FractionalPart

Notations

Traditional name

Fractional part

Traditional notation

$\operatorname{frac}(z)$

Mathematica StandardForm notation

```
FractionalPart[z]
```

Primary definition

04.05.02.0001.01

$\operatorname{frac}(x) = x - n \;/;\; x \in \mathbb{R} \land n \in \mathbb{Z} \land 0 \leq \operatorname{sgn}(x)\,(x - n) < 1 \land x \neq 0$

04.05.02.0002.01

$\operatorname{frac}(z) = \operatorname{frac}(\operatorname{Re}(z)) + i \operatorname{frac}(\operatorname{Im}(z))$

For real z, the function $\operatorname{frac}(z)$ is the fractional part of z.

Examples: $\operatorname{frac}(3.2) = 0.2$, $\operatorname{frac}(3) = 0$, $\operatorname{frac}(-0.2) = -0.2$, $\operatorname{frac}(-2.3) = -0.3$, $\operatorname{frac}\left(\frac{2}{3}\right) = \frac{2}{3}$, $\operatorname{frac}(-\pi) = 3 - \pi$, $\operatorname{frac}\left(-4 - \frac{5}{3} i\right) = -\frac{2 i}{3}$, $\operatorname{frac}\left(\frac{5}{2}\right) = \frac{1}{2}$, $\operatorname{int}\left(\frac{7}{2}\right) = \frac{1}{2}$.

Specific values

Specialized values

04.05.03.0001.01

$\operatorname{frac}(x) = 0 \;/;\; x \in \mathbb{Z}$

04.05.03.0002.01

$\operatorname{frac}(i\, x) = 0 \;/;\; x \in \mathbb{Z}$

04.05.03.0003.01

$\operatorname{frac}(x + i\, y) = \operatorname{frac}(x) + i \operatorname{frac}(y) \;/;\; x \in \mathbb{R} \land y \in \mathbb{R}$

Values at fixed points

04.05.03.0004.01

$\operatorname{frac}(0) = 0$

04.05.03.0005.01

$\text{frac}(1) = 0$

04.05.03.0006.01

$\text{frac}(-1) = 0$

04.05.03.0007.01

$\text{frac}(i) = 0$

04.05.03.0008.01

$\text{frac}(-i) = 0$

04.05.03.0009.01

$\text{frac}(2) = 0$

04.05.03.0010.01

$\text{frac}(-3) = 0$

04.05.03.0011.01

$\text{frac}(-\pi) = 3 - \pi$

04.05.03.0012.01

$$\text{frac}\left(-\frac{27}{10}\right) = -\frac{7}{10}$$

04.05.03.0013.01

$\text{frac}(-3.4) = -0.4$

04.05.03.0014.01

$$\text{frac}\left(\frac{23}{10} - i\,e\right) = \frac{3}{10} - i\,(e - 2)$$

Values at infinities

04.05.03.0015.01

$\text{frac}(\infty) \in (0, 1)$

04.05.03.0016.01

$\text{frac}(-\infty) \in (-1, 0)$

04.05.03.0017.01

$\text{frac}(i\,\infty) \in (0, i)$

04.05.03.0018.01

$\text{frac}(-i\,\infty) \in (-i, 0)$

04.05.03.0019.01

$\text{frac}(\tilde{\infty}) \in (0, 1)$

General characteristics

Domain and analyticity

frac(z) is a nonanalytical function; it is a piecewise continuous function which is defined over the whole complex z-plane.

04.05.04.0001.01

$z \longrightarrow \text{frac}(z) :: \mathbb{C} \longrightarrow \mathbb{C}$

Symmetries and periodicities

Parity

frac(z) is an odd function.

04.05.04.0002.01

$\text{frac}(-z) = -\text{frac}(z)$

Mirror symmetry

04.05.04.0003.01

$\text{frac}(\bar{z}) = \overline{\text{frac}(z)}$

Periodicity

No periodicity

Sets of discontinuity

The function frac(z) is a piecewise continuous function with unit jumps on the lines $\text{Re}(z) = k \vee \text{Im}(z) = l /; k, l \in \mathbb{Z}, k \neq 0, l \neq 0$.

The function frac(z) is continuous from the right on the intervals $(k - i\infty, k + i\infty)$, $k \in \mathbb{N}^+$, and from the left on the intervals $(-k - i\infty, -k + i\infty)$, $k \in \mathbb{N}^+$.

The function frac(z) is continuous from above on the intervals $(-\infty + i k, \infty + i k)$, $k \in \mathbb{N}^+$, and from below on the intervals $(-\infty - i k, \infty - i k)$, $k \in \mathbb{N}^+$.

04.05.04.0004.01

$$\mathcal{DS}_z(\text{frac}(z)) = \{\{\{(k - i\infty, k + i\infty), -1\} /; k \in \mathbb{N}^+\}, \{\{(-k - i\infty, -k + i\infty), 1\} /; k \in \mathbb{N}^+\}, \{\{(i k - \infty, i k + \infty), -i\} /; k \in \mathbb{N}^+\}, \{\{(-i k - \infty, -i k + \infty), i\} /; k \in \mathbb{N}^+\}\}$$

04.05.04.0005.01

$$\lim_{\epsilon \to +0} \text{frac}(z + \epsilon) = \text{frac}(z) /; \text{Re}(z) \in \mathbb{N}^+$$

04.05.04.0006.01

$$\lim_{\epsilon \to +0} \text{frac}(z - \epsilon) = \text{frac}(z) /; -\text{Re}(z) \in \mathbb{N}^+$$

04.05.04.0007.01

$$\lim_{\epsilon \to +0} \text{frac}(z + \epsilon) = \text{frac}(z) - 1 /; -\text{Re}(z) \in \mathbb{N}^+$$

04.05.04.0008.01

$$\lim_{\epsilon \to +0} \text{frac}(z - \epsilon) = \text{frac}(z) + 1 /; \text{Re}(z) \in \mathbb{N}^+$$

04.05.04.0009.01

$$\lim_{\epsilon \to +0} \text{frac}(z + i\epsilon) = \text{frac}(z) /; \text{Im}(z) \in \mathbb{N}^+$$

04.05.04.0010.01

$$\lim_{\epsilon \to +0} \text{frac}(z - i\epsilon) = \text{frac}(z) /; -\text{Im}(z) \in \mathbb{N}^+$$

04.05.04.0011.01

$$\lim_{\epsilon\to+0} \operatorname{frac}(z + i\,\epsilon) = \operatorname{frac}(z) - i \;/;\; -\operatorname{Im}(z) \in \mathbb{N}^+$$

04.05.04.0012.01

$$\lim_{\epsilon\to+0} \operatorname{frac}(z - i\,\epsilon) = \operatorname{frac}(z) + i \;/;\; \operatorname{Im}(z) \in \mathbb{N}^+$$

Series representations

Exponential Fourier series

04.05.06.0001.01

$$\operatorname{frac}(x) = -\frac{1}{\pi} \sum_{k=1}^{\infty} \frac{\sin(2\,\pi\,k\,x)}{k} + \theta(x) - \frac{1}{2} \;/;\; x \in \mathbb{R} \bigwedge x \notin \mathbb{Z}$$

Other series representations

04.05.06.0002.01

$$\operatorname{frac}\left(\frac{m}{n}\right) = \frac{1}{2} \operatorname{sgn}\left(\frac{m}{n}\right) - \frac{1}{2\,n} \sum_{k=1}^{n-1} \sin\left(\frac{2\,\pi\,k\,m}{n}\right) \cot\left(\frac{\pi\,k}{n}\right) \;/;\; m \in \mathbb{Z} \bigwedge n \in \mathbb{Z} \bigwedge \frac{m}{n} \notin \mathbb{Z} \bigwedge n > 1$$

Transformations

Transformations and argument simplifications

Argument involving basic arithmetic operations

04.05.16.0001.01

$$\operatorname{frac}(-z) = -\operatorname{frac}(z)$$

04.05.16.0002.01

$$\operatorname{frac}(i\,z) = i\,\operatorname{frac}(z)$$

04.05.16.0003.01

$$\operatorname{frac}(-i\,z) = -i\,\operatorname{frac}(z)$$

04.05.16.0004.01

$$\operatorname{frac}(n + z) = \operatorname{frac}(z) + \theta(n + z) - \theta(z) \;/;\; n \in \mathbb{Z} \bigwedge z \notin \mathbb{Z}$$

Argument involving related functions

04.05.16.0005.01

$$\operatorname{frac}(\operatorname{frac}(z)) = \operatorname{frac}(z)$$

04.05.16.0006.01

$$\operatorname{frac}(z - \operatorname{frac}(z)) = 0$$

04.05.16.0007.01

$$\operatorname{frac}(\operatorname{int}(z)) = 0$$

04.05.16.0008.01

$$\operatorname{frac}(\lfloor z \rceil) = 0$$

04.05.16.0009.01

$$\operatorname{frac}(\lfloor z \rfloor) = 0$$

04.05.16.0010.01

$\text{frac}(\lceil z \rceil) = 0$

04.05.16.0013.01

$\text{frac}(\text{quotient}(m, n)) = 0$

04.05.16.0014.01

$\text{frac}(\text{quotient}(m, n)) = 0$

Addition formulas

04.05.16.0011.01

$\text{frac}(n + z) = \text{frac}(z) + \theta(n + z) - \theta(z) \;/;\; n \in \mathbb{Z} \bigwedge z \notin \mathbb{Z}$

Multiple arguments

04.05.16.0012.01

$$\text{frac}(n\, z) = \text{frac}(z)\, n - n\, \text{sgn}(\chi_{\mathbb{Z}}(z) + \theta(z)) + \text{sgn}(\chi_{\mathbb{Z}}(n\, z) + \theta(z)) - \sum_{k=0}^{n-1} k\, \theta\left(z \bmod 1 - \frac{k}{n}\right)\left(1 - \theta\left(z \bmod 1 - \frac{k+1}{n}\right)\right) + n - 1 \;/;\; n \in \mathbb{N} \land z \in \mathbb{R}$$

Complex characteristics

Real part

04.05.19.0001.01

$\text{Re}(\text{frac}(x + i\, y)) = \text{frac}(x)$

04.05.19.0006.01

$\text{Re}(\text{frac}(z)) = \text{frac}(\text{Re}(z))$

Imaginary part

04.05.19.0002.01

$\text{Im}(\text{frac}(x + i\, y)) = \text{frac}(y)$

04.05.19.0007.01

$\text{Im}(\text{frac}(z)) = \text{frac}(\text{Im}(z))$

Absolute value

04.05.19.0003.01

$|\text{frac}(x + i\, y)| = \sqrt{\text{frac}(x)^2 + \text{frac}(y)^2}$

04.05.19.0008.01

$|\text{frac}(z)| = \sqrt{\text{frac}(\text{Im}(z))^2 + \text{frac}(\text{Re}(z))^2}$

Argument

04.05.19.0004.01

$\arg(\text{frac}(x + i\, y)) = \tan^{-1}(\text{frac}(x), \text{frac}(y))$

04.05.19.0009.01

$\arg(\text{frac}(z)) = \tan^{-1}(\text{frac}(\text{Re}(z)), \text{frac}(\text{Im}(z)))$

Conjugate value

04.05.19.0005.01

$$\overline{\text{frac}(x + i\, y)} = \text{frac}(x) - i\, \text{frac}(y)$$

04.05.19.0010.01

$$\overline{\text{frac}(z)} = \text{frac}(\text{Re}(z)) - i\, \text{frac}(\text{Im}(z))$$

Signum value

04.05.19.0011.01

$$\text{sgn}(\text{frac}(x + i\, y)) = \frac{\text{frac}(x) + i\, \text{frac}(y)}{\sqrt{\text{frac}(x)^2 + \text{frac}(y)^2}}$$

04.05.19.0012.01

$$\text{sgn}(\text{frac}(z)) = \frac{\text{frac}(z)}{|\text{frac}(z)|}$$

Differentiation

Low-order differentiation

04.05.20.0001.01

$$\frac{\partial\, \text{frac}(x)}{\partial x} = 1$$

In a distributional sense for $x \in \mathbb{R}$.

04.05.20.0002.01

$$\frac{\partial\, \text{frac}(x)}{\partial x} = x - \sum_{k=-\infty}^{\infty} \delta_{k,0}\, \delta(x - k)$$

Fractional integro-differentiation

04.05.20.0003.01

$$\frac{\partial^{\alpha}\, \text{frac}(z)}{\partial z^{\alpha}} = \frac{\alpha\, z^{1-\alpha}}{\Gamma(2-\alpha)} + \frac{\text{frac}(z)\, z^{-\alpha}}{\Gamma(1-\alpha)}$$

Integration

Indefinite integration

Involving only one direct function

04.05.21.0001.01

$$\int \text{frac}(z)\, dz = z\, \text{frac}(z) - \frac{z^2}{2}$$

Involving one direct function and elementary functions

Involving power function

04.05.21.0002.01

$$\int z^{\alpha-1}\,\mathrm{frac}(z)\,dz = \frac{z^{\alpha}\,\mathrm{frac}(z)}{\alpha} - \frac{z^{\alpha+1}}{\alpha\,(\alpha+1)}$$

04.05.21.0003.01

$$\int \frac{\mathrm{frac}(z)}{z}\,dz = z\,(1-\log(z)) + \mathrm{frac}(z)\log(z)$$

Definite integration

For the direct function itself

In the following formulas $a \in \mathbb{R}$.

04.05.21.0004.01

$$\int_0^n \mathrm{frac}(t)\,dt = \frac{n}{2} \;/;\; n \in \mathbb{N}$$

04.05.21.0005.02

$$\int_0^a \mathrm{frac}(t)\,dt = \frac{1}{2}\left(\mathrm{frac}(a)^2 - \mathrm{sgn}(a)\,\mathrm{frac}(a) + |a|\right)$$

David W. Cantrell (2006)

04.05.21.0006.01

$$\int_0^a t^{\alpha-1}\,\mathrm{frac}(t)\,dt = \frac{a^{\alpha+1}}{\alpha+1} - \frac{1}{\alpha}\left((a-\mathrm{frac}(a))\,a^{\alpha} - \zeta(-\alpha) + \zeta(-\alpha,\, a-\mathrm{frac}(a)+1)\right) \;/;\; \mathrm{Re}(\alpha) > -1$$

04.05.21.0007.01

$$\int_0^1 \mathrm{frac}\left(\frac{1}{t}\right)dt = 1-\gamma$$

04.05.21.0008.01

$$\int_a^\infty t^{\alpha-1}\,\mathrm{frac}(t)\,dt = \frac{a^{\alpha+1}}{\alpha\,(\alpha+1)} - \frac{1}{\alpha}\left(a^{\alpha}\,\mathrm{frac}(a) - \zeta(-\alpha,\, a-\mathrm{frac}(a)+1)\right) \;/;\; \mathrm{Re}(\alpha) < 0$$

04.05.21.0009.01

$$\int_1^\infty t^{\alpha-1}\,\mathrm{frac}(t)\,dt = \frac{1}{\alpha\,(\alpha+1)} - \frac{1-\zeta(-\alpha)}{\alpha} \;/;\; \mathrm{Re}(\alpha) < 0 \land \alpha \neq -1$$

04.05.21.0010.01

$$\int_1^\infty \frac{\mathrm{frac}(t)}{t^2}\,dt = 1-\gamma$$

04.05.21.0011.01

$$\int_0^\infty t^{\alpha-1}\,\mathrm{frac}(t)\,dt = \frac{\zeta(-\alpha)}{\alpha} \;/;\; -1 < \mathrm{Re}(\alpha) < 0$$

04.05.21.0012.01

$$\int_{-a}^a \mathrm{frac}(t)\,dt = 0$$

04.05.21.0013.01

$$\int_0^{a b}\left(\operatorname{frac}\left(\frac{t}{a}\right)-\frac{1}{2}\right)\left(\operatorname{frac}\left(\frac{t}{b}\right)-\frac{1}{2}\right)dt = \frac{a\,b\,\gcd(a,\,b)}{12\,\operatorname{lcm}(a,\,b)}\;/;\; a\in\mathbb{Z}\bigwedge b\in\mathbb{Z}$$

Integral transforms

Fourier exp transforms

04.05.22.0001.01

$$\mathcal{F}_t[\operatorname{frac}(t)]\,(z) = \frac{i}{\sqrt{2\,\pi}\;z} - \frac{i}{\sqrt{2\,\pi}}\sum_{k=1}^{\infty}\frac{\delta(2\,k\,\pi - z) - \delta(2\,\pi\,k + z)}{k}$$

Fourier cos transforms

04.05.22.0002.01

$$\mathcal{F}c_t[\operatorname{frac}(t)]\,(z) = \frac{1}{\sqrt{2\,\pi}\;z^2}\left(z\cot\left(\frac{z}{2}\right) - 2\right) + \sqrt{\frac{\pi}{2}}\;\delta(z)$$

Fourier sin transforms

04.05.22.0003.01

$$\mathcal{F}s_t[\operatorname{frac}(t)]\,(z) = \frac{1}{\sqrt{2\,\pi}\;z} - \frac{1}{\sqrt{2\,\pi}}\sum_{k=1}^{\infty}\frac{\delta(2\,k\,\pi - z) - \delta(2\,\pi\,k + z)}{k}$$

Laplace transforms

04.05.22.0004.01

$$\mathcal{L}_t[\operatorname{frac}(t)]\,(z) = \frac{1}{z^2}\left(1 - \frac{z}{e^z - 1}\right)\;/;\;\operatorname{Re}(z) > 0$$

Mellin transforms

04.05.22.0005.01

$$\mathcal{M}_t[\operatorname{frac}(t)]\,(z) = \frac{\zeta(-z)}{z}\;/;\; -1 < \operatorname{Re}(z) < 0$$

Representations through equivalent functions

With related functions

With Floor

For real arguments

04.05.27.0008.01

$$\operatorname{frac}(x) = x - \lfloor x\rfloor\;/;\; x\in\mathbb{R}\land x > 0 \lor x\in\mathbb{Z}$$

04.05.27.0009.01

$$\operatorname{frac}(x) = x - \lfloor x\rfloor - 1\;/;\; x\in\mathbb{R}\bigwedge x < 0\bigwedge x\notin\mathbb{Z}$$

04.05.27.0010.01

$$\operatorname{frac}(x) = x - \lfloor x \rfloor - 1 + \operatorname{sgn}(\chi_{\mathbb{Z}}(x) + \theta(x)) \;/;\; x \in \mathbb{R}$$

For complex arguments

04.05.27.0011.01

$$\operatorname{frac}(z) = z - \lfloor z \rfloor \;/;\; \operatorname{Re}(z) \geq 0 \land \operatorname{Im}(z) \geq 0 \lor z \in \mathbb{Z} \lor i\, z \in \mathbb{Z}$$

04.05.27.0012.01

$$\operatorname{frac}(z) = z - \lfloor z \rfloor - 1 \;/;\; z \in \mathbb{R} \bigwedge z < 0 \bigwedge z \notin \mathbb{Z} \bigvee \operatorname{Re}(z) < 0 \land \operatorname{Im}(z) > 0$$

04.05.27.0013.01

$$\operatorname{frac}(z) = z - \lfloor z \rfloor - i \;/;\; i\, z \in \mathbb{R} \bigwedge i\, z > 0 \bigwedge i\, z \notin \mathbb{Z} \bigvee \operatorname{Re}(z) > 0 \land \operatorname{Im}(z) < 0$$

04.05.27.0014.01

$$\operatorname{frac}(z) = z - \lfloor z \rfloor - 1 - i \;/;\; \operatorname{Re}(z) < 0 \land \operatorname{Im}(z) < 0$$

04.05.27.0001.01

$$\operatorname{frac}(z) = z - \lfloor z \rfloor - 1 - i + \operatorname{sgn}(\chi_{\mathbb{Z}}(\operatorname{Re}(z)) + \theta(\operatorname{Re}(z))) + i \operatorname{sgn}(\chi_{\mathbb{Z}}(\operatorname{Im}(z)) + \theta(\operatorname{Im}(z)))$$

With Round

For real arguments

04.05.27.0015.01

$$\operatorname{frac}(x) = x - \left\lfloor x - \frac{1}{2} \right\rceil \;/;\; x \in \mathbb{R} \bigwedge x \geq 0 \bigwedge \frac{x+1}{2} \notin \mathbb{Z}$$

04.05.27.0016.01

$$\operatorname{frac}(x) = x - \left\lfloor x - \frac{1}{2} \right\rceil - 1 \;/;\; x \in \mathbb{R} \land x < 0 \bigvee \frac{x+1}{2} \in \mathbb{Z}$$

04.05.27.0017.01

$$\operatorname{frac}(x) = x - \left\lfloor x - \frac{1}{2} \right\rceil - 1 - \chi_{\mathbb{Z}}\left(\frac{x+1}{2}\right) + \operatorname{sgn}(\chi_{\mathbb{Z}}(x) + \theta(x)) \;/;\; x \in \mathbb{R}$$

For complex arguments

04.05.27.0003.01

$$\operatorname{frac}(z) = z - \left\lfloor z - \frac{1+i}{2} \right\rceil - 1 - i - \chi_{\mathbb{Z}}\left(\frac{\operatorname{Re}(z)+1}{2}\right) - i\, \chi_{\mathbb{Z}}\left(\frac{\operatorname{Im}(z)+1}{2}\right) + \operatorname{sgn}(\chi_{\mathbb{Z}}(\operatorname{Re}(z)) + \theta(\operatorname{Re}(z))) + i \operatorname{sgn}(\chi_{\mathbb{Z}}(\operatorname{Im}(z)) + \theta(\operatorname{Im}(z)))$$

With Ceiling

For real arguments

04.05.27.0018.01

$$\operatorname{frac}(x) = x - \lceil x \rceil + 1 \;/;\; x \in \mathbb{R} \bigwedge x > 0 \bigwedge x \notin \mathbb{Z}$$

04.05.27.0019.01

$$\operatorname{frac}(x) = x - \lceil x \rceil \;/;\; x \in \mathbb{R} \land x \leq 0 \lor x \in \mathbb{Z}$$

04.05.27.0020.01

$$\operatorname{frac}(x) = x - \lceil x \rceil - \operatorname{sgn}(\chi_{\mathbb{Z}}(-x) + \theta(-x)) + 1 \;/;\; x \in \mathbb{R}$$

For complex arguments

04.05.27.0021.01

$$\text{frac}(z) = z - \lceil z \rceil + \text{sgn}(\chi_{\mathbb{Z}}(\text{Re}(z)) + \theta(\text{Re}(z))) - i\,(1 - \text{sgn}(\chi_{\mathbb{Z}}(\text{Im}(z)) + \theta(\text{Im}(z))) - \theta(-\chi_{\mathbb{Z}}(\text{Im}(z))) + \theta(\chi_{\mathbb{Z}}(\text{Re}(z)) - 1))$$

04.05.27.0002.01

$$\text{frac}(z) = z + \lceil -z \rceil - 1 - i + \text{sgn}(\chi_{\mathbb{Z}}(\text{Re}(z)) + \theta(\text{Re}(z))) + i\,\text{sgn}(\chi_{\mathbb{Z}}(\text{Im}(z)) + \theta(\text{Im}(z)))$$

With IntegerPart

04.05.27.0004.01

$$\text{frac}(z) = z - \text{int}(z)$$

With Mod

For real arguments

04.05.27.0022.01

$$\text{frac}(x) = x \bmod 1 \;/;\; x \in \mathbb{R} \land x > 0 \lor x \in \mathbb{Z}$$

04.05.27.0023.01

$$\text{frac}(x) = x \bmod 1 - 1 \;/;\; x \in \mathbb{R} \bigwedge x < 0 \bigwedge x \notin \mathbb{Z}$$

04.05.27.0024.01

$$\text{frac}(x) = x \bmod 1 - 1 + \text{sgn}(\chi_{\mathbb{Z}}(x) + \theta(x)) \;/;\; x \in \mathbb{R}$$

For complex arguments

04.05.27.0025.01

$$\text{frac}(z) = z \bmod 1 \;/;\; \text{Re}(z) \geq 0 \land \text{Im}(z) \geq 0 \lor z \in \mathbb{Z} \lor i\,z \in \mathbb{Z}$$

04.05.27.0026.01

$$\text{frac}(z) = z \bmod 1 - 1 \;/;\; z \in \mathbb{R} \bigwedge z < 0 \bigwedge z \notin \mathbb{Z} \bigvee \text{Re}(z) < 0 \land \text{Im}(z) > 0$$

04.05.27.0027.01

$$\text{frac}(z) = z \bmod 1 - i \;/;\; i\,z \in \mathbb{R} \bigwedge i\,z > 0 \bigwedge i\,z \notin \mathbb{Z} \bigvee \text{Re}(z) > 0 \land \text{Im}(z) < 0$$

04.05.27.0028.01

$$\text{frac}(z) = z \bmod 1 - 1 - i \;/;\; \text{Re}(z) < 0 \land \text{Im}(z) < 0$$

04.05.27.0005.01

$$\text{frac}(z) = z \bmod 1 - 1 - i + i\,\text{sgn}(\chi_{\mathbb{Z}}(\text{Im}(z)) + \theta(\text{Im}(z))) + \text{sgn}(\chi_{\mathbb{Z}}(\text{Re}(z)) + \theta(\text{Re}(z)))$$

With Quotient

For real arguments

04.05.27.0029.01

$$\text{frac}(x) = x - \text{quotient}(x,\, 1) \;/;\; x \in \mathbb{R} \land x > 0 \lor x \in \mathbb{Z}$$

04.05.27.0030.01

$$\text{frac}(x) = x - \text{quotient}(x,\, 1) - 1 \;/;\; x \in \mathbb{R} \bigwedge x < 0 \bigwedge x \notin \mathbb{Z}$$

04.05.27.0031.01

$$\text{frac}(x) = x - \text{quotient}(x,\, 1) - 1 + \text{sgn}(\chi_{\mathbb{Z}}(x) + \theta(x)) \;/;\; x \in \mathbb{R}$$

For complex arguments

04.05.27.0032.01

$$\mathrm{frac}(z) = z - \mathrm{quotient}(z, 1) /; \mathrm{Re}(z) \geq 0 \land \mathrm{Im}(z) \geq 0 \lor z \in \mathbb{Z} \lor i z \in \mathbb{Z}$$

04.05.27.0033.01

$$\mathrm{frac}(z) = z - \mathrm{quotient}(z, 1) - 1 /; z \in \mathbb{R} \bigwedge z < 0 \bigwedge z \notin \mathbb{Z} \bigvee \mathrm{Re}(z) < 0 \land \mathrm{Im}(z) > 0$$

04.05.27.0034.01

$$\mathrm{frac}(z) = z - \mathrm{quotient}(z, 1) - i /; i z \in \mathbb{R} \bigwedge i z > 0 \bigwedge i z \notin \mathbb{Z} \bigvee \mathrm{Re}(z) > 0 \land \mathrm{Im}(z) < 0$$

04.05.27.0035.01

$$\mathrm{frac}(z) = z - \mathrm{quotient}(z, 1) - 1 - i /; \mathrm{Re}(z) < 0 \land \mathrm{Im}(z) < 0$$

04.05.27.0006.01

$$\mathrm{frac}(z) = z - \mathrm{quotient}(z, 1) - 1 - i + i \,\mathrm{sgn}(\chi_{\mathbb{Z}}(\mathrm{Im}(z)) + \theta(\mathrm{Im}(z))) + \mathrm{sgn}(\chi_{\mathbb{Z}}(\mathrm{Re}(z)) + \theta(\mathrm{Re}(z)))$$

With elementary functions

04.05.27.0007.01

$$\mathrm{frac}(z) = -\frac{\tan^{-1}(\cot(\pi z))}{\pi} + \mathrm{sgn}(\theta(z)) - \frac{1}{2} /; z \in \mathbb{R} \bigwedge z \notin \mathbb{Z}$$

Zeros

04.05.30.0001.01

$$\mathrm{frac}(z) = 0 /; \mathrm{Re}(z) \in \mathbb{Z} \lor \mathrm{Im}(z) \in \mathbb{Z}$$

Theorems

Distribution of some sequences

For irrational α, the sequences $\mathrm{frac}(n\,\alpha)$, $n = 1, 2, \ldots$ and $\mathrm{frac}\left(\frac{1}{2\pi} \sum_{k=2}^{n} \frac{1}{\sqrt{k}}\right)$, $n = 1, 2, \ldots$ are homogeneously distributed.

Copyright