EllipticPi

Notations

Traditional name

Complete elliptic integral of the third kind

Traditional notation

$\Pi(n \mid m)$

Mathematica StandardForm notation

`EllipticPi[n, m]`

Primary definition

08.03.02.0001.01

$$\Pi(n \mid m) = \Pi\left(n; \frac{\pi}{2} \,\middle|\, m\right)$$

Specific values

Specialized values

For fixed *n*

08.03.03.0001.01

$$\Pi(n \mid 0) = \frac{\pi}{2\sqrt{1-n}}$$

08.03.03.0002.01

$$\Pi(n \mid 1) = -\frac{\infty}{\operatorname{sgn}(n-1)}$$

08.03.03.0003.01

$$\Pi(n \mid n) = \frac{E(n)}{1-n}$$

For fixed *m*

08.03.03.0004.01

$$\Pi(0 \mid m) = K(m)$$

08.03.03.0005.01

$$\Pi(1 \mid m) = \tilde{\infty}$$

Values at infinities

08.03.03.0006.01

$\Pi(\infty \mid m) == 0$

08.03.03.0007.01

$\Pi(-\infty \mid m) == 0$

08.03.03.0008.01

$\Pi(n \mid \infty) == 0$

08.03.03.0009.01

$\Pi(n \mid -\infty) == 0$

General characteristics

Domain and analyticity

$\Pi(n \mid m)$ is an analytical function of n and m which is defined over $\mathbb{C}^2$.

08.03.04.0001.01

$(n * m) \longrightarrow \Pi(n \mid m) :: (\mathbb{C} \otimes \mathbb{C}) \longrightarrow \mathbb{C}$

Symmetries and periodicities

Mirror symmetry

08.03.04.0002.01

$\Pi(\overline{n} \mid \overline{m}) == \overline{\Pi(n \mid m)}$

Periodicity

The function $\Pi(n \mid m)$ is not periodic.

Poles and essential singularities

With respect to *m*

The function $\Pi(n \mid m)$ does not have poles and essential singularities with respect to m.

08.03.04.0003.01

$\mathcal{Sing}_m(\Pi(n \mid m)) == \{\}$

With respect to *n*

The function $\Pi(n \mid m)$ does not have poles and essential singularities with respect to n.

08.03.04.0004.01

$\mathcal{Sing}_n(\Pi(n \mid m)) == \{\}$

Branch points

With respect to *m*

For fixed n, the function $\Pi(n \mid m)$ has two branch points at $m == 1$ and $m == \tilde{\infty}$.

08.03.04.0005.01

$$\mathcal{BP}_m(\Pi(n \mid m)) = \{1, \tilde{\infty}\}$$

08.03.04.0006.01

$$\mathcal{R}_m(\Pi(n \mid m), 1) = 2$$

08.03.04.0007.01

$$\mathcal{R}_m(\Pi(n \mid m), \tilde{\infty}) = 2$$

With respect to *n*

For fixed m, the function $\Pi(n \mid m)$ has two branch points at $n = 1$ and $n = \tilde{\infty}$.

08.03.04.0008.01

$$\mathcal{BP}_n(\Pi(n \mid m)) = \{1, \tilde{\infty}\}$$

08.03.04.0009.01

$$\mathcal{R}_n(\Pi(n \mid m), 1) = 2$$

08.03.04.0010.01

$$\mathcal{R}_n(\Pi(n \mid m), \tilde{\infty}) = 2$$

Branch cuts

With respect to *n*

For fixed generic m, the function $\Pi(n \mid m)$ is a single-valued function on the n-plane cut along the interval $(1, \infty)$, where it is continuous from below.

08.03.04.0011.01

$$\mathcal{BC}_n(\Pi(n \mid m), n) = \{\{(1, \infty), i\}\}$$

08.03.04.0012.01

$$\lim_{\epsilon \to +0} \Pi(n + i\,\epsilon \mid m) = \frac{1}{\sqrt{m}}\, \Pi\!\left(\frac{n}{m}; \sin^{-1}\!\left(\sqrt{m}\right) \,\middle|\, \frac{1}{m}\right) /;\ n > 1$$

08.03.04.0013.01

$$\lim_{\epsilon \to +0} \Pi(n + i\,\epsilon \mid m) = \Pi(n \mid m) + \frac{\pi}{\sqrt{1-n}\,\sqrt{1-\frac{m}{n}}} /;\ n > 1$$

08.03.04.0014.01

$$\lim_{\epsilon \to +0} \Pi(n - i\,\epsilon \mid m) = \Pi(n \mid m) /;\ n > 1$$

With respect to *m*

For fixed generic n, the function $\Pi(n \mid m)$ is a single-valued function on the m-plane cut along the interval $(1, \infty)$, where it is continuous from below.

For fixed real $n > 1$, the function $\Pi(n \mid m)$ is a single-valued function on the m-plane cut along the interval $(-\infty, 1)$. It is continuous from below along the interval $(-\infty, 0)$ and it continuous from above along the interval $(0, 1)$.

08.03.04.0015.01

$$\mathcal{BC}_m(\Pi(n \mid m), m) = \{\{(1, \infty), i\}\}$$

08.03.04.0016.01

$$\lim_{\epsilon\to+0}\Pi(n\mid m+i\,\epsilon)=\Pi(n\mid m)+\frac{2\,i\,m}{m-n}\,\Pi\!\left(\frac{n\,(1-m)}{n-m}\,\middle|\,1-m\right)/;\,m>1$$

08.03.04.0017.01

$$\lim_{\epsilon\to+0}\Pi(n\mid m+i\,\epsilon)=\frac{1}{\sqrt{m}}\,\Pi\!\left(\frac{n}{m}\,\middle|\,\frac{1}{m}\right)+\frac{i\,m}{m-n}\,\Pi\!\left(\frac{n\,(1-m)}{n-m}\,\middle|\,1-m\right)/;\,m>1$$

08.03.04.0018.01

$$\lim_{\epsilon\to+0}\Pi(n\mid m+i\,\epsilon)=\Pi(n\mid m)\,/;\,n>1\land m<0$$

08.03.04.0019.01

$$\lim_{\epsilon\to+0}\Pi(n\mid m-i\,\epsilon)=\Pi(n\mid m)\,/;\,m>1\lor(n>1\land 0<m<1)$$

08.03.04.0020.01

$$\lim_{\epsilon\to+0}\Pi(n\mid m-i\,\epsilon)=\frac{1}{\sqrt{m}}\,\Pi\!\left(\frac{n}{m}\,\middle|\,\frac{1}{m}\right)-\frac{i\,m}{m-n}\,\Pi\!\left(\frac{n\,(1-m)}{n-m}\,\middle|\,1-m\right)/;\,m>1$$

08.03.04.0021.01

$$\lim_{\epsilon\to+0}\Pi(n\mid m-i\,\epsilon)=\Pi(n\mid m)+\frac{2\,i\,\pi}{\sqrt{\frac{m}{n}-m+n-1}}\,/;\,n>1\land m<0$$

08.03.04.0022.01

$$\lim_{\epsilon\to+0}\Pi(n\mid m+i\,\epsilon)=\Pi(n\mid m)+\frac{2\,i\,\pi}{\sqrt{\frac{m}{n}-m+n-1}}\,/;\,n>1\land 0<m<1$$

Series representations

Generalized power series

Expansions at generic point $n == n_0$

For the function itself

08.03.06.0008.01

$$\Pi(n\mid m)\propto\Pi(n_0\mid m)+\frac{1}{2\,(m-n_0)\,(n_0-1)}\left(E(m)-\frac{m-n_0^2}{n_0}\,\Pi(n_0\mid m)+K(m)\left(\frac{m}{n_0}-1\right)\right)(n-n_0)+$$
$$\frac{1}{8\,(m-n_0)^2\,(n_0-1)^2\,n_0^2}\big(-E(m)\,n_0\,(2\,n_0\,m+m+n_0\,(2-5\,n_0))+K(m)\left((1-4\,n_0)\,m^2+3\,n_0\,(3\,n_0-1)\,m+n_0^2\,(2-5\,n_0)\right)+$$
$$\Pi(n_0\mid m)\left(3\,n_0^4+2\,m\,(2-5\,n_0)\,n_0+m^2\,(4\,n_0-1)\right)\big)\,(n-n_0)^2+\ldots\,/;\,(n\to n_0)$$

08.03.06.0009.01

$$\Pi(n \mid m) \propto \Pi(n_0 \mid m) + \frac{1}{2\,(m-n_0)\,(n_0-1)} \left(E(m) - \frac{m-n_0^2}{n_0}\,\Pi(n_0 \mid m) + K(m)\left(\frac{m}{n_0}-1\right)\right)(n-n_0) + \frac{1}{8\,(m-n_0)^2\,(n_0-1)^2\,n_0^2}\left(-E(m)\,n_0\,(2\,n_0\,m+m+n_0\,(2-5\,n_0)) + K(m)\left((1-4\,n_0)\,m^2+3\,n_0\,(3\,n_0-1)\,m+n_0^2\,(2-5\,n_0)\right) + \Pi(n_0 \mid m)\left(3\,n_0^4+2\,m\,(2-5\,n_0)\,n_0+m^2\,(4\,n_0-1)\right)\right)(n-n_0)^2 + O\left((n-n_0)^3\right)$$

08.03.06.0010.01

$$\Pi(n \mid m) = \frac{\pi}{2} \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k}{k!}\, F_1\left(k+\frac{1}{2};\, k+1,\, \frac{1}{2};\, k+1;\, n_0,\, m\right)(n-n_0)^k$$

08.03.06.0011.01

$$\Pi(z \mid m) = \sum_{k=0}^{\infty} \frac{1}{k!}\, \Pi^{(k,0)}(n_0 \mid m)\,(n-n_0)^k$$

08.03.06.0012.01

$$\Pi(n \mid m) \propto \Pi(n_0 \mid m)\,(1 + O(n-n_0))$$

Expansions on branch cuts

08.03.06.0013.01

$$\Pi(n \mid m) \propto \left\lfloor \frac{\arg(x-n)}{2\,\pi} \right\rfloor \frac{\pi\, i}{\sqrt{\frac{(x-1)\,(x-m)}{x}}}\, e^{-i\,\pi\left(\left\lfloor \frac{\pi-\arg\left(\frac{n-x}{x-m}+1\right)-\arg\left(\frac{x-m}{x}\right)}{2\,\pi} \right\rfloor + \left\lfloor \frac{\pi-\arg(n-m)}{2\,\pi} \right\rfloor\right)} \left(1 + \frac{\left(x^2-m\right)(n-x)}{2\,(m-x)\,(x-1)\,x} + \frac{\left(3\,x^4+2\,m\,(2-5\,x)\,x+m^2\,(4\,x-1)\right)(n-x)^2}{8\,(m-x)^2\,(x-1)^2\,x^2} + \ldots\right) + \Pi\,(x \mid m) + \frac{\pi}{4}\, F_1\left(\frac{3}{2};\, 2,\, \frac{1}{2};\, 2;\, x,\, m\right)(n-x) + \frac{3\,\pi}{16}\, F_1\left(\frac{5}{2};\, 3,\, \frac{1}{2};\, 3;\, x,\, m\right)(n-x)^2 + \ldots\ /;\ (n \to x) \land x \in \mathbb{R} \land x > 1$$

08.03.06.0014.01

$$\Pi(n \mid m) = \left\lfloor \frac{\arg(x-n)}{2\,\pi} \right\rfloor \frac{\pi\, i}{\sqrt{n-1}\, \sqrt{1-\frac{m}{n}}} + \frac{1}{2}\,\pi \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k}{k!}\, F_1\left(k+\frac{1}{2};\, k+1,\, \frac{1}{2};\, k+1;\, x,\, m\right)(n-x)^k\ /;\ x \in \mathbb{R} \land x > 1$$

08.03.06.0015.01

$$\Pi(n \mid m) = \frac{\pi}{2} \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k}{k!}\, F_1\left(k+\frac{1}{2};\, k+1,\, \frac{1}{2};\, k+1;\, x,\, m\right)(n-x)^k + \frac{\pi\, i}{\sqrt{\frac{(x-1)\,(x-m)}{x}}} \left\lfloor \frac{\arg(x-n)}{2\,\pi} \right\rfloor e^{-i\,\pi\left(\left\lfloor \frac{-\arg\left(\frac{n-x}{x-m}+1\right)-\arg\left(\frac{x-m}{x}\right)+\pi}{2\,\pi} \right\rfloor + \left\lfloor \frac{\pi-\arg(n-m)}{2\,\pi} \right\rfloor\right)} \sum_{k=0}^{\infty} \sum_{j=0}^{k} \sum_{i=0}^{k-j} \frac{(-1)^k \left(-\frac{1}{2}\right)_j \left(\frac{1}{2}\right)_i \left(\frac{1}{2}\right)_{-i-j+k}}{i!\,(-i-j+k)!\,j!}\,(x-1)^{i+j-k}\, x^{-j}\,(x-m)^{-i}\,(n-x)^k\ /;\ x \in \mathbb{R} \land x > 1$$

08.03.06.0016.01

$$\Pi(n\mid m)\propto \Pi(x\mid m)\,(1+O(n-x))+\left\lfloor \frac{\arg(x-n)}{2\pi}\right\rfloor \frac{\pi\, i}{\sqrt{\frac{(x-1)\,(x-m)}{x}}}\, e^{-i\pi\left(\left\lfloor \frac{\pi-\arg\left(\frac{n-x}{x-m}+1\right)-\arg\left(\frac{x-m}{x}\right)}{2\pi}\right\rfloor+\left\lfloor \frac{\pi-\arg(n-m)}{2\pi}\right\rfloor\right)}(1+O(n-x))\ /;$$

$$(n\to x)\land x\in\mathbb{R}\land x>1$$

Expansions at $n == 0$

08.03.06.0017.01

$$\Pi(n\mid m)\propto K(m)+\frac{\pi}{4}\,{}_2F_1\left(\frac{3}{2},\frac{1}{2};2;m\right)n+\frac{3\pi}{16}\,{}_2F_1\left(\frac{5}{2},\frac{1}{2};3;m\right)n^2+\ldots\ /;\ (n\to 0)$$

08.03.06.0018.01

$$\Pi(n\mid m)\propto K(m)-\frac{E(m)-K(m)}{m}\,n-\frac{2\,(m+1)\,E(m)-(m+2)\,K(m)}{3\,m^2}\,n^2+O\left(n^3\right)$$

08.03.06.0019.01

$$\Pi(n\mid m)=\frac{1}{\sqrt{1-m}}\sum_{k=0}^{\infty}\frac{\left(\frac{n}{1-m}\right)^k}{\left(\frac{1}{2}\right)_k}\,K^{(k)}\left(\frac{m}{m-1}\right)$$

08.03.06.0004.01

$$\Pi(n\mid m)==\frac{\pi}{2}\sum_{k=0}^{\infty}\frac{n^k\left(\frac{1}{2}\right)_k}{k!}\,{}_2F_1\left(k+\frac{1}{2},\frac{1}{2};k+1;m\right)\ /;\ |n|<1$$

08.03.06.0020.01

$$\Pi(n\mid m)\propto K(m)\,(1+O(n))$$

Expansions at $n == 1$

08.03.06.0021.01

$$\Pi(n\mid m)\propto \frac{\pi}{2\sqrt{1-\frac{m}{n}}\,\sqrt{1-n}}+\frac{E(m)}{m-1}+K(m)+$$

$$\frac{(m+1)\,E(m)+(m-1)\,K(m)}{3\,(m-1)^2}\,(n-1)-\frac{\left(2\,m^2-7\,m-3\right)E(m)-\left(m^2+2\,m-3\right)K(m)}{15\,(m-1)^3}\,(n-1)^2+\ldots\ /;\ (n\to 1)$$

08.03.06.0022.01

$$\Pi(n\mid m)\propto \frac{\pi}{2\sqrt{1-m}\,\sqrt{1-n}}\,(-1)^{\left\lfloor \frac{1}{2}-\frac{\arg\left(\frac{n-m}{n\,(1-m)}\right)}{2\pi}-\frac{\arg(1-m)}{2\pi}\right\rfloor}\left(1-\frac{m}{2\,(1-m)}\,(n-1)+\frac{3\,m^2}{8\,(1-m)^2}\,(n-1)^2+\ldots\right)+\frac{E(m)}{m-1}+K(m)+$$

$$\frac{(m+1)\,E(m)+(m-1)\,K(m)}{3\,(m-1)^2}\,(n-1)-\frac{\left(2\,m^2-7\,m-3\right)E(m)-\left(m^2+2\,m-3\right)K(m)}{15\,(m-1)^3}\,(n-1)^2+\ldots\ /;\ (n\to 1)$$

08.03.06.0023.01

$$\Pi(n\mid m)\propto \frac{\pi}{2\sqrt{1-\frac{m}{n}}\,\sqrt{1-n}}-\frac{m\pi}{4}\left({}_2F_1\left(\frac{3}{2},\frac{3}{2};2;m\right)-{}_2F_1\left(\frac{3}{2},\frac{5}{2};2;m\right)(n-1)+{}_2F_1\left(\frac{3}{2},\frac{7}{2};2;m\right)(n-1)^2\right)+O\left((n-1)^3\right)$$

08.03.06.0024.01

$$\Pi(n \mid m) = \frac{\pi\,(-1)^{\left\lfloor \frac{1}{2} - \frac{\arg(1-m)}{2\pi} - \frac{\arg\left(\frac{n-m}{n(1-m)}\right)}{2\pi} \right\rfloor}}{2\sqrt{1-m}\,\sqrt{\frac{n-m}{n(1-m)}}\,\sqrt{1-n}} - \frac{\pi m}{4} \sum_{k=0}^{\infty} (-1)^k\, {}_2F_1\!\left(\frac{3}{2}, k+\frac{3}{2}; 2; m\right)(n-1)^k$$

08.03.06.0025.01

$$\Pi(n \mid m) = \frac{\pi}{2\sqrt{1-\frac{m}{n}}\,\sqrt{1-n}} - \frac{\pi m}{4} \sum_{k=0}^{\infty} (-1)^k\, {}_2F_1\!\left(\frac{3}{2}, k+\frac{3}{2}; 2; m\right)(n-1)^k$$

08.03.06.0026.01

$$\Pi(n \mid m) \propto \frac{\pi}{2\sqrt{1-\frac{m}{n}}\,\sqrt{1-n}} + \left(\frac{E(m)}{m-1} + K(m)\right)(1 + O(n-1))$$

Expansions at $n == \infty$

08.03.06.0027.01

$$\Pi(n \mid m) \propto \frac{\pi}{2\sqrt{-n}}\left(1 + \frac{m+1}{2n} + \frac{3m^2+2m+3}{8n^2} + \ldots\right) + \frac{1}{4n}\left(4(E(m)-K(m)) - \frac{4(m+2)K(m) - 8(m+1)E(m)}{3\,n} + \frac{4\left(\left(8m^2+7m+8\right)E(m) - \left(4m^2+3m+8\right)K(m)\right)}{15\,n^2} + \ldots\right) /; (|n| \to \infty)$$

08.03.06.0028.01

$$\Pi(n \mid m) \propto \frac{\pi}{2\sqrt{-n}}\left(1 + \frac{m+1}{2n} + \frac{3m^2+2m+3}{8n^2} + O\left(\frac{1}{n^3}\right)\right) + \frac{1}{4n}\left(4(E(m)-K(m)) - \frac{4(m+2)K(m) - 8(m+1)E(m)}{3\,n} + \frac{4\left(\left(8m^2+7m+8\right)E(m) - \left(4m^2+3m+8\right)K(m)\right)}{15\,n^2} + O\left(\frac{1}{n^3}\right)\right)$$

08.03.06.0029.01

$$\Pi(n \mid m) = \frac{\pi}{2\sqrt{-n}} \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k}{k!}\, {}_2F_1\!\left(\frac{1}{2}, -k; \frac{1}{2}-k; m\right) n^{-k} - \frac{\pi m}{4n} \sum_{k=0}^{\infty} \frac{\left(\frac{3}{2}\right)_k m^k}{(k+1)!}\, {}_2F_1\!\left(\frac{1}{2}, k+\frac{3}{2}; k+2; m\right) n^{-k}$$

08.03.06.0030.01

$$\Pi(n \mid m) \propto \frac{\pi}{2\sqrt{-n}}\left(1 + O\left(\frac{1}{n}\right)\right) + \frac{E(m)-K(m)}{n}\left(1 + O\left(\frac{1}{n}\right)\right) /; (|n| \to \infty)$$

Expansions at generic point $m == m_0$

For the function itself

08.03.06.0031.01

$$\Pi(n\mid m)\propto\Pi(n\mid m_0)+\frac{1}{2(n-m_0)}\left(\frac{1}{m_0-1}E(m_0)+\Pi(n\mid m_0)\right)(m-m_0)+\frac{1}{2}\left(\frac{1}{4(m_0-1)m_0(m_0-n)}K(m_0)+\frac{4m_0^2-(n+2)m_0-n}{4(m_0-1)^2m_0(m_0-n)^2}E(m_0)+\frac{3}{4(m_0-n)^2}\Pi(n\mid m_0)\right)(m-m_0)^2+\ldots\;/;\;(m\to m_0)$$

08.03.06.0032.01

$$\Pi(n\mid m)\propto\Pi(n\mid m_0)+\frac{1}{2(n-m_0)}\left(\frac{1}{m_0-1}E(m_0)+\Pi(n\mid m_0)\right)(m-m_0)+\frac{1}{2}\left(\frac{1}{4(m_0-1)m_0(m_0-n)}K(m_0)+\frac{4m_0^2-(n+2)m_0-n}{4(m_0-1)^2m_0(m_0-n)^2}E(m_0)+\frac{3}{4(m_0-n)^2}\Pi(n\mid m_0)\right)(m-m_0)^2+O\left((m-m_0)^3\right)$$

08.03.06.0033.01

$$\Pi(n\mid m)=\frac{\pi}{2}\sum_{k=0}^{\infty}\frac{\left(\frac{1}{2}\right)_k^2}{(k!)^2}F_1\left(k+\frac{1}{2};1,k+\frac{1}{2};k+1;n,m_0\right)(m-m_0)^k$$

08.03.06.0034.01

$$\Pi(z\mid m)=\sum_{k=0}^{\infty}\frac{1}{k!}\Pi^{(0,k)}(n\mid m_0)(m-m_0)^k$$

08.03.06.0035.01

$$\Pi(n\mid m)\propto\Pi(n\mid m_0)(1+O(m-m_0))$$

Expansions on branch cuts

08.03.06.0036.01

$$\Pi(n\mid m)=\frac{1}{\sqrt{x}}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{\left(\frac{1}{2}\right)_{j+k}x^{-j}}{j!\,k!\,(2j+2k+1)}F_1\left(j+k+\frac{1}{2};1,\frac{1}{2};j+k+\frac{3}{2};\frac{n}{x},\frac{1}{x}\right)(m-x)^j+\frac{e^{-i\pi\left\lfloor\frac{\arg(x-m)}{2\pi}\right\rfloor}i\sqrt{x-1}}{(n-1)x}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^j\left(k+\frac{1}{2}\right)_j\left(\frac{1}{2}\right)_k}{j!\,k!}x^{-j-k}F_1\left(\frac{1}{2};k+1,1;\frac{3}{2};1-\frac{1}{x},\frac{n(x-1)}{(n-1)x}\right)(m-x)^j\;/;\;x\in\mathbb{R}\land x>1$$

08.03.06.0037.01

$$\Pi(n\mid m)\propto\left(\frac{1}{\sqrt{x}}\sum_{k=0}^{\infty}\frac{\left(\frac{1}{2}\right)_k}{k!\,(2k+1)}F_1\left(k+\frac{1}{2};1,\frac{1}{2};k+\frac{3}{2};\frac{n}{x},\frac{1}{x}\right)+\frac{e^{-i\pi\left\lfloor\frac{\arg(x-m)}{2\pi}\right\rfloor}i\sqrt{x-1}}{(n-1)x}\sum_{k=0}^{\infty}\frac{\left(\frac{1}{2}\right)_k x^{-k}}{k!}F_1\left(\frac{1}{2};k+1,1;\frac{3}{2};1-\frac{1}{x},\frac{n(x-1)}{(n-1)x}\right)\right)(1+O(m-x))\;/;\;(m\to x)\land x\in\mathbb{R}\land x>1$$

Expansions at $m == 0$

08.03.06.0038.01

$$\Pi(n\mid m)\propto\frac{\pi}{2\sqrt{1-n}}-\frac{\pi}{4\sqrt{1-n}\,n}\left(\sqrt{1-n}-1\right)m-\frac{3\pi}{32n^2}\left(n-\frac{2}{\sqrt{1-n}}+2\right)m^2+\ldots\;/;\;(m\to 0)$$

08.03.06.0039.01

$$\Pi(n\mid m)\propto\frac{\pi}{2\sqrt{1-n}}-\frac{\pi}{4\sqrt{1-n}\,n}\left(\sqrt{1-n}-1\right)m-\frac{3\pi}{32\,n^2}\left(n-\frac{2}{\sqrt{1-n}}+2\right)m^2+O\left(m^3\right)$$

08.03.06.0040.01

$$\Pi(n\mid m)=\frac{1}{2}\pi\sum_{k=0}^{\infty}\frac{\left(\frac{1}{2}\right)_k^2}{(k!)^2}\left(\frac{n^{-k}\,k!}{\sqrt{1-n}\,\left(\frac{1}{2}\right)_k}-\frac{2\,k}{n}\sum_{j=0}^{k-1}\frac{(1-k)_j}{\left(\frac{3}{2}\right)_j}\left(1-\frac{1}{n}\right)^j\right)m^k\;/;\;|m|<1$$

08.03.06.0005.01

$$\Pi(n\mid m)==\frac{\pi}{2}\sum_{k=0}^{\infty}\frac{\left(\frac{1}{2}\right)_k^2}{(k!)^2}\,{}_2F_1\left(1,\,k+\frac{1}{2};\,k+1;\,n\right)m^k\;/;\;|m|<1$$

08.03.06.0041.01

$$\Pi(n\mid m)\propto\frac{\pi}{2\sqrt{1-n}}\,(1+O(m))$$

Expansions at *m* == 1

08.03.06.0042.01

$$\Pi(n\mid m)\propto\frac{1}{2}\left(-\frac{\log(1-m)}{1-n}\left(1+\frac{(n+1)\,(m-1)}{4\,(n-1)}-\frac{3\left(n^2-6\,n-3\right)(m-1)^2}{64\,(n-1)^2}+\ldots\right)+\frac{\sqrt{n}\,\left(\log\left(\sqrt{n}+1\right)-\log\left(1-\sqrt{n}\right)\right)-4\log(2)}{n-1}-\frac{\left(2\,n\log(2)+2\log(2)+\sqrt{n}\,\left(\log\left(1-\sqrt{n}\right)-\log\left(\sqrt{n}+1\right)\right)-1\right)(m-1)}{2\,(n-1)^2}+\frac{1}{64\,(n-1)^3}\left(-5\,n^2+12\,n+24\left(\log\left(\sqrt{n}+1\right)-\log\left(1-\sqrt{n}\right)\right)\sqrt{n}+12\,((n-6)\,n-3)\log(2)+21\right)(m-1)^2+\ldots\right)\;/;\;(m\to 1)$$

08.03.06.0043.01

$$\Pi(n\mid m)=\frac{1}{2}\left(-\log(1-m)\sum_{k=0}^{\infty}\frac{(1-m)^k\left(\frac{1}{2}\right)_k^2}{(k!)^2}\,{}_2F_1\left(1,\,k+\frac{1}{2};\,\frac{1}{2};\,n\right)+\sum_{k=0}^{\infty}\frac{(1-m)^k\left(\frac{1}{2}\right)_k^2}{(k!)^2}\left(2\,\psi(k+1)-\psi\left(k+\frac{1}{2}\right)\right){}_2F_1\left(1,\,k+\frac{1}{2};\,\frac{1}{2};\,n\right)-\sum_{k=0}^{\infty}\frac{(1-m)^k\left(\frac{1}{2}\right)_k^2}{(k!)^2}\sum_{j=0}^{\infty}\frac{\left(k+\frac{1}{2}\right)_j\,\psi\left(k+j+\frac{1}{2}\right)n^j}{\left(\frac{1}{2}\right)_j}\right)\;/;\;|m-1|<1$$

08.03.06.0044.01

$$\Pi(n \mid m) = \frac{1}{2}\left(\sum_{k=0}^{\infty} \frac{(1-m)^k \left(\frac{1}{2}\right)_k^2}{(k!)^2} \sum_{j=0}^{\infty} \frac{\left(k+\frac{1}{2}\right)_j \left(2\,\psi(k+1) - \psi\left(k+\frac{1}{2}\right) - \psi\left(k+j+\frac{1}{2}\right)\right) n^j}{\left(\frac{1}{2}\right)_j} - \log(1-m) \sum_{j=0}^{\infty} \frac{(1-m)^j \left(\frac{1}{2}\right)_j^2}{(j!)^2} \, {}_2F_1\left(1,\, j+\frac{1}{2};\, \frac{1}{2};\, n\right)\right) /;\ |m-1| < 1$$

08.03.06.0045.01

$$\Pi(n \mid m) \propto \frac{\log(1-m)}{2\,(n-1)}\,(1+O(m-1)) + \frac{\sqrt{n}\,\left(\log\left(\sqrt{n}+1\right) - \log\left(1-\sqrt{n}\right)\right) - 4\log(2)}{2\,(n-1)}\,(1+O(m-1))$$

Expansions at $m == \infty$

08.03.06.0046.01

$$\Pi(n \mid m) \propto \frac{\log(-m)}{2\sqrt{-m}}\left(1 + \frac{2n+1}{4m} + \frac{3\left(8n^2+4n+3\right)}{64m^2} + \ldots\right) + \frac{1}{2\sqrt{-m}}\left(4\log(2) + \frac{2\log(2) + n\,(4\log(2)-1) - 1}{2m} - \frac{28n^2 + 26n - 12\left(8n^2+4n+3\right)\log(2) + 21}{64m^2} + \ldots\right) + \frac{\sqrt{n}\,\sin^{-1}\left(\sqrt{n}\right)}{\sqrt{1-n}\,\sqrt{-m}}\left(1 + \frac{n}{2m} + \frac{3n^2}{8m^2} + \ldots\right) /;\ (|m| \to \infty)$$

08.03.06.0047.01

$$\Pi(n \mid m) \propto \frac{\log(-m)}{2\sqrt{-m}}\left(1 + \frac{2n+1}{4m} + \frac{3\left(8n^2+4n+3\right)}{64m^2} + O\left(\frac{1}{m^3}\right)\right) + \frac{1}{2\sqrt{-m}}\left(4\log(2) + \frac{2\log(2) + n\,(4\log(2)-1) - 1}{2m} - \frac{28n^2 + 26n - 12\left(8n^2+4n+3\right)\log(2) + 21}{64m^2} + O\left(\frac{1}{m^3}\right)\right) + \frac{\sqrt{n}\,\sin^{-1}\left(\sqrt{n}\right)}{\sqrt{1-n}\,\sqrt{-m}}\left(1 + \frac{n}{2m} + \frac{3n^2}{8m^2} + O\left(\frac{1}{m^3}\right)\right)$$

08.03.06.0048.01

$$\Pi(n \mid m) = \frac{\sqrt{n}\,\sin^{-1}\left(\sqrt{n}\right)}{\sqrt{1-n}\,\sqrt{-m}} \sum_{k=0}^{\infty} \frac{n^k \left(\frac{1}{2}\right)_k}{k!}\, m^{-k} + \frac{\log(-m)}{2\sqrt{-m}} \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k^2}{(k!)^2}\, {}_2F_1\left(1,\, -k;\, \frac{1}{2} - k;\, n\right) m^{-k} + \frac{1}{2\sqrt{-m}}\left(\log(16) + \sum_{k=1}^{\infty} \frac{\left(\frac{1}{2}\right)_k^2 m^{-k}}{k!} \left(\frac{(-n)^k}{\left(\frac{1}{2}-k\right)_k}\left(\log(16) - \sum_{i=0}^{k-1} \frac{2}{i+k} + \frac{1}{k}\right) + \sum_{j=0}^{k-1} \frac{(-n)^j}{\left(\frac{1}{2}-k\right)_j (k-j)!}\left(\log(16) - \sum_{i=0}^{-j+k-1} \frac{2}{i-j+k} - \sum_{i=0}^{k-1} \frac{2}{i+k} + \frac{1}{k-j} + \frac{1}{k}\right)\right)\right)$$

08.03.06.0049.01

$$\Pi(n \mid m) = \frac{\sqrt{n}\,\sin^{-1}(\sqrt{n})}{\sqrt{1-n}\,\sqrt{-m}} \sum_{k=0}^{\infty} \frac{n^k \left(\frac{1}{2}\right)_k}{k!} m^{-k} + \frac{\log(-m)}{2\sqrt{-m}} \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k^2}{(k!)^2} \, {}_2F_1\left(1, -k; \frac{1}{2}-k; n\right) m^{-k} + \frac{1}{2\sqrt{-m}} \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k^2 m^{-k}}{k!} \sum_{j=0}^{k} \frac{(-n)^j \left(-\psi\left(j-k+\frac{1}{2}\right) + \psi(k+1) - \psi\left(k+\frac{1}{2}\right) + \psi(-j+k+1)\right)}{\left(\frac{1}{2}-k\right)_j (k-j)!}$$

08.03.06.0050.01

$$\Pi(n \mid m) \propto \frac{\log(-16\,m)}{2\sqrt{-m}} \left(1 + O\left(\frac{1}{m}\right)\right) + \frac{\sqrt{n}\,\sin^{-1}(\sqrt{n})}{\sqrt{1-n}\,\sqrt{-m}} \left(1 + O\left(\frac{1}{m}\right)\right)$$

Expansions at $\{m, n\} == \{0, 0\}$

08.03.06.0001.02

$$\Pi(n \mid m) \propto \frac{\pi}{2}\left(1 + \frac{m}{4} + \frac{9\,m^2}{64} + \frac{25\,m^3}{256} + \frac{n}{2} + \frac{3\,m\,n}{16} + \frac{15\,m^2\,n}{128} + \frac{3\,n^2}{8} + \frac{5\,m\,n^2}{32} + \frac{5\,n^3}{16} + \ldots\right) /; (m \to 0) \land (n \to 0)$$

08.03.06.0002.01

$$\Pi(n \mid m) == \frac{\pi}{2} \sum_{k=0}^{\infty} \sum_{j=0}^{k} \frac{(2\,k)!\,(2\,j)!\,m^j\,n^{k-j}}{4^k\,4^j\,k!^2\,j!^2} /; |m| < 1 \land |n| < 1$$

08.03.06.0003.01

$$\Pi(n \mid m) == \frac{\pi}{2} F_{1\times 0\times 0}^{1\times 1\times 1}\left(\begin{matrix} \frac{1}{2}; \frac{1}{2}; 1; \\ 1;;; \end{matrix} \; m, n\right)$$

Residue representations

08.03.06.0006.01

$$\Pi(n \mid m) == \frac{1}{2} \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \operatorname{res}_{s,t}\left(\frac{\Gamma\left(\frac{1}{2}-s-t\right)\Gamma(s)\,\Gamma(1-s)\,\Gamma(t)\,\Gamma\left(\frac{1}{2}-t\right)}{\Gamma(1-s-t)} (-n)^{-s} (-m)^{-t}\right)(-j, -k)$$

Other series representations

08.03.06.0007.01

$$\Pi(n \mid m) == \sqrt{\frac{n}{(m-n)(n-1)}}\, K(m) \left(\frac{2\,i\,\pi}{K(m)} \sum_{k=1}^{\infty} \frac{q(m)^k}{1-q(m)^{2k}} \sin\left(\frac{k\,\pi}{K(m)} \operatorname{sn}^{-1}\left(\sqrt{\frac{n}{m}} \,\middle|\, m\right)\right) + \sqrt{\frac{(m-n)(n-1)}{n}}\right) /; -1 \le n \le 1 \land -1 \le m \le 1$$

Integral representations

On the real axis

Of the direct function

08.03.07.0001.01

$$\Pi(n \mid m) = \int_0^{\frac{\pi}{2}} \frac{1}{\left(1 - n \sin^2(t)\right) \sqrt{1 - m \sin^2(t)}} \, dt$$

08.03.07.0002.01

$$\Pi(n \mid m) = \int_0^1 \frac{1}{\left(1 - n t^2\right) \sqrt{1 - t^2} \sqrt{1 - m t^2}} \, dt$$

08.03.07.0005.01

$$\Pi(n \mid m) = \frac{1}{2} \int_0^1 \frac{1}{\sqrt{t} \, (1 - n t) \sqrt{1 - t} \sqrt{1 - m t}} \, dt$$

08.03.07.0003.01

$$\Pi(n \mid m) = \int_0^{K(m)} \frac{1}{1 - n \operatorname{sn}(t \mid m)^2} \, dt$$

Contour integral representations

08.03.07.0004.01

$$\Pi(n \mid m) = \frac{1}{2 (2 \pi i)^2} \int_{\mathcal{L}^*} \int_{\mathcal{L}} \frac{\Gamma\left(\frac{1}{2} - s - t\right) \Gamma(s) \, \Gamma(1 - s) \, \Gamma(t) \, \Gamma\left(\frac{1}{2} - t\right)}{\Gamma(1 - s - t)} (-n)^{-s} (-m)^{-t} \, ds \, dt$$

Differential equations

Ordinary linear differential equations and wronskians

For the direct function itself

With respect to *n*

08.03.13.0002.01

$$2 (n - 1) (m - n) n \frac{\partial^3 w(n)}{\partial n^3} + \left(-13 n^2 + 8 m n + 8 n - 3 m\right) \frac{\partial^2 w(n)}{\partial n^2} + 4 (m - 4 n + 1) \frac{\partial w(n)}{\partial n} - 2 w(n) = 0 \; /; \; w(n) = \Pi(n \mid m)$$

With respect to *m*

08.03.13.0001.01

$$8 (m - 1) m (m - n) \frac{\partial^3 w(m)}{\partial m^3} + 4 \left(11 m^2 - 6 n m - 7 m + 2 n\right) \frac{\partial^2 w(m)}{\partial m^2} + 6 (7 m - n - 2) \frac{\partial w(m)}{\partial m} + 3 w(m) = 0 \; /; \; w(m) = \Pi(n \mid m)$$

Identities

Functional identities

08.03.17.0001.01

$$\Pi(n \mid m) = \frac{1}{(1 - n) \sqrt{1 - m}} \Pi\left(\frac{n}{n - 1} \,\middle|\, \frac{m}{m - 1}\right) /; \; |\arg(1 - m)| < \pi \bigwedge |\arg(1 - n)| < \pi$$

08.03.17.0002.01

$$\Pi(n \mid m) = \frac{\pi}{2\left(\sqrt{1-n}\,\sqrt{1-\frac{m}{n}}\right)}\left(1-\sqrt{\frac{1}{1-n}}\,\sqrt{1-n}\right)+$$
$$\frac{1}{1-n}\sqrt{\frac{1}{1-m}}\,\Pi\!\left(\frac{n}{n-1}\,\middle|\,\frac{m}{m-1}\right)-\left(1-\sqrt{\frac{1}{1-m}}\,\sqrt{1-m}\right)\frac{i\,m}{m-n}\,\Pi\!\left(\frac{n(1-m)}{n-m}\,\middle|\,1-m\right)$$

08.03.17.0003.01

$$\Pi(n \mid m) = K(m) - \Pi\!\left(\frac{m}{n}\,\middle|\,m\right) + \frac{i\,\pi}{2\sqrt{\frac{m}{n}-m+n-1}} \;/;\; (0<n<1 \land 0<m<1) \bigvee (0<n<1 \bigwedge 0<\arg(m)\le\pi)$$

08.03.17.0004.01

$$\Pi(n \mid m) = K(m) - \Pi\!\left(\frac{m}{n}\,\middle|\,m\right) - \frac{i\,\pi}{2\sqrt{\frac{m}{n}-m+n-1}} \;/;\; (0<n<1 \land m>1) \lor (0<n<1 \land \operatorname{Im}(m)<0)$$

Differentiation

Low-order differentiation

With respect to *n*

08.03.20.0001.01

$$\frac{\partial \Pi(n \mid m)}{\partial n} = \frac{1}{2(m-n)(n-1)}\left(E(m)+\frac{m-n}{n}K(m)+\frac{n^2-m}{n}\Pi(n \mid m)\right)$$

08.03.20.0002.01

$$\frac{\partial^2 \Pi(n \mid m)}{\partial n^2} = \frac{(5n-2)n+m(1-4n)}{4(m-n)(n-1)^2 n^2}K(m) - \frac{m(2n+1)+n(2-5n)}{4(m-n)^2(n-1)^2 n}E(m) + \frac{3n^4+2m(2-5n)n+(4n-1)m^2}{4(m-n)^2(n-1)^2 n^2}\Pi(n \mid m)$$

With respect to *m*

08.03.20.0003.01

$$\frac{\partial \Pi(n \mid m)}{\partial m} = \frac{1}{2(n-m)}\left(\frac{E(m)}{m-1}+\Pi(n \mid m)\right)$$

08.03.20.0004.01

$$\frac{\partial^2 \Pi(n \mid m)}{\partial m^2} = \frac{4m^2-(n+2)m-n}{4(m-1)^2 m(m-n)^2}E(m) + \frac{1}{4(m-1)m(m-n)}K(m) + \frac{3}{4(m-n)^2}\Pi(n; z \mid m)$$

Symbolic differentiation

With respect to *n*

08.03.20.0005.02

$$\frac{\partial^p \Pi(n \mid m)}{\partial n^p} = \frac{\pi\left(\frac{1}{2}\right)_p}{2}\sum_{k=0}^{\infty}\frac{n^k\left(p+\frac{1}{2}\right)_k}{k!}\,{}_2F_1\!\left(k+p+\frac{1}{2},\frac{1}{2};k+p+1;m\right) \;/;\; p\in\mathbb{N}$$

08.03.20.0011.01

$$\frac{\partial^p \Pi(n \mid m)}{\partial n^p} == \frac{\pi \left(\frac{1}{2}\right)_p}{2} F_1\left(p + \frac{1}{2}; p+1, \frac{1}{2}; p+1; n, m\right) /; p \in \mathbb{N}$$

With respect to *m*

08.03.20.0006.02

$$\frac{\partial^p \Pi(n \mid m)}{\partial m^p} == \frac{\pi \left(\frac{1}{2}\right)_p^2}{2} \sum_{k=0}^{\infty} \frac{m^k \left(p + \frac{1}{2}\right)_k^2}{k! \, (k+p)!} \, {}_2F_1\left(1, k+p+\frac{1}{2}; k+p+1; n\right) /; p \in \mathbb{N}$$

08.03.20.0012.01

$$\frac{\partial^p \Pi(n \mid m)}{\partial m^p} == \frac{\pi \left(\frac{1}{2}\right)_p^2}{2 \, p!} F_1\left(p + \frac{1}{2}; 1, p + \frac{1}{2}; p+1; n, m\right) /; p \in \mathbb{N}$$

Fractional integro-differentiation

With respect to *n*

08.03.20.0007.01

$$\frac{\partial^\alpha \Pi(n \mid m)}{\partial n^\alpha} == \frac{\pi \, n^{-\alpha}}{2} \tilde{F}_{1\times 0\times 1}^{1\times 1\times 2}\left(\begin{matrix} \frac{1}{2}; \frac{1}{2}; 1, 1; \\ 1;; 1-\alpha; \end{matrix} \; m, n\right)$$

08.03.20.0008.01

$$\frac{\partial^\alpha \Pi(n \mid m)}{\partial n^\alpha} == \frac{\sqrt{\pi} \; n^{-\alpha}}{2} \sum_{k=0}^{\infty} \frac{n^k \, \Gamma\left(k + \frac{1}{2}\right)}{\Gamma(k - \alpha + 1)} \, {}_2F_1\left(k + \frac{1}{2}, \frac{1}{2}; k+1; m\right)$$

With respect to *m*

08.03.20.0009.01

$$\frac{\partial^\alpha \Pi(n \mid m)}{\partial m^\alpha} == \frac{\pi \, m^{-\alpha}}{2} \tilde{F}_{1\times 1\times 0}^{1\times 2\times 1}\left(\begin{matrix} \frac{1}{2}; \frac{1}{2}, 1; 1; \\ 1; 1-\alpha; ; \end{matrix} \; m, n\right)$$

08.03.20.0010.01

$$\frac{\partial^\alpha \Pi(n \mid m)}{\partial m^\alpha} == \frac{m^{-\alpha}}{2} \sum_{k=0}^{\infty} \frac{m^k \, \Gamma\left(k + \frac{1}{2}\right)^2}{\Gamma(k - \alpha + 1) \, k!} \, {}_2F_1\left(1, k + \frac{1}{2}; k+1; n\right)$$

Integration

Indefinite integration

Involving only one direct function with respect to *n*

08.03.21.0001.01

$$\int \Pi(n \mid m) \, dn == \frac{\sqrt{\pi} \; n}{2} \sum_{k=0}^{\infty} \frac{n^k \, \Gamma\left(k + \frac{1}{2}\right)}{(k+1)!} \, {}_2F_1\left(k + \frac{1}{2}, \frac{1}{2}; k+1; m\right)$$

08.03.21.0002.01

$$\int \Pi(n \mid m)\, dn = \frac{\pi n}{2} F_{1\times 1\times 0}^{1\times 2\times 1}\left(\begin{matrix} \frac{1}{2}; \frac{1}{2}, 1; 1; \\ 1;; 2; \end{matrix} \, m, n \right)$$

Involving only one direct function with respect to *m*

08.03.21.0003.01

$$\int \Pi(n \mid m)\, dm = 2\,(E(m) - K(m) + (m - n)\,\Pi(n \mid m))$$

Involving one direct function and elementary functions with respect to *m*

Involving power function

08.03.21.0004.01

$$\int m\, \Pi\left(n \mid m^2\right) dm = E\left(m^2\right) - K\left(m^2\right) + \left(m^2 - n\right) \Pi\left(n \mid m^2\right)$$

Representations through more general functions

Through hypergeometric functions of two variables

08.03.26.0006.01

$$\Pi(n \mid m) = \frac{\pi}{2} F_1\left(\frac{1}{2}; 1, \frac{1}{2}; 1; n, m\right)$$

08.03.26.0001.01

$$\Pi(n \mid m) = \frac{\pi}{2} F_{1\times 0\times 0}^{1\times 1\times 1}\left(\begin{matrix} \frac{1}{2}; 1; \frac{1}{2}; \\ 1;;; \end{matrix} \, n, m \right)$$

Through Meijer *G*

Classical cases

08.03.26.0002.01

$$\Pi\left(1 - \frac{1}{\sqrt{z}} \,\middle|\, 1 - \frac{1}{z}\right) = \frac{\sqrt{z} + 1}{2} K\left(1 - \frac{1}{z}\right) - \frac{1}{4\pi}\left(K\left(1 - \frac{1}{z}\right) G_{2,2}^{2,2}\left(z \,\middle|\, \begin{matrix} \frac{1}{2}, \frac{3}{2} \\ 0, 1 \end{matrix}\right) + E\left(1 - \frac{1}{z}\right) G_{2,2}^{2,2}\left(z \,\middle|\, \begin{matrix} \frac{3}{2}, \frac{3}{2} \\ 1, 1 \end{matrix}\right)\right) /;\, z \notin (-\infty, -1)$$

Through other functions

Involving incomplete elliptic integrals

08.03.26.0003.01

$$\Pi(n \mid m) = \Pi\left(n; \frac{\pi}{2} \,\middle|\, m\right)$$

08.03.26.0004.01

$$\Pi(n \mid m) = \frac{1}{\sqrt{m}} \Pi\left(\frac{n}{m}; \sin^{-1}\left(\sqrt{m}\right) \,\middle|\, \frac{1}{m}\right)$$

Involving some hypergeometric-type functions

08.03.26.0005.01

$$\Pi(n \mid m) = \frac{\pi}{2} F_1\left(\frac{1}{2}; \frac{1}{2}, 1; 1; m, n\right)$$

Representations through equivalent functions

With related functions

08.03.27.0001.01

$$\Pi(n \mid m) = K(m) - \frac{\tan(\phi)}{\sqrt{1-n}} \left(E(m) F(\phi \mid m) - K(m) E(\phi \mid m)\right) /; \phi = \sin^{-1}\left(\sqrt{\frac{n}{m}}\right) \bigwedge 0 < n < 1 \bigwedge 0 < m < 1$$

History

– A. M. Legendre (1811)

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