\wp^{-1}(z; g_2, g_3)$ does not have poles and essential singularities with respect to z.

09.22.04.0005.01

$$\mathcal{S}ing_{z}\left(\wp^{-1}(z; g_2, g_3)\right) = \{\}$$

Branch points

With respect to g_3

For fixed z, g_2, the function $\wp^{-1}(z; g_2, g_3)$ has two branch points: $g_3 = 4 z^3 - g_2 z$, $g_3 = \tilde{\infty}$.

09.22.04.0006.01

$$\mathcal{BP}_{g_3}\left(\wp^{-1}(z; g_2, g_3)\right) = \left\{4 z^3 - g_2 z, \tilde{\infty}\right\}$$

09.22.04.0007.01

$$\mathcal{R}_{g_3}\left(\wp^{-1}(z; g_2, g_3), 4 z^3 - g_2 z\right) = 2$$

09.22.04.0008.01

$$\mathcal{R}_{g_3}\left(\wp^{-1}(z; g_2, g_3), \tilde{\infty}\right) = \log$$

With respect to g_2

For fixed z, g_3, the function $\wp^{-1}(z; g_2, g_3)$ has two branch points: $g_2 = 4 z^2 - \frac{g_3}{z}$ $g_2 = \tilde{\infty}$.

09.22.04.0009.01

$$\mathcal{BP}_{g_2}\left(\wp^{-1}(z; g_2, g_3)\right) = \left\{4 z^2 - \frac{g_3}{z}, \tilde{\infty}\right\}$$

09.22.04.0010.01

$$\mathcal{R}_{g_2}\left(\wp^{-1}(z; g_2, g_3), 4 z^2 - \frac{g_3}{z}\right) = 2$$

09.22.04.0011.01

$$\mathcal{R}_{g_2}\left(\wp^{-1}(z; g_2, g_3), \tilde{\infty}\right) = \log$$

With respect to z

For fixed g_2, g_3, the function $\wp^{-1}(z; g_2, g_3)$ has four branch points: $z = \left(z; 4 z^3 - g_2 z - g_3\right)_k^{-1} /; k \in \{1, 2, 3\}$, $z = \tilde{\infty}$.

09.22.04.0012.01

$$\mathcal{BP}_z\left(\wp^{-1}(z; g_2, g_3)\right) = \left\{\left(z; 4 z^3 - g_2 z - g_3\right)_1^{-1}, \left(z; 4 z^3 - g_2 z - g_3\right)_2^{-1}, \left(z; 4 z^3 - g_2 z - g_3\right)_3^{-1}, \tilde{\infty}\right\}$$

09.22.04.0013.01

$$\mathcal{R}_z\left(\wp^{-1}(z; g_2, g_3), \left(z; 4 z^3 - g_2 z - g_3\right)_k^{-1}\right) = 2 /; k \in \{1, 2, 3\}$$

09.22.04.0014.01

$$\mathcal{R}_z\left(\wp^{-1}(z; g_2, g_3), \tilde{\infty}\right) = \log$$

Branch cuts

Branch cut locations: complicated

Series representations

Generalized power series

Expansions at generic point $z = z_0$

For the function itself

09.22.06.0002.01

$$\wp^{-1}(z; g_2, g_3) \propto \wp^{-1}(z_0; g_2, g_3) + \frac{1}{\wp'\left(\wp^{-1}(z_0; g_2, g_3); g_2, g_3\right)} (z - z_0) - \frac{12 z_0^2 - g_2}{4 \wp'\left(\wp^{-1}(z_0; g_2, g_3); g_2, g_3\right)^3} (z - z_0)^2 + \frac{\left(\left(g_2 - 12 z_0^2\right)^2 - 16 z_0 \wp'\left(\wp^{-1}(z_0; g_2, g_3); g_2, g_3\right)^2\right)}{8 \wp'\left(\wp^{-1}(z_0; g_2, g_3); g_2, g_3\right)^5} (z - z_0)^3 + \frac{-4 \wp'\left(\wp^{-1}(z_0; g_2, g_3); g_2, g_3\right)^4 + 18 \left(12 z_0^2 - g_2\right) z_0 \wp'\left(\wp^{-1}(z_0; g_2, g_3); g_2, g_3\right)^2 - 5 \left(6 z_0^2 - \frac{g_2}{2}\right)^3}{8 \wp'\left(\wp^{-1}(z_0; g_2, g_3); g_2, g_3\right)^7} (z - z_0)^4 + \left(35 \left(g_2 - 12 z_0^2\right)^4 - 1440 z_0 \wp'\left(\wp^{-1}(z_0; g_2, g_3); g_2, g_3\right)^2 \left(g_2 - 12 z_0^2\right)^2 + 384 \left(30 z_0^2 - g_2\right) \wp'\left(\wp^{-1}(z_0; g_2, g_3); g_2, g_3\right)^4\right) \Big/ \left(640 \wp'\left(\wp^{-1}(z_0; g_2, g_3); g_2, g_3\right)^9\right) (z - z_0)^5 + \ldots /; (z \to z_0)$$

09.22.06.0003.01

$$\wp^{-1}(z; g_2, g_3) \propto \wp^{-1}(z_0; g_2, g_3) + \frac{1}{\sqrt{4 z_0^3 - g_2 z_0 - g_3}} (z - z_0) - \frac{12 z^2 - g_2}{4 \left(4 z_0^3 - g_2 z_0 - g_3\right)^{3/2}} (z - z_0)^2 + \frac{g_2^2 - 8 z (6 z - 5 z_0) g_2 + 16 z \left(15 z^3 - 10 z_0^3 + g_3\right)}{8 \left(4 z_0^3 - g_2 z_0 - g_3\right)^{5/2}} (z - z_0)^3 - \frac{1}{64 \left(4 z_0^3 - g_2 z_0 - g_3\right)^{7/2}} \left(-5 g_2^3 + 4 \left(150 z^2 + 7 z_0 (5 z_0 - 24 z)\right) g_2^2 - 16 \left(525 z^4 - 168 z_0^3 z + 12 g_3 z + 70 z_0^4 - 7 \left(60 z^3 + g_3\right) z_0\right) g_2 + 32 \left(840 z^6 + 60 g_3 z^3 + 70 z_0^6 - 14 \left(60 z^3 + g_3\right) z_0^3 + g_3^2\right)\right) (z - z_0)^4 + \frac{1}{640 \left(4 z_0^3 - g_2 z_0 - g_3\right)^{9/2}} \left(35 g_2^4 - 48 \left(175 z^2 + 9 z_0 (7 z_0 - 25 z)\right) g_2^3 + 96 \left(252 z_0^4 - 450 z z_0^3 + 945 z^2 z_0^2 - 18 \left(175 z^3 + g_3\right) z_0 + 25 z \left(98 z^3 + g_3\right)\right) g_2^2 - 384 \left(4900 z^6 + 175 g_3 z^3 + 1890 z_0^4 z^2 - 135 \left(28 z^3 + g_3\right) z_0 z^2 + 126 z_0^6 - 18 \left(175 z^3 + g_3\right) z_0^3 + g_3^2\right) g_2 + 11520 z^2 \left(385 z^6 + 28 g_3 z^3 + 126 z_0^6 - 18 \left(28 z^3 + g_3\right) z_0^3 + g_3^2\right)\right) (z - z_0)^5 + \ldots \;/;\; (z \to z_0)$$

09.22.06.0004.01

$$\wp^{-1}(z; g_2, g_3) \propto \wp^{-1}(z_0; g_2, g_3) \left(1 + O(z - z_0)\right)$$

09.22.06.0005.01

$$\wp^{-1}(z; g_2, g_3) \propto \wp^{-1}(z_0; g_2, g_3) \left(1 + O(z - z_0)\right)$$

Expansions at $z == 0$

09.22.06.0001.01

$$\wp^{-1}(z; g_2, g_3) == \sqrt{\frac{1}{z}} + \frac{1}{40} g_2 \left(\frac{1}{z}\right)^{5/2} + \frac{1}{56} g_3 \left(\frac{1}{z}\right)^{7/2} + \frac{1}{384} g_2^2 \left(\frac{1}{z}\right)^{9/2} + \frac{3}{704} g_2 g_3 \left(\frac{1}{z}\right)^{11/2} + \frac{5 g_2^3 + 24 g_3^2}{13312} \left(\frac{1}{z}\right)^{13/2} + \frac{g_2^2 g_3}{1024} \left(\frac{1}{z}\right)^{15/2} + \frac{5 \left(7 g_2^4 + 96 g_3^2 g_2\right)}{557056} \left(\frac{1}{z}\right)^{17/2} + \frac{5 \left(7 g_3 g_2^3 + 8 g_3^3\right)}{155648} \left(\frac{1}{z}\right)^{19/2} + \frac{3 g_2^5 + 80 g_3^2 g_2^2}{262144} \left(\frac{1}{z}\right)^{21/2} + O\left(\left(\frac{1}{z}\right)^{23/2}\right)$$

Integral representations

On the real axis

Of the direct function

09.22.07.0001.01

$$\wp^{-1}(z; g_2, g_3) == \int_{\infty}^{z} \frac{1}{\sqrt{4 t^3 - g_2 t - g_3}} \, dt \;/;\; z \in \mathbb{R} \bigwedge \operatorname{Re}\left(4 z^3 - g_2 z - g_3\right) > 0$$

Differential equations

Ordinary nonlinear differential equations

09.22.13.0001.01

$$\left(4 z^3 - g_2 z - g_3\right) w'(z)^2 - 1 == 0 \;/;\; w(z) == \wp^{-1}(z; g_2, g_3)$$

Identities

Functional identities

09.22.17.0001.01

$$\wp^{-1}(z_1; g_2, g_3) + \wp^{-1}(z_2; g_2, g_3) == \wp^{-1}(z_3; g_2, g_3) \;/;$$
$$\sqrt{4 z_3^3 - g_2 z_3 - g_3}\,(z_2 - z_1) + \sqrt{4 z_1^3 - g_2 z_1 - g_3}\,(z_3 - z_2) + \sqrt{4 z_2^3 - g_2 z_2 - g_3}\,(z_1 - z_3) == 0$$

Differentiation

Low-order differentiation

09.22.20.0001.01

$$\frac{\partial \wp^{-1}(z; g_2, g_3)}{\partial z} == \frac{1}{\wp'\left(\wp^{-1}(z; g_2, g_3); g_2, g_3\right)}$$

09.22.20.0002.01

$$\frac{\partial \wp^{-1}(z; g_2, g_3)}{\partial z} == \frac{1}{\sqrt{4 z^3 - g_2 z - g_3}}$$

09.22.20.0003.01

$$\frac{\partial^2 \wp^{-1}(z; g_2, g_3)}{\partial z^2} == \frac{g_2 - 12 z^2}{2 \wp'\left(\wp^{-1}(z; g_2, g_3); g_2, g_3\right)^3}$$

Symbolic differentiation

09.22.20.0004.01

$$\frac{\partial^n \wp^{-1}(z; g_2, g_3)}{\partial z^n} == \frac{\delta_{n-1}}{\sqrt{4 z^3 - g_2 z - g_3}} + \wp^{-1}(z; g_2, g_3)\, \delta_n + \sum_{m=1}^{n-1} \frac{1}{m!} \left(\frac{1}{2} - m\right)_m \sum_{j=0}^{m-1} (-1)^j \binom{m}{j} \left(4 z^3 - g_2 z - g_3\right)^{j-m-\frac{1}{2}}$$
$$\sum_{k_1=0}^{m-j} \sum_{k_2=0}^{m-j} \sum_{k_3=0}^{m-j} (-1)^{n+k_2+k_3-1}\, \delta_{m-j, k_1+k_2+k_3}\, (k_1 + k_2 + k_3; k_1, k_2, k_3)\, 4^{k_1} g_2^{k_2} g_3^{k_3} (-3 k_1 - k_2)_{n-1}\, z^{-n+3 k_1+k_2+1} \;/;\; n \in \mathbb{N}$$

Representations through more general functions

Through other functions

Involving some hypergeometric-type functions

09.22.26.0001.01

$$\wp^{-1}(z; g_2, g_3) == -\frac{1}{\sqrt{z - r_1}} F_1\left(\frac{1}{2}; \frac{1}{2}, \frac{1}{2}; \frac{3}{2}; \frac{r_2 - r_1}{z - r_1}, \frac{r_3 - r_1}{z - r_1}\right) /;$$
$$4 z^3 - g_2 z - g_3 == 4 (z - r_1)(z - r_2)(z - r_3) \bigwedge \mathrm{Re}\left(4 z^3 - g_2 z - g_3\right) > 0$$

Representations through equivalent functions

With inverse function

09.22.27.0001.01

$\wp(\wp^{-1}(z; g_2, g_3); g_2, g_3) = z$

History

– L. Euler (1761)

– J.-L. Lagrange (1769)

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