EllipticThetaPrime4

Notations

Traditional name

Derivative of the Jacobi theta function ϑ_4

Traditional notation

$\vartheta_4'(z, q)$

Mathematica StandardForm notation

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EllipticThetaPrime[4, z, q]
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Primary definition

09.08.02.0001.01

$$\vartheta_4'(z, q) = -4 \sum_{k=1}^{\infty} (-1)^k \, k \, q^{k^2} \sin(2 \, k \, z) \, /; \, |q| < 1$$

Specific values

Specialized values

For fixed *z*

09.08.03.0001.01

$$\vartheta_4'(z, 0) = 0$$

For fixed *q*

09.08.03.0002.01

$$\vartheta_4'(0, q) = 0$$

09.08.03.0004.01

$$\vartheta_4'\left(-\frac{\pi}{4}, q\right) = -4 \, \eta\left(-\frac{4 \, i \log(q)}{\pi}\right)^3$$

09.08.03.0005.01

$$\vartheta_4'\left(\frac{\pi}{4}, q\right) = 4 \, \eta\left(-\frac{4 \, i \log(q)}{\pi}\right)^3$$

09.08.03.0003.01

$$\vartheta_4'\left(\frac{m\pi}{2}, q\right) == 0 \,/;\, m \in \mathbb{Z}$$

09.08.03.0006.01

$$\vartheta_4'\left(\frac{\pi m}{2} + \frac{\pi}{4}, q\right) == 4\,(-1)^m\,\eta\left(-\frac{4\,i\log(q)}{\pi}\right)^3 \,/;\, m \in \mathbb{Z}$$

09.08.03.0007.01

$$\vartheta_4'(-i\log(q), q) == \frac{2\,i}{q}\sqrt{\frac{2}{\pi}}\,\sqrt[4]{1-q^{-1}(q)}\,\sqrt{K\left(q^{-1}(q)\right)}$$

09.08.03.0008.01

$$\vartheta_4'\left(-\frac{i\log(q)}{2}, q\right) == \frac{i}{\sqrt[4]{q}}\,2\,\eta\left(-\frac{i\log(q)}{\pi}\right)^3$$

09.08.03.0009.01

$$\vartheta_4'\left(\frac{1}{2}(\pi - i\log(q)), q\right) == -i\,\frac{1}{\sqrt[4]{q}}\sqrt{\frac{2}{\pi}}\,\sqrt[4]{q^{-1}(q)}\,\sqrt{K\left(q^{-1}(q)\right)}$$

09.08.03.0010.01

$$\vartheta_4'(\pi m + i\,n\log(q), q) == (-1)^n\,2\,n\,i\,q^{-n^2}\sqrt{\frac{2}{\pi}}\,\sqrt[4]{1-q^{-1}(q)}\,\sqrt{K\left(q^{-1}(q)\right)} \,/;\, \{m, n\} \in \mathbb{Z}$$

General characteristics

Domain and analyticity

$\vartheta_4'(z, q)$ is an analytic function of z and q for $z, q \in \mathbb{C}$ and $|q| < 1$.

09.08.04.0001.01

$$(4 * z * q) \longrightarrow \vartheta_4'(z, q) :: (\{4\} \otimes \mathbb{C} \otimes \mathbb{C}) \longrightarrow \mathbb{C}$$

Symmetries and periodicities

Parity

$\vartheta_4'(z, q)$ is an odd function with respect to z.

09.08.04.0002.01

$$\vartheta_4'(-z, q) == -\vartheta_4'(z, q)$$

09.08.04.0003.02

$$\vartheta_4'(z, -q) == \vartheta_3'(z, q)$$

Mirror symmetry

09.08.04.0004.01

$$\vartheta_4'(\bar{z}, \bar{q}) == \overline{\vartheta_4'(z, q)}$$

Periodicity

$\vartheta_4(z, q)$, considered as a function of z, has a period of π.

09.08.04.0013.01

$$\vartheta_4'(z+\pi, q) = \vartheta_4'(z, q)$$

09.08.04.0005.01

$$\vartheta_4'(z+\pi, q) = \vartheta_4'(z, q) /; m \in \mathbb{Z}$$

Poles and essential singularities

With respect to *q*

The function $\vartheta_4'(z, q)$ does not have poles and essential singularities inside of the unit circle $|q| < 1$.

09.08.04.0006.01

$$\mathcal{S}ing_q(\vartheta_4'(z, q)) = \{\}$$

With respect to *z*

09.08.04.0007.01

$$\mathcal{S}ing_z(\vartheta_4'(z, q)) = \{\}$$

Branch points

With respect to *q*

The function $\vartheta_4'(z, q)$ does not have branch points with respect to q.

09.08.04.0008.01

$$\mathcal{BP}_q(\vartheta_4'(z, q)) = \{\}$$

With respect to *z*

The function $\vartheta_4'(z, q)$ does not have branch points with respect to z.

09.08.04.0009.01

$$\mathcal{BP}_z(\vartheta_4'(z, q)) = \{\}$$

Branch cuts

With respect to *q*

The function $\vartheta_4'(z, q)$ does not have branch cuts with respect to q.

09.08.04.0010.01

$$\mathcal{BC}_q(\vartheta_4'(z, q)) = \{\}$$

With respect to *z*

The function $\vartheta_4'(z, q)$ does not have branch cuts with respect to z.

09.08.04.0011.01

$$\mathcal{BC}_z(\vartheta_4'(z, q)) = \{\}$$

Natural boundary of analyticity

The unit circle $|q| = 1$ is the natural boundary of the region of analyticity.

09.08.04.0012.01
$$\mathcal{AB}_z(\vartheta_4'(z,q)) = \{e^{i(-\pi,\pi)}\}$$

Series representations

q-series

Expansions at generic point $q = q_0$

09.08.06.0005.01
$$\vartheta_4'(z,q) \propto \vartheta_4'(z,q_0) + \vartheta_4^{(1,1)}(z,q_0)(q-q_0) + \frac{\vartheta_4^{(1,2)}(z,q_0)}{2}(q-q_0)^2 + \frac{\vartheta_4^{(1,3)}(z,q_0)}{6}(q-q_0)^3 + O\left((q-q_0)^4\right)$$

09.08.06.0006.01
$$\vartheta_4'(z,q) = \sum_{k=0}^{\infty} \frac{\vartheta_4^{(1,k)}(z,q_0)}{k!}(q-q_0)^k$$

09.08.06.0007.01
$$\vartheta_4'(z,q) \propto \vartheta_4'(z,q_0)\,(1+O(q-q_0))$$

Expansions at $q = 0$

09.08.06.0008.01
$$\vartheta_4'(z,q) \propto 4\sin(2z)\,q - 8\sin(4z)\,q^4 + 12\sin(6z)\,q^9 - 16\sin(8z)\,q^{16} + \ldots \;/;\; (q\to 0)$$

09.08.06.0001.01
$$\vartheta_4'(z,q) = -4\sum_{k=1}^{\infty}(-1)^k\,k\,q^{k^2}\sin(2kz) \;/;\; |q|<1$$

09.08.06.0002.01
$$\vartheta_4'(z,q) = 2i\sum_{k=-\infty}^{\infty}(-1)^k q^{k^2}\,k\,e^{2kiz} \;/;\; |q|<1$$

09.08.06.0009.01
$$\vartheta_4'(z,q) \propto 4q\sin(2z)\left(1+O\left(q^3\right)\right) \;/;\; (q\to 0)$$

Expansions at $q = 1$

09.08.06.0010.01
$$\vartheta_4'(z,q) \propto -\frac{2\sqrt{\pi}\,i}{(q-1)^{3/2}}\,e^{-i\pi\left\lfloor -\frac{\arg(q-1)}{2\pi}\right\rfloor}\left(1+\frac{3(q-1)}{4}-\frac{1}{32}(q-1)^2+\frac{3}{128}(q-1)^3+\ldots\right)e^{\frac{4z^2+\pi^2}{4\log(q)}}$$
$$\left(\pi\sinh\left(\frac{\pi z}{\log(q)}\right)+2z\cosh\left(\frac{\pi z}{\log(q)}\right)+e^{\frac{2\pi^2}{\log(q)}}\left(3\pi\sinh\left(\frac{3\pi z}{\log(q)}\right)+2z\cosh\left(\frac{3\pi z}{\log(q)}\right)\right)+\ldots\right) \;/;\; (q\to 1)\land |q|<1$$

09.08.06.0011.01

$$\vartheta_4'(z,q) = -\frac{6\sqrt{\pi}\,i}{(q-1)^{3/2}}\,e^{-i\pi\left\lfloor -\frac{\arg(q-1)}{2\pi}\right\rfloor}$$
$$\sum_{k=0}^{\infty}\binom{k+\frac{3}{2}}{k}\sum_{j=0}^{k}\frac{(-1)^j}{2j+3}\binom{k}{j}p_{j,k}\,(q-1)^k\,e^{\frac{4z^2+\pi^2}{4\log(q)}}\sum_{m=0}^{\infty}e^{\frac{m(m+1)\pi^2}{\log(q)}}\left((2m+1)\pi\sinh\left(\frac{(2m+1)\pi z}{\log(q)}\right)+2z\cosh\left(\frac{(2m+1)\pi z}{\log(q)}\right)\right)/;$$
$$(|q|<1\land|q-1|<1)\bigwedge c_k=\frac{(-1)^{k-1}}{k+1}\bigwedge p_{j,0}=1\bigwedge p_{j,k}=-\frac{1}{k}\sum_{m=1}^{k}(j\,m-k+m)\,c_m\,p_{j,k-m}\bigwedge k\in\mathbb{N}^+$$

09.08.06.0012.01

$$\vartheta_4'(z,q)\propto-\frac{2\sqrt{\pi}\,i}{(q-1)^{3/2}}\,e^{-i\pi\left\lfloor -\frac{\arg(q-1)}{2\pi}\right\rfloor}(1+O(q-1))\,e^{\frac{4z^2+\pi^2}{4\log(q)}}$$
$$\left(\pi\sinh\left(\frac{\pi z}{\log(q)}\right)+2z\cosh\left(\frac{\pi z}{\log(q)}\right)+O\left(e^{\frac{2\pi^2}{\log(q)}}\left(3\pi\sinh\left(\frac{3\pi z}{\log(q)}\right)+2z\cosh\left(\frac{3\pi z}{\log(q)}\right)\right)\right)\right)/;\ |q|<1$$

Other *q*-series representations

09.08.06.0003.01

$$\frac{\vartheta_4'(z,q)}{\vartheta_4(z,q)}=4\sum_{k=1}^{\infty}\frac{q^k}{1-q^{2k}}\sin(2kz)$$

Other series representations

09.08.06.0013.01

$$\vartheta_4'(z,q)=-\frac{2\sqrt{\pi}}{(-\log(q))^{3/2}}\,e^{\frac{4z^2+\pi^2}{4\log(q)}}\left(2z\sum_{k=0}^{\infty}e^{\frac{k(k+1)\pi^2}{\log(q)}}\cosh\left(\frac{(2k+1)\pi z}{\log(q)}\right)+\pi\sum_{k=0}^{\infty}e^{\frac{k(k+1)\pi^2}{\log(q)}}(2k+1)\sinh\left(\frac{(2k+1)\pi z}{\log(q)}\right)\right)$$

09.08.06.0004.01

$$\vartheta_4'(z,q)=-\frac{2\,i^{3/2}}{\tau^{3/2}}\sum_{n=-\infty}^{\infty}\left(n+\frac{z}{\pi}-\frac{1}{2}\right)\exp\left(-\frac{\pi i}{\tau}\left(\frac{z}{\pi}+n-\frac{1}{2}\right)^2\right)/;\ q=e^{i\pi\tau}$$

Differential equations

Partial differential equations

The elliptic theta functions satisfy the one-dimensional heat equation:

09.08.13.0001.01

$$\frac{\partial\vartheta_4'(z,q)}{\partial\tau}=-\frac{\pi i}{4}\,\frac{\partial^2\vartheta_4'(z,q)}{\partial z^2}/;\ q=e^{i\pi\tau}$$

09.08.13.0002.01

$$4q\,\frac{\partial\vartheta_4'(z,q)}{\partial q}+\frac{\partial^2\vartheta_4'(z,q)}{\partial z^2}=0$$

Transformations

Transformations and argument simplifications

Argument involving basic arithmetic operations

09.08.16.0001.01

$$\vartheta_4'(z,q) = -\frac{e^{\frac{4z^2+\pi^2}{4\log(q)}}\sqrt{\pi}}{\sqrt[4]{e^{\frac{\pi^2}{\log(q)}}}\,(-\log(q))^{3/2}}\left(2\,z\,\vartheta_2\left(\frac{i\,\pi\,z}{\log(q)},\,e^{\frac{\pi^2}{\log(q)}}\right)+i\,\pi\,\vartheta_2'\left(\frac{i\,\pi\,z}{\log(q)},\,e^{\frac{\pi^2}{\log(q)}}\right)\right)$$

Differentiation

Low-order differentiation

With respect to *z*

09.08.20.0001.01

$$\frac{\partial\vartheta_4'(z,q)}{\partial z} = 8\sum_{k=1}^{\infty}(-1)^{k-1}\,k^2\,q^{k^2}\cos(2\,k\,z)\;/;\;|q|<1$$

09.08.20.0002.01

$$\frac{\partial^2\vartheta_4'(z,q)}{\partial z^2} = 16\sum_{k=1}^{\infty}(-1)^{k}\,q^{k^2}\,k^3\sin(2\,k\,z)\;/;\;|q|<1$$

With respect to *q*

09.08.20.0003.01

$$\frac{\partial\vartheta_4'(z,q)}{\partial q} = 4\sum_{k=1}^{\infty}(-1)^{k-1}\,k^3\,q^{k^2-1}\sin(2\,k\,z)\;/;\;|q|<1$$

09.08.20.0004.01

$$\frac{\partial^2\vartheta_4'(z,q)}{\partial q^2} = \frac{4}{q^2}\sum_{k=2}^{\infty}(-1)^{k-1}\,q^{k^2}\,k^3\left(k^2-1\right)\sin(2\,k\,z)\;/;\;|q|<1$$

Symbolic differentiation

With respect to *z*

09.08.20.0005.01

$$\frac{\partial^n\vartheta_4'(z,q)}{\partial z^n} = 2^{n+2}\sum_{k=1}^{\infty}(-1)^{k-1}\,q^{k^2}\,k^{n+1}\sin\left(\frac{\pi\,n}{2}+2\,k\,z\right)\;/;\;|q|<1\bigwedge n\in\mathbb{N}^+$$

With respect to *q*

09.08.20.0006.01

$$\frac{\partial^n\vartheta_4'(z,q)}{\partial q^n} = 4\sum_{k=1}^{\infty}(-1)^{k-1}\,q^{k^2-n}\,k\left(k^2-n+1\right)_n\sin(2\,k\,z)\;/;\;|q|<1\bigwedge n\in\mathbb{N}^+$$

Fractional integro-differentiation

With respect to *z*

09.08.20.0007.01

$$\frac{\partial^\alpha \vartheta_4'(z, q)}{\partial z^\alpha} = 2^{\alpha+2} \sqrt{\pi}\, z^{1-\alpha} \sum_{k=1}^{\infty} (-1)^{k-1} k^2 q^{k^2} {}_1\tilde{F}_2\left(1; \frac{3-\alpha}{2}, 1-\frac{\alpha}{2}; -k^2 z^2\right) /; |q| < 1$$

With respect to q

09.08.20.0008.01

$$\frac{\partial^\alpha \vartheta_4'(z, q)}{\partial q^\alpha} = 4\, q^{-\alpha} \sum_{k=1}^{\infty} \frac{(-1)^{k-1} k\, q^{k^2}\, \Gamma(k^2+1) \sin(2 k z)}{\Gamma(k^2-\alpha+1)} /; |q| < 1$$

Integration

Indefinite integration

Involving only one direct function

09.08.21.0001.01

$$\int \vartheta_4'(z, q)\, dz = \vartheta_4(z, q)$$

Involving only one direct function with respect to q

09.08.21.0002.01

$$\int \vartheta_4'(z, q)\, dq = 4 \sum_{k=1}^{\infty} \frac{(-1)^{k-1} k\, q^{k^2+1} \sin(2 k z)}{k^2+1} /; |q| < 1$$

Representations through equivalent functions

With related functions

Involving theta functions

Involving $\vartheta_1(z, q)$

09.08.27.0005.01

$$\vartheta_4'(z, q) = \frac{\vartheta_4(z, q)}{\vartheta_1(z, q)} \vartheta_1'(z, q) - \vartheta_4(0, q)^2 \frac{\vartheta_2(z, q)\, \vartheta_3(z, q)}{\vartheta_1(z, q)}$$

09.08.27.0006.01

$$\vartheta_4'(z, q) = e^{i z} \sqrt[4]{q} \left(i\, \vartheta_1'\left(z - \frac{\pi \tau}{2}, q\right) - \vartheta_1\left(z - \frac{\pi \tau}{2}, q\right)\right) /; q = e^{-i \pi \tau}$$

09.08.27.0007.01

$$\vartheta_4'(z, q) = (-1)^m e^{i (2 m+1) z} q^{\left(m+\frac{1}{2}\right)^2} \left(i\, \vartheta_1'\left(z - \frac{1}{2}(2 m+1) \pi \tau, q\right) - (2 m+1)\, \vartheta_1\left(z - \frac{1}{2}(2 m+1) \pi \tau, q\right)\right) /; m \in \mathbb{Z} \bigwedge q = e^{-i \pi \tau}$$

09.08.27.0008.01

$$\vartheta_4'(z, q) = e^{-i z} \sqrt[4]{q} \left(\vartheta_1\left(z + \frac{1}{2} i \log(q), q\right) + i\, \vartheta_1'\left(z + \frac{1}{2} i \log(q), q\right)\right)$$

09.08.27.0009.01

$$\vartheta_4'(z,q)=(-1)^m e^{-i(2m+1)z} q^{m^2+m+\frac{1}{4}}\left((2m+1)\vartheta_1\left(z+\frac{1}{2}i(2m+1)\log(q),q\right)+i\vartheta_1'\left(z+\frac{1}{2}i(2m+1)\log(q),q\right)\right)/;m\in\mathbb{Z}$$

Involving $\vartheta_2(z,q)$

09.08.27.0010.01

$$\vartheta_4'(z,q)=-e^{-iz}\sqrt[4]{q}\left(\vartheta_2\left(z-\frac{1}{2}\pi(\tau+1),q\right)+i\vartheta_2'\left(z-\frac{1}{2}\pi(\tau+1),q\right)\right)/;q=e^{i\pi\tau}$$

09.08.27.0011.01

$$\vartheta_4'(z,q)=e^{i(2m+1)z} q^{m^2+m+\frac{1}{4}}\left(i\vartheta_2'\left(z+\frac{1}{2}(2m+1)\pi(\tau-1),q\right)-(2m+1)\vartheta_2\left(z+\frac{1}{2}(2m+1)\pi(\tau-1),q\right)\right)/;m\in\mathbb{Z}\bigwedge q=e^{i\pi\tau}$$

09.08.27.0012.01

$$\vartheta_4'(z,q)=e^{-iz}\sqrt[4]{q}\left(\vartheta_2\left(z+\frac{1}{2}i\log(q)-\frac{\pi}{2},q\right)+i\vartheta_2'\left(z+\frac{1}{2}i\log(q)-\frac{\pi}{2},q\right)\right)$$

09.08.27.0013.01

$$\vartheta_4'(z,q)=e^{i(2m+1)z} q^{m^2+m+\frac{1}{4}}\left((2m+1)\vartheta_2\left(z-\frac{1}{2}(2m+1)(i\log(q)+\pi),q\right)-i\vartheta_2'\left(z-\frac{1}{2}(2m+1)(i\log(q)+\pi),q\right)\right)/;m\in\mathbb{Z}$$

Involving $\vartheta_3'(z,q)$

09.08.27.0004.01

$$\vartheta_4'(z,q)=\vartheta_3'\left(z-\frac{\pi}{2},q\right)$$

09.08.27.0014.01

$$\vartheta_4'(z,q)=\vartheta_3'\left(z+\frac{\pi}{2},q\right)$$

09.08.27.0015.01

$$\vartheta_4'(z,q)=\vartheta_3'\left(\frac{1}{2}\pi(2m+1)+z,q\right)/;m\in\mathbb{Z}$$

Involving $\vartheta_4(z,q)$

09.08.27.0001.02

$$\vartheta_4'\left(z,e^{i\pi\tau}\right)=2e^{-i(2z-\pi\tau)}i\vartheta_4\left(z-\pi\tau,e^{i\pi\tau}\right)-e^{-i(2z-\pi\tau)}\vartheta_4'\left(z-\pi\tau,e^{i\pi\tau}\right)/;\operatorname{Im}(\tau)>0$$

09.08.27.0016.01

$$\vartheta_4'(z,q)=-e^{2iz}q(2i\vartheta_4(z+\pi\tau,q)+\vartheta_4'(z+\pi\tau,q))/;q=e^{i\pi\tau}$$

09.08.27.0017.01

$$\vartheta_4'(z,q)=(-1)^n e^{2inz}q^{n^2}(2in\vartheta_4(z+n\pi\tau,q)+\vartheta_4'(z+n\pi\tau,q))/;n\in\mathbb{Z}\bigwedge q=e^{i\pi\tau}$$

09.08.27.0018.01

$$\vartheta_4'(z,q)=(-1)^n e^{2inz}q^{n^2}(2in\vartheta_4(z+\pi m+n\pi\tau,q)+\vartheta_4'(z+\pi m+n\pi\tau,q))/;\{m,n\}\in\mathbb{Z}\bigwedge q=e^{i\pi\tau}$$

09.08.27.0019.01

$$\vartheta_4'(z,q)=-e^{2iz}q(2i\vartheta_4(z-i\log(q),q)+\vartheta_4'(z-i\log(q),q))/;q=e^{i\pi\tau}$$

09.08.27.0020.01

$$\vartheta_4'(z,q)=e^{-2iz}q(2i\vartheta_4(z+i\log(q),q)-\vartheta_4'(z+i\log(q),q))/;q=e^{i\pi\tau}$$

09.08.27.0021.01

$$\vartheta_4'(z, q) == (-1)^n e^{-2 i n z} q^{n^2} (-2 i n \vartheta_4(z + i n \log(q), q) + \vartheta_4'(z + i n \log(q), q)) /; n \in \mathbb{Z} \bigwedge q == e^{i \pi \tau}$$

09.08.27.0022.01

$$\vartheta_4'(z, q) == (-1)^n e^{-2 i n z} q^{n^2} (-2 i n \vartheta_4(z + \pi m + i n \log(q), q) + \vartheta_4'(z + \pi m + i n \log(q), q)) /; \{m, n\} \in \mathbb{Z} \bigwedge q == e^{i \pi \tau}$$

Zeros

09.08.30.0001.01

$$\vartheta_4'(z, 0) == 0$$

09.08.30.0002.02

$$\vartheta_4'(0, q) == 0$$

09.08.30.0003.01

$$\vartheta_4'\left(\frac{m \pi}{2}, q\right) == 0 /; m \in \mathbb{Z}$$

Theorems

Solution set of the Halphen equations

The functions $w_1(z) == 2 \frac{\partial \log(\vartheta_4(0, e^{i \pi \tau}))}{\partial \tau} \bigwedge w_2(z) == 2 \frac{\partial \log(\vartheta_2(0, e^{i \pi \tau}))}{\partial \tau} \bigwedge w_3(z) == 2 \frac{\partial \log(\vartheta_3(0, e^{i \pi \tau}))}{\partial \tau}$ are a solution set of the Halphen equations

$$w_1'(z) == w_1(z) (w_2(z) + w_3(z)) - w_2(z) w_3(z) \bigwedge w_2'(z) == w_2(z) (w_1(z) + w_3(z)) - w_1(z) w_3(z) \bigwedge w_3'(z) == w_3(z) (w_1(z) + w_2(z)) - w_1(z) w_2(z).$$

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