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General Identities

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Notations

Traditional name

Abstract sufficiently smooth (analytic or piecewise differentiable) function

Traditional notation

$f(z)$

Mathematica StandardForm notation

`f[z]`

Primary definition

$f(z)$

The arbitrary function $f(z)$ in this document is defined for all complex z or for some region of $\mathbb{C}$ (if it is described explicitly). In the majority of cases, it is assumed to be an analytical function of the variable z (and sufficiently fast decaying if needed for the convergence of integrals and sums).

General characteristics

Domain and analyticity

In the majority of cases, $f(z)$ is an analytical function of z, which is defined in the whole complex z plane (if special restrictions are not shown).

$z \longrightarrow f(z) :: \mathbb{C} \longrightarrow \mathbb{C}$

Symmetries and periodicities

$$f(-z) = f(z) \;/;\; f_o(z) = 0 \bigwedge f(z) = f_e(z) + f_o(z) \bigwedge f_e(z) = \frac{f(z) + f(-z)}{2} \bigwedge f_o(z) = \frac{f(z) - f(-z)}{2}$$

This formula is the condition for a function to be even. For example, the function $f(z) = \cos(z)$ is an even function.

$$f(-z) == -f(z) \;/;\; f_e(z) == 0 \bigwedge f(z) == f_e(z) + f_o(z) \bigwedge f_e(z) == \frac{f(z) + f(-z)}{2} \bigwedge f_o(z) == \frac{f(z) - f(-z)}{2}$$

This formula is the condition for a function to be odd. For example, the function $f(z) = \sin(z)$ is an odd function.

$f(z + m\rho) == f(z) /; m \in \mathbb{Z}$

This formula reflects periodicity of function $f(z)$ (if f is periodic with period ρ). The analytic function $f(z)$ is called periodic if there exists a complex constant $\rho \neq 0$, such that $f(z+\rho) == f(z) /; z \in \mathbb{C}$. The number ρ (with minimal value $|\rho|$) is called the period of the function $f(z)$. For example, the functions $f(z) = \sin(z)$ and $f(z) = \tan(z)$ have periods $\rho = 2\pi$ and $\rho = \pi$ accordingly.

$\rho = 2\pi$

Series representations

General remarks

There are three main possibilities to represent an arbitrary function $f(z)$ as an infinite sum of simple functions. The first is the *power series* expansion and its two important generalizations, the *Laurent series* and the *Puiseux series*. The second is the *q-series* and *Dirichlet series* (general and periodic), and the third is the *Fourier series* (exponential, trigonometric, and generalized Fourier series by the orthogonal systems). These representations provide very general convenient methods for studying a wide range of functions.

The terms of a power series expansion or its generalizations include power functions in the form $(z - z_0)^k$ or $(z - z_0)^{k/m}$; the terms in the *q-series* include expressions like $q^k /; q == \phi(z)$; the terms in a Dirichlet series include exponential functions in the form $\exp(-\lambda_k s)$. The Fourier trigonometric series usually provide expansions in terms of $\cos(k x)$ and $\sin(k x)$ (for trigonometric series) and in terms of $e^{i k x}$ (for exponential series). Generalized Fourier series provide expansions in terms of other orthogonal systems of functions, such as the classical orthogonal polynomials. With each of these methods, you can express in closed form an enormous number of non-elementary functions in terms of only simple elementary functions.

Generalized power series

Expansions at $z == z_0$

For the function itself

$$f(z) \leftrightarrow \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!} (z - z_0)^k /; |z - z_0| < R \leq \infty$$

The *Taylor series*, first investigated by B. Taylor in 1715, gives the above Taylor expansion for an arbitrary function $f(z)$ at a finite point $z == z_0$.

The sign $\leftrightarrow$ means that the Taylor series should not be taken as purely equal to $f(z)$, since it may not converge everywhere on the complex plane. The series converges absolutely in some disk of radius R centered on z_0, where R is called the *radius of convergence*. On the disk $|z - z_0| < R$, you can exchange the sign $\leftrightarrow$ for $=\!=$. You can state for most functions that $R = \lim_{k\to\infty} |c_k|^{-1/k}$ /; $c_k = \frac{f^{(k)}(z_0)}{k!}$. The Taylor series coefficients c_k can also be evaluated by the Cauchy integral formula $c_k = \frac{1}{2\pi i} \int_{C_\rho} \frac{f(t)}{(t-z_0)^{k+1}} \, dt$ /; $C_\rho =\!= |t - z_0| =\!= \rho < R$.

There are then three cases for R: either R is infinite, R is zero, or R is some finite positive number greater than 0.

If $R = 0$, this series converges only at the point z_0, and the Taylor series offers little analytical benefit. For example, the series $\sum_{k=0}^{\infty} k!\,(z - z_0)^k$ has no disk of convergence around z_0, since the coefficient $k!$ makes the terms grow infinitely large no matter how small $|z - z_0|$ becomes.

If $R = \infty$, the series converges in the entire plane. In such cases it either represents an entire transcendental function—as, for example, the series $\sum_{k=0}^{\infty} (z - z_0)^k / k!$ does for the exponential function $\exp(z - z_0)$—or it contains only a finite number of terms and therefore represents a polynomial.

If $0 < R < \infty$, the sum of the series defines a regular analytic function having at least one singular point on the circle $|z - z_0| =\!= R$. There may be finitely or infinitely many singular points on the circle, but there must be at least one. The power series $\sum_{k=1}^{\infty} (z - z_0)^k / k = -\log(z_0 + 1 - z)$, for example, has only one singular point, at $z = z_0 + 1$. Other power series have dense sets of singular points on the circle, such as the series $\sum_{k=0}^{\infty} z^{k!}$, which has many singular points on the unit circle, the edge of its natural region of analyticity. The same holds true, with the same region of analyticity, for the series representation of the logarithm of `InverseEllipticNomeQ`: $\log\left(\frac{q^{-1}(z)}{16}\, z\right) = \sum_{k=1}^{\infty} \log\left(\left(\frac{1+z^{2k}}{1+z^{2k+1}}\right)^8\right)$.

Inside the region of convergence of the series, you can exchange the sign $\leftrightarrow$ for $=\!=$ in the preceding Taylor expansion for an arbitrary function $f(z)$ at a finite point z_0:

$$f(z) =\!= \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!} (z - z_0)^k \text{ /; } |z - z_0| < R \leq \infty$$

Infinite Taylor series expansion can be approximated by the truncated version, the finite Taylor polynomial expansions. But in these cases, instead of the equality sign $=\!=$, you will use the sign $\propto$ and an ellipsis … or an explicit Landau O term, for example:

$$f(z) \propto f(z_0) + f'(z_0)(z - z_0) + \frac{1}{2} f''(z_0)(z - z_0)^2 + \frac{1}{6} f^{(3)}(z_0)(z - z_0)^3 + \ldots \text{ /; } (z \to z_0)$$

$$f(z) \propto \sum_{k=0}^{n} \frac{f^{(k)}(z_0)}{k!} (z - z_0)^k + O((z - z_0)^n)$$

The Landau O term $O((z - z_0)^n)$ in the preceding relation means that the next expression is bounded near point $z =\!= z_0$:

$$\left|(z-z_0)^{-n}\left(f(z)-\sum_{k=0}^{n}\frac{f^{(k)}(z_0)\,(z-z_0)^k}{k!}\right)\right| < \text{const}$$

For compositions with elementary functions

Including power functions

$$(a\,f(z)^r)^b = \exp\left(2\,\pi\,i\,b\left(r\left\lfloor\frac{\pi-\arg(c)-\operatorname{Im}(m\log(z-z_0))}{2\,\pi}\right\rfloor + \left\lfloor\frac{\pi-\arg(a\,c^r)-\operatorname{Im}(m\,r\log(z-z_0))}{2\,\pi} - r\left\lfloor\frac{\pi-\arg(c)-\operatorname{Im}(m\log(z-z_0))}{2\,\pi}\right\rfloor\right\rfloor + \left\lfloor\frac{\pi-\arg(a\,c^r)-\arg\left(\frac{z}{a\,c^r}\right)}{2\,\pi}\right\rfloor\right)\right) (a\,c^r)^b\,(z-z_0)^{m\,r\,b}\sum_{k=0}^{\infty}\frac{(-1)^k\,(-b)_k}{k!}\left(c^{-r}\left(\frac{f(z)}{(z-z_0)^m}\right)^r-1\right)^k \;/;\; (z\to z_0)\bigwedge\lim_{z\to z_0}\frac{f(z)}{(z-z_0)^m} = c\bigwedge c\neq 0$$

Including logarithmic and power functions

$$\log(a\,f(z)^r) = 2\,i\,\pi\left(r\left\lfloor\frac{\pi-\arg(c)-\operatorname{Im}(m\log(z-z_0))}{2\,\pi}\right\rfloor + \left\lfloor\frac{\pi-\arg(a\,c^r)-\operatorname{Im}(m\,r\log(z-z_0))}{2\,\pi} - r\left\lfloor\frac{\pi-\arg(c)-\operatorname{Im}(m\log(z-z_0))}{2\,\pi}\right\rfloor\right\rfloor + \left\lfloor\frac{\pi-\arg(a\,c^r)-\arg\left(\frac{z}{a\,c^r}\right)}{2\,\pi}\right\rfloor\right) + \log(a\,c^r) + m\,r\log(z-z_0) + \sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}\left(c^{-r}\left(\frac{f(z)}{(z-z_0)^m}\right)^r-1\right)^k \;/;\; (z\to z_0)\bigwedge\lim_{z\to z_0}\frac{f(z)}{(z-z_0)^m} = c\bigwedge c\neq 0$$

$$\log\left(a\left(q^{f(z)}\right)^r\right) = 2\,i\,\pi\left(r\left\lfloor\frac{\pi-\operatorname{Im}(\log(q)\,f(z_0))}{2\,\pi}\right\rfloor + \left\lfloor\frac{\pi-\arg(a)-\operatorname{Im}(r\log(q)\,f(z_0))}{2\,\pi} - \operatorname{Re}(r)\left\lfloor\frac{\pi-\operatorname{Im}(\log(q)\,f(z_0))}{2\,\pi}\right\rfloor\right\rfloor\right) + \log(a) + r\log(q)\sum_{k=0}^{\infty}\frac{f^{(k)}(z_0)\,(z-z_0)^k}{k!} \;/;\; (z\to z_0)$$

StringTake[" |||", {6, 4}] Expansions at *z* == 0

For the function itself

$$f(z)\propto f(0)+f'(0)\,z+\frac{1}{2}\,f''(0)\,z^2+\frac{1}{6}\,f^{(3)}(0)\,z^3+\ldots \;/;\; (z\to z_0)$$

$$f(z)\propto f(0)+f'(0)\,z+\frac{1}{2}\,f''(0)\,z^2+\frac{1}{6}\,f^{(3)}(0)\,z^3+O\left(z^4\right)$$

$$f(z) \leftrightarrow \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} z^k \;/;\; |z| < R \leq \infty$$

$$f(z) \propto f(0)\,(1 + O(z))$$

Laurent series

$$f(z) \leftrightarrow \sum_{k=-\infty}^{\infty} c_k (z - z_0)^k \;/;\; c_k = \frac{1}{2\pi i} \int_{C_\rho} \frac{f(t)}{(t - z_0)^{k+1}}\, dt \bigwedge k \in \mathbb{Z}$$

The *Laurent series*, first studied by P. Laurent in 1843, gives the Laurent expansion for the function $f(z)$ around a finite point z_0, where C_ρ is the circle $|t - z_0| = \rho$ with radius ρ, $r < \rho < R$. Here R is the radius of convergence of its *regular part* $\sum_{k=0}^{\infty} c_k (z - z_0)^k$ and $1/r$ is the radius of convergence of its *principal part* $\sum_{k=-\infty}^{-1} c_k (z - z_0)^k = \sum_{k=1}^{\infty} c_{-k} (z - z_0)^{-k}$. The coefficient c_{-1} of the power $(z - z_0)^{-1}$ in the Laurent expansion of function $f(z)$ is called the *residue* of $f(z)$ at the point z_0:

$$\operatorname{res}_z(f(z))(z_0) = \frac{1}{2\pi i} \int_{C_\rho} f(t)\, dt$$

If $r < R$, then the Laurent series converges absolutely in the ring $r < |z - z_0| < R$, where its sum defines an analytic function. If $r = 0$ and the principal part includes only a finite number of terms (that is, if $c_k = 0$ for $k < -n < 0$ for some $n \in \mathbb{N}^+$), then $f(z)$ has a *pole of order n* at the point $z = z_0$.

For example, the function $\cot(z)$ has a pole of order 1 (also called a *simple pole*) at the point $z = 0$, since it has only one term in the principal part of its Laurent series expansion:

$$\cot(z) = \frac{1}{z} + \sum_{k=1}^{\infty} \frac{(-1)^k\, 2^{2k}\, B_{2k}}{(2k)!} z^{2k-1} \propto \frac{1}{z} - \frac{z}{3} - \frac{z^3}{45} + \cdots$$

If $r = 0$ and the principal part includes an infinite number of terms, this analytic function has an *essential singularity* at the point $z = z_0$. For example, the sum $\sum_{k=-\infty}^{0} z^k \big/ (-k)! = \exp(1/z)$ has an essential singularity at the point $z = 0$.

Puiseux series

$$f(z) \leftrightarrow \sum_{k=-\infty}^{\infty} c_k (z - z_0)^{k/m}$$

The *Puiseux series*, first studied by V.-A. Puiseux in 1850, expand $f(z)$ around a finite point z_0 as $f(z) \leftrightarrow \sum_{k=-\infty}^{\infty} c_k\, \varphi_k(z - z_0)$, with $\varphi_k(w)$ usually chosen as $\varphi_k(w) = w^{k/m}$. (Other choices for $\varphi_k(w)$ are possible through the use of iterated logarithms of the form $\varphi_k(w) = w^{k/m} \log^r(w)$, $\varphi_k(w) = w^{k/m} \log^r(w) \log^s(\log(w))$, and so on.) Choosing $\varphi_k(w) = w^{k/m}$ gives the Puiseux series for algebraic bivariate functions (because such functions should not include logarithms like $\log^r(w)$).

If this series contains only a finite number of nonzero coefficients c_k with negative indices k, the point z_0 is an *algebraic branch point* of order $m-1$; otherwise it is a *transcendental branch point*, such as the point $z=0$ for the function $\sqrt{\exp(1/z)}$.

Puiseux series are widely used for describing solutions of differential equations near singular points and, in particular, for representations of elementary and special functions near their singular points. The named elementary functions can have rather complicated behaviors near their singular points.

$$\cos^{-1}(z) = \sqrt{2}\,\sqrt{1-z}\,\sum_{k=0}^{\infty}\frac{(-1)^k\left(\frac{1}{2}\right)_k(z-1)^k}{2^k\,(2k+1)\,k!}\;/;\;|z-1|<1$$

The point $z=1$ is an algebraic branch point of order 1 for the function $\cos^{-1}(z) = \left(\sqrt{1-z}\,/\sqrt{z-1}\right)\log\left(z+\sqrt{z-1}\,\sqrt{z+1}\right)$. This function has the Puiseux representation for its fundamental branch near point $z=1$.

A similar representation occurs near the branch point $z=-1$.

$$\cos^{-1}(z) = \frac{\pi}{2}-\frac{z}{2\sqrt{-z^2}}\left(\log\left(-4z^2\right)-\frac{1}{2}\sum_{k=1}^{\infty}\frac{\left(\frac{1}{2}\right)_k z^{-2k}}{k\,k!}\right)/;\;|z|>1$$

This expansion near the branch point $z=\infty$ shows that $\cos^{-1}(z)$ has a logarithmic-type singularity that can provide infinitely many values depending on the direction of approach of the variable z to ∞.

$$z^z = \sum_{k=0}^{\infty}\frac{z^k\log^k(z)}{k!}$$

This Puiseux series for z^z near the point $z=0$ includes powers of logarithmic functions.

The Puiseux series for $\log^2(\log(z))$ near the point $z=0$ coincides with itself.

$$W(z) = \log(z)-\log(\log(z))-\sum_{k=0}^{\infty}\frac{(-1)^k}{\log^k(z)}\sum_{j=1}^{k}\frac{S_k^{(-j+k+1)}\log^j(\log(z))}{j!}\propto$$

$$\log(z)-\log(\log(z))+\frac{\log(\log(z))}{\log(z)}+\frac{\log^2(\log(z))-\log(\log(z))}{2\log^2(z)}+\cdots\;/;\;(|z|\to\infty)$$

This Puiseux series for the `ProductLog` function $W(1/z)$ near the point $z=\infty$ has a more complicated structure involving iterated logarithms. The complicated structure is to be expected from inverse functions like `ProductLog`.

q-series

$$f(z)\leftrightarrow\sum_{k=-\infty}^{\infty}c_k\,q^k\;/;\;q=\phi(z)$$

This q-series is widely used in applications. In the case $q == \phi(z) == e^{i z}$, $c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)\, e^{-i k t}\, dt$, it coincides with exponential Fourier series.

In very particular cases, q-series can coincide with some trigonometric functions. For example, if $q == \phi(z) == e^{i z}$, $c_{-1} == c_1 == \frac{1}{2}$, and all other c_k are zero, then $f(z) == \cos(z)$.

In more complicated cases, q-series can define different special functions, for example elliptic theta functions:

$$\vartheta_3(w, q) == 1 + \sum_{k=-\infty}^{\infty} \begin{cases} 2\cos\left(2\sqrt{k}\, w\right) & \sqrt{k} \in \mathbb{Z} \bigwedge k > 0 \\ 0 & \text{True} \end{cases} q^k.$$

Dirichlet series

General Dirichlet series

$$f(z) \leftrightarrow \sum_{k=0}^{\infty} a_k \exp(-\lambda_k z) \;/;\; \mathrm{Re}(z) > c \land c = \lim_{k\to\infty} \frac{\log(|a_k|)}{\lambda_k}$$

If all coefficients $\lambda_k > 0$, this Dirichlet series for the function $f(z)$ converges in some open half-plane $\mathrm{Re}(z) > c$. In this case, for many Dirichlet series the abscissa of absolute convergence can be found by the formula $c == \lim_{k\to\infty} \frac{\log(|a_k|)}{\lambda_k}$.

In the more general case $0 < |\lambda_0| \leq |\lambda_1| \leq \ldots$, the Dirichlet series absolutely converges in some open convex domain. In both cases, the sum of the Dirichlet series is an analytic function in the domain of convergence.

For example, if $\lambda_k = 2k$ and $a_k = 1$ or $a_k = (-1)^k$, you have, respectively, the Dirichlet series:

$$\sum_{k=0}^{\infty} e^{-2kz} == e^z \frac{\mathrm{csch}(z)}{2} \;/;\; \mathrm{Re}\,(z) > 0$$

$$\sum_{k=0}^{\infty} (-1)^k e^{-2kz} == -\frac{e^z \,\mathrm{sech}(z)}{2} \;/;\; \mathrm{Re}(z) > 0$$

In the case $\lambda_k = \log(k)$, you have the *ordinary Dirichlet series* $f(z) \leftrightarrow \sum_{k=1}^{\infty} (a_k / k^z)$, which in its simplest case ($a_k = 1$, $\mathrm{Re}(z) > 1$) is the Riemann zeta function $\zeta(z) = \sum_{k=1}^{\infty} (1/k^z)$. Other interesting examples of the Dirichlet series include the series for elliptic theta functions. For instance, if $a_k = \frac{1}{2} q^{k^2}$ and $\lambda_k = \mp 2 k i$, you have the representations:

$$\sum_{k=1}^{\infty} q^{k^2} \cos(2kz) == \frac{1}{2} \sum_{k=1}^{\infty} q^{k^2} e^{2kiz} + \frac{1}{2} \sum_{k=1}^{\infty} q^{k^2} e^{-2kiz} \;/;\; |q| < 1$$

$$\sum_{k=1}^{\infty} q^{k^2} \cos(2kz) == \frac{1}{2} (\vartheta_3(z, q) - 1) \;/;\; |q| < 1$$

Generalized Fourier series

$$f(x) \leftrightarrow \sum_{k=0}^{\infty} d_k\, \psi_k(x) \;/;\; d_k == \int_a^b \psi_k(t)\, f(t)\, dt \bigwedge k \in \mathbb{N} \bigwedge \int_a^b \psi_m(t)\, \psi_n(t)\, dt = \delta_{m,n}$$

Under some additional conditions (such as piecewise differentiability), this *Fourier series* of an arbitrary function $f(x)$ by the *orthogonal system* $\{\psi_k(x)\}$ with *Fourier coefficients* d_k converges to $f(x)$ on an interval (a, b) at the points of continuity of f, and to $\frac{1}{2}(f(x+0) + f(x-0))$ at the points of discontinuity of f, where $f(x \pm 0) = \lim_{\epsilon \to +0} f(x \pm \epsilon)$.

$$\sum_{k=0}^{\infty} d_k^2 \leq \int_a^b f(t)^2\, dt \;/;\; d_k == \int_a^b \psi_k(t)\, f(t)\, dt \bigwedge k \in \mathbb{N} \bigwedge \int_a^b \psi_m(t)\, \psi_n(t)\, dt = \delta_{m,n}$$

The inequality takes place for all orthogonal systems $\{\psi_k(x)\}$; it is called *Bessel's inequality*. If it can be transformed into *Parseval's equality*:

$$\sum_{k=0}^{\infty} d_k^2 == \int_a^b f(t)^2\, dt \;/;\; d_k == \int_a^b \psi_k(t)\, f(t)\, dt \bigwedge k \in \mathbb{N} \bigwedge \int_a^b \psi_m(t)\, \psi_n(t)\, dt = \delta_{m,n}$$

The corresponding system $\{\psi_k(x)\}$ is called the *complete system*.

The best-known examples of orthogonal systems are the classical orthogonal polynomials and the trigonometric system including $\cos(k\, x)$ and $\sin(k\, x)$.

Generalized Fourier series through classical orthogonal polynomials

$$f(x) \leftrightarrow \sum_{k=0}^{\infty} d_k\, \psi_k(x) \;/;\; d_k == \int_a^b \psi_k(t)\, f(t)\, dt \bigwedge k \in \mathbb{N} \bigwedge \int_a^b \psi_m(t)\, \psi_n(t)\, dt = \delta_{m,n}$$

For example, any sufficiently smooth function $f(x)$ can be expanded in the Hermite orthogonal system $\{H_n(x)\}_{n=0,1,\ldots}$ or in other orthogonal systems with corresponding weight factors as a generalized Fourier series, with its sum converging to $f(x)$ almost everywhere in corresponding intervals of variable x. The following represents this type of Fourier expansion through all classical orthogonal systems of polynomials.

$$f(x) == \sum_{n=0}^{\infty} d_n\, \psi_n(x) \;/;\; d_n == \int_{-\infty}^{\infty} \psi_n(t)\, f(t)\, dt \bigwedge \psi_n(x) == \frac{1}{\sqrt{\sqrt{\pi}\, 2^n\, n!}}\, e^{-\frac{x^2}{2}}\, H_n(x) \bigwedge x \in \mathbb{R}$$

$$f(x) = \sum_{n=0}^{\infty} d_n\, \psi_n(x) \;/;\; d_n = \int_0^{\infty} \psi_n(t)\, f(t)\, dt \bigwedge \psi_n(x) = e^{-\frac{x}{2}}\, L_n(x) \bigwedge x > 0$$

$$f(x) = \sum_{n=0}^{\infty} d_n\, \psi_n(x) \;/;\; d_n = \int_0^{\infty} \psi_n(t)\, f(t)\, dt \bigwedge \psi_n(x) = \frac{\sqrt{n!}}{\sqrt{\Gamma(n+\lambda+1)}}\, x^{\lambda/2}\, e^{-\frac{x}{2}}\, L_n^{\lambda}(x) \bigwedge x > 0$$

$$f(x) = \sum_{n=0}^{\infty} d_n\, \psi_n(x) \;/;\; d_n = \int_{-1}^{1} \psi_n(t)\, f(t)\, dt \bigwedge \psi_n(x) = \sqrt{\frac{1}{2}(2\, n+1)}\; P_n(x) \bigwedge -1 < x < 1$$

$$f(x) = \sum_{n=0}^{\infty} d_n\, \psi_n(x) \,/;\, d_n = \int_{-1}^{1} \psi_n(t)\, f(t)\, dt \bigwedge \psi_n(x) = \frac{\sqrt{\frac{2}{\pi}}}{\left(\sqrt{2} - 1\right)\delta_n + 1} \left(1 - x^2\right)^{-\frac{1}{4}} T_n(x) \bigwedge -1 < x < 1$$

$$f(x) = \sum_{n=0}^{\infty} d_n\, \psi_n(x) \,/;\, d_n = \int_{-1}^{1} \psi_n(t)\, f(t)\, dt \bigwedge \psi_n(x) = \sqrt{\frac{2}{\pi}}\, \sqrt[4]{1 - x^2}\, U_n(x) \bigwedge -1 < x < 1$$

$$f(x) = \sum_{n=0}^{\infty} d_n\, \psi_n(x) \,/;\, d_n = \int_{-1}^{1} \psi_n(t)\, f(t)\, dt \bigwedge \psi_n(x) = \frac{2^{\lambda - \frac{1}{2}} \sqrt{n!}\, \sqrt{n + \lambda}\, \Gamma(\lambda)}{\sqrt{\pi}\, \sqrt{\Gamma(n + 2\lambda)}} \left(1 - x^2\right)^{\frac{2\lambda - 1}{4}} C_n^{\lambda}(x) \bigwedge -1 < x < 1$$

$$f(x) = \sum_{n=0}^{\infty} d_n\, \psi_n(x) \,/;\, d_n = \int_{-1}^{1} \psi_n(t)\, f(t)\, dt \bigwedge$$

$$\psi_n(x) = \frac{2^{-\frac{a+b+1}{2}} \sqrt{n!}\, \sqrt{a + b + 2n + 1}\, \sqrt{\Gamma(a + b + n + 1)}}{\sqrt{\Gamma(a + n + 1)}\, \sqrt{\Gamma(b + n + 1)}} (1 - x)^{a/2} (x + 1)^{b/2} P_n^{(a,b)}(x) \bigwedge -1 < x < 1$$

Exponential Fourier series

$$f(x) \leftrightarrow \sum_{k=-\infty}^{\infty} c_k\, e^{i k x} \,/;\, c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)\, e^{-i k t}\, dt$$

This exponential Fourier series expansion can be transformed into a trigonometric Fourier series by using the Euler formula $e^{i z} = \cos(z) + i \sin(z)$.

Following is an example of the exponential Fourier series for the simple function $f(x) = x \,/;\, -\pi < x < \pi$.

$$x = \sum_{k=-\infty}^{\infty} c_k\, e^{i k x} \,/;\, c_k = \begin{cases} 0 & k = 0 \\ \frac{i (k \pi \cos(k \pi) - \sin(k \pi))}{k^2 \pi} & \text{True} \end{cases} \bigwedge -\pi < x < \pi$$

Outside the interval $-\pi < x < \pi$, the sum forms a periodic function and the following expansion holds for all real x:

$$x = 2\pi \left\lfloor \frac{x + \pi}{2\pi} \right\rfloor + \begin{cases} -\pi & \frac{x - \pi}{2\pi} \in \mathbb{Z} \\ 0 & \text{True} \end{cases} + \sum_{k=-\infty}^{\infty} c_k\, e^{i k x} \,/;\, c_k = \begin{cases} 0 & k = 0 \\ \frac{i (k \pi \cos(k \pi) - \sin(k \pi))}{k^2 \pi} & \text{True} \end{cases} \bigwedge x \in \mathbb{R}$$

Trigonometric Fourier series

$$f(x) \leftrightarrow \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos(k x) + b_k \sin(k x)) \,/;\, a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos(k t)\, dt \bigwedge b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin(k t)\, dt$$

This series expansion into a trigonometric system first appeared in an 1807 paper of J. Fourier, who used a similar expansion on the interval $(0, 2\pi)$. However, L. Euler had discovered similar formulas for Fourier coefficients in 1777.

An arbitrary interval (a, b) can be transformed into the interval $(-\pi, \pi)$ by changing the variable $x \to x(b-a)/(2\pi) + (a+b)/2$. This allows you to write the following Fourier series expansion of the arbitrary function $f(x)$ on an arbitrary interval (a, b):

$$f(x) \leftrightarrow \frac{a_0}{2} + \sum_{k=1}^{\infty} \left(a_k \cos\left(k \left(x - \frac{a+b}{2}\right) \frac{2\pi}{b-a}\right) + b_k \sin\left(k \left(x - \frac{a+b}{2}\right) \frac{2\pi}{b-a}\right)\right) /;$$

$$a_k = \frac{2}{b-a} \int_a^b f(t) \cos\left(k \left(t - \frac{a+b}{2}\right) 2 \frac{\pi}{b-a}\right) dt \bigwedge b_k = \frac{2}{b-a} \int_a^b f(t) \sin\left(k \left(t - \frac{a+b}{2}\right) 2 \frac{\pi}{b-a}\right) dt.$$

In the preceding formulas the coefficients should vanish at infinity $a_k \to 0$, $b_k \to 0$ as $k \to \infty$.

In the internal points x, $a < x < b$, where the piecewise differentiable function $f(x)$ is continuous, the preceding sum of the Fourier series is equal to $f(x)$ and you can change the symbol $\leftrightarrow$ to $=$.

At the points of discontinuity, this Fourier sum is equal to $\frac{1}{2}\,(f(x+0) + f(x-0))$.

Outside the interval (a, b), the Fourier sum is a periodic function with period $b-a$, but the behavior of this sum is not necessarily related to the actual behavior of the function $f(x)$ outside the interval of expansion (a, b).

For example, let $f(x) = x$ and $(a, b) = (-1, 1)$. Then $a_k = 0$, $b_k = 2 \frac{\sin(k\pi)}{k^2 \pi^2} - \frac{2\cos(k\pi)}{k\pi}$, $k \in \mathbb{N}$, and:

$$f(x) \leftrightarrow \sum_{k=1}^{\infty} \left(\frac{2\sin(k\pi)}{k^2 \pi^2} - \frac{2\cos(k\pi)}{k\pi} \right) \sin(k\pi x) = \frac{i}{\pi} \log\left(e^{-\pi i x}\right)$$

Inspection shows that the Fourier sum $\frac{i}{\pi} \log\left(e^{-\pi i x}\right)$ has period 2 and coincides with x on the interval $(-1, 1)$. But at the endpoint $x = 1$ the Fourier sum is equal to 0, since you are evaluating $\sum_{k=1}^{\infty} \left(\frac{2\sin(k\pi)}{k^2 \pi^2} - \frac{2\cos(k\pi)}{k\pi} \right) \sin(k\pi)$ with $\sin(k\pi) = 0$. This result coincides with $\frac{1}{2} \left(\lim_{x \to 1-0} \frac{i}{\pi} \log\left(e^{-\pi i x}\right) + \lim_{x \to 1+0} \frac{i}{\pi} \log\left(e^{-\pi i x}\right) \right) = \frac{1}{2}\,(1-1)$.

Asymptotic series expansions

Expansions at $z = \tilde{\infty}$

$$\left(f(z) \propto g(z) \sum_{k=0}^{\infty} c_k z^{-k} /; c_0 \neq 0 \bigwedge (|z| \to \infty) \right) = \left(\left| \frac{f(z) - g(z) \sum_{k=0}^{n} c_k z^{-k}}{g(z)\, z^{-n-1}} \right| < |c_{n+1}| /; |z| > R \right)$$

This asymptotic series expansion of the function $f(z)$ at $\tilde{\infty}$ includes the main term $c_0\, g(z) /; c_0 \neq 0$ and other terms of the form $g(z)\, c_k\, z^{-k}$. The corresponding formal asymptotic series $\sum_{k=0}^{\infty} c_k z^{-k}$ is, by definition, formed in such a way that the following inequality holds for all $n \in \mathbb{N}$ and sufficiently large $|z|$: $\left| \frac{f(z) - g(z) \sum_{k=0}^{n} c_k z^{-k}}{g(z)\, z^{-n-1}} \right| < |c_{n+1}|$. This asymptotic series can be a divergent or convergent series. If this series converges, it coincides with the Taylor power series expansion at infinity.

For example, the confluent hypergeometric function $U(a, b, z)$ has the following asymptotic series expansion through the divergent series ${}_2F_0$:

$$U(a, b, z) \propto z^{-a}\, {}_2F_0\left(a, a-b+1; ; -\frac{1}{z}\right) /; (|z| \to \infty)$$

The relation takes place because the following inequality holds for each $n \in \mathbb{N}$ and sufficiently large $|z|$:

$$\left| \frac{U(a, b, z) - z^{-a} \sum_{k=0}^{n} \frac{(-1)^k (a)_k (a-b+1)_k z^{-k}}{k!}}{z^{-a-n-1}} \right| < \text{const} /; |z| > R$$

The function Ei(z) has a rather interesting asymptotic expansion at $\tilde{\infty}$ through the following divergent series:

$$\text{Ei}(z) \propto \frac{e^z}{z} \sum_{k=0}^{\infty} \frac{k!}{z^k} + \pi\, i\, \text{sgn}(\text{Im}(z)) /; (|z| \to \infty)$$

The next example includes a convergent series in an asymptotic expansion. So, you can use the sign $==$ instead of $\propto$:

$$\sin^{-1}(z) == \frac{z}{2\sqrt{-z^2}} \left(\log\left(-4 z^2\right) - \sum_{k=1}^{\infty} \frac{\left(\frac{1}{2}\right)_k z^{-2k}}{k\, k!} \right) /; |z| > 1$$

In the particular case $n == 0$, the first generic formula for asymptotic expansion can be rewritten in the following form:

$$\left(f(z) \propto g(z) \left(c_0 + O\left(\frac{1}{z}\right) \right) /; c_0 \neq 0 \bigwedge (|z| \to \infty) \right) == (|z\,(f(z) - c_0\, g(z))| < c_1 /; |z| > R)$$

Expansions at $z == z_0$

$$\left(f(z) \propto g(z) \sum_{k=0}^{\infty} c_k (z - z_0)^k /; c_0 \neq 0 \bigwedge (z \to z_0) \right) == \left(\left| \frac{f(z) - g(z) \sum_{k=0}^{n} c_k (z - z_0)^k}{g(z)\, (z - z_0)^{n+1}} \right| < |c_{n+1}| /; |z - z_0| < \epsilon \right)$$

This asymptotic series expansion of the function $f(z)$ at $z == z_0$ includes the main term $c_0\, g(z) /; c_0 \neq 0$ and other terms $g(z)\, c_k\, (z - z_0)^k$. The corresponding formal asymptotic series $\sum_{k=0}^{\infty} c_k\, (z - z_0)^k$ by definition is formed in such a way that the following inequality holds for all $n \in \mathbb{N}$ and sufficiently small $|z - z_0|$: $\left| \frac{f(z) - g(z) \sum_{k=0}^{n} c_k (z - z_0)^k}{g(z) (z - z_0)^{n+1}} \right| < |c_{n+1}|$. This asymptotic series can be a divergent or convergent series. If this series converges, it coincides with the Taylor power series expansion at the point $z == z_0$.

For example, the functions $\sec^{-1}(z)$ and $\sin^{-1}(z)$ have the following asymptotic expansions near the points $z == 0$ and $z == 1$ through convergent series. So, you can use the sign $==$ instead of $\propto$ here:

$$\sec^{-1}(z) == \frac{\pi}{2} + \frac{1}{2} z \sqrt{-\frac{1}{z^2}} \left(\log\left(-\frac{4}{z^2}\right) - \sum_{k=1}^{\infty} \frac{\left(\frac{1}{2}\right)_k z^{2k}}{k\, k!} \right) /; |z| < 1$$

$$\sin^{-1}(z) == \frac{\pi}{2} - \sqrt{2}\, \sqrt{1-z} \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k (1-z)^k}{2^k\, (2k+1)\, k!} /; |z - 1| < 2$$

In the particular case $n == 0$, the first generic formula for asymptotic expansion can be rewritten in the following form:

$$(f(z) \propto g(z)\,(c_0 + O(z - z_0)) \,/;\, c_0 \neq 0 \bigwedge (z \to z_0)) == \left(\left|\frac{f(z) - c_0\, g(z)}{z - z_0}\right| < c_1 \,/;\, |z - z_0| < \epsilon\right)$$

Residue representations

$$f(z) = \sum_{k=0}^{\infty} \operatorname{res}_s(g(s))\,(a\,k + b)$$

For many analytic functions $f(z)$, you can establish so-called residue representations through infinite sums of residues of another analytic function $g(z)$ in the points $s == a\,k + b$. The residue of the analytic function $g(z)$ in the pole of order m can be calculated by the formula:

$$\operatorname{res}_z(g(z))\,(z_0) = \frac{1}{(m-1)!} \lim_{z \to z_0} \frac{\partial^{m-1}\,(g(z)\,(z - z_0)^m)}{\partial z^{m-1}} \,/;\, m \in \mathbb{N}^+$$

If the function $g(z)$ can be represented as a quotient $g(z) == \frac{\phi(z)}{\eta(z)}$, where the functions $\phi(z)$ and $\eta(z)$ are analytic at the point $z == z_0$ and z_0 is the simple root of the equation $\eta(z) == 0$, then the following formula holds:

$$\operatorname{res}\left(\frac{\phi(z)}{\eta(z)}, \{z, z_0\}\right) = \frac{\phi(z)}{\eta'(z_0)} \,/;\, \phi(z_0) \neq 0 \bigwedge \eta(z_0) = 0 \bigwedge \eta'(z_0) \neq 0$$

Very often, residue representations appear from the theory of the Meijer G function. The majority of functions of the hypergeometric type can be equivalently defined as corresponding to infinite sums of residues from products of ratios of gamma functions $\Gamma(\alpha_k\, s + \beta_k)$ on power function z^s.

For example, the following residue representation formula for a logarithm function takes place:

$$\log(z+1) == \sum_{j=1}^{\infty} \operatorname{res}_s\left(\frac{\Gamma(-s)^2\, z^{-s}}{\Gamma(1-s)}\, \Gamma(s+1)\right)(-j) \,/;\, |z| < 1$$

Integral representations

Fourier integral representations

$$f(x) \leftrightarrow \int_0^{\infty} (a(t)\cos(t\,x) + b(t)\sin(t\,x))\, dt \,/;\, a(t) == \frac{1}{\pi}\int_{-\infty}^{\infty} f(\tau)\cos(t\,\tau)\, d\tau \bigwedge b(t) == \frac{1}{\pi}\int_{-\infty}^{\infty} f(\tau)\sin(t\,\tau)\, d\tau$$

The *Fourier integral* is the continuous analogue of a Fourier series. This formulas can be derived from the Fourier series expansion of the function $f(x)$ on interval $(-l, l)$ as $l \to \infty$.

The substitution of $a(t)$ and $b(t)$ into the integral gives the following Fourier integral formulas:

$$f(x) \leftrightarrow \frac{1}{\pi}\int_0^{\infty}\int_{-\infty}^{\infty} f(\tau)\cos(t\,(x - \tau))\, d\tau\, dt$$

$$f(x) \leftrightarrow \frac{1}{\pi} \lim_{R\to\infty} \int_{-\infty}^{\infty} \frac{f(\tau)\sin(R\,(x-\tau))}{x-\tau}\, d\tau.$$

The first definition can be rewritten in exponential form, leading to exponential direct and inverse Fourier transforms:

$$f(x) \leftrightarrow \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathcal{F}_\tau[f(\tau)](t)\, e^{-i t x}\, dt \,/;\quad \mathcal{F}_\tau[f(\tau)](t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\tau)\, e^{i t \tau}\, d\tau.$$

If the function $f(x)$ is absolutely integrable on the real axis, you have the following equality:

$$\frac{1}{2}\,(f(x+0)+f(x-0)) = \lim_{R\to\infty} \frac{1}{\sqrt{2\pi}} \int_{-R}^{R} \mathcal{F}_\tau[f(\tau)](t)\, e^{-i t x}\, dt.$$

Product representations

$$f(z) = \prod_{k=0}^{\infty} P_k(f(z))^{\frac{\log\left(\frac{z}{z_0}\right)}{k!}} \,/;\, P_0(f(z)) = f(z) \bigwedge P_k(f(z)) = \exp\left(z\, \frac{\partial \log(P_{k-1}(f(z)))}{\partial z}\right)$$

Transformations

Transformations and argument simplifications

Argument involving related functions (compositions)

$$f(g(z)) = \sum_{k=0}^{\infty} c_k\, z^k \,/;\, \left(c_k = \sum_{j=0}^{\infty} a_j\, b_0^j\, p_{j,k} \bigwedge p_{j,0} = 1 \bigwedge p_{j,k} = \frac{1}{b_0\, k} \sum_{m=1}^{k} (j\, m + m - k)\, b_m\, p_{j,k-m} \bigwedge k \in \mathbb{N}^+\right) \bigwedge$$
$$\left(f(\zeta) = \sum_{k=0}^{\infty} a_k\, \zeta^k \,/;\, |\zeta| < r_1\right) \bigwedge \left(g(z) = \sum_{k=0}^{\infty} b_k\, z^k \,/;\, |z| < r_2 \wedge b_0 \neq 0\right) \bigwedge |g(z)| < r_1$$

This formula shows how to generate the series expansion of a general composition $f(g(z))$ at the point $z = 0$, using the known series expansions for each of the functions $f(z)$ and $g(z)$ at $z = 0$.

$$f(g(z)) == \sum_{k=0}^{\infty} c_k z^k \; /;$$

$$\left(c_k == \sum_{i_1+2 i_2+\cdots+k i_k=k} (i_1 + i_2 + \cdots + i_k;\, i_1,\, i_2,\, \ldots,\, i_k)\, a_{i_1+i_2+\ldots+i_k} b_1^{i_1} b_2^{i_2} \ldots b_k^{i_k} \bigwedge c_0 == a_0 \bigwedge c_1 == a_1 b_1 \bigwedge c_2 == a_1 b_2 + a_2 b_1^2 \bigwedge \right.$$

$$\left. c_3 == a_1 b_3 + 2\, a_2 b_1 b_2 + a_3 b_1^3 \bigwedge c_4 == a_1 b_4 + 2\, a_2 b_1 b_3 + a_2 b_2^2 + 3\, a_3 b_1^3 b_2 + a_4 b_1^4 \bigwedge \ldots \right) \bigwedge$$

$$\left(f(\zeta) == \sum_{k=0}^{\infty} a_k \zeta^k \; /;\, |\zeta| < r_1 \right) \bigwedge \left(g(z) = \sum_{k=0}^{\infty} b_k z^k \; /;\, |z| < r_2 \wedge b_0 == 0 \right) \bigwedge |g(z)| < r_1$$

This formula shows how to generate series expansion of the general composition $f(g(z))$ at the point $z == 0$, using known series expansions for each function $f(z)$ and $g(z)$ at zero.

Products, sums, and powers of the direct function

Products of the direct function

$$f(z)\, g(z) = \sum_{k=0}^{\infty} \left(\sum_{n=0}^{k} a_n b_{k-n} \right) z^k \; /;\, \left(f(z) == \sum_{k=0}^{\infty} a_k z^k \; /;\, |z| < r_1 \right) \bigwedge \left(g(z) == \sum_{k=0}^{\infty} b_k z^k \; /;\, |z| < r_2 \right) \bigwedge |z| < r == \min(r_1,\, r_2)$$

This formula shows how to multiply power series of two functions $f(z)$ and $g(z)$ at the point $z == 0$.

$$f(z)\, g(z) = \sum_{k=0}^{n} \left(\sum_{j=0}^{k} a_j b_{k-j} \right) z^k + z^{n+1} \sum_{k=0}^{\infty} \left(\sum_{j=0}^{n} a_{j+k+1} b_{n-j} \right) z^k \; /;\, \left(f(z) = \sum_{k=0}^{\infty} a_k z^k \; /;\, |z| < r \right) \bigwedge \left(g(z) = \sum_{k=0}^{n} b_k z^k \; /;\, |z| < r \right)$$

This formula shows how to multiply power series of two functions $f(z)$ and $g(z)$ at the point $z == 0$.

$$f(z)\, g(z)\, h(z) = \sum_{k=0}^{\infty} \left(\sum_{n=0}^{k} \sum_{i=0}^{n} a_{k-n} c_i b_{n-i} \right) z^k \; /;$$

$$\left(f(z) = \sum_{k=0}^{\infty} a_k z^k \; /;\, |z| < r_1 \right) \bigwedge \left(g(z) = \sum_{k=0}^{\infty} b_k z^k \; /;\, |z| < r_2 \right) \bigwedge \left(h(z) = \sum_{k=0}^{\infty} c_k z^k \; /;\, |z| < r_3 \right) \bigwedge |z| < r == \min(r_1,\, r_2,\, r_3)$$

This formula shows how to multiply power series of three functions $f(z)$, $g(z)$ and $h(z)$ at the point $z == 0$.

Ratios of the direct function

$$\frac{1}{g(z)} = \frac{1}{b_0} \sum_{k=0}^{\infty} (k+1) \sum_{r=0}^{k} \frac{(-1)^r}{r+1} \binom{k}{r} p_{r,k} z^k \; /;$$

$$g(z) = \sum_{k=0}^{\infty} b_k z^k \bigwedge b_0 \neq 0 \bigwedge p_{j,0} = 1 \bigwedge p_{j,k} == \frac{1}{b_0 k} \sum_{m=1}^{k} (j\, m + m - k)\, b_m\, p_{j,k-m} \bigwedge k \in \mathbb{N}^+$$

This formula represents the reciprocal of a power series for function $g(z)$ at the point $z == 0$.

$$\frac{f(z)}{g(z)} = \sum_{k=0}^{\infty} q_k z^k /; a_k = \sum_{j=0}^{k} b_j q_{k-j} \bigwedge f(z) = \sum_{k=0}^{\infty} a_k z^k \bigwedge g(z) = \sum_{k=0}^{\infty} b_k z^k$$

This formula represents the ratio of a power series for functions $f(z)$ and $g(z)$ at the point $z == 0$.

$$\frac{f(z)}{g(z)} = \frac{1}{b_0} \sum_{k=0}^{\infty} \sum_{j=0}^{k} (j+1) a_{k-j} \sum_{r=0}^{j} \frac{(-1)^r}{r+1} \binom{j}{r} p_{r,j} z^k /;$$

$$f(z) = \sum_{k=0}^{\infty} a_k z^k \bigwedge g(z) = \sum_{k=0}^{\infty} b_k z^k \bigwedge b_0 \neq 0 \bigwedge p_{j,0} = 1 \bigwedge p_{j,k} = \frac{1}{b_0 k} \sum_{m=1}^{k} (j m + m - k) b_m p_{j,k-m} \bigwedge k \in \mathbb{N}^+$$

This formula represents the ratio of a power series for functions $f(z)$ and $g(z)$ at the point $z == 0$.

$$\frac{f(z) g(z)}{h(z)} = \frac{1}{c_0} \sum_{k=0}^{\infty} \sum_{j=0}^{k} (j+1) d_{k-j} \sum_{r=0}^{j} \frac{(-1)^r \binom{j}{r}}{r+1} p_{r,j} z^k /;$$

$$\left(f(z) = \sum_{k=0}^{\infty} a_k z^k /; |z| < r_1\right) \bigwedge \left(g(z) = \sum_{k=0}^{\infty} b_k z^k /; |z| < r_3\right) \bigwedge \left(h(z) = \sum_{k=0}^{\infty} c_k z^k \bigwedge c_0 \neq 0 /; |z| < r_2\right) \bigwedge$$

$$|z| < r = \min(r_1, r_2, r_3) \bigwedge d_k = \sum_{n=0}^{k} a_n b_{k-n} \bigwedge p_{j,0} = 1 \bigwedge p_{j,k} = \frac{1}{c_0 k} \sum_{m=1}^{k} (j m - k + m) c_m p_{j,k-m} \bigwedge k \in \mathbb{N}^+$$

This formula represents the ratio of a power series for functions $f(z)$, $g(z)$ and $h(z)$, at the point $z == 0$.

Sums of the direct function

$$f(z) \pm g(z) = \sum_{k=0}^{\infty} (a_k \pm b_k) z^k /; \left(f(z) = \sum_{k=0}^{\infty} a_k z^k /; |z| < r_1\right) \bigwedge \left(g(z) = \sum_{k=0}^{\infty} b_k z^k /; |z| < r_2\right) \bigwedge |z| < r = \min(r_1, r_2)$$

This formula represents the summation property for the power series of the functions $f(z)$ and $g(z)$ at the point $z == 0$.

Powers of the direct function

$$f(z)^2 = \sum_{k=0}^{\infty} \left(\sum_{n=0}^{k} a_n a_{k-n}\right) z^k /; \left(f(z) = \sum_{k=0}^{\infty} a_k z^k /; |z| < r\right)$$

This formula represents the series squared property of the functions $f(z)$ at the point $z == 0$.

$$f(z)^n = \sum_{k=0}^{\infty} p_k z^k /; \left(f(z) = \sum_{k=0}^{\infty} a_k z^k /; |z| < r_1\right) \bigwedge p_0 = a_0^n \bigwedge a_0 \neq 0 \bigwedge p_k = \frac{1}{a_0 k} \sum_{j=1}^{k} (n j + j - k) a_j p_{k-j} \bigwedge k \in \mathbb{N}^+ \bigwedge |z| < r_1$$

This formula represents the n^{th} integer power of the series for the functions $f(z)$ at the point $z == 0$.

$$f(z)^{\alpha} == e^{2 i \alpha \pi \left\lfloor \frac{1}{2} - \frac{\arg(a_0)}{2\pi} - \frac{1}{2\pi} \arg\left(\frac{f(z)}{a_0}\right) \right\rfloor} a_0^{\alpha} \alpha \sum_{k=0}^{\infty} \binom{k-\alpha}{k} \sum_{j=0}^{k} \frac{(-1)^j}{\alpha - j} \binom{k}{j} p_{j,k} z^k \;/;$$

$$f(z) == \sum_{k=0}^{\infty} a_k z^k \bigwedge a_0 \neq 0 \bigwedge p_{j,0} = 1 \bigwedge p_{j,k} == \frac{1}{a_0 k} \sum_{m=1}^{k} (j m + m - k) a_m p_{j,k-m} \bigwedge k \in \mathbb{N}^+$$

This formula represents the arbitrary α^{th} power of the series for the functions $f(z)$ at the point $z == 0$.

$$\left(z^{\beta} f(z)\right)^{\alpha} == e^{2 i \alpha \pi \left\lfloor \frac{1}{2} - \frac{\arg(a_0)}{2\pi} - \frac{\operatorname{Im}(\beta \log(z))}{2\pi} - \frac{1}{2\pi} \arg\left(\frac{f(z)}{a_0}\right) \right\rfloor} a_0^{\alpha} \alpha z^{\alpha \beta} \sum_{k=0}^{\infty} \binom{k-\alpha}{k} \sum_{j=0}^{k} \frac{(-1)^j}{\alpha - j} \binom{k}{j} p_{j,k} z^k \;/;$$

$$f(z) == \sum_{k=0}^{\infty} a_k z^k \bigwedge a_0 \neq 0 \bigwedge p_{j,0} = 1 \bigwedge p_{j,k} == \frac{1}{a_0 k} \sum_{m=1}^{k} (j m + m - k) a_m p_{j,k-m} \bigwedge k \in \mathbb{N}^+$$

This formula represents the generic arbitrary power of the series for the functions $f(z)$ at the point $z == 0$.

Related transformations

Inversion

$$f^{(-1)}(z) = \sum_{k=0}^{\infty} c_k z^k \;/;$$

$$\left(c_0 = 0 \bigwedge c_1 = \frac{1}{a_1} \bigwedge c_k = -\frac{1}{a_1} \sum_{j=2}^{k+1} a_j \sum_{i_1=0}^{j} \sum_{i_2=0}^{j} \ldots \sum_{i_k=0}^{j} \delta_{j, \sum_{p=1}^{k} i_p} \delta_{k, \sum_{q=1}^{k} q\, i_q} (i_1 + i_2 + \ldots + i_k; i_1, i_2, \ldots, i_k) \prod_{p=1}^{k-1} c_p^{i_p} \bigwedge \right.$$
$$\left. c_2 = -\frac{a_2}{a_1^3} \bigwedge c_3 = \frac{2 a_2^2 - a_1 a_3}{a_1^5} \bigwedge c_4 = -\frac{5 a_2^3 - 5 a_1 a_3 a_2 + a_1^2 a_4}{a_1^7} \bigwedge \ldots \right) \bigwedge$$
$$\left(f(\zeta) = \sum_{k=0}^{\infty} a_k \zeta^k \bigwedge a_0 = 0 \bigwedge a_1 \neq 0\right) \bigwedge f\left(f^{(-1)}(z)\right) = z$$

This formula represents the series expansion of the inverse function $f^{(-1)}(z)$ through the known power series of the direct function $f(z)$ at the point $z == 0$.

$$f^{(-1)}(z) = \sum_{k=1}^{\infty} c_k z^k \;/; \left(v_k == \sum_{n_2 + 2 n_3 + \cdots + (k-1) n_k = k-1} \frac{n_1!}{k a_1^k} (k + n_1 + n_2 + \cdots + n_k - 1; k-1, n_1, n_2, \ldots, n_k) \left(-\frac{a_2}{a_1}\right)^{n_2} \cdots \left(-\frac{a_k}{a_1}\right)^{n_k}\right) \bigwedge$$
$$\left(f(\zeta) = \sum_{k=0}^{\infty} a_k \zeta^k \bigwedge a_0 = 0 \bigwedge a_1 \neq 0\right) \bigwedge f\left(f^{(-1)}(z)\right) = z$$

This formula represents the series expansion of the inverse function $f^{(-1)}(z)$ through the known power series of direct function $f(z)$ at the point $z == 0$.

Re-expansions in different points

$$f(z) == f(z_0) + \sum_{k=1}^{\infty}\sum_{j=0}^{\infty} a_{j+k} \binom{j+k}{k} z_0^j (z-z_0)^k \;/;\; f(z) == \sum_{k=0}^{\infty} a_k z^k$$

This formula shows how to generate the series expansion of the function $f(z)$ at the point $z == z_0$, if this function is presented through the power series at the point $z == 0$.

$$f(z) == \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} a_{j+k} \binom{j+k}{k} (-z_0)^j z^k \;/;\; f(z) == \sum_{k=0}^{\infty} a_k (z-z_0)^k$$

This formula shows how to generate the series expansion of the function $f(z)$ at the point $z == 0$, if this function is presented through the power series at the point $z == z_0$.

$$f(z) == \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} a_{j+k} \binom{j+k}{j} (z_1 - z_0)^k (z-z_1)^j \;/;\; f(z) == \sum_{k=0}^{\infty} a_k (z-z_0)^k$$

This formula shows how to generate the series expansion of the function $f(z)$ at the point $z == z_1$, if this function is presented through the power series at the point $z == z_0$.

Expansions through classical orthogonal polynomials

$$f(z) == f(z_0) + \sum_{k=1}^{\infty}\sum_{j=0}^{\infty} a_{j+k} \binom{j+k}{k} z_0^j (z-z_0)^k \;/;\; f(z) == \sum_{k=0}^{\infty} a_k z^k$$

This formula shows how to generate the series expansion of the function $f(z)$ at the point $z == z_0$, if this function is presented through the power series at the point $z == 0$.

$$f(z) == \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} a_{j+k} \binom{j+k}{k} (-z_0)^j z^k \;/;\; f(z) == \sum_{k=0}^{\infty} a_k (z-z_0)^k$$

This formula shows how to generate the series expansion of the function $f(z)$ at the point $z == 0$, if this function is presented through the power series at the point $z == z_0$.

$$f(z) == \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} a_{j+k} \binom{j+k}{j} (z_1 - z_0)^k (z-z_1)^j \;/;\; f(z) == \sum_{k=0}^{\infty} a_k (z-z_0)^k$$

This formula shows how to generate the series expansion of the function $f(z)$ at the point $z == z_1$, if this function is presented through the power series at the point $z == z_0$.

Determinants

$$\left| \begin{pmatrix} 1 & a_1 & a_1^2 & \ldots & a_1^{n-1} \\ 1 & a_2 & a_2^2 & \ldots & a_2^{n-1} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 1 & a_n & a_n^2 & \ldots & a_n^{n-1} \end{pmatrix} \right| = \prod_{j=1}^{n} \prod_{i=j+1}^{n} (a_i - a_j)$$

This determinant is called the Vandermonde determinant.

Complex characteristics

Real part

$$\operatorname{Re}(f(x+i\,y)) = \frac{f(x+i\,y)+f(x-i\,y)}{2}$$

$$\operatorname{Re}(f(x+i\,y)) = \frac{1}{2}\left(f\left(x-x\sqrt{-\frac{y^2}{x^2}}\right)+f\left(x+x\sqrt{-\frac{y^2}{x^2}}\right)\right)$$

Imaginary part

$$\operatorname{Im}(f(x+i\,y)) = \frac{f(x+i\,y)-f(x-i\,y)}{2\,i}$$

$$\operatorname{Im}(f(x+i\,y)) = \frac{x}{2\,y}\sqrt{-\frac{y^2}{x^2}}\left(f\left(x-x\sqrt{-\frac{y^2}{x^2}}\right)-f\left(x+x\sqrt{-\frac{y^2}{x^2}}\right)\right)$$

Absolute value

$$|f(x+i\,y)| = \sqrt{f(x+i\,y)\,f(x-i\,y)}$$

$$|f(x+i\,y)| = \sqrt{f\left(x-x\sqrt{-\frac{y^2}{x^2}}\right)f\left(x+x\sqrt{-\frac{y^2}{x^2}}\right)}$$

Argument

$$\arg(f(x+i\,y)) = \tan^{-1}\left(\frac{f(x+i\,y)+f(x-i\,y)}{2},\ \frac{f(x+i\,y)-f(x-i\,y)}{2\,i}\right)$$

$$\arg(f(x+i\,y)) = \tan^{-1}\left(\frac{1}{2}\left(f\left(x-x\sqrt{-\frac{y^2}{x^2}}\right)+f\left(x+x\sqrt{-\frac{y^2}{x^2}}\right)\right),\ \frac{x}{2\,y}\sqrt{-\frac{y^2}{x^2}}\left(f\left(x-x\sqrt{-\frac{y^2}{x^2}}\right)-f\left(x+x\sqrt{-\frac{y^2}{x^2}}\right)\right)\right)$$

Conjugate value

$$\overline{f(x+i\,y)} = f(x-i\,y)$$

$$\overline{f(z)} = f(\bar{z})$$

$$\overline{f(x+i\,y)} = \frac{1}{2}\left(f\left(\sqrt{-\frac{y^2}{x^2}}\;x+x\right)+f\left(x-x\sqrt{-\frac{y^2}{x^2}}\right)\right)-\frac{i\,x}{2\,y}\sqrt{-\frac{y^2}{x^2}}\left(f\left(x-x\sqrt{-\frac{y^2}{x^2}}\right)-f\left(x+x\sqrt{-\frac{y^2}{x^2}}\right)\right)$$

Differentiation

Low-order differentiation

Derivatives of the first order

$$f'(z) == \lim_{\epsilon \to 0} \frac{f(z+\epsilon) - f(z)}{\epsilon}$$

This limit defines the derivative of a function f at the point z, if it exists.

$$\frac{\partial (c\, f(z))}{\partial z} == c\, \frac{\partial f(z)}{\partial z}$$

This formula reflects the property that a constant factor can be pulled out of the differentiation.

$$\frac{\partial (f(z) \pm g(z))}{\partial z} == \frac{\partial f(z)}{\partial z} \pm \frac{\partial g(z)}{\partial z}$$

This formula reflects the property that the derivative of a sum (and difference) is equal to the sum (and difference) of the derivatives.

$$\frac{\partial (f(z)\, g(z))}{\partial z} == \frac{\partial f(z)}{\partial z}\, g(z) + f(z)\, \frac{\partial g(z)}{\partial z}$$

The *product rule for differentiation* shows that the derivative of a product is equal to the derivative of the first function multiplied by the second function plus the derivative of the second function multiplied by the first function.

$$\frac{\partial (f(z)\, g(z)\, h(z))}{\partial z} = \frac{\partial h(z)}{\partial z}\, f(z)\, g(z) + \frac{\partial f(z)}{\partial z}\, h(z)\, g(z) + \frac{\partial g(z)}{\partial z}\, f(z)\, h(z)$$

The *product rule for differentiation* shows how to evaluate the derivative of the product of three functions.

$$\frac{\partial \left(\prod_{k=1}^{n} f_k(z)\right)}{\partial z} = \sum_{j=1}^{n} \frac{1}{f_j(z)}\, \frac{\partial f_j(z)}{\partial z} \prod_{k=1}^{n} f_k(z)$$

The *product rule for differentiation* shows how to evaluate the derivative of the product of n functions.

$$\frac{\partial}{\partial z}\, \frac{f(z)}{g(z)} == \frac{1}{g(z)^2} \left(g(z)\, \frac{\partial f(z)}{\partial z} - f(z)\, \frac{\partial g(z)}{\partial z} \right)$$

The *quotient rule for differentiation* shows that the derivative of the ratio is equal to the derivative of the numerator multiplied by the denominator minus the derivative of the denominator multiplied by the numerator, divided by the square of the denominator.

$$\frac{\partial \frac{f(z)\, h(z)}{g(z)}}{\partial z} = \frac{g(z)\, h(z)\, f'(z) - f(z)\, h(z)\, g'(z) + f(z)\, g(z)\, h'(z)}{g(z)^2}$$

The *quotient rule for differentiation* has been generalized to the case when the numerator is the product of two functions.

$$\frac{\partial \frac{f(z)}{g(z)\,h(z)}}{\partial z} = \frac{g(z)\,h(z)\,f'(z) - f(z)\,h(z)\,g'(z) - f(z)\,g(z)\,h'(z)}{g(z)^2\,h(z)^2}$$

The *quotient rule for differentiation* is generalized to the case when the denominator is the product of two functions.

$$\frac{\partial}{\partial z}\sum_{k=0}^{\infty} a_k(z) = \sum_{k=0}^{\infty} \frac{\partial a_k(z)}{\partial z}$$

This formula shows that the derivative of the sum is equal to the sum of the derivatives. For an infinite sum it is true under some restrictions on $a_k(z)$, which ensure the convergence of the series.

$$\frac{\partial f(z)}{\partial z} = \sum_{k=1}^{\infty} a_k\,k\,z^{k-1} \;/;\; |z| < r \bigwedge \left(f(z) = \sum_{k=0}^{\infty} a_k\,z^k \;/;\; |z| < r\right)$$

This formula shows that the derivative of a power series is equal to the corresponding sum of the derivatives. It is true inside the corresponding circle of convergence with radius r.

$$\frac{\partial f(g(z))}{\partial z} = f'(g(z))\,g'(z)$$

This *chain rule for differentiation* shows that the derivative of composition $f(g(z))$ is equal to the derivative of the outer function f in the point $g(z)$, multiplied by the derivative of the inner function g.

$$\frac{\partial f(g(z),\,h(z))}{\partial z} = g'(z)\,f^{(1,0)}(g(z),\,h(z)) + h'(z)\,f^{(0,1)}(g(z),\,h(z))$$

This *chain rule for partial differentiation* generalizes the previous chain rule for differentiation in the case of a function with two variables $f(u,\,v)\;/;\;u = g(z) \bigwedge v = h(z)$.

$$\frac{\partial f(g_1(z),\,g_2(z),\,\ldots,\,g_n(z))}{\partial z} = \sum_{k=1}^{n} g_k'(z)\,f^{(0,\ldots,1,\ldots,0)}(g_1(z),\,\ldots,\,g_k(z),\,\ldots,\,g_n(z))$$

This *chain rule for partial differentiation* generalizes the chain rule for differentiation in the case of a function with several variables $f(u_1,\,u_2,\,\ldots,\,u_n)\;/;\;u_k = g_k(z) \bigwedge 1 \le k \le n$.

$$\frac{\partial f^{(-1)}(z)}{\partial z} = \frac{1}{f'\left(f^{(-1)}(z)\right)}$$

This formula shows that the derivative of the inverse function $f^{(-1)}(z)$ is equal to the reciprocal of the derivative of the direct function f in the point $f^{(-1)}(z)$.

$$\frac{\partial \frac{\partial f(z)}{\partial z}}{\partial z} = \frac{\partial^2 f(z)}{\partial z^2}$$

This formula shows that the composition of the first derivatives is equal to the derivative of the second order.

$$\frac{\partial}{\partial z}\int f(z)\,dz == f(z)$$

This formula shows that the derivative of an indefinite integral produces the original function (the derivative is the inverse operation to the indefinite integration).

$$\frac{\partial}{\partial z}\int_a^z f(t)\,dt == f(z)$$

This formula shows that the derivative of a definite integral with respect to the upper limit produces the original function.

$$\frac{\partial}{\partial z}\int_z^b f(t)\,dt == -f(z)$$

This formula shows that the derivative of a definite integral with respect to the low limit gives the original function with a negative sign.

$$\frac{\partial}{\partial z}\int_a^b f(t, z)\,dt == \int_a^b f^{(0,1)}(t, z)\,dt$$

This formula shows that the order of differentiation and definite integration can be changed if the limits of the integral do not depend on the variable of differentiation.

$$\frac{\partial}{\partial z}\int_{g(z)}^{h(z)} f(t, z)\,dt == \int_{g(z)}^{h(z)} f^{(0,1)}(t, z)\,dt - f(g(z), z)\,g'(z) + f(h(z), z)\,h'(z)$$

This formula reflects the general rule of differentiating an integral when its limits and its integrand depend on the variable of differentiation.

Derivatives of the second order

$$\frac{\partial^2 f(z)}{\partial z^2} = \lim_{\epsilon\to 0}\frac{f(z) - 2 f(z+\epsilon) + f(z+2\epsilon)}{\epsilon^2}$$

This limit defines the second derivative of a function f at the point z, if it exists.

$$\frac{\partial^2 f(g(z))}{\partial z^2} = f''(g(z))\,g'(z)^2 + f'(g(z))\,g''(z)$$

This formula shows how to evaluate the second derivative of a general composition $f(g(z))$.

Symbolic differentiation

Definition

$$f^{(n)}(z) == \lim_{\epsilon\to 0}\frac{1}{\epsilon^n}\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k} f(z+k\,\epsilon)$$

This limit defines the n^{th}-order derivative of a function f at the point z, if it exists.

Converting to finite differences and back

$$\frac{\partial^n f(z)}{\partial z^n} = \sum_{k=0}^{\infty} \frac{h^{-n}}{(n+1)_k} S_{k+n}^{(n)} \Delta_{z,h}^{n+k} f(z) \,/;\, \Delta_{z,h}^{k} f(z) == \Delta_{z,h}^{k-1}(\Delta_{z,h} f(z)) \bigwedge \Delta_{z,h} f(z) == f(z+h) - f(z)$$

$$\Delta_{z,h}^{n} f(z) = \sum_{k=0}^{\infty} \frac{h^{j+n}}{(n+1)_k} \mathcal{S}_{k+n}^{(n)} \frac{\partial^{k+n} f(z)}{\partial z^{k+n}} \,/;\, \Delta_{z,h}^{k} f(z) == \Delta_{z,h}^{k-1}(\Delta_{z,h} f(z)) \bigwedge \Delta_{z,h} f(z) == f(z+h) - f(z)$$

Products

$$\partial_{\{z,n\}}(f(z)\, g(z)) == \sum_{k=0}^{n} \binom{n}{k} \frac{\partial^k f(z)}{\partial z^k} \frac{\partial^{n-k} g(z)}{\partial z^{n-k}}$$

This rule is called the *binomial differentiation rule* for the n^{th}-order derivative.

$$\frac{\partial^n \left(\prod_{k=1}^{3} f_k(z)\right)}{\partial z^n} == \sum_{n_1=0}^{n} \sum_{n_2=0}^{n} \sum_{n_3=0}^{n} \delta_{n,n_1+n_2+n_3} \,(n_1+n_2+n_3;\, n_1,\, n_2,\, n_3) \prod_{k=1}^{3} \frac{\partial^{n_k} f_k(z)}{\partial z^{n_k}}$$

This rule is called the *multinomial differentiation rule* for the n^{th}-order derivative.

$$\frac{\partial^n}{\partial z^n} \prod_{k=1}^{m} f_k(z) == \sum_{n_1=0}^{n} \sum_{n_2=0}^{n} \ldots \sum_{n_m=0}^{n} \delta_{n,\sum_{k=1}^{m} n_k} \,(n_1+n_2+\ldots+n_m;\, n_1,\, n_2,\, \ldots,\, n_m) \prod_{k=1}^{m} \frac{\partial^{n_k} f_k(z)}{\partial z^{n_k}}$$

This rule is called the *multinomial differentiation rule* for the n^{th}-order derivative.

Ratios

$$\frac{\partial^n \frac{f(z)}{g(z)}}{\partial z^n} == n! \sum_{k=0}^{n} \frac{\partial^{n-k} f(z)}{\partial z^{n-k}} \sum_{j=0}^{k} \frac{(-1)^j (k+1)\, g(z)^{-j-1}}{(j+1)!\,(n-k)!\,(k-j)!} \frac{\partial^k g(z)^j}{\partial z^k} \,/;\, n \in \mathbb{N}$$

This formula shows how to evaluate an n^{th}-order derivative for the quotient of two functions.

$$\frac{\partial^n \frac{f(z)}{g(z)}}{\partial z^n} == \frac{\frac{\partial^n f(z)}{\partial z^n}}{g(z)} + n! \sum_{k=1}^{n} \frac{\partial^{n-k} f(z)}{\partial z^{n-k}} \sum_{j=1}^{k} \frac{(-1)^j (k+1)}{(j+1)!\,(n-k)!\,(k-j)!}\, g(z)^{-j-1} \frac{\partial^k g(z)^j}{\partial z^k} \,/;\, n \in \mathbb{N}$$

This formula shows how to evaluate an n^{th}-order derivative for the quotient of two functions.

$$\frac{\partial^n \frac{f(z)}{g(z)}}{\partial z^n} == \frac{\frac{\partial^n f(z)}{\partial z^n}}{g(z)} + n! \sum_{m=1}^{n} \frac{\partial^{n-m} f(z)}{\partial z^{n-m}} \sum_{q=1}^{m} \frac{(-1)^q (m+1)}{(q+1)!\,(n-m)!\,(m-q)!}\, g(z)^{-q-1}$$

$$\sum_{k(1)=0}^{m} \sum_{k(2)=0}^{m-k(1)} \ldots \sum_{k(q-1)=0}^{m-\sum_{j=1}^{q-2} k(j)} \left(\prod_{i=1}^{q-1} \binom{m - \sum_{j=1}^{i-1} k(j)}{k(p)} \right) \left(\prod_{i=1}^{q-1} \frac{\partial^{k(i)} g(z)}{\partial z^{k(i)}} \right) \frac{\partial^{m-\sum_{j=1}^{q-1} k(j)} g(z)}{\partial z^{m-\sum_{j=1}^{q-1} k(j)}} \,/;\, n \in \mathbb{N}$$

This formula shows how to evaluate an n^{th}-order derivative for the quotient of two functions.

$$\frac{\partial^n \frac{f(z)}{g(z)}}{\partial z^n} == \frac{\frac{\partial^n f(z)}{\partial z^n}}{g(z)} + n! \sum_{k=1}^{n} \frac{\partial^{n-k} f(z)}{\partial z^{n-k}}$$

$$\sum_{m=1}^{k} \frac{(-1)^m (k+1)}{(m+1)! (n-k)! (k-m)!} g(z)^{-m-1} \sum_{n_1=0}^{k} \sum_{n_2=0}^{k} \dots \sum_{n_m=0}^{k} \delta_{k, \sum_{k=1}^{m} n_k} (n_1 + n_2 + \dots + n_m; n_1, n_2, \dots, n_m) \prod_{k=1}^{m} \frac{\partial^{n_k} g(z)}{\partial z^{n_k}} /; n \in \mathbb{N}$$

This formula shows how to evaluate an n^{th}-order derivative for the quotient of two functions.

Power

$$\frac{\partial^n f(z)^a}{\partial z^n} == a \binom{n-a}{n} \sum_{k=0}^{n} \frac{(-1)^k}{a-k} \binom{n}{k} f(z)^{a-k} \frac{\partial^n f(z)^k}{\partial z^n} /; n \in \mathbb{N}$$

This formula shows how to evaluate an n^{th}-order derivative of the power $f(z)^a$.

$$\frac{\partial^n f(z)^a}{\partial z^n} == \delta_n f(z)^a + a \binom{n-a}{n} \sum_{m=1}^{n} \frac{(-1)^m}{a-m} \binom{n}{m} f(z)^{a-m} \frac{\partial^n f(z)^m}{\partial z^n} /; n \in \mathbb{N}$$

This formula shows how to evaluate an n^{th}-order derivative of the power $f(z)^a$.

$$\frac{\partial^n f(z)^a}{\partial z^n} == f(z)^a \delta_n +$$

$$a \binom{n-a}{n} \sum_{m=1}^{n} \frac{(-1)^m}{a-m} \binom{n}{m} f(z)^{a-m} \sum_{k(1)=0}^{n} \sum_{k(2)=0}^{n-k(1)} \dots \sum_{k(m-1)=0}^{n-\sum_{j=1}^{m-2} k(j)} \left(\prod_{p=1}^{m-1} \binom{n - \sum_{j=1}^{p-1} k(j)}{k(p)} \right) \left(\prod_{i=1}^{m-1} \frac{\partial^{k(i)} f(z)}{\partial z^{k(i)}} \right) \frac{\partial^{n-\sum_{j=1}^{m-1} k(j)} f(z)}{\partial z^{n-\sum_{j=1}^{m-1} k(j)}} /; n \in \mathbb{N}$$

This formula shows how to evaluate an n^{th}-order derivative of the power $f(z)^a$.

$$\frac{\partial^n f(z)^a}{\partial z^n} == \delta_n f(z)^a + a \binom{n-a}{n} \sum_{m=1}^{n} \frac{(-1)^m}{a-m} \binom{n}{m} f(z)^{a-m} \sum_{n_1=0}^{n} \sum_{n_2=0}^{n} \dots \sum_{n_m=0}^{n} \delta_{n, \sum_{k=1}^{m} n_k} (n_1 + n_2 + \dots + n_m; n_1, n_2, \dots, n_m) \prod_{k=1}^{m} \frac{\partial^{n_k} f(z)}{\partial z^{n_k}} /;$$
$$n \in \mathbb{N}$$

This formula shows how to evaluate an n^{th}-order derivative of the power $f(z)^a$.

Positive integer powers

$$\frac{\partial^n f(z)^2}{\partial z^n} == \sum_{k=0}^{n} \binom{n}{k} f^{(n-k)}(z) f^{(k)}(z) /; n \in \mathbb{N}$$

This formula shows how to evaluate an n^{th}-order derivative of the square $f(z)^2$.

$$\frac{\partial^n f(z)^3}{\partial z^n} == \sum_{k_1=0}^{n} \sum_{k_2=0}^{n-k_1} \binom{n}{k_1} \binom{n-k_1}{k_2} f^{(k_1)}(z) f^{(n-k_1-k_2)}(z) f^{(k_2)}(z) /; n \in \mathbb{N}$$

This formula shows how to evaluate an n^{th}-order derivative of the square $f(z)^3$.

$$\frac{\partial^n f(z)^3}{\partial z^n} = \sum_{k_1=0}^{n}\sum_{k_2=0}^{n}\sum_{k_3=0}^{n} \delta_{n,k_1+k_2+k_3} (k_1+k_2+k_3; k_1, k_2, k_3) \prod_{i=1}^{3} \frac{\partial^{k_i} f(z)}{\partial z^{k_i}} /; n \in \mathbb{N}$$

This formula shows how to evaluate an n^{th}-order derivative of the cube $f(z)^3$.

$$\frac{\partial^n f(z)^4}{\partial z^n} = \sum_{k_1=0}^{n}\sum_{k_2=0}^{n-k_1}\sum_{k_3=0}^{n-k_1-k_2} \binom{n}{k_1}\binom{n-k_1}{k_2}\binom{n-k_1-k_2}{k_3} f^{(k_1)}(z)\, f^{(k_2)}(z)\, f^{(n-k_1-k_2-k_3)}(z)\, f^{(k_3)}(z) /; n \in \mathbb{N}$$

This formula shows how to evaluate an n^{th}-order derivative of the fourth power $f(z)^4$.

$$\frac{\partial^n f(z)^4}{\partial z^n} = \sum_{k_1=0}^{n}\sum_{k_2=0}^{n}\sum_{k_3=0}^{n}\sum_{k_4=0}^{n} \delta_{n,k_1+k_2+k_3+k_4} (k_1+k_2+k_3+k_4; k_1, k_2, k_3, k_4) \prod_{i=1}^{4} \frac{\partial^{k_i} f(z)}{\partial z^{k_i}} /; n \in \mathbb{N}$$

This formula shows how to evaluate an n^{th}-order derivative of the fourth power $f(z)^4$.

$$\frac{\partial^n f(z)^m}{\partial z^n} = \sum_{k(1)=0}^{n}\sum_{k(2)=0}^{n-k(1)} \cdots \sum_{k(m-1)=0}^{n-\sum_{j=1}^{m-2} k(j)} \left(\prod_{p=1}^{m-1}\binom{n-\sum_{j=1}^{p-1} k(j)}{k(p)}\right)\left(\prod_{i=1}^{m-1}\frac{\partial^{k(i)} f(z)}{\partial z^{k(i)}}\right) \frac{\partial^{n-\sum_{j=1}^{m-1} k(j)} f(z)}{\partial z^{n-\sum_{j=1}^{m-1} k(j)}} /; m \in \mathbb{N}^+ \bigwedge n \in \mathbb{N}$$

This formula shows how to evaluate an n^{th}-order derivative of the general integer power $f(z)^m /; m \in \mathbb{N}^+$.

$$\frac{\partial^n f(z)^m}{\partial z^n} = \sum_{n_1=0}^{n}\sum_{n_2=0}^{n} \cdots \sum_{n_m=0}^{n} \delta_{n,\sum_{k=1}^{m} n_k} (n_1+n_2+\ldots+n_m; n_1, n_2, \ldots, n_m) \prod_{k=1}^{m} \frac{\partial^{n_k} f(z)}{\partial z^{n_k}} /; m \in \mathbb{N}^+ \bigwedge n \in \mathbb{N}$$

This formula shows how to evaluate an n^{th}-order derivative of the general integer power $f(z)^m /; m \in \mathbb{N}^+$.

Negative integer powers

$$\frac{\partial^n \frac{1}{f(z)}}{\partial z^n} = (n+1)\sum_{k=0}^{n} \frac{(-1)^k}{k+1}\binom{n}{k} f(z)^{-k-1} \frac{\partial^n f(z)^k}{\partial z^n} /; n \in \mathbb{N}^+$$

This formula shows how to evaluate an n^{th}-order derivative of the reciprocal $1/f(z)$.

$$\frac{\partial^n \frac{1}{f(z)}}{\partial z^n} = \frac{\delta_n}{f(z)} + (n+1)\sum_{k=1}^{n} \frac{(-1)^k}{k+1}\binom{n}{k} f(z)^{-k-1} \frac{\partial^n f(z)^k}{\partial z^n} /; n \in \mathbb{N}$$

This formula shows how to evaluate an n^{th}-order derivative of the reciprocal $1/f(z)$.

$$\frac{\partial^n \frac{1}{f(z)}}{\partial z^n} = \frac{\delta_n}{f(z)} + (n+1)\sum_{m=1}^{n} \frac{(-1)^m}{m+1}\binom{n}{m} f(z)^{-m-1} \sum_{k(1)=0}^{n}\sum_{k(2)=0}^{n-k(1)} \cdots \sum_{k(m-1)=0}^{n-\sum_{j=1}^{m-2} k(j)} \left(\prod_{p=1}^{m-1}\binom{n-\sum_{j=1}^{p-1} k(j)}{k(p)}\right)\left(\prod_{i=1}^{m-1}\frac{\partial^{k(i)} f(z)}{\partial z^{k(i)}}\right) \frac{\partial^{n-\sum_{j=1}^{m-1} k(j)} f(z)}{\partial z^{n-\sum_{j=1}^{m-1} k(j)}} /; n \in \mathbb{N}$$

This formula shows how to evaluate an n^{th}-order derivative of the reciprocal $1/f(z)$.

$$\frac{\partial^n \frac{1}{f(z)}}{\partial z^n} = \frac{\delta_n}{f(z)} + (n+1)\sum_{m=1}^{n} \frac{(-1)^m}{m+1}\binom{n}{m} f(z)^{-m-1} \sum_{n_1=0}^{n}\sum_{n_2=0}^{n} \dots \sum_{n_m=0}^{n} \delta_{n,\sum_{k=1}^{m} n_k} (n_1+n_2+\dots+n_m;\, n_1,\, n_2,\, \dots,\, n_m) \prod_{k=1}^{m} \frac{\partial^{n_k} f(z)}{\partial z^{n_k}} \;/;\; n \in \mathbb{N}$$

This formula shows how to evaluate an n^{th}-order derivative of the reciprocal $1/f(z)$.

Log from function

$$\frac{\partial^n \log(f(z))}{\partial z^n} = \delta_n \log(f(z)) + \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k\, f(z)^k}\binom{n}{k} \frac{\partial^n f(z)^k}{\partial z^n} \;/;\; n \in \mathbb{N}$$

This formula shows how to evaluate an n^{th}-order derivative of the composition with logarithm $\log(f(z))$.

$$\frac{\partial^n \log(f(z))}{\partial z^n} = \delta_n \log(f(z)) + \sum_{m=1}^{n} \frac{(-1)^{m-1}}{m}\binom{n}{m} f(z)^{-m} \sum_{k(1)=0}^{n}\sum_{k(2)=0}^{n-k(1)} \dots \sum_{k(m-1)=0}^{n-\sum_{j=1}^{m-2} k(j)} \left(\prod_{p=1}^{m-1} \binom{n-\sum_{j=1}^{p-1} k(j)}{k(p)}\right)\left(\prod_{i=1}^{m-1} \frac{\partial^{k(i)} f(z)}{\partial z^{k(i)}}\right) \frac{\partial^{n-\sum_{j=1}^{m-1} k(j)} f(z)}{\partial z^{n-\sum_{j=1}^{m-1} k(j)}} \;/;\; n \in \mathbb{N}$$

This formula shows how to evaluate an n^{th}-order derivative of the composition with logarithm $\log(f(z))$.

This formula shows how to evaluate an n^{th}-order derivative of the composition with logarithm $\log(f(z))$.

Exp from function

$$\frac{\partial^n e^{g(z)}}{\partial z^n} = e^{g(z)} \sum_{m=0}^{n} \frac{1}{m!} \sum_{j=0}^{m} (-1)^j \binom{m}{j} g(z)^j \frac{\partial^n g(z)^{m-j}}{\partial z^n} \;/;\; n \in \mathbb{N}$$

This formula shows how to evaluate an n^{th}-order derivative of the composition with exponential function $\exp(f(z))$.

$$\frac{\partial^n e^{g(z)}}{\partial z^n} = e^{g(z)} \left(\delta_n + \sum_{m=1}^{n} \frac{1}{m!} \sum_{j=0}^{m} (-1)^j \binom{m}{j} g(z)^j \frac{\partial^n g(z)^{m-j}}{\partial z^n}\right) \;/;\; n \in \mathbb{N}$$

This formula shows how to evaluate an n^{th}-order derivative of the composition with exponential function $\exp(f(z))$.

$$\frac{\partial^n e^{g(z)}}{\partial z^n} = e^{g(z)} \sum_{m=1}^{n} \sum_{k_1+2k_2+\dots+n k_n=n} \delta_{m,\sum_{r=1}^{n} k_r} \frac{n!}{\prod_{j=1}^{n} j!^{k_j}\, k_j!} \prod_{j=1}^{n} g^{(j)}(z)^{k_j} \;/;\; n \in \mathbb{N}^+$$

This formula shows how to evaluate an n^{th}-order derivative of the composition with exponential function $\exp(f(z))$.

$$\frac{\partial^n e^{g(z)}}{\partial z^n} = e^{g(z)} \sum_{m=0}^{n} \frac{1}{m!} \sum_{q=0}^{m} (-1)^q \binom{m}{q} g(z)^q \left(\delta_n \delta_{m-q} - g^{(n)}(z)\, \delta_{m-q} + \sum_{k(1)=0}^{n}\sum_{k(2)=0}^{n-k(1)} \dots \sum_{k(m-q-1)=0}^{n-\sum_{j=1}^{m-q-2} k(j)} \left(\prod_{p=1}^{m-q-1} \binom{n-\sum_{j=1}^{p-1} k(j)}{k(p)}\right)\left(\prod_{i=1}^{m-q-1} \frac{\partial^{k(i)} g(z)}{\partial z^{k(i)}}\right) \frac{\partial^{n-\sum_{j=1}^{m-q-1} k(j)} g(z)}{\partial z^{n-\sum_{j=1}^{m-q-1} k(j)}}\right) \;/;\; n \in \mathbb{N}$$

This formula shows how to evaluate an n^{th}-order derivative of the composition with exponential function $\exp(f(z))$.

$$\frac{\partial^n e^{g(z)}}{\partial z^n} == e^{g(z)} \left(\delta_n + \sum_{m=1}^{n} \frac{1}{m!} \sum_{j=0}^{m} (-1)^j \binom{m}{j} g(z)^j \sum_{n_1=0}^{n} \sum_{n_2=0}^{n} \ldots \sum_{n_{m-j}=0}^{n} \delta_{n, \sum_{k=1}^{m-j} n_k} \left(n_1 + n_2 + \ldots + n_{m-j}; n_1, n_2, \ldots, n_{m-j}\right) \prod_{k=1}^{m-j} \frac{\partial^{n_k} f(z)}{\partial z^{n_k}} \right) /;$$
$$n \in \mathbb{N}$$

This formula shows how to evaluate an n^{th}-order derivative of the composition with exponential function $\exp(f(z))$.

Function from power

$$\frac{\partial^n f(z^a)}{\partial z^n} == \sum_{k=0}^{n} \sum_{j=0}^{k} \frac{(-1)^j (a k - a j - n + 1)_n f^{(k)}(z^a)}{j! (k-j)! z^{n-a k}} /; n \in \mathbb{N}$$

This formula shows how to evaluate an n^{th}-order derivative of the composition with power function $f(z^a)$.

$$\frac{\partial^n f(z^2)}{\partial z^n} == \sum_{k=0}^{n} \frac{(2k - n + 1)_{2(n-k)}}{(n-k)! (2z)^{n-2k}} f^{(k)}(z^2) /; n \in \mathbb{N}$$

This formula shows how to evaluate an n^{th}-order derivative of the composition $f(z^2)$.

$$\frac{\partial^n f\left(\frac{1}{z}\right)}{\partial z^n} == f\left(\frac{1}{z}\right) \delta_n + (-1)^n \sum_{k=1}^{n} \frac{(n-1)!}{(k-1)! z^{k+n}} \binom{n}{k} f^{(k)}\left(\frac{1}{z}\right) /; n \in \mathbb{N}$$

This formula shows how to evaluate the derivative of the n^{th}-order of the composition $f(1/z)$.

$$\frac{\partial^n f(\sqrt{z})}{\partial z^n} == \sum_{k=0}^{n} \frac{(-1)^{n-k} (k)_{2(n-k)}}{(n-k)! \left(2\sqrt{z}\right)^{2n-k}} f^{(k)}(\sqrt{z}) /; n \in \mathbb{N}$$

This formula shows how to evaluate an n^{th}-order derivative of the composition $f(\sqrt{z})$.

Function from exponent

$$\frac{\partial^n f(e^z)}{\partial z^n} == \sum_{k=0}^{n} e^{k z} \mathcal{S}_n^{(k)} f^{(k)}(e^z) /; n \in \mathbb{N}^+$$

This formula shows how to evaluate an n^{th}-order derivative of the composition $f(e^z)$.

$$\frac{\partial^n f(a^z)}{\partial z^n} == \log^n(a) \sum_{k=0}^{n} a^{k z} \mathcal{S}_n^{(k)} f^{(k)}(a^z) /; n \in \mathbb{N}$$

This formula shows how to evaluate an n^{th}-order derivative of the composition $f(a^z)$.

$$\frac{\partial^n f(e^{1/z})}{\partial z^n} == f(e^{1/z}) \delta_n + (-1)^n \sum_{k=1}^{n} \frac{(n-1)! \binom{n}{k}}{(k-1)! z^{k+n}} \sum_{m=0}^{k} e^{m/z} \mathcal{S}_k^{(m)} f^{(m)}(e^{1/z}) /; n \in \mathbb{N}$$

This formula shows how to evaluate an n^{th}-order derivative of the composition $f\left(e^{1/z}\right)$.

$$\frac{\partial^n f\left(c^{b/z}\right)}{\partial z^n} = f\left(c^{b/z}\right)\delta_n + (-1)^n \sum_{k=1}^{n} \frac{(n-1)!\binom{n}{k}(b\log(c))^k}{(k-1)!\,z^{k+n}} \sum_{m=0}^{k} c^{\frac{m b}{z}}\,\mathcal{S}_k^{(m)}\, f^{(m)}\left(c^{b/z}\right) /;\, n \in \mathbb{N}$$

This formula shows how to evaluate an n^{th}-order derivative of the composition $f\left(c^{b/z}\right)$.

$$\frac{\partial^n f\left(e^{\sqrt{z}}\right)}{\partial z^n} = \sum_{k=0}^{n} \frac{(-1)^{n-k}(k)_{2(n-k)}}{(n-k)!\left(2\sqrt{z}\right)^{2n-k}} \sum_{m=0}^{k} e^{m\sqrt{z}}\,\mathcal{S}_k^{(m)}\, f^{(m)}\left(e^{\sqrt{z}}\right) /;\, n \in \mathbb{N}$$

This formula shows how to evaluate an n^{th}-order derivative of the composition $f\left(e^{\sqrt{z}}\right)$.

$$\frac{\partial^n f\left(e^{z^2}\right)}{\partial z^n} = \sum_{k=0}^{n} \frac{(2k-n+1)_{2(n-k)}}{(n-k)!\,(2z)^{n-2k}} \sum_{m=0}^{k} e^{m z^2}\,\mathcal{S}_k^{(m)}\, f^{(m)}\left(e^{z^2}\right) /;\, n \in \mathbb{N}$$

This formula shows how to evaluate an n^{th}-order derivative of the composition $f\left(e^{z^2}\right)$.

$$\frac{\partial^n f\left(e^{z^a}\right)}{\partial z^n} = \sum_{k=0}^{n}\sum_{j=0}^{k} \frac{(-1)^j(a k - a j - n + 1)_n}{j!\,(k-j)!\,z^{n-a k}} \sum_{m=0}^{k} e^{m z^a}\,\mathcal{S}_k^{(m)}\, f^{(m)}\left(e^{z^a}\right) /;\, n \in \mathbb{N}$$

This formula shows how to evaluate an n^{th}-order derivative of the composition $f\left(e^{z^a}\right)$.

$$\frac{\partial^n f\left(c^{b\sqrt{z}}\right)}{\partial z^n} = \sum_{k=0}^{n} \frac{(-1)^{n-k}(k)_{2(n-k)}(b\log(c))^k}{(n-k)!\left(2\sqrt{z}\right)^{2n-k}} \sum_{m=0}^{k} c^{m b\sqrt{z}}\,\mathcal{S}_k^{(m)}\, f^{(m)}\left(c^{b\sqrt{z}}\right) /;\, n \in \mathbb{N}$$

This formula shows how to evaluate an n^{th}-order derivative of the composition $f\left(c^{b\sqrt{z}}\right)$.

$$\frac{\partial^n f\left(c^{b z^2}\right)}{\partial z^n} = \sum_{k=0}^{n} \frac{(2k-n+1)_{2(n-k)}(b\log(c))^k}{(n-k)!\,(2z)^{n-2k}} \sum_{m=0}^{k} c^{m b z^2}\,\mathcal{S}_k^{(m)}\, f^{(m)}\left(c^{b z^2}\right) /;\, n \in \mathbb{N}$$

This formula shows how to evaluate an n^{th}-order derivative of the composition $f\left(c^{b z^2}\right)$.

$$\frac{\partial^n f\left(c^{b z^a}\right)}{\partial z^n} = \sum_{k=0}^{n}\sum_{j=0}^{k} \frac{(-1)^j(-a j + a k - n + 1)_n(b\log(c))^k}{j!\,(k-j)!\,z^{n-a k}} \sum_{m=0}^{k} c^{m b z^a}\,\mathcal{S}_k^{(m)}\, f^{(m)}\left(c^{b z^a}\right) /;\, n \in \mathbb{N}$$

This formula shows how to evaluate an n^{th}-order derivative of the composition $f\left(c^{b z^a}\right)$.

Function from trigonometric functions

$$\frac{\partial^n f(\sin(z))}{\partial z^n} = f(\sin(z))\,\delta_n + \sum_{m=1}^{n} \frac{1}{m!} \left(\sum_{j=0}^{m-1} \binom{m}{j} \sum_{l=0}^{m-j} (-1)^j\, 2^{j-m} \sin^j(z)\, (j+2l-m)^n \exp\!\left(-\frac{i}{2}\,(n\pi - (j+2l-m)(\pi - 2z))\right) \binom{m-j}{l} f^{(m)}(\sin(z)) \right) /;\, n \in \mathbb{N}$$

This formula shows how to evaluate an n^{th}-order derivative of the composition $f(\sin(z))$.

$$\frac{\partial^n f(\sin(z))}{\partial z^n} = \sum_{m=1}^{n} \frac{f^{(m)}(\sin(z))}{m!} \left(\frac{\partial^n (\sin(y) - \sin(z))^m}{\partial y^n} \,/.\, \{y \to z\} \right) /;\, n \in \mathbb{N}^+$$

This formula shows how to evaluate an n^{th}-order derivative of the composition $f(\sin(z))$.

$$\frac{\partial^n f(\cos(z))}{\partial z^n} = f(\cos(z))\,\delta_n + i^n \sum_{m=1}^{n} \frac{1}{m!} \left(\sum_{j=0}^{m-1} (-1)^j \binom{m}{j} \left(\sum_{l=0}^{m-j} 2^{j-m} \cos^j(z)\, (j+2l-m)^n\, e^{(j+2l-m)\,i\,z} \binom{m-j}{l} \right) f^{(m)}(\cos(z)) \right) /;\, n \in \mathbb{N}$$

This formula shows how to evaluate an n^{th}-order derivative of the composition $f(\cos(z))$.

$$\frac{\partial^n f(\cos(z))}{\partial z^n} = \sum_{m=1}^{n} \frac{f^{(m)}(\cos(z))}{m!} \left(\frac{\partial^n (\cos(y) - \cos(z))^m}{\partial y^n} \,/.\, \{y \to z\} \right) /;\, n \in \mathbb{N}^+$$

This formula shows how to evaluate an n^{th}-order derivative of the composition $f(\cos(z))$.

Function from hyperbolic functions

$$\frac{\partial^n f(\sinh(z))}{\partial z^n} = f(\sinh(z))\,\delta_n + (-1)^n \sum_{m=1}^{n} \frac{1}{m!} \sum_{j=0}^{m-1} \binom{m}{j} \sum_{l=0}^{m-j} (-1)^{j-l}\, 2^{j-m} \sinh^j(z)\, (j+2l-m)^n\, e^{-(j+2l-m)z} \binom{m-j}{l} f^{(m)}(\sinh(z)) /;\, n \in \mathbb{N}^+$$

This formula shows how to evaluate an n^{th}-order derivative of the composition $f(\sinh(z))$.

$$\frac{\partial^n f(\sinh(z))}{\partial z^n} = \sum_{m=1}^{n} \left(\frac{\partial^n (\sinh(y) - \sinh(z))^m}{\partial y^n} \,/.\, \{y \to z\} \right) \frac{f^{(m)}(\sinh(z))}{m!} /;\, n \in \mathbb{N}^+$$

This formula shows how to evaluate an n^{th}-order derivative of the composition $f(\sinh(z))$.

$$\frac{\partial^n f(\cosh(z))}{\partial z^n} = f(\cosh(z))\,\delta_n + \sum_{m=1}^{n} \frac{1}{m!} \left(\sum_{j=0}^{m-1} (-1)^j \binom{m}{j} \sum_{i=0}^{m-j} 2^{j-m} \cosh^j(z)\, (2i+j-m)^n\, e^{(2i+j-m)z} \binom{m-j}{i} f^{(m)}(\cosh(z)) \right) /;\, n \in \mathbb{N}$$

This formula shows how to evaluate an n^{th}-order derivative of the composition $f(\cosh(z))$.

$$\frac{\partial^n f(\cosh(z))}{\partial z^n} = \sum_{m=1}^{n} \frac{f^{(m)}(\cosh(z))}{m!} \left(\frac{\partial^n (\cosh(y) - \cosh(z))^m}{\partial y^n} \,/.\, \{y \to z\} \right) /;\, n \in \mathbb{N}^+$$

This formula shows how to evaluate an n^{th}-order derivative of the composition $f(\cosh(z))$.

General compositions

$$\frac{\partial^n f(g(z))}{\partial z^n} = \sum_{m=0}^{n} \frac{1}{m!} \left(\sum_{j=0}^{m} (-1)^j \binom{m}{j} g(z)^j \frac{\partial^n g(z)^{m-j}}{\partial z^n} \right) f^{(m)}(g(z)) \;/;\; n \in \mathbb{N}$$

This formula shows how to evaluate an n^{th}-order derivative of the general composition $f(g(z))$.

$$\frac{\partial^n f(g(z))}{\partial z^n} = f(g(z))\, \delta_n + \sum_{m=1}^{n} \frac{1}{m!} \left(\sum_{j=0}^{m} (-1)^j \binom{m}{j} g(z)^j \frac{\partial^n g(z)^{m-j}}{\partial z^n} \right) f^{(m)}(g(z)) \;/;\; n \in \mathbb{N}$$

This formula shows how to evaluate an n^{th}-order derivative of the general composition $f(g(z))$.

$$\frac{\partial^n f(g(z))}{\partial z^n} = \sum_{m=1}^{n} \sum_{k_1 + 2 k_2 + \ldots + n k_n = n} \delta_{m, \sum_{r=1}^{n} k_r} \frac{f^{(m)}(g(z))\, n!}{\prod_{j=1}^{n} j!^{k_j}\, k_j!} \prod_{j=1}^{n} g^{(j)}(z)^{k_j} \;/;\; n \in \mathbb{N}^+$$

This formula is called *Faá di Bruno's formula*.

$$\frac{\partial^n f(g(z))}{\partial z^n} = \sum_{m=0}^{n} \frac{1}{m!} \sum_{q=0}^{m} (-1)^q \binom{m}{q} g(z)^q \Bigg(\delta_n\, \delta_{m-q} - g^{(n)}(z)\, \delta_{m-q} +$$
$$\sum_{k(1)=0}^{n} \sum_{k(2)=0}^{n-k(1)} \ldots \sum_{k(m-q-1)=0}^{n-\sum_{j=1}^{m-q-2} k(j)} \left(\prod_{p=1}^{m-q-1} \binom{n - \sum_{j=1}^{p-1} k(j)}{k(p)} \right) \left(\prod_{i=1}^{m-q-1} \frac{\partial^{k(i)} g(z)}{\partial z^{k(i)}} \right) \frac{\partial^{n-\sum_{j=1}^{m-q-1} k(j)} g(z)}{\partial z^{n-\sum_{j=1}^{m-q-1} k(j)}} \Bigg) f^{(m)}(g(z)) \;/;\; n \in \mathbb{N}$$

This formula, *Faá di Bruno's relation*, shows how to evaluate an n^{th}-order derivative of the general composition $f(g(z))$.

$$\frac{\partial^n f(g(z))}{\partial z^n} = f(g(z))\, \delta_n +$$
$$\sum_{m=1}^{n} \frac{1}{m!} f^{(m)}(g(z)) \sum_{j=0}^{m} (-1)^j \binom{m}{j} g(z)^j \sum_{n_1=0}^{n} \sum_{n_2=0}^{n} \ldots \sum_{n_{m-j}=0}^{n} \delta_{n, \sum_{k=1}^{m-j} n_k} (n_1 + n_2 + \ldots + n_{m-j};\, n_1, n_2, \ldots, n_{m-j}) \prod_{k=1}^{m-j} \frac{\partial^{n_k} f(z)}{\partial z^{n_k}} \;/;\; n \in \mathbb{N}$$

This formula shows how to evaluate an n^{th}-order derivative of the general composition $f(g(z))$.

General power exponential compositions

$$\frac{\partial^n f(z)^{g(z)}}{\partial z^n} =$$
$$f(z)^{g(z)} \Bigg(\delta_n + \sum_{m=1}^{n} \frac{1}{m!} \sum_{j=0}^{m-1} (-1)^j \binom{m}{j} (g(z) \log(f(z)))^j \sum_{k(1)=1}^{n} \sum_{k(2)=0}^{n} \ldots \sum_{k(m-j)=0}^{n} \delta_{n, \sum_{v=1}^{m-j} k(v)} (k(1) + k(2) + \ldots + k(m-j);\, k(1),\, k($$
$$2), \ldots, k(m-j)) \prod_{i=1}^{m-j} \sum_{s=0}^{k(i)} \binom{k(i)}{s} \Bigg(\delta_s \log(f(z)) + \sum_{h=1}^{s} \frac{(-1)^{h-1}}{h\, f(z)^h} \binom{s}{h} \sum_{q(1)=0}^{s} \sum_{q(2)=0}^{s} \ldots \sum_{q(h)=0}^{s} \delta_{s, \sum_{u=1}^{h} q(u)}$$
$$(q(1) + q(2) + \ldots + q(h);\, q(1), q(2), \ldots, q(h)) \prod_{i=1}^{h} \frac{\partial^{q(i)} f(z)}{\partial z^{q(i)}} \Bigg) \frac{\partial^{k(i)-s} g(z)}{\partial z^{k(i)-s}} \Bigg) \;/;\; n \in \mathbb{N}$$

This formula shows how to evaluate an n^{th}-order derivative of the general power exponential composition $f(z)^{g(z)}$.

$$\frac{\partial^n z^z}{\partial z^n} = z^z \left(\delta_n + \sum_{m=1}^{n} \frac{1}{m!} \sum_{j=0}^{m-1} (-1)^j \binom{m}{j} (z \log(z))^j \sum_{k(1)=0}^{n} \sum_{k(2)=0}^{n} \ldots \sum_{k(m-j)=0}^{n} \delta_{n,\sum_{v=1}^{m-j} k(v)} \, (k(1)+k(2)+\ldots+k(m-j); k(1), k(2), \ldots, k(m-j)) \prod_{i=1}^{m-j} \left(\left(k(i)\, S_{k(i)-1}^{(1)} + S_{k(i)}^{(1)}\right) z^{1-k(i)} + \log(z) \left(k(i)\, S_{k(i)-1}^{(0)} + z\, S_{k(i)}^{(0)}\right)\right)\right) /; n \in \mathbb{N}$$

This formula shows how to evaluate an n^{th}-order derivative of the general power exponential composition z^z.

Inverse function

$$f^{(-1)^{(n)}}(z) == \delta_n \, f^{(-1)}(z) + \left(f'\left(f^{(-1)}(z)\right)\right)^{-n} \sum_{j_2=0}^{n} \ldots \sum_{j_n=0}^{n} \delta_{\sum_{i=2}^{n}(i-1)\, j_i, n-1} \, (-1)^{\sum_{i=2}^{n} j_i} \left(n + \sum_{i=2}^{n} j_i - 1\right)! \prod_{i=2}^{n} \frac{1}{j_i!} \left(\frac{f^{(i)}\left(f^{(-1)}(z)\right)}{i!\, f'\left(f^{(-1)}(z)\right)}\right)^{j_i} /; n \in \mathbb{N}$$

This formula shows how to evaluate an n^{th}-order derivative of the inverse function $f^{(-1)}(z)$.

Repeated derivatives

$$\underbrace{z \frac{\partial}{\partial z}\left(\ldots\left(z \frac{\partial}{\partial z}\right.\right.}_{n \text{ times}} (f(z))\Big)\Big) == \sum_{k=0}^{n} \mathcal{S}_n^{(k)} z^k \frac{\partial^k f(z)}{\partial z^k} /; n \in \mathbb{N}$$

This formula shows how to apply n times the operation $z \frac{\partial}{\partial z}$ to the function $f(z)$.

$$\underbrace{(z-a) \frac{\partial}{\partial z}\left(\ldots\left((z-a) \frac{\partial}{\partial z}\right.\right.}_{n \text{ times}} (f(z))\Big)\Big) == \sum_{k=0}^{n} \mathcal{S}_n^{(k)} (z-a)^k \frac{\partial^k f(z)}{\partial z^k} /; n \in \mathbb{N}$$

This formula shows how to apply n times the operation $(z-a) \frac{\partial}{\partial z}$ to the function $f(z)$.

Fractional integro-differentiation

$$\frac{\partial^\alpha f(z)}{\partial z^\alpha} = \begin{cases} \dfrac{\partial^n \left(\int_0^z \frac{f(t)\,(z-t)^{n+\alpha-1}}{\Gamma(n+\alpha)}\, dt\right)}{\partial z^n} & n = \lfloor \mathrm{Re}(-\alpha) \rfloor + 1 \land \mathrm{Re}(-\alpha) \leq 0 \\ \int_0^z \frac{f(t)\,(z-t)^{-\alpha-1}}{\Gamma(-\alpha)}\, dt & \text{True} \end{cases}$$

The α^{th} fractional integro-derivative of the function $f(z)$ with respect to z is defined by the preceding formula, where the integration in *Mathematica* should be performed with the option `GenerateConditions->False`: `Integrate[f[t] (z-t)^(α+n-1)/Gamma[α+n], {t, 0, z}, GenerateConditions → False]`. This definition supports the Riemann-Liouville-Hadamard fractional left-sided integro-differentiation at the point 0.

$$\frac{\partial^\alpha f(z)}{\partial z^\alpha} = \int_0^z \frac{f(t)\,(z-t)^{-\alpha-1}}{\Gamma(-\alpha)}\, dt /; \mathrm{Re}(-\alpha) > 0$$

This formula for the α^{th} fractional integro-derivative represents the fractional integral of the function $f(z)$ with respect to z. This integral is called the *Abel integral*.

$$\frac{\partial^\alpha f(z)}{\partial z^\alpha} = \frac{\partial^n\left(\int_0^z \frac{f(t)\,(z-t)^{n+\alpha-1}}{\Gamma(n+\alpha)}\,dt\right)}{\partial z^n} \;/;\; n = \lfloor \operatorname{Re}(-\alpha) \rfloor + 1 \land \operatorname{Re}(-\alpha) \leq 0$$

This formula for the α^{th} fractional integro-derivative actually represents the fractional derivative of the function $f(z)$ with respect to z. This derivative includes the composition of the corresponding usual n^{th} derivative of order $n = \lfloor \operatorname{Re}(-\alpha) \rfloor + 1$ and an Abel integral.

$$\frac{\partial^\alpha \left(\sum_{k=0}^\infty c_k\, z^k\right)}{\partial z^\alpha} = \sum_{k=0}^\infty \frac{k!\, c_k\, z^{k-\alpha}}{\Gamma(k-\alpha+1)}$$

This formula shows how to evaluate the α^{th} fractional integro-derivative of the analytical function near the point $z == 0$.

$$\frac{\partial^\alpha \left(\log^n(z) \sum_{k=-\infty}^\infty c_k\, z^{a+k}\right)}{\partial z^\alpha} = \sum_{k=-\infty}^\infty c_k\, \mathcal{FC}_{\log}^{(\alpha)}(z,\, a+k,\, n)\, z^{k+a-\alpha} \;/;\; n \in \mathbb{N}$$

This formula shows how to evaluate the α^{th} fractional integro-derivative of a function having Laurent series expansion, multiplied on $\log^n(z)\, z^a$ near the point $z == 0$.

$$\frac{\partial^\alpha f(z)}{\partial z^\alpha} = \sum_{k=0}^\infty \sum_{j=0}^\infty \frac{k!\, a_{j+k} \binom{j+k}{k} (-z_0)^j\, z^{k-\alpha}}{\Gamma(k-\alpha+1)} \;/;\; f(z) = \sum_{k=0}^\infty a_k\, (z-z_0)^k$$

This formula shows that for the evaluation of the α^{th} fractional integro-derivative of the analytical function $f(z)$ near the point $z == z_0$, you need to re-expand this function in a series near the point $z == 0$ and then evaluate the corresponding integro-derivative.

Integration

Indefinite integration

For the direct function

$$\int f'(z)\, dz == f(z)$$

Integration of the derivative gives the original function.

$$\int (f(z) \pm g(z))\, dz == \int f(z)\, dz \pm \int g(z)\, dz$$

The integral of the sum gives the sum of the integrals.

$$\int c\, f(z)\, dz == c \int f(z)\, dz$$

The constant factor can be placed outside of the integral.

$$\int f(g(z))\, g'(z)\, dz == \int f(w)\, dw \,/;\, w == g(z)$$

This formula reflects the *changing variables rule* in the integral.

$$\int f(z)\, g'(z)\, dz == f(z)\, g(z) - \int g(z)\, f'(z)\, dz$$

This formula reflects the *integration by parts* rule.

$$\int f(z) \frac{\partial^n g(z)}{\partial z^n}\, dz = (-1)^n \int \frac{\partial^n f(z)}{\partial z^n}\, g(z)\, dz + \sum_{k=0}^{n-1} (-1)^k \frac{\partial^k f(z)}{\partial z^k} \frac{\partial^{n-k-1} g(z)}{\partial z^{n-k-1}}$$

This formula reflects the *generalized integration by parts* rule.

$$\int f(z)\, dz = \sum_{k=0}^{\infty} \int u_k(z)\, dz \,/;\, f(z) = \sum_{k=0}^{\infty} u_k(z)$$

The integral of the sum is equal to the sum of the integrals of the summands (under some restrictions for convergence of the occurring infinite series).

$$\int f(z)\, dz == \sum_{k=0}^{\infty} \frac{a_k\, z^{k+1}}{k+1} \,/;\, |z| < r \bigwedge \left(f(z) == \sum_{k=0}^{\infty} a_k\, z^k \,/;\, |z| < r \right)$$

The integral from the power series is equal to the sum of the integrals from each term of the series (inside some circle of convergence).

Repeated indefinite integration

$$\underbrace{\int dz \left(\ldots \left(\int dz\, (f(z)) \right) \right)}_{n \text{ times}} == \int_0^z \frac{(z-t)^{n-1}}{(n-1)!}\, f(t)\, dt$$

This formula reflects repeated indefinite integration, where the integration in *Mathematica* should be performed with the option `GenerateConditions -> False`: `Integrate[(z-t)^(n-1)/(n-1)! f[t], {t, 0, z}, GenerateConditions → False]`.

$$\underbrace{\int \frac{1}{z}\, dz \left(\ldots \left(\int \frac{1}{z}\, dz\, (f(z)) \right) \right)}_{n \text{ times}} == \int_0^z \frac{(\log(z) - \log(t))^{n-1}}{t\,(n-1)!}\, f(t)\, dt$$

This formula reflects repeated indefinite integration, where the integration in *Mathematica* should be performed with the option `GenerateConditions -> False`: `Integrate[(Log[z]-Log[t])^(n-1)/(t (n-1)!) f[t], {t, 0, z}, GenerateConditions → False]`.

Definite integration

For the direct function

$$\int_a^b f(t)\,dt = \sum_{j=0}^{\infty} \frac{(b-a)^{2k+1}}{4^k\,(2k+1)!}\, f^{(2k)}\!\left(\frac{b-a}{2}\right)$$

$$\int_a^b f'(t)\,dt = f(b) - f(a)$$

$$\int_a^b (f(t) \pm g(t))\,dt = \int_a^b f(t)\,dt \pm \int_a^b g(t)\,dt$$

$$\int_a^b c\, f(t)\,dt = c\int_a^b f(t)\,dt$$

$$\int f(z)\, g'(z)\,dz = f(z)\, g(z) - \int g(z)\, f'(z)\,dz$$

$$\int_a^b f(g(t))\, g'(t)\,dt = \int_{g(a)}^{g(b)} f(\tau)\,d\tau \;/;\; \tau = g(t)$$

$$\int_b^a f(t)\,dt = -\int_a^b f(t)\,dt$$

$$\int_a^b f(t)\,dt = \int_a^c f(t)\,dt + \int_c^b f(t)\,dt$$

$$\int_a^b f(t)\,dt + \int_b^c f(t)\,dt = \int_a^c f(t)\,dt$$

$$\int_a^b f(t)\,dt = \sum_{k=0}^{\infty} \frac{a_k\left(b^{k+1} - a^{k+1}\right)}{k+1} \;/;\; f(z) = \sum_{k=0}^{\infty} a_k\, z^k$$

$$\int_b^a f(t)\, g(t)\,dt = f(\tau)\int_a^b g(t)\,dt \;/;\; \tau \in (a, b) \bigwedge g[t] \geq 0 \;/;\; t \in (a, b)$$

$$\int_b^a f(t)\, g(t)\,dt = \mu \int_a^b g(t)\,dt \;/;\; \min(f(t)) \leq \mu \leq \max(f(t))$$

This formula reflects *the first mean value theorem*.

$$\int_b^a f(t)\, g(t)\,dt = f(a)\int_a^{\alpha} g(t)\,dt \;/;\; \alpha \in (a, b) \bigwedge f'(t) < 0 \bigwedge f(t) \geq 0 \;/;\; t \in (a, b)$$

This formula reflects *the second mean value theorem*.

$$\int_b^a f(t)\, g(t)\,dt = f(b)\int_{\beta}^b g(t)\,dt \;/;\; \beta \in (a, b) \bigwedge f'(t) > 0 \bigwedge f(t) \geq 0 \;/;\; t \in (a, b)$$

$$\int_b^a f(t)\, g(t)\,dt = f(a)\int_a^{\alpha} g(t)\,dt + f(b)\int_{\alpha}^b g(t)\,dt \;/;\; \alpha \in (a, b) \bigwedge f'(t) > 0 \;/;\; t \in (a, b)$$

$$\int_{\alpha}^{\beta}\int_{a}^{b} f(t, z)\,dt\,dz == \int_{a}^{b}\int_{\alpha}^{\beta} f(t, z)\,dz\,dt$$

$$\int_{a}^{b}\int_{a}^{b} \frac{f(t, \tau)}{(t-x)(\tau-t)}\,dt\,d\tau == \int_{a}^{b}\int_{a}^{b} \frac{f(t, \tau)}{(t-x)(\tau-t)}\,d\tau\,dt - \pi^2 f(x, x) /; x \in (a, b)$$

This formula is called *the Poincaré-Bertrand formula.*

$$\int_{0}^{\beta}\int_{0}^{z} f(t, z)\,dt\,dz == \int_{0}^{\beta}\int_{t}^{\beta} f(t, z)\,dz\,dt$$

$$\int_{a}^{b}\left(\sum_{k=0}^{\infty} a_k(z)\right)dz == \sum_{k=0}^{\infty}\int_{a}^{b} a_k(z)\,dz$$

Orthogonality

See Generalized Fourier series in the section Series representations.

Cauchy integrals

$$\Phi(z) == \frac{1}{2\pi i}\int_{L} \frac{\varphi(t)}{t-z}\,dt$$

This formula is called *the Cauchy-type integral* along the piecewise smooth contour L (which can be closed or opened). The *density function* $\varphi(\tau)$ must be continuous along L, but it must also meet a more stringent test known as the *Hölder condition*. The function $\varphi(t)$ satisfies the Hölder condition if, for two arbitrary points $t_1,\ t_2$ on the curve, $|\varphi(t_2) - \varphi(t_1)| < A\,|t_2 - t_1|^{\lambda}$ for some positive constants A and $0 < \lambda \leq 1$. The Cauchy-type integral is analytic everywhere on the complex plane except on the contour L itself, which is a singular line for this integral. Since the integral contains a factor (called the kernel) in the form of $1/(\tau - z)$, it diverges at $\tau == z$ for any z lying on L.

$$\frac{1}{2\pi i}\int_{L} \frac{\varphi(t)}{t-z}\,dt = \begin{cases} \varphi(z) & z \in D^+ \\ 0 & z \in D^- \end{cases}$$

This formula is called the *Cauchy integral formula* for the Cauchy integral. It is valid if L is a closed, smooth contour enclosing the region D^+ on the complex plane, and the function $\varphi(z)$ is analytic over D^+, continuous over $D^+ \bigcup L$, and D^- represents the region outside of L.

Singular integrals

$$\Phi(z) == \frac{1}{2\pi i}\int_{L} \frac{\varphi(t)}{t-z}\,dt /; z \in L$$

In this Cauchy-type integral, the singular point z belongs to the contour L and under the integrand function has a nonintegrable singularity. That is why this improper integral is called a *singular integral*. It can be evaluated, however, if a small neighborhood around this singularity $t == z$ is removed from the path of integration. The corresponding limit, as the size of the neighborhood shrinks to zero, is the *Cauchy principal value* of this divergent integral. For example:

$$\mathcal{P}\int_{a}^{b} \frac{\varphi(t)}{t-x}\,dt == \lim_{\epsilon\to 0}\left(\int_{a}^{x-\epsilon} \frac{\varphi(t)}{t-x}\,dt + \int_{x+\epsilon}^{b} \frac{\varphi(t)}{t-x}\,dt\right) /; a < x < b$$

In this example, $\mathcal{P}\int$ represents the Cauchy principal value, and the contour L is simply a straight segment on the real axis from a to b; in other words, $L = (a, b)$. Rather than integrating from a through the point $x \in L$ to the point b, you can integrate on the intervals $(a, x - \epsilon)$ and $(x + \epsilon, b)$ and then add these results to arrive at a value. By taking the limit of this calculation as $\epsilon \to 0$, you can state the principal value.

$$\mathcal{P}\int_L \frac{\varphi(t)}{t - t_0}\, dt == \lim_{\varrho \to 0} \int_{L-l} \frac{\varphi(t)}{t - t_0}\, dt \;/;\; t_0 \in L$$

This formula represents the Cauchy principal value of singular curvilinear integrals by the curve L with a circular neighborhood l, centered on t_0 and of radius ϱ, removed.

Sokhotskii formulas

General

$$\Phi(z) == \frac{1}{2\pi i} \int_L \frac{\varphi(t)}{t - z}\, dt$$

$$\Phi(z) = \begin{cases} \Phi^+[z] & z \in D^+ \\ \Phi^-[z] & z \in D^- \end{cases}$$

This formula represents the function $\Phi(z)$ as a piecewise analytic function in the case when L is a closed, smooth contour enclosing the region D^+ on the complex plane and D^- represents the region outside of L. If $t_0 \in L$, the values Φ^+ (t_0) and Φ^- (t_0) can be defined as the following limits:

$$\Phi^+(\tau_0) == \lim_{z \to t_0} \Phi(z) \;/;\; z \in D^+ \bigwedge t_0 \in L$$

$$\Phi^-(\tau_0) == \lim_{z \to t_0} \Phi(z) \;/;\; z \in D^- \bigwedge t_0 \in L$$

If L is an open contour with endpoints a and b, you can add an additional arbitrary curve segment connecting b to a (with the sense that $\Phi^+(\tau)$ corresponds to the limiting value from the left) and assign $\varphi(\tau) = 0$ along this new segment. This extension allows you to apply the definitions for Φ^+ and Φ^- to open contours L.

$$\Phi^+(t_0) == \frac{\varphi(t_0)}{2} + \frac{1}{2\pi i} \mathcal{P}\int_L \frac{\varphi(t)}{t - t_0}\, dt \;/;\; t_0 \in L$$

$$\Phi^-(t_0) == -\frac{\varphi(t_0)}{2} + \frac{1}{2\pi i} \mathcal{P}\int_L \frac{\varphi(t)}{t - t_0}\, dt \;/;\; t_0 \in L$$

These two formulas are called the *Sokhotskii formulas*. They were first derived by Y. V. Sokhotskii in 1873. Because they were later given a more rigorous treatment by J. Plemelj in 1908, they are often referred to as the Plemelj formulas. Sometimes the name Sokhotskii–Plemelj formulas is used.

$$\Phi^+(t_0) + \Phi^-(t_0) == \frac{1}{\pi i} \mathcal{P}\int_L \frac{\varphi(t)}{t - t_0}\, dt$$

$$\Phi^+(t_0) - \Phi^-(t_0) == \varphi(t_0)$$

The second of these formulas can be obtained from the Sokhotskii formulas by addition and subtraction.

In particular, if $\varphi(z)$ is analytical over $D^+ \bigcup L$, then $\Phi^+(t_0) == \varphi(t_0)$ and $\Phi^-(t_0) == 0$.

If the contour L is a finite or infinite segment of the real axis, $L = (a, b)$, these formulas hold for all $a < x < b$, and so $\Phi^+(x) = \lim_{\epsilon \to +0} \Phi(x + i\,\epsilon)$, $\Phi^-(x) = \lim_{\epsilon \to -0} \Phi(x - i\,\epsilon)$. Thus $\Phi(z)$ is an analytic function with a jump discontinuity at L, and the size of the jump is determined by the Sokhotskii formulas.

Example: The exponential integral Ei

$$\Phi(z) = \frac{1}{2\pi i} \int_0^\infty \frac{e^{-t}}{t - z}\, dt$$

After evaluation of this integral, you get:

$$\Phi(z) == \frac{e^{-z}}{2\pi i} \begin{cases} \pi i - \mathrm{Ei}(z) & \mathrm{Im}(z) > 0 \\ -\pi i - \mathrm{Ei}(z) & \mathrm{Im}(z) < 0 \\ -\mathrm{Ei}(z) & \mathrm{Im}(z) = 0 \end{cases}$$

For arbitrary $x > 0$, the Sokhotskii formulas give the following values:

$$\Phi^+(x) == \frac{e^{-x}}{2} + \frac{-e^{-x}\,\mathrm{Ei}(x)}{2\pi i}$$

$$\Phi^+(x) == -\frac{e^{-x}}{2} + \frac{-e^{-x}\,\mathrm{Ei}(x)}{2\pi i}$$

$$\Phi^+(x) + \Phi^-(x) == \frac{1}{\pi i}\, \mathcal{P} \int_0^\infty \frac{e^{-\tau}}{\tau - x}\, d\tau$$

$$\Phi^+(x) + \Phi^-(x) == -\frac{e^{-x}\,\mathrm{Ei}(x)}{\pi i}$$

$$\Phi^+(x) - \Phi^-(x) == e^{-x}$$

It is important to note that for $-\infty < z < 0$, the function $\Phi(z)$ is analytic and its limit values taken from either side of the real axis should agree with each other. This gives the relations:

$$\lim_{\epsilon \to 0^+} (\pi i - \mathrm{Ei}(x + i\,\epsilon)) = -\mathrm{Ei}(x)$$

$$\lim_{\epsilon \to 0^+} (-\pi i - \mathrm{Ei}(x - i\,\epsilon)) = -\mathrm{Ei}(x)$$

This leads to the following behavior of $\mathrm{Ei}(x)$:

$$\lim_{\epsilon \to 0^+} \mathrm{Ei}(x + i\,\epsilon) = \mathrm{Ei}(x) + \pi i$$

$$\lim_{\epsilon \to 0^+} \mathrm{Ei}(x - i\,\epsilon) = \mathrm{Ei}(x) - \pi i$$

Example: Beta-type integral

$$\Phi(z) == \frac{1}{2\pi\, i} \int_0^\infty \frac{t^{\alpha-1}}{t-z}\, dt$$

This integral can be called a beta-type integral. It can be evaluated by the following formulas:

$$\Phi(z) == \frac{1}{2\pi\, i} \begin{cases} \pi\,(-z)^{\alpha-1}\csc(\alpha\,\pi) & 0 < \mathrm{Re}(\alpha) < 1 \bigwedge \arg(z) \neq 0 \\ -\pi\, z^{\alpha-1}\cot(\alpha\,\pi) & 0 < \mathrm{Re}(\alpha) < 1 \wedge z > 0 \end{cases}$$

For arbitrary $x > 0$, the Sokhotskii formulas give the following results when you take into account the fact that, as $\epsilon \to +0$, $(-z)^{\alpha-1}$is continuous for $-z = -x + i\,\epsilon$ but has a jump of size $e^{-i\,\alpha\,\pi}$ compared to when it is approached from the other side, $-z = -x - i\,\epsilon$.

$$\Phi^+(x) = \lim_{\epsilon\to 0^+} \frac{(-x - i\,\epsilon)^{\alpha-1}\csc(\pi\,\alpha)}{2\,i} = -\frac{x^{\alpha-1}\, e^{-i\,\alpha\,\pi}\csc(\alpha\,\pi)}{2\,i} = \frac{x^{\alpha-1}}{2} + \frac{i}{2}\cot(\alpha\,\pi)\, x^{\alpha-1}$$

$$\Phi^-(x) = \lim_{\epsilon\to 0^+} \frac{(-x + i\,\epsilon)^{\alpha-1}\csc(\pi\,\alpha)}{2\,i} = -\frac{x^{\alpha-1}\, e^{i\,\alpha\,\pi}\csc(\alpha\,\pi)}{2\,i} = -\frac{x^{\alpha-1}}{2} + \frac{i}{2}\cot(\alpha\,\pi)\, x^{\alpha-1}$$

$$\Phi^+(x) + \Phi^-(x) = -\frac{x^{\alpha-1}\, e^{-i\,\alpha\,\pi}\csc(\alpha\;\pi)}{2\;i} - \frac{x^{\alpha-1}\, e^{i\,\alpha\,\pi}\csc(\alpha\;\pi)}{2\;i} == i\; x^{\alpha-1}\cot(\alpha\;\pi) == \frac{1}{\pi\; i}\int_0^\infty \frac{t^{\alpha-1}}{t-x}\, dt$$

$$\Phi^+(x) - \Phi^-(x) = \frac{x^{\alpha-1}\, e^{i\,\alpha\,\pi}\csc(\alpha\,\pi)}{2\,i} - \frac{x^{\alpha-1}\, e^{-i\,\alpha\,\pi}\csc(\alpha\,\pi)}{2\,i} = x^{\alpha-1}$$

Integral transforms

Exponential Fourier transform

Definition

$$\mathcal{F}_t[f(t)](z) == \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty f(t)\, e^{i\,t\,z}\, dt$$

This formula is the definition of the exponential Fourier transform of the function f with respect to the variable t. If the integral does not converge, the value of $\mathcal{F}_t[f(t)](z)$ is defined in the sense of generalized functions for functions $f(t)$ that do not grow faster than polynomials at $\pm\infty$.

Properties

Linearity

$$\mathcal{F}_t[a\, f(t) + b\, g(t)](z) = a\,(\mathcal{F}_t[f(t)](z)) + b\,(\mathcal{F}_t[g(t)](z))$$

This formula reflects the linearity of the exponential Fourier transform.

Reflection

$$\mathcal{F}_t[f(-t)](z) = \mathcal{F}_t[f(t)](-z)$$

This formula is called the *reflection property* of the exponential Fourier transform.

Dilation

$$\mathcal{F}_t[f(a\,t)](z) = \frac{1}{|a|}\,\mathcal{F}_t[f(t)]\left(\frac{z}{a}\right) /;\, a \in \mathbb{R} \land a \neq 0$$

This formula reflects the *scaling* or *dilation property* of the exponential Fourier transform.

Shifting or translation

$$\mathcal{F}_t[f(t-a)](z) = e^{i\,a\,z}\,(\mathcal{F}_t[f(t)](z)) /;\, a \in \mathbb{R}$$

This formula reflects the *shifting* or *translation property* of the exponential Fourier transform.

Modulation

$$\mathcal{F}_t\left[e^{i\,a\,t}\,f(t)\right](z) = \mathcal{F}_t[f(t)](a+z) /;\, a \in \mathbb{R}$$

This formula reflects the *modulation property* of the exponential Fourier transform.

$$\mathcal{F}_t[\cos(b\,t)\,f(a\,t)](z) = \frac{i}{2\,|a|}\left(\mathcal{F}_t[f(t)]\left(\frac{z-b}{a}\right) - \mathcal{F}_t[f(t)]\left(\frac{z+b}{a}\right)\right) /;\, a \in \mathbb{R} \bigwedge b \in \mathbb{R}$$

This formula reflects the *modulation property* of the exponential Fourier transform.

$$\frac{i}{2\,|a|}\left(\mathcal{F}_t[f(t)]\left(\frac{z-b}{a}\right) - \mathcal{F}_t[f(t)]\left(\frac{z+b}{a}\right)\right) /;\, a \in \mathbb{R} \bigwedge b \in \mathbb{R}$$

This formula reflects the *modulation property* of the exponential Fourier transform.

Power scaling

$$\mathcal{F}_t[t^n\,f(t)](z) = i^n\,\frac{\partial^n(\mathcal{F}_t[f(t)](z))}{\partial z^n} /;\, n \in \mathbb{N}^+$$

The *power scaling property* shows that multiplication of a function by t^n corresponds to the n^{th} derivative of the exponential Fourier transform.

$$\mathcal{F}_t\left[\frac{f(t)}{t}\right](z) = i\int_a^\infty \mathcal{F}_t[f(t)](z)\,dz + \frac{1}{\sqrt{2\,\pi}}\int_{-\infty}^\infty \frac{f(t)\,e^{i\,a\,t}}{t}\,dt$$

This formula shows that multiplication of a function by t^{-1} corresponds to integration of the exponential Fourier transform.

Multiplication

$$\mathcal{F}_t[f(t)\,g(t)](z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(\mathcal{F}_\eta[f(\eta)](z-\tau)\right)\left(\mathcal{F}_\eta[g(\eta)](\tau)\right) d\tau$$

The *multiplication property* shows that the exponential Fourier transform of a product gives the convolution of the exponential Fourier transform divided by $\sqrt{2\pi}$.

Conjugation

$$\mathcal{F}_t[\overline{f(t)}](z) = \overline{(\mathcal{F}_t[f(t)](-z))}$$

This formula reflects the *conjugation property* of the exponential Fourier transform.

Derivative

$$\mathcal{F}_t\left[f^{(n)}(t)\right](z) = (-i\,z)^n\,\mathcal{F}_t[f(t)](z)\;/;\;\lim_{|t|\to\infty} f^{(k)}(t) = 0 \bigwedge 0 \le k \le n-1$$

The *derivative property* shows that the exponential Fourier transform of the n^{th} derivative gives the product of the power function on the exponential Fourier transform.

Integral

$$\mathcal{F}_t\left[\int_a^t f(\tau)\,d\tau\right](z) = \frac{i}{z}\,(\mathcal{F}_t[f(t)](z)) + g(a)\,\delta(z)$$

This formula shows that the exponential Fourier transform of an integral gives the product of the power function and the exponential Fourier transform plus an expression that includes a Dirac delta function.

Parseval identity

$$\int_{-\infty}^{\infty} f(t)\,\overline{g(t)}\,dt = \int_{-\infty}^{\infty} (\mathcal{F}_\tau[f(\tau)](t))\,\overline{(\mathcal{F}_\tau[g(\tau)](t))}\,dt$$

This formula is called the *Parseval identity*.

$$\int_{-\infty}^{\infty} |f(t)|^2\,dt = \int_{-\infty}^{\infty} |\mathcal{F}_\tau[f(\tau)](t)|^2\,dt$$

This Bessel’s equality follows from the Parseval identity, when $g(t) == f(t)$.

Convolution theorem

$$\mathcal{F}_t\left[\int_{-\infty}^{\infty} f(t-\tau)\,g(\tau)\,d\tau\right](z) = \sqrt{2\pi}\,(\mathcal{F}_t[f(t)](z))\,(\mathcal{F}_t[g(t)](z))$$

This *Fourier convolution theorem* or convolution (Faltung) theorem for the exponential Fourier transform shows that the Fourier transform of a convolution is equal to the product of the Fourier transform multiplied by $\sqrt{2\pi}$.

Relations with other integral transforms

With inverse exponential Fourier transform

$$\mathcal{F}_t\left[\mathcal{F}_\tau^{-1}[f(\tau)](t)\right](z) = f(z)$$

This formula reflects the relation between direct and inverse exponential Fourier transforms. In the point $z == z_0$, where $f(z)$ has a jump discontinuity the composition of the inverse and direct exponential Fourier transforms converges to the mean $\frac{1}{2}\left(\lim_{z \to z_0^+} f(z) + \lim_{z \to z_0^-} f(z)\right)$.

With Fourier cosine and sine transforms

$$\mathcal{F}_t[f(t)](z) = \mathcal{F}c_t\left[\frac{f(t)+f(-t)}{2}\right](z) + i\,\mathcal{F}s_t\left[\frac{f(t)-f(-t)}{2}\right](z)$$

This formula shows how the exponential Fourier transform can be represented through cosine and sine Fourier transforms from even and odd parts.

With Laplace transform

$$\mathcal{F}_t[f(t)](z) == \frac{1}{\sqrt{2\pi}}\mathcal{L}_t[f(t)](-i z) + \frac{1}{\sqrt{2\pi}}\mathcal{L}_t[f(-t)](i z)$$

This formula shows how the exponential Fourier transform can be represented through the Laplace transform.

With Mellin transform

$$\mathcal{F}_t[f(t)](z) = \frac{1}{\sqrt{2\pi}}\mathcal{M}_t[f(-\log(t))](-i z)$$

This formula shows how the exponential Fourier transform can be represented through the Mellin transform.

With Z-transform

$$\mathcal{F}_t\left[\sum_{n=-\infty}^{\infty} f(n)\,\delta(t-n)\right](x) = \mathcal{Z}_n[f(n)]\left(e^{i x}\right)$$

This formula shows how the exponential Fourier transform can be represented through the Z-transform.

Inverse exponential Fourier transform

Definition

$$\mathcal{F}_t^{-1}[f(t)](z) == \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\,e^{-i t z}\,dt$$

This formula is the definition of the inverse exponential Fourier transform of the function f with respect to the variable t. If the integral does not converge, the value of $\mathcal{F}_t^{-1}[f(t)](z)$ is defined in the sense of generalized functions.

Relations with other integral transforms

With exponential Fourier transform

$$\mathcal{F}_t^{-1}[f(t)](z) = \mathcal{F}_t[f(t)](-z)$$

This *near-equivalence identity* shows that the inverse exponential Fourier transform in the point z coincides with the direct Fourier transform in the point $-z$.

$$\mathcal{F}_t^{-1}[f(t)](z) = \mathcal{F}_t[f(-t)](z)$$

This *near-equivalence identity* shows that the inverse exponential Fourier transform in the point z coincides with the direct Fourier transform in the point $-z$.

$$\mathcal{F}_t\left[\mathcal{F}_\tau^{-1}[f(\tau)](t)\right](z) = f(z)$$

This formula reflects the relation between the direct and inverse exponential Fourier transforms. In the point $z = z_0$, where $f(z)$ has a jump discontinuity, the composition of inverse and direct exponential Fourier transforms converges to the mean $\frac{1}{2}\left(\lim_{z \to z_0^+} f(z) + \lim_{z \to z_0^-} f(z)\right)$.

Multiple exponential Fourier transform

Definition

$$\mathcal{F}_{\{t_1,t_2\}}[f(t_1, t_2)](z_1, z_2) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t_1, t_2)\, e^{i(t_1 z_1 + t_2 z_2)}\, dt_1\, dt_2$$

This formula is the definition of the double exponential Fourier transform of the function f with respect to the variables t_1, t_2. If the integral does not converge, the value of $\mathcal{F}_{\{t_1,t_2\}}[f(t_1, t_2)](z_1, z_2)$ is defined in the sense of generalized functions.

$$\mathcal{F}_{\{t_1,t_2,\ldots,t_n\}}[f(t_1, t_2, \ldots, t_n)](z_1, z_2, \ldots, z_n) = \frac{1}{(2\pi)^{\frac{n}{2}}} \underbrace{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty}}_{n-\text{times}} f(t_1, t_2, \ldots, t_n)\, e^{i(t_1 z_1 + t_2 z_2 + \ldots + t_n z_n)}\, dt_n \ldots dt_2\, dt_1$$

This formula is the definition of the multiple exponential Fourier transform of the function $f(t_1, t_2, \ldots, t_n)$ with respect to the variables $t_1, t_2, \ldots, t_n$ over $\mathbb{R}^n$. If this integral does not converge, the value of $\mathcal{F}_{\{t_1,t_2,\ldots,t_n\}}[f(t_1, t_2, \ldots, t_n)](z_1, z_2, \ldots, z_n)$ is defined in the sense of generalized functions.

Inverse multiple exponential Fourier transform

Definition

$$\mathcal{F}_{\{t_1,t_2\}}^{-1}[f(t_1, t_2)](z_1, z_2) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t_1, t_2)\, e^{-i(t_1 z_1 + t_2 z_2)}\, dt_1\, dt_2$$

This formula is the definition of the inverse double exponential Fourier transform of the function f with respect to the variables t_1, t_2. If this integral does not converge, the value of $\mathcal{F}^{-1}_{\{t_1,t_2\}}[f(t_1, t_2)](z_1, z_2)$ is defined in the sense of generalized functions.

$$\mathcal{F}^{-1}_{\{t_1,t_2,\ldots,t_n\}}[f(t_1, t_2, \ldots, t_n)](z_1, z_2, \ldots, z_n) = \frac{1}{(2\pi)^{\frac{n}{2}}} \underbrace{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\ldots\int_{-\infty}^{\infty} f(t_1, t_2, \ldots, t_n)\, e^{-i(t_1 z_1 + t_2 z_2 + \ldots + t_n z_n)}\, dt_n \ldots dt_2\, dt_1}_{n-\text{times}}$$

This formula is the definition of the inverse multiple exponential Fourier transform of the function $f(t_1, t_2, \ldots, t_n)$ with respect to the variables $t_1, t_2, \ldots, t_n$ over $\mathbb{R}^n$. If this integral does not converge, the value of $\mathcal{F}^{-1}_{\{t_1,t_2,\ldots,t_n\}}[f(t_1, t_2, \ldots, t_n)](z_1, z_2, \ldots, z_n)$ is defined in the sense of generalized functions.

Relations with other integral transforms

With multiple exponential Fourier transform

$$\mathcal{F}^{-1}_{\{t_1,t_2,\ldots,t_n\}}[f(t_1, t_2, \ldots, t_n)](z_1, z_2, \ldots, z_n) = \mathcal{F}_{\{t_1,t_2,\ldots,t_n\}}[f(t_1, t_2, \ldots, t_n)](-z_1, -z_2, \ldots, -z_n)$$

This *near-equivalence identity* shows that the inverse multiple exponential Fourier transform in the point z coincides with the direct multiple exponential Fourier transform at the point $-z$.

Fourier transform (continuous → discrete)

Definition

$$\mathcal{F}cd_{L;t}[f(t)](k) = \frac{1}{\sqrt{L}} \int_0^L f(t)\, e^{-\frac{2\pi i k t}{L}}\, dt \;/;\; k \in \mathbb{Z}$$

General properties

Linearity

$$\mathcal{F}cd_{L;t}[a\, f(t) + b\, g(t)](k) = a\, \mathcal{F}cd_{L;t}[f(t)](k) + b\, \mathcal{F}cd_{L;t}[g(t)](k)$$

Reflection

$$\mathcal{F}cd_{L;t}[f(-t)](k) = \mathcal{F}cd_{L;t}[f(t)](-k)$$

Dilation

$$\mathcal{F}cd_{L;t}[f(m\, t)](k) = \begin{cases} \left(\mathcal{F}cd_{L;t}[f(t)]\left(\frac{k}{m}\right)\right) & m \mid k \\ 0 & \text{True} \end{cases} \;/;\; m \in \mathbb{N}^+$$

Shifting or translation

$$\mathcal{F}cd_{L;t}[f(t-a)](k) = e^{-\frac{2\pi i k a}{L}}\, \mathcal{F}cd_{L;t}[f(t)](k)$$

Modulation

$$\mathcal{F}cd_{L;t}\left[e^{\frac{2\pi i m t}{L}} f(t)\right](k) = \mathcal{F}cd_{L;t}[f(t)](k-m) \;/;\; m \in \mathbb{Z}$$

Multiplication

$$\mathcal{F}cd_{L;t}[f(t)\,g(t)](k) = \frac{1}{\sqrt{L}} \sum_{j=-\infty}^{\infty} (\mathcal{F}cd_{L;t}[f(t)](j))\,(\mathcal{F}cd_{L;t}[g(t)](k-j))$$

Conjugation

$$\mathcal{F}cd_{L;t}[\overline{f(t)}](k) = \overline{\mathcal{F}cd_{L;t}[f(t)](-k)}$$

Derivative

$$\mathcal{F}cd_{L;t}\left[\frac{\partial f(t)}{\partial t}\right](k) = \frac{2\pi i k}{L} \mathcal{F}cd_{L;t}[f(t)](k)$$

Grouping

$$\mathcal{F}cd_{L;t}\left[\sum_{j=0}^{m-1} f\left(\frac{t - j L}{m}\right)\right](k) = m\,(\mathcal{F}cd_{L;t}[f(t)](k\,m)) \;/;\; m \in \mathbb{N}^+$$

Summation

$$\mathcal{F}cd_{L;t}\left[\sum_{j=-\infty}^{\infty} f(t - j L)\right](k) = \frac{1}{L}\left(\mathcal{F}cd_{L;t}[f(t)]\left(\frac{k}{L}\right)\right)$$

Parseval identity

$$\int_0^L f(\tau)\,\overline{g(\tau)}\,d\tau = \sum_{j=-\infty}^{\infty} (\mathcal{F}cd_{L;t}[f(t)](j))\,(\overline{\mathcal{F}cd_{L;t}[g(t)](j)})$$

Convolution theorem

$$\mathcal{F}cd_{L;t}\left[\int_0^L f(\tau)\,g(t-\tau)\,d\tau\right](k) = \sqrt{L}\,(\mathcal{F}cd_{L;t}[f(t)](k))\,(\mathcal{F}cd_{L;t}[g(t)](k))$$

Relations with other integral transforms

With inverse Fourier transform (continuous → discrete)

$$\mathcal{F}cd_{L;t}\left[\mathcal{F}cd_{L;k}^{-1}[f(k)](t)\right](k) = f(k)$$

Inverse Fourier transform (continuous → discrete)

Definition

$$\mathcal{F}cd_{L;k}^{-1}[f(k)](x) = \frac{1}{\sqrt{L}} \sum_{k=-\infty}^{\infty} f(k)\, e^{\frac{2\pi i k x}{L}}$$

Relations with other integral transforms

With Fourier transform (continuous → discrete)

$$\mathcal{F}cd_{L;t}\left[\mathcal{F}cd_{L;k}^{-1}[f(k)](t)\right](k) = f(k)$$

Fourier transform (discrete → continuous)

Definition

$$\mathcal{F}dc_{L;k}[f(k)](x) = \frac{1}{\sqrt{L}} \sum_{k=-\infty}^{\infty} f(k)\, e^{-\frac{2\pi i k x}{L}}$$

General properties

Linearity

$$\mathcal{F}dc_{L;k}[a\, f(k) + b\, g(k)](x) = a\, \mathcal{F}dc_{L;k}[f(k)](x) + b\, \mathcal{F}dc_{L;k}[g(k)](x)$$

Reflection

$$\mathcal{F}dc_{L;k}[f(-k)](x) = \mathcal{F}dc_{L;k}[f(k)](-x)$$

Dilation

$$\mathcal{F}dc_{L;k}[f(k\, m)](x) = \frac{1}{m} \sum_{j=0}^{m-1} \left(\mathcal{F}dc_{L;k}[f(k)]\left(\frac{x - j\, L}{m}\right)\right) /;\, m \in \mathbb{N}^{+}$$

Shifting or translation

$$\mathcal{F}dc_{L;k}[f(k-m)](x) = e^{-\frac{2\pi i x m}{L}}\, \mathcal{F}dc_{L;k}[f(k)](x) /;\, m \in \mathbb{Z}$$

Modulation

$$\mathcal{F}dc_{L;k}\left[e^{\frac{2\pi i a k}{L}}\, f(k)\right](x) = \mathcal{F}dc_{L;k}[f(k)](x-a)$$

Power scaling

$$\mathcal{F}dc_{L;k}[k\, f(k)](x) = -\frac{L}{2\pi i}\, \frac{\partial(\mathcal{F}dc_{L;k}[f(k)](x))}{\partial x}$$

Multiplication

$$\mathcal{F}dc_{L;k}[f(k)\, g(k)](x) = \frac{1}{\sqrt{L}} \int_0^L (\mathcal{F}dc_{L;k}[f(k)](t))\, (\mathcal{F}dc_{L;k}[g(k)](x-t))\, dt$$

Conjugation

$$\mathcal{F}dc_{L;k}[\overline{f(k)}](x) = \overline{\mathcal{F}dc_{L;k}[f(k)](-x)}$$

Sampling

$$\mathcal{F}dc_{L;k}\left[f\left(\frac{k}{L}\right)\right](x) = \sum_{j=-\infty}^{\infty} (\mathcal{F}dc_{L;k}[f(k)](x - j\, L))$$

Zero packing

$$\mathcal{F}dc_{L;k}\left[\begin{cases} f\left(\frac{k}{m}\right) & m \mid k \\ 0 & \text{True} \end{cases}\right](x) = (\mathcal{F}dc_{L;k}[f(k)](m\, x))\ /;\ m \in \mathbb{N}^+$$

Parseval identity

$$\sum_{j=-\infty}^{\infty} f(j)\, \overline{g(j)} = \int_0^L (\mathcal{F}dc_{L;k}[f(k)](t))\, (\overline{\mathcal{F}dc_{L;k}[g(k)](t)})\, dt$$

Convolution theorem

$$\mathcal{F}dc_{L;k}\left[\sum_{j=-\infty}^{\infty} f(j)\, g(k-j)\right](x) = \sqrt{L}\, (\mathcal{F}dc_{L;k}[f(k)](x))\, (\mathcal{F}dc_{L;k}[g(k)](x))$$

Relations with other integral transforms

With inverse Fourier transform (discrete → continuous)

$$\mathcal{F}dc_{L;k}\left[\mathcal{F}dc_{L;t}^{-1}[f(t)](k)\right](x) = f(x)$$

Inverse Fourier transform (discrete → continuous)

Definition

$$\mathcal{F}dc_{L;t}^{-1}[f(t)](k) = \frac{1}{\sqrt{L}} \int_0^L f(t)\, e^{\frac{2\pi i k t}{L}}\, dt\ /;\ k \in \mathbb{Z}$$

Relations with other integral transforms

With Fourier transform (discrete → continuous)

$$\mathcal{F}dc_{L;k}\left[\mathcal{F}dc_{L;t}^{-1}[f(t)](k)\right](x) = f(x)$$

Fourier transform (discrete → discrete)

Definition

$$\mathcal{F}dd_{m;k}[f(k)](n) = \frac{1}{\sqrt{m}} \sum_{k=0}^{m-1} f(k)\, e^{-\frac{2\pi i k n}{m}} \;/;\; m \in \mathbb{N}^+ \bigwedge n \in \mathbb{Z}$$

General properties

Linearity

$$\mathcal{F}dd_{m;k}[a\, f(k) + b\, g(k)](n) = a\, \mathcal{F}dd_{m;k}[f(k)](n) + b\, \mathcal{F}dd_{m;k}[g(k)](n) \;/;\; m \in \mathbb{N}^+ \bigwedge n \in \mathbb{Z}$$

Reflection

$$\mathcal{F}dd_{m;k}[f(-k)](n) = \mathcal{F}dd_{m;k}[f(k)](-n) \;/;\; m \in \mathbb{N}^+ \bigwedge n \in \mathbb{Z}$$

Dilation

$$\mathcal{F}dd_{m;k}[f(k\, j)](n) = \sum_{i=0}^{j-1} (\mathcal{F}dd_{m;k}[f(k)](n - i\, m)) \;/;\; m \in \mathbb{N}^+ \bigwedge n \in \mathbb{Z} \bigwedge j \in \mathbb{N}^+$$

$$\mathcal{F}dd_{m;k}[f(k\, j)](n) = (\mathcal{F}dd_{m;k}[f(k)](r\, n)) \;/;\; m \in \mathbb{N}^+ \bigwedge n \in \mathbb{Z} \bigwedge j \in \mathbb{Z} \bigwedge \gcd(j, m) = 1 \bigwedge j\, r = 1 \bmod m$$

$$\mathcal{F}dd_{m;k}[f(k\, j)](n) = \begin{cases} \sum_{i=0}^{j-1} \left(\mathcal{F}dd_{m;r}[f(r)]\left(\frac{n - i\, m}{j}\right)\right) & j \mid n \\ 0 & \text{True} \end{cases} \;/;\; m \in \mathbb{N}^+ \bigwedge n \in \mathbb{Z} \bigwedge j \in \mathbb{N}^+ \bigwedge j \mid m$$

Shifting or translation

$$\mathcal{F}dd_{m;k}[f(k - j)](n) = e^{-\frac{2\pi i j n}{m}}\, \mathcal{F}dd_{m;k}[f(k)](n) \;/;\; m \in \mathbb{N}^+ \bigwedge n \in \mathbb{Z} \bigwedge j \in \mathbb{Z}$$

Modulation

$$\mathcal{F}dd_{m;k}\left[e^{\frac{2\pi i k j}{m}} f(k)\right](n) = \mathcal{F}dd_{m;k}[f(k)](n - j) \;/;\; m \in \mathbb{N}^+ \bigwedge n \in \mathbb{Z} \bigwedge j \in \mathbb{Z}$$

Multiplication

$$\mathcal{F}dd_{m;k}[f(k)\, g(k)](n) = \frac{1}{\sqrt{m}} \sum_{j=0}^{m-1} (\mathcal{F}dd_{m;k}[f(k)](j))\, (\mathcal{F}dd_{m;k}[g(k)](n - j)) \;/;\; m \in \mathbb{N}^+ \bigwedge n \in \mathbb{Z} \bigwedge j \in \mathbb{Z}$$

Conjugation

$$\mathcal{F}dd_{m;k}[\overline{f(k)}](n) = \overline{\mathcal{F}dd_{m;k}[f(k)](-n)}$$

Repeat

$$\mathcal{F}dd_{m;k}[f(k)](n) = \begin{cases} \mathcal{F}dd_{m;k}[f(k)]\left(\frac{n}{j}\right) & j \mid n \\ 0 & \text{True} \end{cases} /; m \in \mathbb{N}^+ \bigwedge n \in \mathbb{Z} \bigwedge j \in \mathbb{Z}$$

Zero packing

$$\mathcal{F}dd_{m;k}\left[\begin{cases} f\left(\frac{k}{j}\right) & j \mid k \\ 0 & \text{True} \end{cases}\right](n) = \frac{1}{j}\,\mathcal{F}dd_{m;k}[f(k)](n) /; m \in \mathbb{N}^+ \bigwedge n \in \mathbb{Z} \bigwedge j \in \mathbb{Z}$$

Summation

$$\mathcal{F}dd_{m;k}\left[\sum_{i=0}^{j-1} f(k - i\,m)\right](n) = j\,(\mathcal{F}dd_{m;k}[f(k)](j\,n)) /; m \in \mathbb{N}^+ \bigwedge n \in \mathbb{Z} \bigwedge j \in \mathbb{Z}$$

Parseval identity

$$\sum_{j=0}^{m-1} f(j)\,\overline{g(j)} = \sum_{j=0}^{m-1} (\mathcal{F}dd_{m;k}[f(k)](j))\,(\overline{\mathcal{F}dd_{m;k}[g(k)](j)}) /; m \in \mathbb{N}^+ \bigwedge n \in \mathbb{Z}$$

Convolution theorem

$$\mathcal{F}dd_{m;k}\left[\sum_{j=0}^{m-1} f(j)\,g(k-j)\right](n) = \sqrt{m}\,(\mathcal{F}dd_{m;k}[f(k)](n))\,(\mathcal{F}dd_{m;k}[g(k)](n)) /; m \in \mathbb{N}^+ \bigwedge n \in \mathbb{Z}$$

Relations with other integral transforms

With inverse Fourier transform (discrete → discrete)

$$\mathcal{F}dd_{m;k}\left[\mathcal{F}dd_{m;j}^{-1}[f(j)](k)\right](n) = f(n) /; m \in \mathbb{N}^+ \bigwedge n \in \mathbb{Z}$$

Inverse Fourier transform (discrete → discrete)

Definition

$$\mathcal{F}dd_{m;k}^{-1}[f(k)](n) = \frac{1}{\sqrt{m}} \sum_{k=0}^{m-1} f(k)\,e^{\frac{2\pi i k n}{m}} /; m \in \mathbb{N}^+ \bigwedge n \in \mathbb{Z}$$

Relations with other integral transforms

With Fourier transform (discrete → discrete)

$$\mathcal{F}dd_{m,k}\left[\mathcal{F}dd_{m;j}^{-1}[f(j)](k)\right](n) = f(n) \;/;\; m \in \mathbb{N}^+ \land n \in \mathbb{Z}$$

Fourier cosine transform

Definition

$$\mathcal{F}c_t[f(t)](z) == \sqrt{\frac{2}{\pi}} \int_0^\infty f(t)\cos(t\,z)\,dt$$

This formula is the definition of the Fourier cosine transform of the function f with respect to the variable t. If the integral does not converge, the value of $\mathcal{F}c_t[f(t)](z)$ is defined in the sense of generalized functions.

General properties

Linearity

$$\mathcal{F}c_t[a\,f(t) + b\,g(t)](z) == a\,\mathcal{F}c_t[f(t)](z) + b\,\mathcal{F}c_t[g(t)](z)$$

This formula reflects the linearity of the Fourier cosine transform.

Scaling

$$\mathcal{F}c_t[f(a\,t)](z) = \frac{1}{a}\,\mathcal{F}c_t[f(t)]\left(\frac{z}{a}\right) \;/;\; a > 0$$

This formula reflects the *scaling property* of the Fourier cosine transform.

Modulation

$$\mathcal{F}c_t[\cos(b\,t)\,f(a\,t)](z) = \frac{1}{2\,a}\left(\mathcal{F}c_t[f(t)]\left(\frac{z+b}{a}\right) + \mathcal{F}c_t[f(t)]\left(\frac{z-b}{a}\right)\right) \;/;\; a > 0 \land b > 0$$

This formula reflects the *modulation property* of the Fourier cosine transform.

$$\mathcal{F}c_t[\sin(b\,t)\,f(a\,t)](z) = \frac{1}{2\,a}\left(\mathcal{F}s_t[f(t)]\left(\frac{z+b}{a}\right) - \mathcal{F}s_t[f(t)]\left(\frac{z-b}{a}\right)\right) \;/;\; a > 0 \land b > 0$$

This formula reflects the *modulation property* of the Fourier cosine transform.

Parity

$$\mathcal{F}c_t\left[\frac{f(t+a) + f(-t-a)}{2} - \frac{f(t-a) - f(a-t)}{2}\right](z) == \sin(a\,z)\,(\mathcal{F}s_t[f(t)](z)) \;/;\; a \in \mathbb{R}$$

This formula shows how the Fourier cosine transform can be applied to the difference between the even part of $f(t+a)$ and the odd part of $f(t-a)$.

$$\mathcal{F}c_t\left[\frac{f(t+a)-f(-t-a)}{2}+\frac{f(t-a)+f(a-t)}{2}\right](z) = \cos(a\,z)\,(\mathcal{F}c_t[f(t)](z))\ /;\ a \in \mathbb{R}$$

This formula shows how the Fourier cosine transform can be applied to the sum of the even part of $f(t-a)$ and the odd part of $f(t+a)$.

Power scaling

$$\mathcal{F}c_t\left[t^{2n} f(t)\right](z) = (-1)^n \frac{\partial^{2n}(\mathcal{F}c_t[f(t)](z))}{\partial z^{2n}}\ /;\ n \in \mathbb{N}^+$$

This formula shows that multiplication of a function by t^{2n} corresponds to the $2n^{\text{th}}$ derivative of the Fourier cosine transform.

$$\mathcal{F}c_t\left[t^{2n+1} f(t)\right](z) = (-1)^n \frac{\partial^{2n+1}(\mathcal{F}s_t[f(t)](z))}{\partial z^{2n+1}}\ /;\ n \in \mathbb{N}$$

This formula shows that multiplication of a function by t^{2n+1} corresponds to the $(2n+1)^{\text{th}}$ derivative of the Fourier sine transform.

Derivative

$$\mathcal{F}c_t\left[f^{(2n)}(t)\right](z) = (-1)^n z^{2n}\,\mathcal{F}c_t[f(t)](z) - \sqrt{\frac{2}{\pi}} \sum_{k=0}^{n-1} (-1)^k z^{2k} f^{(2n-2k-1)}(0)\ /;\ \lim_{t\to\infty} f^{(k)}(t) = 0 \bigwedge 0 \le k \le 2n-1 \bigwedge n \in \mathbb{N}^+$$

This formula shows that the Fourier cosine transform of an even-order derivative gives the product of the power function with the Fourier cosine transform plus some even polynomial.

$$\mathcal{F}c_t\left[f^{(2n+1)}(t)\right](z) = (-1)^n z^{2n+1}\,\mathcal{F}s_t[f(t)](z) - \sqrt{\frac{2}{\pi}} \sum_{k=0}^{n-1} (-1)^k z^{2k} f^{(2n-2k)}(0)\ /;\ \lim_{t\to\infty} f^{(k)}(t) = 0 \bigwedge 0 \le k \le 2n \bigwedge n \in \mathbb{N}$$

This formula shows that the Fourier cosine transform of an odd-order derivative gives the product of a power function with the Fourier sine transform plus some even polynomial.

Convolution related

$$\mathcal{F}c_t\left[\int_0^\infty f(\tau)\,(g(t+\tau)+g(|t-\tau|))\,d\tau\right](z) = \sqrt{2\pi}\ (\mathcal{F}c_t[f(t)](z))\,(\mathcal{F}c_t[g(t)](z))$$

This formula shows that the Fourier cosine transform of a convolution gives the product of Fourier cosine transforms multiplied by $\sqrt{2\pi}$.

Integral

$$\mathcal{F}c_t\left[\int_t^\infty f(\tau)\,d\tau\right](z) = \frac{1}{z}\,(\mathcal{F}s_t[f(t)](z))$$

This formula shows that the Fourier cosine transform of an indefinite integral with a variable lower limit gives the product of the Fourier sine transforms by $1/z$.

Limit at infinity

$$\lim_{z\to\infty} \left(\mathcal{F}c_t[f(t)](z)\right) = 0$$

This Riemann–Lebesgue theorem shows that the Fourier cosine transform $\mathcal{F}c_t[f(t)](z)$ converges to zero as z tends to infinity for some classes of the function $f(t)$.

Relations with other integral transforms

With inverse Fourier cosine transform

$$\mathcal{F}c_t\left[\mathcal{F}c_\tau^{-1}[f(\tau)](t)\right](z) == f(z)$$

This formula reflects the relation between the direct and the inverse Fourier cosine transforms. In the point $z == z_0$, where $f(z)$ has a jump discontinuity, the composition of the inverse and the direct Fourier cosine transforms converges to the mean $\frac{1}{2}\left(\lim_{z\to z_0^+} f(z) + \lim_{z\to z_0^-} f(z)\right)$.

With exponential Fourier transform

$$\mathcal{F}c_t[f(t)](z) == \mathcal{F}c_t[f(-t)\,\theta(-t) + f(t)\,\theta(t)](z)$$

This formula reflects the relation between the direct and the inverse exponential Fourier transforms.

With Laplace transform

$$\mathcal{F}c_t[f(t)](z) == \frac{1}{\sqrt{2\pi}}\,\mathcal{L}_t[f(t)](i z) + \frac{1}{\sqrt{2\pi}}\,\mathcal{L}_t[f(t)](-i z)$$

This formula represents the Fourier cosine transform through Laplace transforms.

Inverse Fourier cosine transform

Definition

$$\mathcal{F}c_t^{-1}[f(t)](z) == \sqrt{\frac{2}{\pi}} \int_0^\infty f(t)\cos(t z)\,dt$$

This formula is the definition of the inverse Fourier cosine transform of the function f with respect to the variable t. If the integral does not converge, the value of $\mathcal{F}c_t^{-1}[f(t)](z)$ is defined in the sense of generalized functions.

Relations with other integral transforms

With Fourier cosine transform

$$\mathcal{F}c_t^{-1}[f(t)](z) == \mathcal{F}c_t[f(t)](z)$$

This formula shows that the inverse Fourier cosine transform coincides with the direct Fourier cosine transform.

Fourier sine transform

Definition

$$\mathcal{F}s_t[f(t)](z) == \sqrt{\frac{2}{\pi}} \int_0^\infty f(t)\sin(t z)\, dt$$

This formula is the definition of the Fourier sine transform of the function f with respect to the variable t. If the integral does not converge, the value of $\mathcal{F}s_t[f(t)](z)$ is defined in the sense of generalized functions.

General properties

Linearity

$$\mathcal{F}s_t[a\, f(t) + b\, g(t)](z) == a\, \mathcal{F}s_t[f(t)](z) + b\, \mathcal{F}s_t[g(t)](z)$$

This formula reflects the linearity of the Fourier sine transform.

Scaling

$$\mathcal{F}s_t[f(a\, t)](z) = \frac{1}{a}\, \mathcal{F}s_t[f(t)]\left(\frac{z}{a}\right) /;\, a > 0$$

This formula reflects the *scaling property* of the Fourier sine transform.

Modulation

$$\mathcal{F}s_t[\cos(b\, t)\, f(a\, t)](z) = \frac{1}{2\, a}\left(\mathcal{F}s_t[f(t)]\left(\frac{z-b}{a}\right) + \mathcal{F}s_t[f(t)]\left(\frac{z+b}{a}\right)\right) /;\, a > 0 \bigwedge b > 0$$

This formula reflects the *modulation property* of the Fourier sine transform.

$$\mathcal{F}s_t[\sin(b\, t)\, f(a\, t)](z) = \frac{1}{2\, a}\left(\mathcal{F}c_t[f(t)]\left(\frac{z-b}{a}\right) - \mathcal{F}c_t[f(t)]\left(\frac{z+b}{a}\right)\right) /;\, a > 0 \bigwedge b > 0$$

This formula reflects the *modulation property* of the Fourier sine transform.

Parity

$$\mathcal{F}s_t\left[\frac{f(t-a) + f(a-t)}{2} - \frac{f(t+a) - f(-t-a)}{2}\right](z) == \sin(a\, z)\, (\mathcal{F}c_t[f(t)](z)) /;\, a \in \mathbb{R}$$

This formula shows how the Fourier sine transform can be applied to the difference between the even part of $f(t-a)$ and the odd part of $f(t+a)$.

$$\mathcal{F}s_t\left[\frac{f(t+a)+f(-t-a)}{2}+\frac{f(t-a)-f(a-t)}{2}\right](z) = \cos(a\,z)\,(\mathcal{F}s_t[f(t)]\,(z))\;/;\;a\in\mathbb{R}$$

This formula shows how the Fourier sine transform can be applied to a sum of the even part of $f(t+a)$ and the odd part of $f(t-a)$.

Power scaling

$$\mathcal{F}s_t\left[t^{2n}\,f(t)\right](z) = (-1)^n\,\frac{\partial^{2n}(\mathcal{F}s_t[f(t)]\,(z))}{\partial z^{2n}}\;/;\;n\in\mathbb{N}^+$$

This formula shows that multiplication of a function by t^{2n} corresponds to the $2\,n^{\text{th}}$ derivative of the Fourier sine transform.

$$\mathcal{F}s_t\left[t^{2n+1}\,f(t)\right](z) = (-1)^{n+1}\,\frac{\partial^{2n+1}(\mathcal{F}c_t[f(t)]\,(z))}{\partial z^{2n+1}}\;/;\;n\in\mathbb{N}$$

This formula shows that multiplication of a function by t^{2n+1} corresponds to the $(2\,n+1)^{\text{th}}$ derivative of the Fourier cosine transform.

Derivative

$$\mathcal{F}s_t\left[f^{(2n)}(t)\right](z) = (-1)^n\,z^{2n}\,\mathcal{F}s_t[f(t)]\,(z) + \sqrt{\frac{2}{\pi}}\sum_{k=0}^{n-2}(-1)^k\,z^{2k+1}\,f^{(2n-2k-2)}(0)\;/;\;\lim_{t\to\infty}f^{(k)}(t) = 0\bigwedge 0\le k\le 2\,n-1\bigwedge n\in\mathbb{N}^+$$

This formula shows that the Fourier sine transform of an even-order derivative gives the product of the power function and the Fourier sine transform plus some odd-order polynomial.

$$\mathcal{F}s_t\left[f^{(2n+1)}(t)\right](z) = (-1)^{n-1}\,z^{2n+1}\,\mathcal{F}c_t[f(t)]\,(z) + \sqrt{\frac{2}{\pi}}\sum_{k=0}^{n-1}(-1)^k\,z^{2k+1}\,f^{(2n-2k-1)}(0)\;/;\;\lim_{t\to\infty}f^{(k)}(t) = 0\bigwedge 0\le k\le 2\,n\bigwedge n\in\mathbb{N}$$

This formula shows that the Fourier sine transform of an odd-order derivative gives the product of the power function and the Fourier cosine transform plus some odd-order polynomial.

Convolution related

$$\mathcal{F}s_t\left[\int_0^\infty f(\tau)\,(g(|t-\tau|)-g(t+\tau))\,d\tau\right](z) = \sqrt{2\pi}\,(\mathcal{F}s_t[f(t)]\,(z))\,(\mathcal{F}c_t[g(t)]\,(z))$$

This formula shows that the Fourier sine transform of a convolution gives the product of the Fourier sine and the Fourier cosine transforms multiplied by $\sqrt{2\pi}$.

Integral

$$\mathcal{F}s_t\left[\int_t^\infty f(\tau)\,d\tau\right](z) = -\frac{1}{z}\,(\mathcal{F}c_t[f(t)]\,(z)-\mathcal{F}c_t[f(t)]\,(0))$$

This formula shows that the Fourier sine transform of an indefinite integral with a variable lower limit gives the difference of the Fourier cosine transforms in z and 0 multiplied by $-1/z$.

Limit at infinity

$$\lim_{z\to\infty} (\mathcal{F}s_t[f(t)](z)) = 0$$

The Riemann–Lebesgue theorem shows that the Fourier sine transform $\mathcal{F}s_t[f(t)](z)$ converges to zero as z tends to infinity for some classes of function $f(t)$.

Relations with other integral transforms

With inverse Fourier sine transform

$$\mathcal{F}s_t\left[\mathcal{F}s_\tau^{-1}[f(\tau)](t)\right](z) == f(z)$$

This formula reflects the relation between the direct and the inverse Fourier sine transforms. At the point $z == z_0$, where $f(z)$ has a jump discontinuity, the composition of the inverse and the direct Fourier sine transforms converges to the mean $\frac{1}{2}\left(\lim_{z\to z_0^+} f(z) + \lim_{z\to z_0^-} f(z)\right)$.

With exponential Fourier transform

$$\mathcal{F}s_t[f(t)](z) == i\,\mathcal{F}s_t[f(-t)\,\theta(-t) - f(t)\,\theta(t)](z)$$

This formula represents the Fourier sine transform through the exponential Fourier transform.

With Laplace transform

$$\mathcal{F}s_t[f(t)](z) == \frac{i}{\sqrt{2\pi}}\mathcal{L}_t[f(t)](i z) - \frac{i}{\sqrt{2\pi}}\mathcal{L}_t[f(t)](-i z)$$

This formula represents the Fourier sine transform through the Laplace transforms.

Inverse Fourier sine transform

Definition

$$\mathcal{F}s_t^{-1}[f(t)](z) == \sqrt{\frac{2}{\pi}}\int_0^\infty f(t)\sin(t z)\,dt$$

This formula is the definition of the inverse Fourier sine transform of the function f with respect to the variable t. If the integral does not converge, the value of $\mathcal{F}s_t^{-1}[f(t)](z)$ is defined in the sense of generalized functions.

Relations with other integral transforms

With Fourier sine transform

$$\mathcal{F}s_t^{-1}[f(t)](z) = \mathcal{F}s_t[f(t)](z)$$

This formula shows that the inverse Fourier sine transform coincides with the direct Fourier sine transform.

Laplace transform

Definition

$$\mathcal{L}_t[f(t)](z) = \int_0^\infty f(t)\, e^{-t z}\, dt$$

This formula is the definition of the Laplace transform of the function f with respect to the variable t. If the integral does not converge, the value of $\mathcal{L}_t[f(t)](z)$ is defined in the sense of generalized functions.

General properties

Linearity

$$\mathcal{L}_t[a\, f(t) + b\, g(t)](z) = a\, \mathcal{L}_t[f(t)](z) + b\, \mathcal{L}_t[g(t)](z)$$

This formula reflects the linearity of the Laplace transform.

Shift

$$\mathcal{L}_t\left[e^{-a t}\, f(t)\right](z) = \mathcal{L}_t[f(t)](z + a)$$

This *shift theorem* shows that the Laplace transform of a product with an exponential function gives the Laplace transform in the shifted point.

Power scaling

$$\mathcal{L}_t[t\, f(t)](z) = -\frac{\partial^n (\mathcal{L}_t[f(t)](z))}{\partial z^n}$$

This formula shows that differentiation of a Laplace transform corresponds to multiplication of the original function by $-t$.

$$\mathcal{L}_t[t^n\, f(t)](z) = (-1)^n \frac{\partial^n (\mathcal{L}_t[f(t)](z))}{\partial z^n} /;\ n \in \mathbb{N}^+$$

This formula shows that differentiation of a Laplace transform of order n corresponds to multiplication of the original function by $(-t)^n$.

Product

$$\mathcal{L}_t[f(t)\, g(t)](z) = \frac{1}{2 \pi i} \int_{\gamma - i \infty}^{\gamma + i \infty} (\mathcal{L}_t[f(t)](\tau))\, (\mathcal{L}_t[g(t)](z - \tau))\, d\tau /;\ \mathrm{Im}(\gamma) = 0$$

This formula represents the Laplace transform of a product $f(t)$ and $g(t)$ through the contour integral along a vertical line from the corresponding product of Laplace transforms.

Derivative

$$\mathcal{L}_t\left[f^{(n)}(t)\right](z) = z^n\, \mathcal{L}_t[f(t)](z) - \sum_{k=0}^{n-1} z^{n-k-1}\, f^{(k)}(0)$$

This *time differentiation* relation gives the representation for the Laplace transform of the first derivative.

$$\mathcal{L}_t\left[f^{(n)}(t)\right](z) = z^n\, \mathcal{L}_t[f(t)](z) - \sum_{k=0}^{n-1} z^{n-k-1}\, f^{(k)}(0)$$

This *time differentiation* relation gives the representation of the Laplace transform of the n^{th} derivative.

Integral

$$\mathcal{L}_t\left[\int_0^t f(\tau)\, d\tau\right](z) = \frac{1}{z}\, \mathcal{L}_t[f(t)](z)$$

This formula shows that the Laplace transform of an indefinite integral gives the product of the reciprocal function of z by the Laplace transform of the function.

$$\mathcal{L}_t\left[\int_0^t \frac{f(\tau)\,(t-\tau)^{n-1}}{(n-1)!}\, d\tau\right](z) = \frac{1}{z^n}\left(\mathcal{L}_t[f(t)](z)\right) /;\, n \in \mathbb{N}^+$$

This formula shows that the Laplace transform of the repeated indefinite integral $\underbrace{\int_0^x \int_0^t \ldots \int_0^t f(t)\, dt\, dt \ldots dt}_{n-\text{times}} = \int_0^x f(t)\, \frac{(x-t)^{n-1}}{(n-1)!}\, dt$ gives the product of the power function z^{-n} on Laplace transform.

Convolution

$$\mathcal{L}_t\left[\int_0^t f(t-\tau)\, g(\tau)\, d\tau\right](z) = \left(\mathcal{L}_t[f(t)](z)\right)\left(\mathcal{L}_t[g(t)](z)\right)$$

The *convolution theorem* or convolution (Faltung) theorem for the Laplace transform shows that the Laplace transform of a convolution is equal to the product of Laplace transforms of the convoluted functions.

Limit at infinity

$$\lim_{z\to\infty} z\left(\mathcal{L}_t[f(t)](z)\right) = \lim_{t\to 0^+} f(t)$$

The *initial value theorem* shows that limit at infinity of the Laplace transform multiplied by z is the one-sided limit of the initial function at zero.

Limit at zero

$$\lim_{z \to 0} z\,(\mathcal{L}_t[f(t)](z)) = \lim_{t \to \infty} f(t)$$

The *final value theorem* shows that the limit at zero of the Laplace transform multiplied by z is the limit of the initial function at infinity.

Sum

$$\mathcal{L}_t\left[\sum_{k=-\infty}^{\infty} f(t - L\,k)\right](z) = \frac{1}{1 - e^{-z\,L}}\,\mathcal{L}_t[\theta(t)\,\theta(L - t)\,f(t)](z)$$

Relations with other integral transforms

With inverse Laplace transform

$$\mathcal{L}_t\left[\mathcal{L}^{-1}_{\gamma;\tau}[f(\tau)](t)\right](z) == f(z)$$

This formula reflects the relation between the direct and the inverse Laplace transforms.

$$\mathcal{L}_t\left[\mathcal{L}^{-1}_{\tau}[f(\tau)](t)\right](z) == f(z)$$

This formula reflects the relation between the direct and the inverse Laplace transforms.

With exponential Fourier transform

$$\mathcal{L}_t[f(t)](z) = \sqrt{2\,\pi}\,(\mathcal{F}_t[\theta(t)\,f(t)](i\,z))$$

This formula shows how the Laplace transform can be represented through the exponential Fourier transform.

With Mellin transform

$$\mathcal{L}_t[f(t)](z) = \mathcal{M}_t[\theta(1 - t)\,f(-\log(t))](z)$$

This formula shows how the Laplace transform can be represented through the Mellin transform.

With Z-transform

$$\mathcal{L}_t\left[\sum_{n=-\infty}^{\infty} f(n)\,\delta(t - n)\right](z) = \mathcal{Z}_n[f(n)](e^z)$$

This formula shows how the Laplace transform can be represented through the Z-transform.

Inverse Laplace transform

Definition

$$\mathcal{L}^{-1}_{\gamma;t}[f(t)](p) == \frac{1}{2\,\pi\,i}\int_{\gamma - i\,\infty}^{\gamma + i\,\infty} f(t)\,e^{t\,p}\,dt$$

This formula is the definition of the inverse Laplace integral transform of the function f with respect to the variable t.

$$\mathcal{L}_t^{-1}[f(t)](p) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} f(t)\, e^{t p}\, dt$$

This formula is the definition of the inverse Laplace integral transform of the function f with respect to the variable t.

$$\mathcal{L}_{\gamma;t}^{-1}[f(t)](p) = \lim_{k\to\infty}\left(\frac{(-1)^k}{k!}\left(\frac{k}{p}\right)^{k+1} \mathcal{L}_t^{(0,0,k)}[f(t)]\left(\frac{k}{p}\right)\right)$$

This formula is the Post–Widder form of the inverse Laplace integral transform of the function f with respect to the variable t.

Multiple Laplace transform

Definition

$$\mathcal{L}_{\{t_1,t_2\}}[f(t_1, t_2)](z_1, z_2) = \int_0^\infty \int_0^\infty f(t_1, t_2)\, e^{-t_1 z_1 - t_2 z_2}\, dt_1\, dt_2$$

This formula is the definition of the double Laplace transform of the function f with respect to the variables t_1, t_2.

$$\mathcal{L}_{\{t_1,t_2,\ldots,t_n\}}[f(t_1, t_2, \ldots, t_n)](z_1, z_2, \ldots, z_n) = \underbrace{\int_0^\infty \int_0^\infty \ldots \int_0^\infty f(t_1, t_2, \ldots, t_n)\, e^{-t_1 z_1 - t_2 z_2 - \ldots - t_n z_n}\, dt_n \ldots dt_2\, dt_1}_{n-\text{times}}$$

This formula is the definition of multiple Laplace transforms of the function $f(t_1, t_2, \ldots, t_n)$ with respect to the variables $t_1, t_2, \ldots, t_n$ over $\mathbb{R}^n$.

Mellin transform

Definition

$$\mathcal{M}_t[f(t)](z) = \int_0^\infty f(t)\, t^{z-1}\, dt$$

This formula is the definition of the Mellin transform of the function f with respect to the variable t. If the integral does not converge, the value of $\mathcal{M}_t[f(t)](z)$ is defined in the sense of generalized functions. Usually, the integral converges in the strip $\alpha < \operatorname{Re}(z) < \beta$, where α and β depend on the function $f(t)$ and can assume the values $\pm\infty$. For example, $\mathcal{M}_t\left[\theta(t-a)\, t^b\right](z) = -\frac{a^{b+z}}{b+z}$ /; $\operatorname{Re}(z) < -\operatorname{Re}(b)$.

General properties

Linearity

$$\mathcal{M}_t[a\, f(t) + b\, g(t)](z) = a\, \mathcal{M}_t[f(t)](z) + b\, \mathcal{M}_t[g(t)](z)$$

This formula reflects the linearity of the Mellin transform.

Scaling

$$\mathcal{M}_t[f(a\,t)](z) = a^{-z}\,\mathcal{M}_t[f(t)](z) \;/;\; a > 0$$

The operation reflects the scaling of the original variable *t* by a positive number *a* in a Mellin transform.

Power

$$\mathcal{M}_t\left[f\left(\frac{1}{t}\right)\right](z) = \mathcal{M}_t[f(t)](-z)$$

This formula reflects the Mellin transform of the function $f(1/t)$.

$$\mathcal{M}_t[f(t^a)](z) = \frac{1}{|a|}\,\mathcal{M}_t[f(t)]\left(\frac{z}{a}\right) \;/;\; a \neq 0 \land a \in \mathbb{R}$$

The operation provides the Mellin transform of the original variable *t* raised to a real power *a*.

Shifting

$$\mathcal{M}_t[t^a\,f(t)](z) = \mathcal{M}_t[f(t)](z+a)$$

The *shift theorem* gives the Mellin transform of a product of the original function by some power of *t*.

$$\mathcal{M}_t[\log^a(t)\,f(t)](z) = \frac{\partial^k\,\mathcal{M}_t[f(t)](z)}{\partial z^k}$$

The operation gives the Mellin transform of a product of the original function by a power of log(*t*).

Derivative

$$\mathcal{M}_t\left[f^{(n)}(t)\right](z) = (1-z)_n\,\mathcal{M}_t[f(t)](z-n) \;/;\; \lim_{t\to 0} t^{z-k-1}\,f^{(k)}(t) = 0 \bigwedge 0 \leq k \leq n-1 \bigwedge n \in \mathbb{N}^+$$

This formula shows that the Mellin transform of an n^{th} derivative gives the product of a polynomial and the Mellin transform of the function.

$$\mathcal{M}_t\left[\left(t\,\frac{\partial}{\partial t}\right)^n f(t)\right](z) = (-1)^n\,z^n\;\mathcal{M}_t[f(t)](z) \;/;\; n \in \mathbb{N}^+$$

This formula shows that the Mellin transform of $\left(t\,\frac{\partial}{\partial t}\right)^n f(t)$ gives the product of a power function and the Mellin transform.

$$\mathcal{M}_t\left[\left(\frac{\partial}{\partial t}\,t\right)^n f(t)\right](z) = (1-z)^n\;\mathcal{M}_t[f(t)](z) \;/;\; n \in \mathbb{N}^+$$

This formula shows that the Mellin transform of $\left(\frac{\partial}{\partial t}\,t\right)^n f(t)$ gives the product of the power function and the Mellin transform.

$$\mathcal{M}_t\left[\frac{\partial^n (t^n f(t))}{\partial t^n}\right](z) = (1-z)_n \, \mathcal{M}_t[f(t)](z) \;/;\; n \in \mathbb{N}^+$$

This formula shows that the Mellin transform of $\frac{\partial^n (t^n f(t))}{\partial t^n}$ gives the product of a polynomial and the Mellin transform.

$$\mathcal{M}_t\left[t^n f^{(n)}(t)\right](z) = (-1)^n (z)_n \, \mathcal{M}_t[f(t)](z) \;/;\; \lim_{t \to 0} t^{z-k-1} f^{(k)}(t) = 0 \bigwedge 0 \le k \le n-1 \bigwedge n \in \mathbb{N}^+$$

This formula shows that the Mellin transform of $t^n f^{(n)}(t)$ gives the product of a polynomial and the Mellin transform.

Integral

$$\mathcal{M}_t\left[\int_0^t f(\tau)\, d\tau\right](z) = -\frac{1}{z} \mathcal{M}_t[f(t)](z+1)$$

This formula shows that the Mellin transform of an indefinite integral gives the product of $-1/z$ and the Mellin transform in the shifted point.

$$\mathcal{M}_t\left[\int_0^t \ldots \int_0^{\tau_3} \int_0^{\tau_2} f(\tau_1)\, d\tau_1\, d\tau_2 \ldots d\tau_n\right](z) = \frac{(-1)^n}{(z)_n} \mathcal{M}_t[f(t)](z+n)$$

This formula shows how the Mellin transform of a repeated indefinite integral gives the product of a rational function and the Mellin transform in the shifted point.

$$\mathcal{M}_t\left[\int_t^\infty f(\tau)\, d\tau\right](z) = \frac{1}{z} \mathcal{M}_t[f(t)](z+1)$$

This formula shows that the Mellin transform of the indefinite integral with a variable lower limit gives the product of $1/z$ and the Mellin transform in the shift point.

$$\mathcal{M}_t\left[\int_t^\infty \ldots \int_{\tau_3}^\infty \int_{\tau_2}^\infty f(\tau_1)\, d\tau_1\, d\tau_2 \ldots d\tau_n\right](z) = \frac{1}{(z)_n} \mathcal{M}_t[f(t)](z+n)$$

This formula demonstrates how the Mellin transform of a repeated indefinite integral with variable lower limits gives the product of a rational function amd the Mellin transform in the shifted point.

Convolution

$$\mathcal{M}_t\left[\int_0^\infty \frac{1}{\tau} f(\tau)\, g\left(\frac{t}{\tau}\right) d\tau\right](z) = (\mathcal{M}_t[f(t)](z))\,(\mathcal{M}_t[g(t)](z))$$

The *Mellin convolution theorem* shows that the Mellin transform of a Mellin convolution equals the product of the Mellin transforms.

$$\mathcal{M}_t\left[t^a \int_0^\infty \tau^{b-1} f(\tau^c)\, g(\tau^e\, t)\, d\tau\right](z) = \frac{1}{|c|}\left(\mathcal{M}_t[f(t)]\left(\frac{b-e\,(a+z)}{c}\right)\right)(\mathcal{M}_t[g(t)](a+z)) \;/;\; c \neq 0 \land c \in \mathbb{R} \land e \in \mathbb{R}$$

The *generalized Mellin convolution theorem* shows that the Mellin transform of the generalized Mellin convolution is equal to the product of the Mellin transforms.

Parseval

$$\int_0^\infty f(z\,\tau)\,g(\tau)\,d\tau = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}(\mathcal{M}_t[f(t)](s))\,(\mathcal{M}_t[g(t)](1-s))\,z^{-s}\,ds$$

This formula is called Mellin–Parseval's formula.

$$\int_0^\infty f(\tau)\,g(\tau)\,d\tau = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}(\mathcal{M}_t[f(t)](s))\,(\mathcal{M}_t[g(t)](1-s))\,ds$$

This formula is called Parseval's formula.

$$\int_0^\infty \tau^a\,f(b\,\tau^c)\,g(d\,\tau^e)\,d\tau = \frac{b^{\frac{a+1}{c}}}{e\,|c|}\,\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\left(\mathcal{M}_t[f(t)]\left(\frac{a+s+1}{c}\right)\right)\left(\mathcal{M}_t[g(t)]\left(-\frac{s}{e}\right)\right)\left(d^{1/e}\,b^{-\frac{1}{c}}\right)^s ds\;/;\;c\neq 0\bigwedge c\in\mathbb{R}\bigwedge d\in\mathbb{R}$$

This representation can be used for evaluation of the general class of integrals from products of Meijer G functions.

Relations with other integral transforms

With inverse Mellin transform

$$\mathcal{M}_t\left[\mathcal{M}_{\gamma;\tau}^{-1}[f(\tau)](t)\right](s) = f(s)$$

This formula reflects the relation between the direct and the inverse Mellin transforms. The following theorem holds: if an analytical function $f(s)$ satisfies the restriction $|f(s)| < K\,|s|^{-2}$ in the strip $\alpha < \operatorname{Re}(z) < \beta$ with some constant K, then the integral $\mathcal{M}_{\gamma;s}^{-1}[f(s)](t)$ is a continuous function of the variable t and is its Mellin transform in this strip.

With exponential Fourier transform

$$\mathcal{M}_t[f(t)](z) = \sqrt{2\pi}\,\left(\mathcal{F}_t\left[f\left(e^{-t}\right)\right](i\,z)\right)$$

This formula shows how the Mellin transform can be represented through the exponential Fourier transform.

With Fourier cosine and sine transforms

$$\mathcal{M}_t[f(t)](z) = \sqrt{2\pi}\,\mathcal{F}c_t\left[\frac{f(e^{-t})+f(e^t)}{2}\right](i\,z) + i\,\sqrt{2\pi}\,\mathcal{F}s_t\left[\frac{f(e^{-t})-f(e^t)}{2}\right](i\,z)$$

This formula shows how the Mellin transform can be represented through the cosine and the sine Fourier transforms from the even and odd parts of the function.

With Mellin transform

$$\mathcal{L}_t[f(t)](z) = \mathcal{M}_t[\theta(1-t)\, f(-\log(t))](z)$$

This formula shows how the Laplace transform can be represented through the Mellin transform.

Inverse Mellin transform

Definition

$$\mathcal{M}_{\gamma;s}^{-1}[f(s)](t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} f(s)\, t^{-s}\, ds$$

This formula is the definition of the inverse Mellin integral transform of the function f with respect to the variable s. If the integral does not converge, the value of $\mathcal{M}_{\gamma;s}^{-1}[f(s)](t)$ is defined in the sense of generalized functions. The condition on γ usually has the following form: $\alpha < \gamma == \mathrm{Re}(s) < \beta$, which represents a vertical strip of convergence for the integral.

Changing the vertical strip of integration $\alpha < \mathrm{Re}(s) < \beta$ leads to a change in the original function $\mathcal{M}_{\gamma;s}^{-1}[f(s)](t)$. For example, in the case of gamma function $f(s) == \Gamma(s)$ you have $\mathcal{M}_{\gamma;s}^{-1}[\Gamma(s)](t) == e^{-t}$ /; $\gamma > 0$ and $\mathcal{M}_{\gamma,s}^{-1}[\Gamma(s)](t) == e^{-t} - 1$ /; $-1 < \gamma < 0$.

Multiple Mellin transform

Definition

$$\mathcal{M}_{\{t_1,t_2\}}[f(t_1, t_2)](z_1, z_2) = \int_0^\infty \int_0^\infty f(t_1, t_2)\, t_1^{z_1-1}\, t_2^{z_2-1}\, dt_1\, dt_2$$

This formula is the definition of the double Mellin transform of the function $f(t_1, t_2)$ with respect to the variables t_1, t_2.

$$\mathcal{M}_{\{t_1,t_2,\ldots,t_n\}}[f(t_1, t_2, \ldots, t_n)](z_1, z_2, \ldots, z_n) = \underbrace{\int_0^\infty \int_0^\infty \ldots \int_0^\infty f(t_1, t_2, \ldots, t_n)\, t_1^{z_1-1}\, t_2^{z_2-1} \ldots t_n^{z_n-1}\, dt_n \ldots dt_2\, dt_1}_{n-\text{times}}$$

This formula is the definition of the multiple Mellin transform of the function $f(t_1, t_2, \ldots, t_n)$ with respect to the variables $t_1, t_2, \ldots, t_n$.

General properties

Convolution

$$\mathcal{M}_{\{t_1,t_2\}}\left[\int_0^\infty \frac{f(\tau)}{\tau}\, g_1\left(\frac{t_1}{\tau}\right) g_2\left(\frac{t_2}{\tau}\right) d\tau\right](z_1, z_2) == (\mathcal{M}_t[f(t)](z_1+z_2)) \left(\mathcal{M}_{t_1}[g_1(t_1)](z_1)\right) \left(\mathcal{M}_{t_2}[g_2(t_2)](z_2)\right)$$

The *generalized Mellin convolution theorem* shows that the double Mellin transform of a Mellin generalized convolution equals the product of the Mellin transforms in the corresponding points.

$$\mathcal{M}_{\{t_1,t_2,\ldots,t_n\}}\left[\int_0^\infty \frac{f(\tau)}{\tau} \prod_{k=1}^n g_k\left(\frac{t_k}{\tau}\right) d\tau\right](z_1, z_2, \ldots, z_n) == \left(\mathcal{M}_t[f(t)]\left(\sum_{k=1}^n z_k\right)\right)\left(\prod_{k=1}^n \mathcal{M}_{t_k}[g_k(t_k)](z_k)\right)$$

The *generalized Mellin convolution theorem* shows that the multiple Mellin transform of a Mellin generalized convolution equals the product of the Mellin transforms in the corresponding points.

Hankel transform

Definition

$$\mathcal{H}_{\nu;t}[f(t)](z) == \int_0^\infty f(t)\sqrt{t\,z}\,J_\nu(t\,z)\,dt$$

This formula is the definition of the Hankel integral transform of the function f with respect to the variable t. If this integral does not converge, the value of $\mathcal{H}_{\nu;t}[f(t)](z)$ is defined in the sense of generalized functions.

General properties

$$\mathcal{H}_{\nu;t}[\mathcal{H}_{\nu;\tau}[f(\tau)](t)](z) == f(z) \;/;\; \mathrm{Re}(\nu) > -\frac{1}{2}$$

This formula shows that the inverse Hankel integral transform coincides with the direct Hankel integral transform under the restriction $\mathrm{Re}(\nu) > -\frac{1}{2}$.

Hilbert transform

Definition

$$\mathcal{H}_t[f(t)](x) == \frac{1}{\pi}\mathcal{P}\int_{-\infty}^\infty \frac{f(t)}{t-x}\,dt \;/;\; x \in \mathbb{R}$$

This formula is the definition of the Hilbert transform of the function f with respect to the variable t for real x.

Inverse Hilbert transform

Definition

$$\mathcal{H}_t^{-1}[f(t)](x) == -\frac{1}{\pi}\mathcal{P}\int_{-\infty}^\infty \frac{f(t)}{t-x}\,dt \;/;\; x \in \mathbb{R}$$

This formula is the definition of the inverse Hilbert transform of the function f with respect to the variable t for real x. It coincides with the direct Hilbert transform multiplied by -1.

Relations with other integral transforms

With Hilbert transform

$$\mathcal{H}_t^{-1}[f(t)](x) == -\mathcal{H}_t[f(t)](x)$$

Z-transform

Definition

$$\mathcal{Z}_n[f(n)](z) = \sum_{n=0}^{\infty} f(n)\, z^{-n}$$

This formula is the definition of the Z-transform of the function $f(n)$ with respect to the discrete variable n at the complex point z.

General properties

Linearity

$$\mathcal{Z}_n[a\, f(n) + b\, g(n)](z) = a\, (\mathcal{Z}_n[f(n)](z)) + b\, (\mathcal{Z}_n[g(n)](z))$$

This formula reflects the linearity of the Z-transform.

Shifting

$$\mathcal{Z}_n[f(n-m)](z) = z^{-m}\, (\mathcal{Z}_n[f(n)](z)) + z^{-m} \sum_{k=1}^{m} f(-k)\, z^k$$

This formula reflects the s*hifting property* of the Z-transform.

$$\mathcal{Z}_n[f(n-m)](z) = z^{-m}\, (\mathcal{Z}_n[f(n)](z)) \;/;\; f(-1) = f(-2) = \ldots == f(-m) = 0$$

This formula reflects the *shifting property* of the Z-transform.

$$\mathcal{Z}_n[f(m+n)](z) = z^{m}\, (\mathcal{Z}_n[f(n)](z)) - z^{m} \sum_{k=0}^{m-1} f(k)\, z^{-k}$$

This formula reflects the *shifting property* of the Z-transform.

Scaling

$$\mathcal{Z}_n[a^n\, f(n)](z) = \mathcal{Z}_n[f(n)]\left(\frac{z}{a}\right)$$

This formula reflects the scaling property of the Z-transform.

$$\mathcal{Z}_n[n^m\, f(n)](z) = (-1)^m \left(z\, \frac{\partial}{\partial z}\right)^m (\mathcal{Z}_n[f(n)](z)) \;/;\; m \in \mathbb{N}^+$$

This formula shows that multiplication of a function $f(n)$ by n^m leads to repeated differentiation of the Z-transform.

$$\mathcal{Z}_n\left[(n)_m\, f(n)\right](z) = (-z)^m\, \frac{\partial^m (\mathcal{Z}_n[f(n)](z))}{\partial z^m} \;/;\; m \in \mathbb{N}^+$$

This formula shows that multiplication of a function $f(n)$ by $(n)_m$ gives the product of $(-z)^m$ and the m^{th} derivative of the Z-transform.

Product

$$\mathcal{Z}_n[f(n)\,g(n)](z) = \int_L \frac{1}{t}\,(\mathcal{Z}_n[f(n)](t))\left(\mathcal{Z}_n[g(n)]\left(\frac{z}{t}\right)\right) dt$$

The Z-transform of a product of $f(n)$ and $g(n)$ is represented through a contour integral along a simple circle-type contour L encircling the origin $t = 0$ counterclockwise. All the singular points of the function $\mathcal{Z}_n[f(n)](t)$ are located inside the contour. All the singular points of the function $\mathcal{Z}_n[g(n)]\left(\frac{z}{t}\right)$ are located outside the contour.

Parseval

$$\sum_{n=0}^{\infty} f(n)\,g(n) = \int_L \frac{1}{t}\,(\mathcal{Z}_n[f(n)](t))\left(\mathcal{Z}_n[g(n)]\left(\frac{1}{t}\right)\right) dt$$

The *Parseval theorem* follows from the previous relation for $z == 1$. The integration is performed along a simple circle-type contour L encircling the origin $t = 0$ counterclockwise. All the singular points of the function $\mathcal{Z}_n[f(n)](t)$ are located inside the contour. All the singular points of the function $\mathcal{Z}_n[g(n)]\left(\frac{z}{t}\right)$ are located outside the contour.

Correlation

$$\sum_{k=0}^{\infty} f(k)\,g(k-n) = \int_L t^{n-1}\,(\mathcal{Z}_n[f(n)](t))\left(\mathcal{Z}_n[g(n)]\left(\frac{1}{t}\right)\right) dt \;/;\; n \in \mathbb{N}^+$$

The property is called the *cross correlation property* of the Z-transform.

Convolution

$$\mathcal{Z}_n\left[\sum_{k=0}^{\infty} f(k)\,g(n-k)\right](z) = (\mathcal{Z}_n[f(n)](z))\,(\mathcal{Z}_n[g(n)](z)) \;/;\; g(-m) = 0 \bigwedge m \in \mathbb{N}^+$$

The *convolution theorem* for the Z-transform shows that the Z-transform of a convolution sum is equal to the product of the corresponding Z-transforms.

Limit at infinity

$$\lim_{z \to \infty} \mathcal{Z}_n[f(n)](z) = f(0)$$

The analog of the Riemann-Lebesgue theorem shows that the Z-transform $\mathcal{Z}_n[f(n)](z)$ at infinity tends to the initial value $f(0)$.

$$\lim_{z \to \infty} z\,(\mathcal{Z}_n[f(n)](z)) = f(1) \;/;\; f(0) = 0$$

This formula shows that the Z-transform $\mathcal{Z}_n[f(n)](z)$ at infinity behaves as $f(0) + \frac{f(1)}{z}$.

Limit at one

$$\lim_{z\to 1}(z-1)\,(\mathcal{Z}_n[f(n)]\,(z)) = \lim_{n\to\infty} f(n)$$

This formula shows that the expression $(z-1)\,\mathcal{Z}_n[f(n)]\,(z)$ near point $z == 1$ behaves as $\lim_{n\to\infty} f(n)$.

Derivative by parameter

$$\mathcal{Z}_n\left[\frac{\partial f(n, a)}{\partial a}\right](z) = \frac{\partial(\mathcal{Z}_n[f(n, a)]\,(z))}{\partial a}$$

This formula reflects the *differentiation by parameter* property of the Z-transform.

Limit by parameter

$$\mathcal{Z}_n\left[\lim_{a\to a_0} f(n, a)\right](z) = \lim_{a\to a_0} \mathcal{Z}_n[f(n, a)]\,(z)$$

This formula reflects the *evaluation limit by parameter* property of the Z-transform.

Integration by parameter

$$\mathcal{Z}_n\left[\int_a^b f(n, t)\,dt\right](z) = \int_a^b \mathcal{Z}_n[f(n, t)]\,(z)\,dt$$

This formula reflects the *integration by parameter* property of the Z-transform.

Relations with other integral transforms

With inverse Z-transforms

$$\mathcal{Z}_n\left[\mathcal{Z}_t^{-1}[f(t)]\,(n)\right](z) = f(z)$$

This formula reflects the relation between the direct and the inverse Z-transforms.

Inverse Z-transform

Definition

$$\mathcal{Z}_t^{-1}[f(t)]\,(n) == \frac{1}{2\pi i}\int_L f(t)\,t^{n-1}\,dt$$

This formula is the definition of the inverse Z-transform of the function $f(t)$ with respect to the variable t at the discrete point n. The contour integral is performed along a simple circle-type contour L encircling the origin $t = 0$ counterclockwise.

Weber transform

Definition

$$\mathcal{W}_{\nu;t}[f(t)](z) == \int_1^{\infty} t\, f(t)\,(J_\nu(t\,z)\,Y_\nu(z) - Y_\nu(t\,z)\,J_\nu(z))\,dt$$

This formula is the definition of the Weber integral transform of the function f with respect to the variable t.

Relations with other integral transforms

With inverse Weber transform

$$\mathcal{W}_{\nu;t}^{-1}[\mathcal{W}_{\nu;\tau}[f(\tau)](t)](z) = f(z)$$

The formula shows that the composition of the direct and the inverse Weber integral transforms gives the original function in a point of continuity.

Inverse Weber transform

Definition

$$\mathcal{W}_{\nu;t}^{-1}[f(\nu,\, t)](z) = \int_0^{\infty} \frac{t\, f(\nu,\, t)}{J_\nu(t)^2 + Y_\nu(t)^2}\,(J_\nu(t\,z)\,Y_\nu(t) - Y_\nu(t\,z)\,J_\nu(t))\,dt$$

This formula gives the formula for the inverse Weber integral transform.

Summation

Finite summation

$$\sum_{k=0}^{n} a_k = a_1 + a_2 + \ldots + a_n$$

This formula is the definition of the finite sum.

$$\sum_{k=0}^{n} a_k = \sum_{k=0}^{m} a_k + \sum_{k=0}^{n-m-1} a_{k+m+1} \;/;\; m \le n$$

This formula shows how a finite sum can be split into two finite sums.

$$\sum_{k=0}^{n} (\alpha\, a_k) = \alpha \sum_{k=0}^{n} a_k$$

This formula shows that a constant factor in a summand can be taken out of the sum.

$$\sum_{k=0}^{n} a_k \pm \sum_{j=0}^{n} b_j = \sum_{k=0}^{n} a_k \pm b_k$$

This formula reflects the linearity of the finite sums.

$$\sum_{k=0}^{n} \log(a_k) == \log\left(\prod_{k=0}^{n} a_k\right) /; -\pi < \sum_{k=0}^{n} \arg(a_k) \leq \pi$$

This formula represents the concept that the *sum of logs is equal to the log of the product*, which is correct under the given restriction.

$$\sum_{k=0}^{n} \log(z_k) == \log\left(\prod_{k=0}^{n} z_k\right) - 2\pi i \left\lfloor \frac{\pi - \sum_{k=0}^{n} \arg(z_k)}{2\pi} \right\rfloor /; n \in \mathbb{N}$$

This general formula is correct without any restrictions.

$$\frac{a_0}{2} + \sum_{k=1}^{n} (a_k \cos(k x) + b_k \sin(k x)) == \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{f(x-t)}{2\sin\left(\frac{t}{2}\right)} \sin\left(\left(n + \frac{1}{2}\right) t\right) dt /; a_k == \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos(k t) \, dt \bigwedge b_k == \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin(k t) \, dt$$

This formula is called the *Dirichlet formula for a Fourier series*.

$$\sum_{k=0}^{n} a_k = \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} a_{2k} + \sum_{k=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} a_{2k+1}$$

In this formula, the sum is divided into the sums of the even and odd terms.

$$\sum_{k=0}^{n} a_k = \sum_{k=0}^{\left\lfloor \frac{n}{3} \right\rfloor} a_{3k} + \sum_{k=0}^{\left\lfloor \frac{n-1}{3} \right\rfloor} a_{3k+1} + \sum_{k=0}^{\left\lfloor \frac{n-2}{3} \right\rfloor} a_{3k+2}$$

In this formula, the sum of a_k is divided into three sums with the terms a_{3k}, a_{3k+1}, and a_{3k+2}.

$$\sum_{k=0}^{n} a_k = \sum_{k=0}^{\left\lfloor \frac{n}{4} \right\rfloor} a_{4k} + \sum_{k=0}^{\left\lfloor \frac{n-1}{4} \right\rfloor} a_{4k+1} + \sum_{k=0}^{\left\lfloor \frac{n-2}{4} \right\rfloor} a_{4k+2} + \sum_{k=0}^{\left\lfloor \frac{n-3}{4} \right\rfloor} a_{4k+3}$$

In this formula, the sum of a_k is divided into four sums with the terms a_{4k}, a_{4k+1}, a_{4k+2}, and a_{4k+3}.

$$\sum_{k=0}^{n} a_k = \sum_{j=0}^{m-1} \sum_{k=0}^{\left\lfloor \frac{n-j}{m} \right\rfloor} a_{mk+j}$$

In this formula, the sum of a_k is divided into m sums with the terms a_{mk}, $a_{mk+1}, \ldots, a_{mk+m-2}$, and a_{mk+m-1}.

$$\left(\sum_{k=0}^{n} a_k\right) \sum_{j=0}^{n} b_j == \sum_{k=0}^{n} \sum_{j=0}^{n} a_k b_j$$

This formula describes the *multiplication rule* for finite sums.

$$\left(\sum_{k=1}^{n} a_k b_k\right)^2 == \left(\sum_{k=1}^{n} a_k^2\right) \sum_{k=1}^{n} b_k^2 - \sum_{k=1}^{n} \sum_{j=k+1}^{n} \left(a_k b_j - a_j b_k\right)^2$$

This formula is called Lagrange's identity.

Infinite summation (series)

$$\left(\sum_{k=0}^{\infty} a_k == \lim_{n\to\infty}\left(\sum_{k=0}^{n} a_k\right)\right) = \left(\forall_{\epsilon,\epsilon>0}\left(\exists_{\delta,\delta>0}\left(\forall_{n,n>\delta}\left|\sum_{k=0}^{\infty} a_k - \sum_{k=0}^{n} a_k\right| < \epsilon\right)\right)\right)$$

This formula reflects the definition of the convergent infinite sums (series) $\sum_{k=0}^{\infty} a_k$. The sum $\sum_{k=0}^{\infty} a_k$ converges absolutely if $a_k == O(k^{-r})$ /; $r > 1$. If $a_k \to 0$ this series can converge conditionally; for example, $\sum_{k=0}^{\infty} (-1)^k / k^r$ converges conditionally if $0 < r \leq 1$, and absolutely for $r > 1$. If $\lim_{k\to\infty} a_k \neq 0$, the series $\sum_{k=0}^{\infty} a_k$ does not converge (it is a divergent series).

$$\sum_{k=0}^{\infty} a_k = \sum_{k=0}^{m} a_k + \sum_{k=0}^{\infty} a_{k+m+1}$$

This formula shows one way to separate an arbitrary finite sum from an infinite sum.

$$\sum_{k=0}^{\infty} (\alpha\, a_k) = \alpha \sum_{k=0}^{\infty} a_k$$

This formula shows that a constant factor in the summands can be taken out of the sum.

$$\sum_{k=0}^{\infty} a_k \pm \sum_{j=0}^{\infty} b_j = \sum_{k=0}^{\infty} a_k \pm b_k$$

This formula reflects the linearity of summation.

$$\sum_{k=0}^{\infty} \log(a_k) == \log\left(\prod_{k=0}^{\infty} a_k\right) \text{ /; } -\pi < \sum_{k=0}^{\infty} \arg(a_k) \leq \pi$$

This formula reflects the statement that the *sum of the logs is equal to the log of the product*, which is correct under the shown restrictions.

$$\sum_{k=0}^{\infty} \log(z_k) == \log\left(\prod_{k=0}^{\infty} z_k\right) - 2\,\pi\,i \left\lfloor \frac{\pi - \sum_{k=0}^{\infty} \arg(z_k)}{2\,\pi} \right\rfloor$$

This formula is correct if all sums are convergent.

$$\frac{a_0^2}{2} + \sum_{k=1}^{\infty} \left(a_k^2 + b_k^2\right) == \frac{1}{\pi} \int_{-\pi}^{\pi} f(t)^2\, dt \text{ /; } a_k == \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos(k\, t)\, dt \bigwedge b_k == \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin(k\, t)\, dt$$

Parseval's lemma reflects completeness in the trigonometric system $\{\cos(k\, t), \sin(k\, t)\}$.

$$\sum_{k=0}^{\infty} a_k = \sum_{k=0}^{\infty} a_{2k} + \sum_{k=0}^{\infty} a_{2k+1}$$

In this formula, the sum is split into the sums of even and odd terms.

$$\sum_{k=0}^{\infty} a_k = \sum_{k=0}^{\infty} a_{3k} + \sum_{k=0}^{\infty} a_{3k+1} + \sum_{k=0}^{\infty} a_{3k+2}$$

In this formula, the sum of a_k is split into three sums with the terms a_{3k}, a_{3k+1}, and a_{3k+2}.

$$\sum_{k=0}^{\infty} a_k = \sum_{k=0}^{\infty} a_{4k} + \sum_{k=0}^{\infty} a_{4k+1} + \sum_{k=0}^{\infty} a_{4k+2} + \sum_{k=0}^{\infty} a_{4k+3}$$

In this formula, the sum of a_k is split into four sums with the terms a_{4k}, a_{4k+1}, a_{4k+2}, and a_{4k+3}.

$$\sum_{k=0}^{\infty} a_k = \sum_{j=0}^{m-1} \sum_{k=0}^{\infty} a_{mk+j}$$

In this formula, the sum of a_k is split into m sums with the terms a_{mk}, $a_{mk+1}, \ldots, a_{mk+m-2}$, and a_{mk+m-1}.

$$\sum_{k=1}^{\infty} a_k = \sum_{j=1}^{m} \sum_{k=0}^{\infty} a_{j+km} \;/;\; m \in \mathbb{N}^+$$

In this formula, the sum of a_k is split into m sums with the terms $a_{\mathrm{mk}+1}$, $a_{\mathrm{mk}+2}, \ldots, a_{\mathrm{mk}+m-1}$, and a_{mk+m}.

$$\left(\sum_{k=0}^{\infty} a_k\right) \sum_{j=0}^{\infty} b_j = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} a_k b_j$$

This formula describes the *multiplication rule* for a series.

$$\left(\sum_{k=1}^{\infty} a_k b_k\right)^2 = \left(\sum_{k=1}^{\infty} a_k^2\right) \sum_{k=1}^{\infty} b_k^2 - \sum_{k=1}^{\infty} \sum_{j=k+1}^{\infty} (a_k b_j - a_j b_k)^2$$

This formula is called Lagrange's identity.

Double finite summation

$$\sum_{k=0}^{m} \sum_{j=0}^{n} a_{k,j} = \sum_{j=0}^{n} \sum_{k=0}^{m} a_{k,j}$$

This formula reflects the commutativity property of finite double sums over the rectangle $0 \le k \le m$, $0 \le j \le n$.

$$\sum_{k=0}^{n} \sum_{j=0}^{n} a_{k,j} = \sum_{m=0}^{n} b_m \;/;\; b_m = \sum_{j=0}^{m} a_{m,j} + \sum_{j=0}^{m-1} a_{j,m}$$

This formula shows how to rewrite the double sum through a single sum.

$$\sum_{k=0}^{m} \sum_{j=0}^{k} a_{k,j} = \sum_{j=0}^{m} \sum_{k=j}^{m} a_{k,j}$$

This formula shows summation over the triangle $0 \le j \le k \le m$ in a different order.

$$\sum_{k=0}^{m}\sum_{j=0}^{\left\lfloor \frac{k}{2} \right\rfloor} a_{k,j} == \sum_{j=0}^{m}\sum_{k=0}^{m-2j} a_{2j+k,j}$$

This formula reflects summation over the triangle $0 \leq j \leq \left\lfloor \frac{k}{2} \right\rfloor \leq m$ in a different order.

$$\sum_{k=0}^{m}\sum_{j=0}^{\left\lfloor \frac{k-1}{2} \right\rfloor} a_{k,j} = \sum_{j=0}^{m}\sum_{k=0}^{m-2j-1} a_{2j+k+1,j}$$

This formula reflects summation over the triangle $0 \leq j \leq \left\lfloor \frac{k-1}{2} \right\rfloor \leq m$ in a different order.

$$\sum_{k=0}^{m}\sum_{j=k}^{m} a_{k,j} = \sum_{j=0}^{m}\sum_{k=0}^{j} a_{k,j}$$

This formula reflects summation over the triangle $0 \leq k \leq j \leq m$ in a different order.

$$\sum_{k=0}^{m}\sum_{j=k}^{p} a_{k,j} = \sum_{j=0}^{m}\sum_{k=0}^{j} a_{k,j} + \sum_{j=m+1}^{p}\sum_{k=0}^{m} a_{k,j} \text{ /; } p \geq m$$

This formula reflects summation over the trapezium (quadrangle) $0 \leq k \leq m,\ 0 \leq k \leq j \leq p$ in a different order.

$$\sum_{k=0}^{m}\sum_{j=\left\lfloor \frac{k}{2} \right\rfloor}^{p} a_{k,j} = \sum_{j=0}^{\left\lfloor \frac{m}{2} \right\rfloor-1}\sum_{k=0}^{2j+1} a_{k,j} + \sum_{j=\left\lfloor \frac{m}{2} \right\rfloor}^{p}\sum_{k=0}^{m} a_{k,j} \text{ /; } p \geq m$$

This formula reflects summation over the trapezium (quadrangle) $0 \leq k \leq m,\ 0 \leq \left\lfloor \frac{k}{2} \right\rfloor \leq j \leq p$ in a different order.

$$\sum_{k=0}^{m}\sum_{j=\left\lfloor \frac{k+1}{2} \right\rfloor}^{p} a_{k,j} = \sum_{j=0}^{\left\lfloor \frac{m-1}{2} \right\rfloor}\sum_{k=0}^{2j} a_{k,j} + \sum_{j=\left\lfloor \frac{m+1}{2} \right\rfloor}^{p}\sum_{k=0}^{m} a_{k,j} \text{ /; } p \geq m$$

This formula reflects summation over the trapezium (quadrangle) $0 \leq k \leq m,\ 0 \leq \left\lfloor \frac{k+1}{2} \right\rfloor \leq j \leq p$ in a different order.

$$\sum_{k=0}^{m}\sum_{j=2k}^{p} a_{k,j} = \sum_{j=0}^{2m-1}\sum_{k=0}^{\left\lfloor \frac{j}{2} \right\rfloor} a_{k,j} + \sum_{j=2m}^{p}\sum_{k=0}^{m} a_{k,j} \text{ /; } p \geq 2m$$

This formula reflects summation over the trapezium (quadrangle) $0 \leq k \leq m,\ 0 \leq 2k \leq j \leq p$ in a different order.

$$\sum_{k=0}^{m}\sum_{j=rk}^{p} a_{k,j} = \sum_{j=0}^{rm-1}\sum_{k=0}^{\left\lfloor \frac{j}{r} \right\rfloor} a_{k,j} + \sum_{j=rm}^{p}\sum_{k=0}^{m} a_{k,j} \text{ /; } p \geq rm$$

This formula reflects summation over the trapezium (quadrangle) $0 \leq k \leq m,\ 0 \leq rk \leq j \leq p,\ r \in \mathbb{N}^+$ in a different order.

Double infinite summation

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty} a_{k,j} = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} a_{k,j}$$

This formula reflects the commutative property of infinite double sums by the quadrant $0 \leq k,\ 0 \leq j$. It takes place under restrictions like $a_{k,j} == O\left(\left(k^2 + l^2\right)^{-r}\right) /; r > 1$, which provide absolute convergence of this double series.

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty} a_{k,j} = \sum_{m=0}^{\infty} b_m /; b_m = \sum_{j=0}^{m} a_{m,j} + \sum_{j=0}^{m-1} a_{j,m}$$

This formula shows how to rewrite the double sum through a single sum.

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty} a_{k,j} = \sum_{j=0}^{\infty}\sum_{k=0}^{j} a_{k,j-k}$$

This formula shows how to change the order in a double sum.

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty} a_{k,j} = \sum_{j=0}^{\infty}\sum_{k=0}^{j} a_{k,j-k}$$

This formula shows how to change the order in a double sum.

$$\sum_{k=0}^{\infty}\sum_{j=0}^{k} a_{k,j} = \sum_{j=0}^{\infty}\sum_{k=j}^{\infty} a_{k,j}$$

This formula reflects summation over the infinite triangle $0 \leq j \leq k$ in a different order.

$$\sum_{k=0}^{\infty}\sum_{j=0}^{k+m} a_{k,j} = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} a_{k,j} + \sum_{j=m+1}^{\infty}\sum_{k=j-m}^{\infty} a_{k,j}$$

This formula reflects summation over the infinite trapezium $0 \leq j \leq k + m \bigwedge m > 0$ in a different order.

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\left\lfloor \frac{k}{2} \right\rfloor} a_{k,j} == \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} a_{2j+k,j}$$

This formula reflects summation over the infinite triangle $0 \leq j \leq \left\lfloor \frac{k}{2} \right\rfloor$ in a different order.

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\left\lfloor \frac{k-1}{2} \right\rfloor} a_{k,j} = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} a_{2j+k+1,j}$$

This formula reflects summation over the infinite triangle $0 \leq j \leq \left\lfloor \frac{k-1}{2} \right\rfloor$ in a different order.

$$\sum_{k=0}^{\infty}\sum_{j=k}^{\infty}a_{k,j}=\sum_{j=0}^{\infty}\sum_{k=0}^{j}a_{k,j}$$

This formula reflects summation over the infinite triangle $0 \le k \le j$ in a different order.

$$\sum_{k=0}^{m}\sum_{j=k}^{\infty}a_{k,j}=\sum_{j=0}^{m}\sum_{k=0}^{j}a_{k,j}+\sum_{j=m+1}^{\infty}\sum_{k=0}^{m}a_{k,j}$$

This formula reflects summation over the infinite trapezium (quadrangle) $0 \le k \le m,\ 0 \le k \le j$ in a different order.

$$\sum_{k=0}^{m}\sum_{j=\left\lfloor\frac{k}{2}\right\rfloor}^{\infty}a_{k,j}=\sum_{j=0}^{\left\lfloor\frac{m}{2}\right\rfloor-1}\sum_{k=0}^{2j+1}a_{k,j}+\sum_{j=\left\lfloor\frac{m}{2}\right\rfloor}^{\infty}\sum_{k=0}^{m}a_{k,j}$$

This formula shows the summation over the infinite trapezium (quadrangle) $0 \le k \le m \bigwedge 0 \le \left\lfloor\frac{k}{2}\right\rfloor \le j$ in a different order.

$$\sum_{k=0}^{m}\sum_{j=\left\lfloor\frac{k+1}{2}\right\rfloor}^{\infty}a_{k,j}=\sum_{j=0}^{\left\lfloor\frac{m-1}{2}\right\rfloor}\sum_{k=0}^{2j}a_{k,j}+\sum_{j=\left\lfloor\frac{m+1}{2}\right\rfloor}^{\infty}\sum_{k=0}^{m}a_{k,j}$$

This formula shows the summation over the trapezium (quadrangle) $0 \le k \le m \bigwedge 0 \le \left\lfloor\frac{k+1}{2}\right\rfloor \le j$ in a different order.

$$\sum_{k=0}^{m}\sum_{j=2k}^{\infty}a_{k,j}=\sum_{j=0}^{2m-1}\sum_{k=0}^{\left\lfloor\frac{j}{2}\right\rfloor}a_{k,j}+\sum_{j=2m}^{\infty}\sum_{k=0}^{m}a_{k,j}$$

This formula shows the summation over the trapezium (quadrangle) $0 \le k \le m,\ 0 \le 2k \le j$ in a different order.

$$\sum_{k=0}^{m}\sum_{j=rk}^{\infty}a_{k,j}=\sum_{j=0}^{rm-1}\sum_{k=0}^{\left\lfloor\frac{j}{r}\right\rfloor}a_{k,j}+\sum_{j=rm}^{\infty}\sum_{k=0}^{m}a_{k,j}$$

This formula shows summation over the trapezium (quadrangle) $0 \le k \le m,\ 0 \le r\,k \le j,\ r \in \mathbb{N}^+$ in a different order.

Triple infinite summation

$$\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\sum_{k_3=0}^{\infty}a_{k_1,k_2,k_3}=\sum_{k_3=0}^{\infty}\sum_{k_1=0}^{k_3}\sum_{k_2=0}^{k_3-k_1}a_{k_1,k_2,k_3-k_1-k_2}$$

This formula shows how to change the order of summation in a triple sum.

Multidimensional infinite summation

$$\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\sum_{k_3=0}^{\infty}\sum_{k_4=0}^{\infty}a_{k_1,k_2,k_3,k_4}=\sum_{k_4=0}^{\infty}\sum_{k_1=0}^{k_4}\sum_{k_2=0}^{k_4-k_1}\sum_{k_3=0}^{k_4-k_1-k_2}a_{k_1,k_2,k_3,k_4-k_1-k_2-k_3}$$

This formula shows how to change the order of summation in multiple sums.

$$\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\cdots\sum_{k_n=0}^{\infty} a_{k_1,k_2,\ldots,k_n} = \sum_{k_n=0}^{\infty}\sum_{k_1=0}^{k_n}\sum_{k_2=0}^{k_n-k_1}\cdots\sum_{k_{n-1}=0}^{k_n-\sum_{k_j=1}^{n-2}k_j} a_{k_1,k_2,\ldots,k_{n-1},k_n-\sum_{k_j=1}^{n-1}k_j}$$

This formula shows how to change the order of summation in multiple sums.

Products

Finite products

$$\prod_{k=0}^{n} a_k == \exp\left(\sum_{k=0}^{n}\log(a_k)\right)$$

This formula reflects the property that the *product is equal to the exponent from the sum of the logarithms*.

$$\prod_{k=0}^{n} e^{a_k} = e^{\sum_{k=0}^{n} a_k}$$

This formula reflects the property that the *product is equal to the exponent from the sum of the logarithms*.

Infinite products

$$\prod_{k=0}^{\infty} a_k == \exp\left(\sum_{k=0}^{\infty}\log(a_k)\right)$$

This formula reflects the property that the *product is equal to the exponent from the sum of the logarithms*.

$$\prod_{k=0}^{\infty} e^{a_k} = e^{\sum_{k=0}^{\infty} a_k}$$

This formula reflects the property that the *product is equal to the exponent from the sum of the logarithms*.

Operations

Limit operation

$$\left(\lim_{z\to a} f(z) = F\right) = \left(\forall_{\epsilon,\epsilon>0}\left(\exists_{\delta,\delta>0}\left(\forall_{z,0<|z-a|<\delta}\, |f(z)-F| < \epsilon\right)\right)\right)$$

This formula reflects the definition of a limiting value F for a function $f(z)$ at the point $z = a$, when z approaches a in any direction (the so-called epsilon-delta definition): the formula $F \equiv \lim_{z \to a} f(z)$ means that the function $f(z)$ has a limit value F if and only if for all $\epsilon > 0$, there exists $\delta > 0$ such that $|f(z) - F| < \epsilon$ whenever $0 < |z - a| < \delta$. A limiting value F does not always exist, and (if it exists) it does not always coincide with the value of the function at the point $z = a$ (the last value also may not exist). But in the "best situation", when all the values exist and coincide: $F \equiv f(z)$, and the function $f(z)$ is called continuous at the point $z = a$.

$$\left(\lim_{z \to \infty} f(z) = F\right) = \left(\forall_{\epsilon,\epsilon>0} \left(\exists_{\delta,\delta>0} \left(\forall_{z,z>\delta} |f(z) - F| < \epsilon\right)\right)\right)$$

This formula reflects the definition of a limiting value F for a function $f(z)$ at the infinite point $z = \infty$: the formula $F \equiv \lim_{z \to \infty} f(z)$ means that the function $f(z)$ has a limit value F if and only if for all $\epsilon > 0$, there exists $\delta > 0$ such that $|f(z) - F| < \epsilon$ whenever $z > \delta$. A limiting value F may not always exist, and (if it exists) may not always coincide with the value of the function at the point $z = \infty$ (the last value also may not exist). But in the "best situation", when all the values exist and coincide: $F \equiv f(z)$, and the function $f(z)$ is called continuous at the point $z = \infty$.

$$\lim_{\epsilon \to 0} f(z + \epsilon) = f(z) \;/;\; f(z) \in C(\mathbb{C})$$

This limit shows that analytic functions are continuous functions.

$$\lim_{\epsilon \to 0} \frac{f(z + \epsilon) - f(z)}{\epsilon} \equiv f'(z) \;/;\; f(z) \in C^1(\mathbb{C})$$

This limit defines the derivative of function f at the point z, if it exists. For analytic functions this limit exists.

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon^n} \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} f(z + k\,\epsilon) \equiv f^{(n)}(z) \;/;\; f(z) \in C^n(\mathbb{C}) \bigwedge n \in \mathbb{N}^+$$

This limit defines the n^{th} derivative of a function f with an argument z.

$$\lim_{z \to a} \frac{f(z)}{g(z)} = \lim_{z \to a} \frac{f'(z)}{g'(z)} \;/;\; \lim_{z \to a} f(z) = \lim_{z \to a} g(z) = 0 \bigvee \lim_{z \to a} f(z) = \lim_{z \to a} g(z) = \infty \bigvee \lim_{z \to a} f(z) = \lim_{z \to a} g(z) = -\infty$$

L'Hospital's rule appeared in the first textbook on differential calculus, *Treatise of L'Hospital*, in 1696. It allows you to evaluate the limit of the ratio of two functions $\lim_{z \to a} \frac{f(z)}{g(z)}$ through the limit of the ratio of their derivatives $\lim_{z \to a} \frac{f'(z)}{g'(z)}$ in the cases when $\lim_{z \to a} f(z)$ and $\lim_{z \to a} g(z)$ are equal to zero or $\pm\infty$.

$$\left(\lim_{b \to \infty} \int_a^b f(t, z)\, dt = \int_a^\infty f(t, z)\, dt\right) = \left(\forall_{\epsilon(z),\epsilon(z)>0} \left(\exists_{\delta,\delta>0} \left(\forall_{b,b>\delta} \left|\int_a^\infty f(t, z)\, dt - \int_a^b f(t, z)\, dt\right| < \epsilon(z)\right)\right)\right)$$

This formula reflects the definition of the convergent integral $\int_a^\infty f(t, z)\, dt$ at the argument z.

$$\lim_{z \to z_0} \int_a^b f(t, z)\, dt \equiv \int_a^b \lim_{z \to z_0} f(t, z)\, dt$$

This formula reflects the commutativity of the two operations definite integration and limit.

$$\lim_{z\to z_0}\left(\sum_{k=0}^{\infty} a_k(z)\right) = \sum_{k=0}^{\infty} \lim_{z\to z_0} a_k(z)$$

This formula reflects the reordering of the two operations infinite summation and limit.

$$\lim_{z\to z_0} \prod_{k=0}^{\infty} a_k(z) = \prod_{k=0}^{\infty} \lim_{z\to z_0} a_k(z)$$

This formula reflects the reordering of the two operations infinite product and limit.

$$\lim_{k\to\infty} \int_{-\pi}^{\pi} f(t)\sin(k\,t)\,dt = 0$$

The Riemann–Lebesgue lemma shows that sine coefficients of the trigonometric Fourier series tend to zero as $k \to \infty$.

$$\lim_{k\to\infty} \int_{-\pi}^{\pi} f(t)\cos(k\,t)\,dt = 0$$

The Riemann–Lebesgue lemma shows that cosine coefficients of the trigonometric Fourier series tend to zero as $k \to \infty$.

Relations with other functions

With inverse function

$$f\left(f^{(-1)}(z)\right) = z$$

This property is the definition of the inverse function $f^{(-1)}(z)$ and can hold without additional restrictions on z (like $z \in D$, where D is not $\mathbb{C}$) for many named functions. In these situations, $f(z)$ is in most cases free of branch cuts. For example, $\sin\left(\sin^{-1}(z)\right) = z$; here $\sin^{-1}$means $f^{(-1)}$ with $f = \sin$, that is, the inverse sine function (do not confuse this with the reciprocal function $1/\sin$).

Some of the functions f are invertible: their inversions $f^{(-1)}$ can coincide with the original f, but for other values of the parameters. For example, the inverse function for the power function z^a is also the power function $z^{1/a}$, and the relation $\left(z^{1/a}\right)^a = z$ takes place only under the restriction $-\pi < \mathrm{Im}\left(\frac{\log(z)}{a}\right) \le \pi$. In general cases the following relation takes place: $\left(z^{1/a}\right)^a = \exp\left(2\,i\,a\,\pi\left\lfloor \frac{1}{2\pi}\left(\pi - \mathrm{Im}\left(\frac{\log(z)}{a}\right)\right)\right\rfloor\right) z$.

$$f^{(-1)}(f(z)) = z \;/;\; z \in D$$

The last property for the inverse function of the direct function can be valid under special restrictions for z (where typically D is not $\mathbb{C}$). For example, $\cos^{-1}(\cos(z)) = z \;/;\; 0 < \mathrm{Re}(z) < \pi \lor \mathrm{Re}(z) = 0 \land \mathrm{Im}(z) \ge 0 \lor \mathrm{Re}(z) = \pi \land \mathrm{Im}(z) \le 0$.

Inequalities

Algebraic inequalities

$$\left(\sum_{k=1}^{n} a_k\, b_k\right)^2 \leq \left(\sum_{k=1}^{n} a_k^2\right)\left(\sum_{k=1}^{n} b_k^2\right) /;\ a_k \in \mathbb{R} \bigwedge b_j \in \mathbb{R}$$

This inequality is called the Cauchy–Schwarz–Buniakowskyinequality.

$$\left|\sum_{k=1}^{n} a_k\, b_k\right|^2 \leq \left(\sum_{k=1}^{n} |a_k|^2\right)\left(\sum_{k=1}^{n} |b_k|^2\right)$$

This inequality is called the Cauchy–Schwarz–Buniakowskyinequality.

$$\left(\sum_{k=1}^{n} (a_k + b_k)^p\right)^{1/p} \leq \left(\sum_{k=1}^{n} a_k^p\right)^{1/p} + \left(\sum_{k=1}^{n} b_k^p\right)^{1/p} /;\ a_k \in \mathbb{R} \bigwedge b_j \in \mathbb{R} \bigwedge a_k \geq 0 \bigwedge b_j \geq 0 \bigwedge p > 1$$

This inequality is called Minkowski's inequality.

$$\left(\sum_{k=1}^{n} |a_k + b_k|^p\right)^{1/p} \leq \left(\sum_{k=1}^{n} |a_k|^p\right)^{1/p} + \left(\sum_{k=1}^{n} |b_k|^p\right)^{1/p} /;\ p > 1$$

This inequality is called Minkowski's inequality.

$$\left(\sum_{k=1}^{n} a_k^p\right)^{1/p} \left(\sum_{k=1}^{n} b_k^q\right)^{1/q} \geq \sum_{k=1}^{n} a_k\, b_k /;\ a_k \in \mathbb{R} \bigwedge b_j \in \mathbb{R} \bigwedge a_k \geq 0 \bigwedge b_j \geq 0 \bigwedge \frac{1}{q} + \frac{1}{p} == 1 \bigwedge p > 1$$

This inequality is called Hölder's inequality.

$$\left(\sum_{k=1}^{n} |a_k|^p\right)^{1/p} \left(\sum_{k=1}^{n} |b_k|^q\right)^{1/q} \geq \left|\sum_{k=1}^{n} a_k\, b_k\right| /;\ \frac{1}{q} + \frac{1}{p} == 1 \bigwedge p > 1$$

This inequality is called Hölder's inequality.

$$\left(\frac{1}{n}\sum_{k=1}^{n} a_k\right)\left(\frac{1}{n}\sum_{k=1}^{n} b_k\right) \leq \frac{1}{n}\sum_{k=1}^{n} a_k\, b_k /;\ a_k \in \mathbb{R} \bigwedge b_j \in \mathbb{R} \bigwedge a_k \leq a_{k+1} \bigwedge b_k \leq b_{k+1} \bigwedge 1 \leq k \leq n-1$$

This inequality is called Chebyshev's inequality.

$$\frac{a+b}{2} \geq \sqrt{a\, b} /;\ a > 0 \bigwedge b > 0$$

This inequality is called the arithmetic-geometric inequality.

$$\frac{1}{n}\sum_{k=1}^{n} a_k \geq \left(\prod_{k=1}^{n} a_k\right)^{1/n} /;\ a_k \in \mathbb{R} \bigwedge a_k > 0$$

This inequality is called the arithmetic-geometric inequality.

$$\sum_{k=1}^{n}\sum_{j=1}^{n}\left(\left|a_k-b_j\right|^p+\left|a_j-b_k\right|^p-\left|a_j-a_k\right|^p-\left|b_j-b_k\right|^p\right)\geq 0\ /;\ 0<p\leq 2\bigwedge a_k\in\mathbb{R}\bigwedge b_j\in\mathbb{R}$$

$$|a+b|\leq|a|+|b|$$

This inequality is called the triangle inequality.

$$|a-b|\geq||a|-|b||$$

This inequality is called the triangle inequality.

Integral inequalities

$$\left(\int_a^b f(t)\,g(t)\,dt\right)^2\leq\left(\int_a^b f(t)^2\,dt\right)\left(\int_a^b g(t)^2\,dt\right)$$

This inequality is called the Cauchy–Schwarz–Buniakowskyinequality.

$$\int_a^b |f(t)\,g(t)|\,dt\leq\left(\int_a^b |f(t)|^p\,dt\right)^{1/p}\left(\int_a^b |g(t)|^q\,dt\right)^{1/q}\ /;\ \frac{1}{q}+\frac{1}{p}==1\bigwedge p>1$$

This inequality is called Hölder's inequality.

$$\left(\int_a^b |f(t)+g(t)|^p\,dt\right)^{1/p}\leq\left(\int_a^b |f(t)|^p\,dt\right)^{1/p}+\left(\int_a^b |g(t)|^p\,dt\right)^{1/p}\ /;\ p>1$$

This inequality is called Minkowski's inequality.

$$\prod_{k=1}^{n}\int_a^b f_k(t)\,dt\leq(b-a)^{n-1}\int_a^b\prod_{k=1}^{n}f_k(t)\,dt\ /;\ f_k(t)\geq 0\bigwedge f_k'(t)>0\bigwedge 1\leq k\leq n$$

This inequality is called Chebyshev's inequality.

$$a\,b\leq\int_0^a f(t)\,dt+\int_0^b f^{(-1)}[t]\,dt\ /;\ f'(t)>0\bigwedge f(0)==0\bigwedge b\leq f(a)$$

This inequality is called Young's inequality.

$$\int_{b-k}^b f(t)\,dt\leq\int_a^b f(t)\,g(t)\,dt\leq\int_a^{a+k} f(t)\,dt\ /;\ k==\int_a^b g(t)\,dt\bigwedge f'(t)<0\bigwedge f(t)\geq 0\bigwedge 0\leq g(t)\leq 1$$

This inequality is called Steffensen's inequality.

$$\left|\begin{pmatrix}\int_a^b f_1(t)^2\,dt & \int_a^b f_1(t)\,f_2(t)\,dt & \ldots & \int_a^b f_1(t)\,f_n(t)\,dt\\ \int_a^b f_2(t)\,f_1(t)\,dt & \int_a^b f_2(t)^2\,dt & \ldots & \int_a^b f_2(t)\,f_n(t)\,dt\\ \ldots & \ldots & \ldots & \ldots\\ \int_a^b f_n(t)\,f_1(t)\,dt & \int_a^b f_n(t)\,f_2(t)\,dt & \ldots & \int_a^b f_n(t)^2\,dt\end{pmatrix}\right|\geq 0$$

This inequality is called Gram's inequality.

$$\left|\int_a^b f(t)\,g(t)\,dt\right| \leq |f(a)| \max\left(\left|\int_a^\alpha g(t)\,dt\right|, \{a \leq \alpha \leq b\}\right) /;\ f'(t) > 0 \bigwedge f(a)\,f(b) \geq 0 \bigwedge |f(a)| \geq |f(b)|$$

This inequality is called Ostrowski's inequality.

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