HermiteH

Notations

Traditional name

Hermite polynomial

Traditional notation

$H_n(z)$

Mathematica StandardForm notation

HermiteH[n, z]

Primary definition

05.01.02.0001.01

$$H_n(z) = n! \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{(-1)^k (2z)^{n-2k}}{k! (n-2k)!} \,/;\, n \in \mathbb{N}$$

Specific values

Specialized values

For fixed n

05.01.03.0001.01

$$H_n(0) = \frac{2^n \sqrt{\pi}}{\Gamma\left(\frac{1-n}{2}\right)}$$

For fixed z

05.01.03.0002.01

$$H_0(z) = 1$$

05.01.03.0003.01

$$H_1(z) = 2z$$

05.01.03.0004.01

$$H_2(z) = -2 + 4z^2$$

05.01.03.0005.01

$$H_3(z) = -12z + 8z^3$$

05.01.03.0006.01

$$H_4(z) = 12 - 48 z^2 + 16 z^4$$

05.01.03.0007.01

$$H_5(z) = 120 z - 160 z^3 + 32 z^5$$

05.01.03.0008.01

$$H_6(z) = -120 + 720 z^2 - 480 z^4 + 64 z^6$$

05.01.03.0009.01

$$H_7(z) = -1680 z + 3360 z^3 - 1344 z^5 + 128 z^7$$

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$$H_8(z) = 1680 - 13\,440 z^2 + 13\,440 z^4 - 3584 z^6 + 256 z^8$$

05.01.03.0011.01

$$H_9(z) = 30\,240 z - 80\,640 z^3 + 48\,384 z^5 - 9216 z^7 + 512 z^9$$

05.01.03.0012.01

$$H_{10}(z) = -30\,240 + 302\,400 z^2 - 403\,200 z^4 + 161\,280 z^6 - 23\,040 z^8 + 1024 z^{10}$$

Values at infinities

05.01.03.0013.01

$$H_n(\infty) = \infty \;/;\; n > 0$$

05.01.03.0014.01

$$H_n(-\infty) = (-1)^n \infty \;/;\; n > 0$$

General characteristics

Domain and analyticity

The function $H_n(z)$is defined over $\mathbb{N} \otimes \mathbb{C}$. For fixed n, the function $H_n(z)$ is a polynomial in z of degree n.

05.01.04.0001.01

$$(n * z) \longrightarrow H_n(z) :: (\mathbb{N} \otimes \mathbb{C}) \longrightarrow \mathbb{C}$$

Symmetries and periodicities

Parity

05.01.04.0002.01

$$H_n(-z) = (-1)^n H_n(z)$$

Mirror symmetry

05.01.04.0003.01

$$H_n(\bar{z}) = \overline{H_n(z)}$$

Periodicity

No periodicity

Poles and essential singularities

With respect to z

The function $H_n(z)$ is polynomial and has pole of order n at $z = \tilde{\infty}$.

05.01.04.0004.01

$$\mathcal{Sing}_z(H_n(z)) = \{\{\tilde{\infty}, n\}\}$$

Branch points

With respect to z

The function $H_n(z)$ does not have branch points.

05.01.04.0005.01

$$\mathcal{BP}_z(H_n(z)) = \{\}$$

Branch cuts

With respect to z

The function $H_n(z)$ does not have branch cuts.

05.01.04.0006.01

$$\mathcal{BC}_z(H_n(z)) = \{\}$$

Series representations

Generalized power series

Expansions at generic point $z = z_0$

For the function itself

05.01.06.0009.01

$$H_n(z) = H_n(z_0) + 2\,n\,H_{n-1}(z_0)\,(z - z_0) + (n-1)\,n\,H_{n-2}(z_0)\,(z - z_0)^2 + \ldots \,/;\, (z \to z_0)$$

05.01.06.0010.01

$$H_n(z) = H_n(z_0) + 2\,n\,H_{n-1}(z_0)\,(z - z_0) + (n-1)\,n\,H_{n-2}(z_0)\,(z - z_0)^2 + O\left((z - z_0)^3\right)$$

05.01.06.0011.01

$$H_n(z) = \sum_{k=0}^{\infty} 2^k \binom{n}{k} H_{n-k}(z_0)\,(z - z_0)^k$$

05.01.06.0012.01

$$H_n(z) = \sum_{k=0}^{n} \frac{1}{k!} \left(\frac{2^{k+n}\,\pi\,z_0^{-k}}{\Gamma\left(\frac{1-n}{2}\right)}\, {}_2\tilde{F}_2\left(1, -\frac{n}{2}; \frac{1-k}{2}, 1 - \frac{k}{2}; z_0^2\right) - \frac{2^{k+n}\,\pi\,z_0^{1-k}}{\Gamma\left(-\frac{n}{2}\right)}\, {}_2\tilde{F}_2\left(1, \frac{1-n}{2}; 1 - \frac{k}{2}, \frac{3-k}{2}; z_0^2\right) \right) (z - z_0)^k$$

05.01.06.0013.01

$$H_n(z) \propto H_n(z_0)\,(1 + O(z - z_0))$$

Expansions at $z = 0$

For the function itself

05.01.06.0001.02

$$H_n(z) \propto \frac{2^n \sqrt{\pi}}{\Gamma\left(\frac{1-n}{2}\right)} \left(1 - n z^2 - \frac{n(2-n)}{6} z^4 + \ldots\right) - \frac{2^{n+1} \sqrt{\pi}}{\Gamma\left(-\frac{n}{2}\right)} z \left(1 + \frac{1-n}{3} z^2 + \frac{(1-n)(3-n)}{30} z^4 + \ldots\right) /; (z \to 0)$$

05.01.06.0014.01

$$H_n(z) \propto \frac{2^n \sqrt{\pi}}{\Gamma\left(\frac{1-n}{2}\right)} \left(1 - n z^2 - \frac{n(2-n)}{6} z^4 + O\left(z^6\right)\right) - \frac{2^{n+1} \sqrt{\pi}}{\Gamma\left(-\frac{n}{2}\right)} z \left(1 + \frac{1-n}{3} z^2 + \frac{(1-n)(3-n)}{30} z^4 + O\left(z^6\right)\right)$$

05.01.06.0002.01

$$H_n(z) = n! \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{(-1)^k (2z)^{n-2k}}{k! (n-2k)!}$$

05.01.06.0003.01

$$H_n(z) = 2^n \sqrt{\pi} \left(\frac{1}{\Gamma\left(\frac{1-n}{2}\right)} {}_1F_1\left(-\frac{n}{2}; \frac{1}{2}; z^2\right) - \frac{2z}{\Gamma\left(-\frac{n}{2}\right)} {}_1F_1\left(\frac{1-n}{2}; \frac{3}{2}; z^2\right) \right)$$

05.01.06.0004.02

$$H_n(z) \propto \frac{2^n \sqrt{\pi}}{\Gamma\left(\frac{1-n}{2}\right)} \left(1 + O\left(z^2\right)\right) - \frac{2^{n+1} \sqrt{\pi}\, z}{\Gamma\left(-\frac{n}{2}\right)} \left(1 + O\left(z^2\right)\right)$$

05.01.06.0015.01

$$H_n(z) \propto \frac{(-1)^{\left\lfloor \frac{n}{2} \right\rfloor} n!\, 2^{n-2\left\lfloor \frac{n}{2} \right\rfloor} z^{n-2\left\lfloor \frac{n}{2} \right\rfloor}}{\left(n - 2\left\lfloor \frac{n}{2} \right\rfloor\right)! \left\lfloor \frac{n}{2} \right\rfloor!} \left(1 + O\left(z^2\right)\right) /; n \in \mathbb{N}$$

Expansions at $z = \infty$

For the function itself

05.01.06.0005.01

$$H_n(z) = 2^n z^n \left(1 - \frac{(n-1)n}{4z^2} + \frac{(n-3)(n-2)(n-1)n}{32 z^4} - \ldots\right) /; (|z| \to \infty)$$

05.01.06.0016.01

$$H_n(z) \propto 2^n z^n \left(1 - \frac{(n-1)n}{4z^2} + \frac{(n-3)(n-2)(n-1)n}{32 z^4} - O\left(\frac{1}{z^6}\right)\right) /; (|z| \to \infty)$$

05.01.06.0006.01

$$H_n(z) = 2^n z^n \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{(-1)^k \left(-\frac{n}{2}\right)_k \left(\frac{1-n}{2}\right)_k z^{-2k}}{k!}$$

05.01.06.0007.01

$$H_n(z) \propto 2^n z^n {}_2F_0\left(-\frac{n}{2}, \frac{1-n}{2};; -\frac{1}{z^2}\right)$$

05.01.06.0008.02

$$H_n(z) \propto 2^n z^n \left(1 + O\left(\frac{1}{z^2}\right)\right)$$

Expansions at $n == \infty$

05.01.06.0017.01

$$H_n(z) \propto \frac{(-1)^{\lfloor \frac{n}{2} \rfloor} 2^n}{\sqrt{\pi}} e^{\frac{z^2}{2}} \left\lfloor \frac{n}{2} \right\rfloor ! \left\lfloor \frac{n}{2} \right\rfloor^{-\frac{1+(-1)^n}{4}} \left(\cos\left(\frac{\pi (1-(-1)^n)}{4} - \sqrt{2n+1}\, z\right) - \frac{z^3}{4\sqrt{2}\sqrt{\lfloor \frac{n}{2} \rfloor}} \left(\cos\left(\sqrt{2n+1}\, z + \frac{(-1)^n \pi}{4}\right) - \sin\left(\sqrt{2n+1}\, z + \frac{(-1)^n \pi}{4}\right)\right) + \frac{32 z^2 \left(-2z^2 + (-1)^n (z^2-1) - 1\right)}{512 z^2 \lfloor \frac{n}{2} \rfloor} \cos\left(\frac{\pi (1-(-1)^n)}{4} - \sqrt{2n+1}\, z\right) + \frac{z \left(-4z^2 + 2(-1)^n (z^2+2) - 5\right)}{32\sqrt{2} \lfloor \frac{n}{2} \rfloor^{3/2}} \left(\cos\left(\sqrt{2n+1}\, z + \frac{(-1)^n \pi}{4}\right) - \sin\left(\sqrt{2n+1}\, z + \frac{(-1)^n \pi}{4}\right)\right) + \ldots \right) /; (n \to \infty)$$

05.01.06.0018.01

$$H_n(z) \propto n^{n/2} e^{\frac{1}{2} n (\log(2)-1)} \sqrt{2}\, e^{\frac{z^2}{2}} \left(\left(\frac{z^4-2}{24n} + 1\right) \cos\left(\frac{n\pi}{2} - \sqrt{2}\sqrt{n}\, z\right) - \frac{z(z^2-3)}{6\sqrt{2}\sqrt{n}} \sin\left(\frac{n\pi}{2} - \sqrt{2}\sqrt{n}\, z\right)\right)(1 + \ldots) /; (n \to \infty)$$

05.01.06.0019.01

$$H_n(z) \propto \frac{\Gamma(n+1)}{\Gamma\left(\frac{n}{2}+1\right)} e^{\frac{z^2}{2}} \left(\cos\left(z\sqrt{2n+1} - \frac{\pi n}{2}\right) + \frac{z^3}{6\sqrt{2n+1}} \sin\left(z\sqrt{2n+1} - \frac{\pi n}{2}\right)\right)(1 + \ldots) /; (n \to \infty)$$

05.01.06.0020.01

$$H_n(z) \propto \frac{(-1)^{\lfloor \frac{n}{2} \rfloor} 2^n}{\sqrt{\pi}} e^{\frac{z^2}{2}} \left\lfloor \frac{n}{2} \right\rfloor ! \left\lfloor \frac{n}{2} \right\rfloor^{-\frac{1+(-1)^n}{4}} \left(\cos\left(\frac{1}{4}(1-(-1)^n)\pi - 2z\sqrt{\frac{1}{4}(2n+1)}\right) + \sum_{k=1}^{\infty} \left\lfloor \frac{n}{2} \right\rfloor^{-k} \left(\sum_{j=0}^{k} \sum_{r=0}^{k-j} \sum_{s=0}^{k-j-r} \frac{(-1)^{j+r+s} 2^{2j-2k+s}}{j!\, s! \left(\frac{1}{2}\right)_r} \left(1 - \frac{(-1)^n}{2}\right)^s z^{2(k-j-2r-s)} A_{2(k-j-r-s)} \cos\left(\pi\left(\frac{1}{4}((-1)^n - 1) + j - k + r + s\right) + 2\sqrt{\frac{1}{4}(2n+1)}\, z\right) B_j^{\left(1-\frac{(-1)^n}{2}\right)}\left(1 - \frac{(-1)^n}{2}\right) \left(\frac{(-1)^n}{2}\right)_j \left(\frac{1}{4}(1-(-1)^n) - j + k - s\right)_s \left(\frac{1}{4}(1-(-1)^n) - j + k - r - s\right)_r \left(\frac{1}{4}(3-(-1)^n) - j + k - r - s\right)_r \left(\frac{1}{4}(1+(-1)^n) + j - k + r + s\right)_r \left(\frac{1}{4}(3+(-1)^n) + j - k + r + s\right)_r - \frac{2}{z^2} \sum_{j=0}^{k-1} \sum_{r=0}^{k-j-1} \sum_{s=0}^{k-j-r-1} \frac{(-1)^{j+r+s} 2^{2j-2k+s}}{j!\, s! \left(\frac{3}{2}\right)_r} \left(1 - \frac{(-1)^n}{2}\right)^s z^{2(k-j-2r-s)} A_{2(k-j-r-s)-1} B_j^{\left(1-\frac{(-1)^n}{2}\right)}\left(1 - \frac{(-1)^n}{2}\right) \left(\frac{(-1)^n}{2}\right)_j \left(\frac{1}{4}(1-(-1)^n) - j + k - s\right)_s \left(\frac{1}{4}(1-(-1)^n) - j + k - r - s\right)_r$$

$$\left(\frac{1}{4}(3-(-1)^n)-j+k-r-s\right)_r \left(\frac{1}{4}((-1)^n+1)+j-k+r+s\right)_{r+1}$$

$$\left(\frac{1}{4}(3+(-1)^n)+j-k+r+s\right)_{r+1} \sin\left(\pi\left(\frac{1}{4}((-1)^n+1)+j-k+r+s\right)+2\sqrt{\frac{1}{4}(2n+1)}\ z\right)\Bigg)+$$

$$\frac{z}{2\sqrt{\left\lfloor\frac{n}{2}\right\rfloor}} \sum_{k=0}^{\infty}\sum_{j=0}^{k}\sum_{r=0}^{k-j}\sum_{s=0}^{k-j-r} \frac{(-1)^{j+r+s}\, 2^{2j-2k+s}}{j!\, s!\left(\frac{3}{2}\right)_r} \left(1-\frac{(-1)^n}{2}\right)^s z^{2(k-j-2r-s)} \left\lfloor\frac{n}{2}\right\rfloor^{-k} B_j^{\left(1-\frac{(-1)^n}{2}\right)}\left(1-\frac{(-1)^n}{2}\right)$$

$$\left(\frac{(-1)^n}{2}\right)_j \left(\frac{1}{4}(3-(-1)^n)-j+k-s\right)_s \left(\frac{1}{4}(3-(-1)^n)-j+k-r-s\right)_r$$

$$\left(\frac{1}{4}(5-(-1)^n)-j+k-r-s\right)_r \left(\frac{1}{4}((-1)^n-1)+j-k+r+s\right)_r \left(\frac{1}{4}((-1)^n+1)+j-k+r+s\right)_r$$

$$\Bigg((2r+1)\, A_{2(-j+k-r-s)+1} \cos\left(\pi\left(\frac{1}{4}((-1)^n-3)+j-k+r+s\right)+2\sqrt{\frac{1}{4}(2n+1)}\ z\right)-$$

$$\frac{(4j+(-1)^n-4k+8r+4s-1)(4j+(-1)^n-4k+8r+4s+1)}{8z^2}$$

$$A_{2(-j+k-r-s)} \sin\left(\pi\left(\frac{1}{4}((-1)^n-1)+j-k+r+s\right)+2\sqrt{\frac{1}{4}(2n+1)}\ z\right)\Bigg)\Bigg)\Bigg)\ /;$$

$$(n\to\infty) \bigwedge A_0=1 \bigwedge A_1=0 \bigwedge A_2=\frac{(-1)^{n-1}+2}{4} \bigwedge$$

$$A_m = \frac{2m+(-1)^{n-1}-2}{2m} A_{m-2} - \frac{1}{2}\left(4\left\lfloor\frac{n}{2}\right\rfloor+(-1)^{n-1}+2\right) A_{m-3} \bigwedge m\in\mathbb{N}^+$$

05.01.06.0021.01

$$H_n(z) \propto (-1)^{\left\lfloor\frac{n}{2}\right\rfloor}\left(\left\lfloor\frac{n}{2}\right\rfloor+\frac{(-1)^{n-1}}{2}\right)!\, e^{\frac{z^2}{2}}\, 2^n z^{n-2\left\lfloor\frac{n}{2}\right\rfloor} \sum_{k=0}^{\infty} A_k\, 2^{-k} z^{2k}\, {}_0\tilde{F}_1\left(;k+\frac{(-1)^{n-1}}{2}+1;-z^2\left(\frac{(-1)^{n-1}+2}{4}+\left\lfloor\frac{n}{2}\right\rfloor\right)\right) /; (n\to\infty) \bigwedge$$

$$A_0=1 \bigwedge A_1=0 \bigwedge A_2=\frac{(-1)^{n-1}+2}{4} \bigwedge A_m=\frac{2m+(-1)^{n-1}-2}{2m} A_{m-2}-\left(2\left\lfloor\frac{n}{2}\right\rfloor+\frac{(-1)^{n-1}}{2}+1\right) A_{m-3} \bigwedge m\in\mathbb{N}^+$$

05.01.06.0022.01

$$H_n(z) \propto n^{n/2}\, e^{\frac{1}{2}n(\log(2)-1)} \sqrt{2}\, e^{\frac{z^2}{2}} \cos\left(\frac{n\pi}{2}-\sqrt{2}\sqrt{n}\ z\right)(1+\ldots) /; (n\to\infty)$$

Integral representations

On the real axis

Of the direct function

05.01.07.0001.01

$$H_n(z) == \frac{2^{n+1}}{\sqrt{\pi}} e^{z^2} \int_0^\infty e^{-t^2} t^n \cos\left(2 z t - \frac{\pi n}{2}\right) dt$$

05.01.07.0002.01

$$H_n(z) == \frac{e^{3 \pi i n/2} 2^n}{\sqrt{\pi}} \int_{-\infty}^\infty e^{-(t - i z)^2} t^n \, dt$$

05.01.07.0004.01

$$H_n(z) == \frac{2^n}{\sqrt{\pi}} \int_{-\infty}^\infty e^{-t^2} (z + i t)^n \, dt$$

Integral representations of negative integer order

Rodrigues-type formula.

05.01.07.0003.01

$$H_n(z) == (-1)^n e^{z^2} \frac{\partial^n e^{-z^2}}{\partial z^n}$$

Limit representations

05.01.09.0001.01

$$H_n(z) == \lim_{\lambda \to \infty} 2^{n/2} \Gamma(n+1) \lambda^{-\frac{n}{2}} L_n^\lambda\left(\lambda - \sqrt{2 \lambda} \; z\right)$$

05.01.09.0002.01

$$H_n(z) == \Gamma(n+1) \lim_{\lambda \to \infty} \lambda^{-\frac{n}{2}} C_n^\lambda\left(\frac{z}{\sqrt{\lambda}}\right) /; |z| < 1$$

05.01.09.0003.01

$$H_n(z) == 2^n \Gamma(n+1) \lim_{a \to \infty} a^{-\frac{n}{2}} P_n^{(a,a)}\left(\frac{z}{\sqrt{a}}\right)$$

Generating functions

05.01.11.0001.01

$$H_n(z) == n! \left([t^n] \, e^{2 z t - t^2}\right)$$

05.01.11.0002.01

$$H_n(z) == n! \left([t^n] \, \frac{e^{\frac{4 z^2 t}{1 + 4 t}}}{\sqrt{1 + 4 t}}\right) /; 2 n \in \mathbb{N}$$

05.01.11.0003.01

$$H_n(z) == [t^n] \left(\left\lfloor \frac{n}{2} \right\rfloor! \left(4 t^2 + 1\right)^{-3/2} \left(4 t^2 + 2 z t + 1\right) e^{\frac{4 z^2 t^2}{4 t^2 + 1}}\right)$$

05.01.11.0004.01

$$H_n(z) = [t^n]\left(\frac{1}{(c)_{\left\lfloor \frac{n}{2}\right\rfloor}}\left(\left(\left\lfloor \frac{n}{2}\right\rfloor!\left(\frac{1}{2}\right)_{\left\lfloor \frac{n}{2}\right\rfloor}\right)\left({}_1F_1\left(c;\frac{1}{2};\frac{4z^2t^2}{4t^2+1}\right)\left(4t^2+1\right)^{-c}+\frac{32ct^3z^3}{3\left(4t^2+1\right)^{c+2}}\,{}_1F_1\left(c+1;\frac{5}{2};\frac{4z^2t^2}{4t^2+1}\right)+\frac{2zt\left(-8ct^2+4t^2+1\right)}{\left(4t^2+1\right)^{c+1}}\,{}_1F_1\left(c;\frac{3}{2};\frac{4z^2t^2}{4t^2+1}\right)\right)\right)\right)$$

Differential equations

Ordinary linear differential equations and wronskians

For the direct function itself

05.01.13.0005.01

$$w''(z) - 2zw'(z) + 2nw(z) = 0\ /;\ w(z) = c_1 H_n(z) + c_2 e^{z^2} H_{-n-1}(iz)$$

05.01.13.0006.01

$$W_z\left(H_n(z),\ e^{z^2} H_{-n-1}(iz)\right) = -i\,e^{z^2-\frac{i\pi n}{2}}$$

05.01.13.0007.01

$$w''(z) - 2zw'(z) + 2nw(z) = 0\ /;\ w(z) = c_1 H_n(z) + c_2\left(z\,{}_1F_1\left(\frac{1-n}{2};\frac{3}{2};z^2\right)+{}_1F_1\left(-\frac{n}{2};\frac{1}{2};z^2\right)\right)$$

05.01.13.0008.01

$$W_z\left(H_n(z),\ z\,{}_1F_1\left(\frac{1-n}{2};\frac{3}{2};z^2\right)+{}_1F_1\left(-\frac{n}{2};\frac{1}{2};z^2\right)\right) = \frac{2^{n+1}e^{z^2}\sqrt{\pi}}{\Gamma\left(-\frac{n}{2}\right)}+\frac{2^n e^{z^2}\sqrt{\pi}}{\Gamma\left(\frac{1-n}{2}\right)}$$

05.01.13.0001.01

$$w''(z) - 2zw'(z) + 2nw(z) = 0\ /;\ w(z) = c_1 H_n(z) + c_2\,{}_1F_1\left(-\frac{n}{2};\frac{1}{2};z^2\right)$$

05.01.13.0002.02

$$W_z\left(H_n(z),\ {}_1F_1\left(-\frac{n}{2};\frac{1}{2};z^2\right)\right) = \frac{2^{n+1}e^{z^2}\sqrt{\pi}}{\Gamma\left(-\frac{n}{2}\right)}$$

05.01.13.0003.01

$$w''(z) - 2zw'(z) + 2nw(z) = 0\ /;\ w(z) = c_1 H_n(z) + c_2\,z\,{}_1F_1\left(\frac{1-n}{2};\frac{3}{2};z^2\right)$$

05.01.13.0004.02

$$W_z\left(H_n(z),\ z\,{}_1F_1\left(\frac{1-n}{2};\frac{3}{2};z^2\right)\right) = \frac{2^n e^{z^2}\sqrt{\pi}}{\Gamma\left(\frac{1-n}{2}\right)}$$

05.01.13.0009.01

$$w''(z) - \left(2g(z)g'(z) + \frac{g''(z)}{g'(z)}\right)w'(z) + 2n\,g'(z)^2\,w(z) = 0\ /;\ w(z) = c_1 H_n(g(z)) + c_2 e^{g(z)^2} H_{-n-1}(i\,g(z))$$

05.01.13.0010.01

$$W_z\left(H_n(g(z)),\ e^{g(z)^2} H_{-n-1}(i\,g(z))\right) = -i\,e^{-\frac{1}{2}i\pi n}\,e^{g(z)^2}\,g'(z)$$

05.01.13.0011.01

$$w''(z) - \left(2\,g(z)\,g'(z) + \frac{2\,h'(z)}{h(z)} + \frac{g''(z)}{g'(z)}\right) w'(z) + \left(2\,n\,g'(z)^2 + \frac{2\,g(z)\,h'(z)\,g'(z)}{h(z)} + \frac{h'(z)\,g''(z)}{h(z)\,g'(z)} + \frac{2\,h'(z)^2 - h(z)\,h''(z)}{h(z)^2}\right) w(z) = 0 \,/;\; w(z) = c_1\,h(z)\,H_n(g(z)) + c_2\,h(z)\,e^{g(z)^2}\,H_{-n-1}(i\,g(z))$$

05.01.13.0012.01

$$W_z\left(h(z)\,H_n(g(z)),\, h(z)\,e^{g(z)^2}\,H_{-n-1}(i\,g(z))\right) = -i\,e^{-\frac{1}{2}\,i\,\pi\,n}\,e^{g(z)^2}\,h(z)^2\,g'(z)$$

05.01.13.0013.01

$$z^2\,w''(z) - \left(2\,a^2\,r\,z^{2\,r} + r + 2\,s - 1\right) z\,w'(z) + \left(2\,a^2\,r\,(s + r\,n)\,z^{2\,r} + s\,(r+s)\right) w(z) = 0 \,/;\; w(z) = c_1\,z^s\,H_n(a\,z^r) + c_2\,z^s\,e^{a^2\,z^{2\,r}}\,H_{-n-1}(i\,a\,z^r)$$

05.01.13.0014.01

$$W_z\left(z^s\,H_n(a\,z^r),\, z^s\,e^{a^2\,z^{2\,r}}\,H_{-n-1}(i\,a\,z^r)\right) = -i\,a\,e^{a^2\,z^{2\,r} - \frac{i\,\pi\,n}{2}}\,r\,z^{r+2\,s-1}$$

05.01.13.0015.01

$$w''(z) + \left(-\left(2\,a^2\,r^{2\,z} + 1\right)\log(r) - 2\log(s)\right) w'(z) + \left(2\,a^2\log(r)\,(n\log(r) + \log(s))\,r^{2\,z} + \log(s)\,(\log(r) + \log(s))\right) w(z) = 0 \,/;\; w(z) = c_1\,s^z\,H_n(a\,r^z) + c_2\,s^z\,e^{a^2\,r^{2\,z}}\,H_{-n-1}(i\,a\,r^z)$$

05.01.13.0016.01

$$W_z\left(s^z\,H_n(a\,r^z),\, s^z\,e^{a^2\,r^{2\,z}}\,H_{-n-1}(i\,a\,r^z)\right) = -i\,a\,e^{a^2\,r^{2\,z} - \frac{i\,\pi\,n}{2}}\,r^z\,s^{2\,z}\log(r)$$

Transformations

Transformations and argument simplifications

Argument involving basic arithmetic operations

05.01.16.0001.01

$$H_n(-z) = (-1)^n\,H_n(z)$$

Addition formulas

05.01.16.0002.01

$$H_n(z_1 + z_2) = 2^{-\frac{n}{2}} \sum_{k=0}^{n} \binom{n}{k} H_k\left(z_1\,\sqrt{2}\right) H_{n-k}\left(z_2\,\sqrt{2}\right)$$

05.01.16.0003.01

$$H_n(\cos(\alpha)\,z_1 + \sin(\alpha)\,z_2) = n! \sum_{k=0}^{n} \frac{\cos^k(\alpha)\,\sin^{n-k}(\alpha)}{k!\,(n-k)!}\,H_k(z_1)\,H_{n-k}(z_2)$$

Multiple arguments

05.01.16.0004.01

$$H_n(z_1\,z_2) = z_1^n \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{n!}{k!\,(n-2\,k)!} \left(1 - \frac{1}{z_1^2}\right)^k H_{n-2\,k}(z_2)$$

Products, sums, and powers of the direct function

Products of the direct function

05.01.16.0005.01

$$H_n(z)\, H_m(z) = n!\, m! \sum_{k=0}^{\min(n,m)} \frac{2^k\, H_{-2k+m+n}(z)}{k!\, (n-k)!\, (m-k)!}$$

05.01.16.0006.01

$$H_n(z)\, H_m(z) = \sum_{k=0}^{\min(n,m)} \binom{m}{k} \binom{n}{k} H_{-2k+m+n}(z)\, 2^k\, k!$$

Identities

Recurrence identities

Consecutive neighbors

05.01.17.0001.01

$$H_n(z) = \frac{2z}{2(n+1)} H_{n+1}(z) - \frac{1}{2(n+1)} H_{n+2}(z)$$

05.01.17.0002.01

$$H_n(z) = 2z\, H_{n-1}(z) - 2(n-1)\, H_{n-2}(z)$$

Distant neighbors

05.01.17.0006.01

$$H_n(z) = C_m(n, z)\, H_{m+n}(z) - \frac{1}{2(m+n)} C_m(n, z)\, H_{m+n+1}(z) \;/;$$
$$C_0(n, z) = 1 \bigwedge C_1(n, z) = \frac{z}{n+1} \bigwedge C_m(n, z) = \frac{z}{m+n} C_{m-1}(n, z) - \frac{1}{2(m+n-1)} C_{m-2}(n, z) \bigwedge m \in \mathbb{N}^+$$

05.01.17.0007.01

$$H_n(z) = C_m(n, z)\, H_{n-m}(z) - 2(n-m)\, C_m(n, z)\, H_{n-m-1}(z) \;/;$$
$$C_0(n, z) = 1 \wedge C_1(n, z) = 2z \wedge C_m(n, z) = 2z\, C_{m-1}(n, z) - 2(n-m+1)\, C_{m-2}(n, z) \wedge m \in \mathbb{N}^+$$

Functional identities

Relations between contiguous functions

Recurrence relations

05.01.17.0003.01

$$2n\, H_{n-1}(z) + H_{n+1}(z) = 2z\, H_n(z)$$

05.01.17.0004.01

$$H_n(z) = \frac{2n\, H_{n-1}(z) + H_{n+1}(z)}{2z}$$

Normalized recurrence relation

05.01.17.0005.01

$$z\, p(n, z) \equiv p(n+1, z) + \frac{1}{2}\, n\, p(n-1, z) \;/;\; p(n, z) \equiv 2^{-n} H_n(z)$$

Complex characteristics

Real part

05.01.19.0001.01

$$\operatorname{Re}(H_n(x + i\, y)) \equiv \sum_{j=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{(-1)^j\, 2^{2j}\, (-n)_{2j}\, y^{2j}}{(2\, j)!}\, H_{n-2j}(x) \;/;\; x \in \mathbb{R} \bigwedge y \in \mathbb{R}$$

Imaginary part

05.01.19.0002.01

$$\operatorname{Im}(H_n(x + i\, y)) \equiv \sum_{j=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \frac{(-1)^{j-1}\, 2^{2j+1}\, (-n)_{2j+1}\, y^{2j+1}}{(2\, j+1)!}\, H_{n-2j-1}(x) \;/;\; x \in \mathbb{R} \bigwedge y \in \mathbb{R}$$

Argument

05.01.19.0003.01

$$\arg(H_n(x + i\, y)) = \tan^{-1}\left(\sum_{j=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{(-1)^j\, 2^{2j}\, (-n)_{2j}}{(2\, j)!}\, H_{n-2j}(x)\, y^{2j},\; \sum_{j=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \frac{(-1)^{j-1}\, 2^{2j+1}\, (-n)_{2j+1}}{(2\, j+1)!}\, H_{-2j+n-1}(x)\, y^{2j+1} \right) /;\; x \in \mathbb{R} \bigwedge y \in \mathbb{R}$$

Conjugate value

05.01.19.0004.01

$$\overline{H_n(x + i\, y)} \equiv \sum_{j=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{(-1)^j\, 2^{2j}\, (-n)_{2j}}{(2\, j)!}\, H_{n-2j}(x)\, y^{2j} - i \sum_{j=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \frac{(-1)^{j-1}\, 2^{2j+1}\, (-n)_{2j+1}}{(2\, j+1)!}\, H_{-2j+n-1}(x)\, y^{2j+1} \;/;\; x \in \mathbb{R} \bigwedge y \in \mathbb{R}$$

Differentiation

Low-order differentiation

With respect to *z*

Forward shift operator:

05.01.20.0001.01

$$\frac{\partial H_n(z)}{\partial z} \equiv 2\, n\, H_{n-1}(z)$$

05.01.20.0002.01

$$\frac{\partial^2 H_n(z)}{\partial z^2} \equiv 4\, (n-1)\, n\, H_{n-2}(z)$$

Backward shift operator:

05.01.20.0003.01

$$\frac{\partial H_n(z)}{\partial z} - 2\,z\,H_n(z) == -H_{n+1}(z)$$

05.01.20.0004.01

$$\frac{\partial\left(e^{-z^2}\,H_n(z)\right)}{\partial z} == -e^{-z^2}\,H_{n+1}(z)$$

Symbolic differentiation

With respect to z

05.01.20.0005.02

$$\frac{\partial^m H_n(z)}{\partial z^m} == \frac{2^m\,n!}{(n-m)!}\,H_{n-m}(z)\;/;\;m \in \mathbb{N}$$

05.01.20.0006.02

$$\frac{\partial^m H_n(z)}{\partial z^m} == \frac{2^{m+n}\,\pi\,z^{-m}}{\Gamma\left(\frac{1-n}{2}\right)}\,{}_2\tilde{F}_2\left(1, -\frac{n}{2}; \frac{1-m}{2}, 1-\frac{m}{2}; z^2\right) - \frac{2^{m+n}\,\pi\,z^{1-m}}{\Gamma\left(-\frac{n}{2}\right)}\,{}_2\tilde{F}_2\left(1, \frac{1-n}{2}; 1-\frac{m}{2}, \frac{3-m}{2}; z^2\right)\;/;\;m \in \mathbb{N}$$

Fractional integro-differentiation

With respect to z

05.01.20.0007.01

$$\frac{\partial^\alpha H_n(z)}{\partial z^\alpha} == \frac{2^{n+\alpha}\,\pi\,z^{-\alpha}}{\Gamma\left(\frac{1-n}{2}\right)}\,{}_2\tilde{F}_2\left(1, -\frac{n}{2}; \frac{1-\alpha}{2}, 1-\frac{\alpha}{2}; z^2\right) - \frac{2^{n+\alpha}\,\pi\,z^{1-\alpha}}{\Gamma\left(-\frac{n}{2}\right)}\,{}_2\tilde{F}_2\left(1, \frac{1-n}{2}; 1-\frac{\alpha}{2}, \frac{3-\alpha}{2}; z^2\right)$$

Integration

Indefinite integration

Involving only one direct function

05.01.21.0001.01

$$\int H_n(z)\,dz == \frac{1}{2\,(n+1)}\,H_{n+1}(z)$$

05.01.21.0002.01

$$\int H_n(z)\,dz == n!\sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{(-1)^k\,2^{n-2k}\,z^{1-2k+n}}{k!\,(n-2k+1)!}$$

Involving one direct function and elementary functions

Involving power function

05.01.21.0003.01

$$\int z^{\alpha-1}\,H_n(z)\,dz == n!\sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{(-1)^k\,2^{n-2k}\,z^{n-2k+\alpha}}{(n-2k+\alpha)\,k!\,(n-2k)!}$$

05.01.21.0004.01

$$\int z\, H_n(z)\, dz = \frac{1}{3}\, 2^{n-1} \sqrt{\pi} \left(\frac{3\left({}_1F_1\left(-\frac{n}{2}-1; -\frac{1}{2}; z^2\right)-1\right)}{(n+2)\,\Gamma\left(\frac{1-n}{2}\right)} - \frac{4 z^3\, {}_1F_1\left(\frac{1}{2}-\frac{n}{2}; \frac{5}{2}; z^2\right)}{\Gamma\left(-\frac{n}{2}\right)} \right)$$

05.01.21.0005.01

$$\int z^m\, H_n(a\, z)\, dz = 2^{n-1}\, \pi\, z^{m+1} \left(\frac{\Gamma\left(\frac{m+1}{2}\right) {}_2\tilde{F}_2\left(-\frac{n}{2}, \frac{m+1}{2}; \frac{1}{2}, \frac{m+3}{2}; a^2 z^2\right)}{\Gamma\left(\frac{1-n}{2}\right)} - \frac{a\, z\, \Gamma\left(\frac{m}{2}+1\right) {}_2\tilde{F}_2\left(\frac{1}{2}-\frac{n}{2}, \frac{m}{2}+1; \frac{3}{2}, \frac{m}{2}+2; a^2 z^2\right)}{\Gamma\left(-\frac{n}{2}\right)} \right)$$

05.01.21.0006.01

$$\int z^{-n-3}\, H_n(z)\, dz = \frac{1}{\Gamma\left(\frac{1-n}{2}\right)\Gamma\left(-\frac{n}{2}\right)} \left(2^{n-1} \sqrt{\pi}\; z^{-n-2} \left(\Gamma\left(-\frac{n}{2}-1\right) {}_1F_1\left(-\frac{n}{2}-1; \frac{1}{2}; z^2\right) - 2\, z\, \Gamma\left(-\frac{n}{2}-\frac{1}{2}\right) {}_1F_1\left(-\frac{n}{2}-\frac{1}{2}; \frac{3}{2}; z^2\right) \right) \right)$$

Involving exponential function

05.01.21.0007.01

$$\int e^{-z^2}\, H_n(z)\, dz = 2^n \sqrt{\pi} \left(\frac{z\, {}_1F_1\left(\frac{n}{2}+\frac{1}{2}; \frac{3}{2}; -z^2\right)}{\Gamma\left(\frac{1-n}{2}\right)} + \frac{{}_1F_1\left(\frac{n}{2}; \frac{1}{2}; -z^2\right)-1}{n\; \Gamma\left(-\frac{n}{2}\right)} \right)$$

Involving exponential function and a power function

05.01.21.0008.01

$$\int z^{\alpha-1}\, e^{-p\, z}\, H_n(z)\, dz = n! \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{(-1)^{k-1}\, 2^{n-2k}\, z^{n-2k+\alpha}\, (p\, z)^{2k-n-\alpha}\, \Gamma(n-2k+\alpha,\, p\, z)}{k!\, (n-2k)!}$$

05.01.21.0009.01

$$\int z^{\alpha-1}\, e^{-p\, z^2}\, H_n(z)\, dz = n! \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{(-1)^{k-1}\, 2^{n-2k-1}\, z^{n-2k+\alpha} \left(p\, z^2\right)^{k-\frac{n+\alpha}{2}} \Gamma\left(\frac{n+\alpha}{2}-k,\, p\, z^2\right)}{k!\, (n-2k)!}$$

05.01.21.0010.01

$$\int z^m\, e^{-a^2 z^2}\, H_n(a\, z)\, dz = 2^{n-1}\, \pi\, z^{m+1} \left(\frac{\Gamma\left(\frac{m+1}{2}\right) {}_2\tilde{F}_2\left(\frac{n+1}{2}, \frac{m+1}{2}; \frac{1}{2}, \frac{m+3}{2}; -a^2 z^2\right)}{\Gamma\left(\frac{1-n}{2}\right)} - \frac{a\, z\, \Gamma\left(\frac{m}{2}+1\right) {}_2\tilde{F}_2\left(\frac{n}{2}+1, \frac{m}{2}+1; \frac{3}{2}, \frac{m}{2}+2; -a^2 z^2\right)}{\Gamma\left(-\frac{n}{2}\right)} \right)$$

05.01.21.0011.01

$$\int z^{n-2}\, e^{-z^2}\, H_n(z)\, dz = \frac{z^{n-1}\, \sin(\pi\, n)}{\pi} \left(z\, \Gamma\left(\frac{1}{2}-\frac{n}{2}\right) \Gamma(n)\, {}_1F_1\left(\frac{n}{2}; \frac{3}{2}; -z^2\right) + \Gamma\left(1-\frac{n}{2}\right) \Gamma(n-1)\, {}_1F_1\left(\frac{n-1}{2}; \frac{1}{2}; -z^2\right) \right)$$

Involving functions of the direct function and elementary functions

Involving elementary functions of the direct function and elementary functions

Involving powers of the direct function and a power function

05.01.21.0012.01

$$\int z\, H_n(z)^2\, dz = \frac{1}{8}\left(2\, H_n(z)^2 + \frac{H_{n+1}(z)^2}{n+1}\right)$$

Involving powers of the direct function, power and exponential functions

05.01.21.0013.01

$$\int z\, e^{-2z^2}\, H_n(z)^2\, dz = -\frac{1}{4}\, e^{-2z^2}\left(2\, n\, H_{n-1}(z)^2 + H_n(z)^2\right)$$

Involving direct function and Gamma-, Beta-, Erf-type functions

Involving probability integral-type functions

Involving erf

05.01.21.0014.01

$$\int \mathrm{erf}(a\, z)\, H_n(a\, z)\, dz = \frac{1}{2\, a(n+1)}\left(\frac{2}{\sqrt{\pi}}\, e^{-a^2 z^2}\, H_n(a\, z) + \mathrm{erf}(a\, z)\, H_{n+1}(a\, z)\right)$$

Involving erfi

05.01.21.0015.01

$$\int e^{-z^2}\, \mathrm{erfi}(z)\, H_n(z)\, dz = \frac{H_n(z)}{n\sqrt{\pi}} - e^{-z^2}\, \mathrm{erfi}(z)\, H_{n-1}(z)$$

Definite integration

Involving the direct function

05.01.21.0016.01

$$\int_0^\infty t^{\alpha-1}\, e^{-a\, t^2}\, H_n(t)\, dt = n! \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{(-1)^k\, 2^{n-2k-1}\, a^{k-\frac{n+\alpha}{2}}\, \Gamma\left(\frac{n+\alpha}{2}-k\right)}{k!\,(n-2k)!} \;/;\; \mathrm{Re}(a) > 0 \land \mathrm{Re}(\alpha) > 0$$

05.01.21.0017.01

$$\int_{-\infty}^\infty e^{-(z-t)^2}\, H_n(t)\, dt = 2^n \sqrt{\pi}\, z^n$$

Orthogonality:

05.01.21.0018.01

$$\int_{-\infty}^\infty e^{-t^2}\, H_m(t)\, H_n(t)\, dt = 2^n\, n!\, \sqrt{\pi}\, \delta_{n,m}$$

05.01.21.0019.01

$$\int_{-\infty}^\infty e^{-t^2}\, H_l(t)\, H_m(t)\, H_n(t)\, dt = \frac{2^{\frac{1}{2}(l+m+n)}\, l!\, m!\, n!\, \sqrt{\pi}}{\left(\frac{1}{2}(l+m-n)\right)!\left(\frac{1}{2}(-l+m+n)\right)!\left(\frac{1}{2}(l-m+n)\right)!} \;/;$$

$$\frac{1}{2}(l+m+n) \in \mathbb{Z} \bigwedge l+m \geq n \bigwedge m+n \geq l \bigwedge l+n \geq m$$

05.01.21.0020.01

$$\int_{-\infty}^{\infty} e^{-t^2} H_l(t) H_m(t) H_n(t) \, dt == 0 \;/;\; \neg\left(\frac{1}{2}(l+m+n) \in \mathbb{Z} \bigwedge l+m \geq n \bigwedge m+n \geq l \bigwedge l+n \geq m\right)$$

05.01.21.0021.01

$$\int_{-\infty}^{\infty} H_n(z)^2 \log\left(H_n(z)^2\right) e^{-z^2} \, dz == 2^n \, n! \sqrt{\pi} \, \log\left(2^{2n}\right) - 2 \sum_{k=1}^{n} V_n(z_k) \;/;$$

$$H_n(z_k) == 0 \bigwedge V_n(z) == 2^n \, n! \sqrt{\pi} \left(\log(2) + \frac{\gamma}{2} - {}_2F_2\left(1, 1; \frac{3}{2}, 2; -z^2\right) z^2 + \frac{1}{2} \sum_{k=1}^{n} \frac{(-1)^k \, 2^k}{k} \binom{n}{k} {}_1F_1\left(k; \frac{1}{2}; -z^2\right)\right)$$

(entropic integral)

05.01.21.0022.01

$$\int_{-\infty}^{\infty} e^{-\frac{z^2}{2} - \frac{1}{2}(z-\zeta)^2} H_n(z) H_p(z-\zeta) \, dz == \sqrt{\pi} \sqrt{2^n \, n!} \sqrt{2^p \, p!} \sqrt{\frac{n! \, p!}{2^{p-n}}} \, e^{-\frac{\zeta^2}{4}} (-\zeta)^{p-n} \sum_{k=0}^{n} \frac{\left(-\frac{\zeta^2}{2}\right)^k}{k! \, (n-k)! \, (k-n+p)!} \;/;\; n \in \mathbb{N} \land p \in \mathbb{N}$$

05.01.21.0023.01

$$\int_{-\infty}^{\infty} e^{-\frac{z^2}{2} - \frac{1}{2}(z-\zeta)^2} H_n(z) H_p(z-\zeta) \, dz ==$$

$$\frac{\sqrt{\pi} \sqrt{2^n \, n!} \sqrt{2^p \, p!} \sqrt{2^{n-p} \, \Gamma(n+1) \, \Gamma(p+1)}}{\Gamma(n+1)} e^{-\frac{\zeta^2}{4}} (-\zeta)^{p-n} \, {}_1\tilde{F}_1\left(-n; -n+p+1; \frac{\zeta^2}{2}\right) \;/;\; n \in \mathbb{N} \land p \in \mathbb{N}$$

Summation

Finite summation

05.01.23.0001.01

$$\sum_{k=0}^{n} \frac{H_k(x) H_k(y)}{2^k \, k!} == \frac{H_{n+1}(x) H_n(y) - H_n(x) H_{n+1}(y)}{2^{n+1} \, n! \, (x-y)}$$

05.01.23.0002.01

$$\sum_{k=0}^{n} \binom{n}{k} i^k H_{n-k}(x) H_k(y) == 2^n (x + i \, y)^n$$

05.01.23.0003.01

$$\sum_{k=0}^{n} \binom{n}{k} \cos\left(\frac{k\pi}{2}\right) H_{n-k}(x) H_k(y) == 2^n \operatorname{Re}\left((x + i \, y)^n\right)$$

05.01.23.0004.01

$$\sum_{k=0}^{n} \binom{n}{k} \sin\left(\frac{k\pi}{2}\right) H_{n-k}(x) H_k(y) == 2^n \operatorname{Im}\left((x + i \, y)^n\right)$$

05.01.23.0016.01

$$\sum_{k=-m}^{m} (-1)^k \, 2^{-n} \binom{2m}{m-k} H_{n-k}\left(\sqrt{2} \, z\right) H_{k+n}\left(\sqrt{2} \, z\right) == \frac{(2m)! \, (n-m)!}{m!} \sum_{k=m}^{n} \frac{2^{k-n}}{(n-k)!} \binom{k-1}{m-1} H_{n-k}\left(\sqrt{2} \, z\right)^2 \;/;$$

$$n \in \mathbb{N}^+ \bigwedge n \in \mathbb{N}^+ \bigwedge n \geq m$$

Infinite summation

05.01.23.0005.01

$$\sum_{n=0}^{\infty} \frac{H_n(z)\, w^n}{n!} = e^{2 z w - w^2}$$

05.01.23.0006.01

$$\sum_{n=0}^{\infty} \frac{H_{2n}(z)\, w^n}{(2n)!} = e^{-w} \cos\left(2 z \sqrt{-w}\right)$$

05.01.23.0007.01

$$\sum_{n=0}^{\infty} \frac{H_{2n+1}(z)\, w^n}{(2n+1)!} = \frac{e^{-w} \sin\left(2 z \sqrt{-w}\right)}{\sqrt{-w}}$$

05.01.23.0008.01

$$\sum_{n=0}^{\infty} \frac{(c)_n\, H_{2n}(z)\, w^{2n}}{(2n)!} = \left(w^2+1\right)^{-c} {}_1F_1\left(c; \frac{1}{2}; \frac{z^2 w^2}{w^2+1}\right)$$

05.01.23.0009.01

$$\sum_{n=0}^{\infty} \frac{\left(c+\frac{1}{2}\right)_n H_{2n+1}(z)\, w^{2n}}{(2n+1)!} = 2 \left(w^2+1\right)^{-c-\frac{1}{2}} z\, {}_1F_1\left(c+\frac{1}{2}; \frac{3}{2}; \frac{w^2 z^2}{w^2+1}\right)$$

05.01.23.0010.01

$$\sum_{n=0}^{\infty} \frac{H_n(z)\, w^n}{\left\lfloor \frac{n}{2} \right\rfloor !} = \left(4 w^2 + 2 z w + 1\right)\left(4 w^2+1\right)^{-\frac{3}{2}} \exp\left(\frac{4 w^2 z^2}{4 w^2+1}\right)$$

05.01.23.0017.01

$$\sum_{k=0}^{\infty} \frac{(c)_{\left\lfloor \frac{k}{2} \right\rfloor}}{\left\lfloor \frac{k}{2} \right\rfloor ! \left(\frac{1}{2}\right)_{\left\lfloor \frac{k}{2} \right\rfloor}} H_k(z)\, w^k =$$

$$\left(4 w^2+1\right)^{-c} {}_1F_1\left(c; \frac{1}{2}; \frac{4 z^2 w^2}{4 t^2+1}\right) + \frac{32 c\, w^3 z^3}{3 \left(4 w^2+1\right)^{c+2}} {}_1F_1\left(c+1; \frac{5}{2}; \frac{4 z^2 w^2}{4 w^2+1}\right) + \frac{2 z t \left(-8 c w^2 + 4 w^2 + 1\right)}{\left(4 w^2+1\right)^{c+1}} {}_1F_1\left(c; \frac{3}{2}; \frac{4 z^2 w^2}{4 w^2+1}\right)$$

05.01.23.0011.01

$$\sum_{n=0}^{\infty} \frac{H_n(x+n y)\, t^n}{n!} = \frac{e^{2 x \alpha - \alpha^2}}{1 - 2 y \alpha} /; \alpha = -\frac{W(-2 t y)}{2 y} \bigwedge -1 < t < 1 \bigwedge -1 < y < 1$$

05.01.23.0012.01

$$\sum_{n=0}^{\infty} \frac{H_{k+j n}(z)\, w^{k+j n}}{(k+j n)!} = \frac{1}{j} \sum_{l=1}^{j} \exp\left(2 e^{\frac{2 i \pi l}{j}} z w + e^{\frac{4 i \pi l}{j}} \left(-w^2\right) - \frac{2 i \pi k l}{j}\right) /; k \in \mathbb{N} \wedge j \in \mathbb{N}$$

05.01.23.0013.01

$$\sum_{n=0}^{\infty} \frac{H_n(z)\, H_n(z_1)\, w^n}{n!} = \frac{1}{\sqrt{1-4 w^2}} \exp\left(\frac{2 w \left(2 w \left(z^2+z_1^2\right) - 2 z z_1\right)}{4 w^2 - 1}\right) /; |w| < \frac{1}{2}$$

05.01.23.0014.01

$$\sum_{n=0}^{\infty} \frac{H_n(x)\, H_n(y)}{2^n\, n!} = \sqrt{\pi}\; e^{\frac{1}{2}(x^2+y^2)}\, \delta(x-y) \;/;\; x \in \mathbb{R} \bigwedge y \in \mathbb{R}$$

05.01.23.0015.01

$$\sum_{n=0}^{\infty} \binom{m}{n} H_n(z)\, H_{m-n}(z_1) = 2^{m/2}\, H_m\!\left(\frac{z+z_1}{\sqrt{2}}\right)$$

05.01.23.0018.01

$$\sum_{k=0}^{\infty} \frac{(b\,k+1)^{k/2}}{k!} \left(-t\, e^{b\,t^2+2\,a\,z\,t}\right)^k H_k\!\left(\frac{(a\,k+1)}{\sqrt{b\,k+1}}\, z\right) = \frac{e^{-t^2-2\,z\,t}}{2\,b\,t^2+2\,a\,z\,t+1} \;/;\; \left|2\,a\,t\,z\,e^{b\,t^2+2\,a\,z\,t+1}\right| < 1$$

05.01.23.0019.01

$$\sum_{k=0}^{\infty} \frac{(b\,k+1)^{\frac{k-2}{2}}}{k!} \left(-t\, e^{b\,t^2+2\,a\,z\,t}\right)^k H_k\!\left(\frac{(a\,k+1)}{\sqrt{b\,k+1}}\, z\right) = e^{-t^2-2\,z\,t}\; {}_1F_1\!\left(1;\, \frac{b+1}{b};\, \frac{2\,z\,t\,(b-a)}{b}\right) /;\; \left|2\,a\,t\,z\,e^{b\,t^2+2\,a\,z\,t+1}\right| < 1$$

05.01.23.0020.01

$$\sum_{k=0}^{\infty} \frac{(b\,k+1)^{k/2}}{k!\,(a\,k+1)} \left(-t\, e^{b\,t^2+2\,a\,z\,t}\right)^k H_k\!\left(\frac{z\,(a\,k+1)}{\sqrt{b\,k+1}}\right) = e^{-t^2-2\,z\,t}\; {}_1F_1\!\left(1;\, \frac{2\,a+1}{2\,a};\, \frac{2\,t^2\,(a-b)}{a}\right) /;\; \left|2\,a\,t\,z\,e^{b\,t^2+2\,a\,z\,t+1}\right| < 1$$

Multidimensional summation

05.01.23.0021.01

$$2^{-\frac{n^2}{2}} \sum_{m_{1,2}=0}^{\infty} \sum_{m_{1,3}=0}^{\infty} \ldots \sum_{m_{1,n}=0}^{\infty} \sum_{m_{2,3}=0}^{\infty} \ldots \sum_{m_{n-1,n}=0}^{\infty} \frac{\prod_{i=1}^{n}\prod_{j=i+1}^{n} w_{i,j}^{m_{i,j}}}{\prod_{i=1}^{n}\prod_{j=i+1}^{n} m_{i,j}!} H_{\sum_{j=1}^{n} m_{1,j}}\!\left(\frac{z_1}{\sqrt{2}}\right) H_{\sum_{j=1}^{n} m_{2,j}}\!\left(\frac{z_2}{\sqrt{2}}\right) \ldots H_{\sum_{j=1}^{n} m_{n,j}}\!\left(\frac{z_n}{\sqrt{2}}\right) =$$

$$\frac{1}{|W|}\, e^{\frac{1}{2}\left(\sum_{j=0}^{n} z_j^2 - \sum_{j=0}^{n}\sum_{k=0}^{n} z_j\, w_{j,k}\, z_k\right)} \;/;\; n \in \mathbb{N}^+ \land m_{i,i} = 0 \land \left(w_{i,j} = w_{j,i} \;/;\; i \neq j\right) \land w_{i,i} = 1 \bigwedge W = \begin{pmatrix} 1 & w_{1,2} & \ldots & w_{1,n} \\ w_{1,2} & 1 & \ldots & w_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ w_{1,n} & \ldots & w_{n-1,n} & 1 \end{pmatrix}$$

This is far reaching generalization of Mehler's formula—the Kibble–Slepian formula. More compactly it can be written as:

$$\sum_{\mathbf{M}\in\mathcal{M}} 2^{-\frac{n^2}{2}} \frac{\prod_{i=1}^{n}\prod_{j=i+1}^{n} w_{i\,j}^{m_{i\,j}}}{\prod_{i=1}^{n}\prod_{j=i+1}^{n} m_{i\,j}!} H_{s_1}\!\left(\frac{z_1}{\sqrt{2}}\right) H_{s_2}\!\left(\frac{z_2}{\sqrt{2}}\right) \ldots H_{s_n}\!\left(\frac{z_n}{\sqrt{2}}\right) = \frac{1}{\sqrt{\det(\mathbf{W})}} \exp\!\left(\frac{\mathbf{z}^T.\mathbf{z} - \mathbf{z}^T.\mathbf{W}.\mathbf{z}}{2}\right)$$

Here $\mathbf{W}$ is asymmetric $n \times n$ matrix with 1's on the diagonal and indeterminates $w_{i\,j}$, $1 \leq i,\ j \leq n$ else. $\mathbf{z}$ is a vector of length n with indeterminates $z_1, z_2, \ldots, z_n$. $\mathcal{M}$ is the set of all symmetric $n \times n$ matrices $\mathbf{M}$ with elements $m_{i\,j}$ that are 0 on the diagonal and nonnegative integer else and the outer sum runs over all such matrices. Finally $s_i = \sum_{j=1}^{n} m_{i\,j}$.

Operations

Limit operation

05.01.25.0001.01

$$\lim_{n\to\infty} \frac{(-1)^n}{4^n\, n!} H_{2n+1}\left(\frac{z}{2\sqrt{n}}\right) = \frac{2\sin(z)}{\sqrt{\pi}}$$

05.01.25.0002.01

$$\lim_{n\to\infty} \frac{(-1)^n \sqrt{n}}{4^n\, n!} H_{2n}\left(\frac{z}{2\sqrt{n}}\right) = \frac{\cos(z)}{\sqrt{\pi}}$$

Orthogonality, completeness, and Fourier expansions

The set of functions $H_n(x)$, $n = 0, 1, \ldots$, forms a complete, orthogonal (with weight $\frac{1}{2^n\, n!\sqrt{\pi}} e^{-x^2}$) system on the interval $(-\infty, \infty)$.

05.01.25.0003.01

$$\sum_{n=0}^{\infty} \left(\frac{1}{\sqrt{\sqrt{\pi}\; 2^n\, n!}} e^{-\frac{x^2}{2}} H_n(x)\right)\left(\frac{1}{\sqrt{\sqrt{\pi}\; 2^n\, n!}} e^{-\frac{y^2}{2}} H_n(y)\right) = \delta(x-y)$$

05.01.25.0004.01

$$\int_{-\infty}^{\infty} \left(\frac{1}{\sqrt{\sqrt{\pi}\; 2^m\, m!}} e^{-\frac{t^2}{2}} H_m(t)\right)\left(\frac{1}{\sqrt{\sqrt{\pi}\; 2^n\, n!}} e^{-\frac{t^2}{2}} H_n(t)\right) dt = \delta_{n,m}$$

Any sufficiently smooth function $f(x)$ can be expanded in the system $\{H_n(x)\}_{n=0,1,\ldots}$ as a generalized Fourier series, with its sum converging to $f(x)$ almost everywhere.

05.01.25.0005.01

$$f(x) = \sum_{n=0}^{\infty} c_n\, \psi_n(x) \;/;\; c_n = \int_{-\infty}^{\infty} \psi_n(t)\, f(t)\, dt \bigwedge \psi_n(x) = \frac{1}{\sqrt{\sqrt{\pi}\; 2^n\, n!}} e^{-\frac{x^2}{2}} H_n(x) \bigwedge x \in \mathbb{R}$$

Representations through more general functions

Through hypergeometric functions

Involving $_1\tilde{F}_1$

05.01.26.0001.01

$$H_n(z) = 2^n \pi \left(\frac{1}{\Gamma\left(\frac{1-n}{2}\right)}\, {}_1\tilde{F}_1\left(-\frac{n}{2}; \frac{1}{2}; z^2\right) - \frac{z}{\Gamma\left(-\frac{n}{2}\right)}\, {}_1\tilde{F}_1\left(\frac{1-n}{2}; \frac{3}{2}; z^2\right)\right)$$

Involving $_1F_1$

05.01.26.0002.01

$$H_n(z) = 2^n \sqrt{\pi} \left(\frac{1}{\Gamma\left(\frac{1-n}{2}\right)}\, {}_1F_1\left(-\frac{n}{2}; \frac{1}{2}; z^2\right) - \frac{2z}{\Gamma\left(-\frac{n}{2}\right)}\, {}_1F_1\left(\frac{1-n}{2}; \frac{3}{2}; z^2\right)\right)$$

Involving $_pF_q$

05.01.26.0003.01

$$H_n(z) == (2\,z)^n\, {}_2F_0\left(-\frac{n}{2}, \frac{1-n}{2};\, ;\, -\frac{1}{z^2}\right)$$

Involving hypergeometric *U*

05.01.26.0004.01

$$H_n(z) == 2^n\, U\left(-\frac{n}{2}, \frac{1}{2}, z^2\right) /; \operatorname{Re}(z) > 0$$

05.01.26.0005.01

$$H_n(z) == 2^n\, z\, U\left(\frac{1-n}{2}, \frac{3}{2}, z^2\right) /; \operatorname{Re}(z) > 0$$

05.01.26.0006.01

$$H_n(z) == \frac{2^n}{\sqrt{z^2}\,\Gamma\left(\frac{1-n}{2}\right)} \left(\sqrt{\pi}\left(\sqrt{z^2} - z\right) {}_1F_1\left(-\frac{n}{2}; \frac{1}{2}; z^2\right) + z\,\Gamma\left(\frac{1-n}{2}\right) U\left(-\frac{n}{2}, \frac{1}{2}, z^2\right)\right)$$

05.01.26.0007.01

$$H_n(z) == \frac{2^n}{z\,\Gamma\left(\frac{1-n}{2}\right)} \left(\Gamma\left(\frac{1-n}{2}\right) U\left(\frac{1-n}{2}, \frac{3}{2}, z^2\right) z^2 + \sqrt{\pi}\left(z - \sqrt{z^2}\right) {}_1F_1\left(-\frac{n}{2}; \frac{1}{2}; z^2\right)\right)$$

05.01.26.0008.01

$$H_n(z) == \frac{2^n}{\Gamma\left(-\frac{n}{2}\right)} \left(2\sqrt{\pi}\left(\sqrt{z^2} - z\right) {}_1F_1\left(\frac{1-n}{2}; \frac{3}{2}; z^2\right) + \Gamma\left(-\frac{n}{2}\right) U\left(-\frac{n}{2}, \frac{1}{2}, z^2\right)\right)$$

05.01.26.0009.01

$$H_n(z) == \frac{2^n}{\Gamma\left(-\frac{n}{2}\right)} \left(2\sqrt{\pi}\left(\sqrt{z^2} - z\right) {}_1F_1\left(\frac{1-n}{2}; \frac{3}{2}; z^2\right) + \sqrt{z^2}\,\Gamma\left(-\frac{n}{2}\right) U\left(\frac{1-n}{2}, \frac{3}{2}, z^2\right)\right)$$

Through Meijer *G*

Classical cases involving exp

05.01.26.0010.01

$$e^{-z^2}\, H_n(z) == 2^n\, G_{1,2}^{2,0}\left(z^2 \,\middle|\, \begin{matrix} \frac{1-n}{2} \\ 0, \frac{1}{2} \end{matrix}\right) /; \operatorname{Re}(z) > 0$$

05.01.26.0011.01

$$e^{-z}\, H_n\left(\sqrt{z}\right) == 2^n\, G_{1,2}^{2,0}\left(z \,\middle|\, \begin{matrix} \frac{1-n}{2} \\ 0, \frac{1}{2} \end{matrix}\right)$$

Generalized cases for the direct function itself

05.01.26.0012.01

$$H_n(z) == \sqrt{\pi}\, n!\, (-1)^{\left\lfloor \frac{n}{2} \right\rfloor} G_{1,2}^{1,0}\left(z, \frac{1}{2} \,\middle|\, \begin{matrix} \frac{n}{2}+1 \\ \frac{n}{2} - \left\lfloor \frac{n}{2} \right\rfloor, -\frac{n}{2} + \left\lfloor \frac{n}{2} \right\rfloor + \frac{1}{2} \end{matrix}\right)$$

05.01.26.0013.01

$$H_n(z) == \frac{1}{2\sqrt{\pi}\, z} \lim_{m \to n} \frac{1}{\Gamma(-m)}\, G_{1,2}^{2,1}\left(z, \frac{1}{2} \,\middle|\, \begin{matrix} \frac{m+3}{2} \\ 1, \frac{1}{2} \end{matrix}\right)$$

Generalized cases involving exp

05.01.26.0014.01

$$e^{-z^2} H_n(z) = 2^n G_{1,2}^{2,0}\left(z, \frac{1}{2} \middle| \begin{matrix} \frac{1-n}{2} \\ 0, \frac{1}{2} \end{matrix}\right)$$

Through other functions

Involving some hypergeometric-type functions

05.01.26.0018.01

$$H_n(z) = (-1)^{\lfloor \frac{n}{2} \rfloor} 2^n \left\lfloor \frac{n}{2} \right\rfloor! \, z^{n-2\lfloor \frac{n}{2} \rfloor} L_{\lfloor \frac{n}{2} \rfloor}^{\frac{1}{2}(-1)^{n-1}}(z^2)$$

05.01.26.0015.01

$$H_n(z) = \lim_{\lambda \to \infty} 2^{n/2} \Gamma(n+1) \lambda^{-\frac{n}{2}} L_n^{\lambda}\left(\lambda - \sqrt{2\lambda}\, z\right)$$

05.01.26.0016.01

$$H_n(z) = \Gamma(n+1) \lim_{\lambda \to \infty} \lambda^{-\frac{n}{2}} C_n^{\lambda}\left(\frac{z}{\sqrt{\lambda}}\right) /; |z| < 1$$

05.01.26.0017.01

$$H_n(z) = 2^n \Gamma(n+1) \lim_{a \to \infty} a^{-\frac{n}{2}} P_n^{(a,a)}\left(\frac{z}{\sqrt{a}}\right)$$

Representations through equivalent functions

With related functions

05.01.27.0001.01

$$H_n(z) = 2^n \left(\cos\left(\frac{n\pi}{2}\right) \Gamma\left(\frac{n}{2}+1\right) L_{\frac{n}{2}}^{-\frac{1}{2}}(z^2) + z \sin\left(\frac{n\pi}{2}\right) \Gamma\left(\frac{n+1}{2}\right) L_{\frac{n-1}{2}}^{\frac{1}{2}}(z^2)\right)$$

05.01.27.0002.01

$$H_{2n}(z) = (-1)^n 2^{2n} n! \, L_n^{-\frac{1}{2}}(z^2)$$

05.01.27.0003.01

$$H_{2n+1}(z) = (-1)^n 2^{2n+1} n! \, z \, L_n^{\frac{1}{2}}(z^2)$$

Zeros

05.01.30.0001.01

$$\sum_{\substack{k=1 \\ k \neq j}}^{n} \frac{1}{x_j - x_k} = x_j /; H_n(x_k) = 0$$

Theorems

Expansions in generalized Fourier series

$$f(x) == \sum_{k=0}^{\infty} c_k\, \psi_k(x) \,/;\quad c_k == \int_{-\infty}^{\infty} f(t)\, \psi_k(t)\, dt,\ \psi_k(x) == \frac{1}{2^{k/2} \sqrt{k!} \sqrt[4]{\pi}}\, e^{-x^2/2}\, H_k(x),\ k \in \mathbb{N}.$$

Fourier transform eigenfunctions

Hermite polynomials together with their weighting function are eigenfunctions of the Fourier and inverse Fourier transforms:

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i t x}\, e^{-x^2/2}\, H_n(x)\, dx == -i^n\, e^{-t^2/2}\, H_n(t) \,/;\, n \in \mathbb{N}$$

Zeros of Hermite polynomials

For any given interval (a, b), there exists some $n \in \mathbb{N}$ such that $H_n(x)$ has a zero in this interval.

The number of simple graphs

The number of simple graphs with no cycles and n vertices is $H_n(n+1) - n\, H_{n-1}(n+1)$.

History

– P. S. Laplace (1810)
– Ch. Hermite (1864)
– N. J. Sonine (1880)

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