EllipticNomeQ

Notations

Traditional name

Nome

Traditional notation

$q(m)$

Mathematica StandardForm notation

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EllipticNomeQ[m]
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Primary definition

09.53.02.0001.01

$$q(m) \equiv \exp\left(-\frac{\pi K(1-m)}{K(m)}\right)$$

Specific values

Values at fixed points

09.53.03.0001.01

$$q(0) \equiv 0$$

09.53.03.0002.01

$$q\left(\frac{1}{2}\right) \equiv e^{-\pi}$$

09.53.03.0003.01

$$q(1) \equiv 1$$

General characteristics

Domain and analyticity

$q(m)$ is an analytical function of m which is defined over the whole complex m-plane.

09.53.04.0001.01

$$m \longrightarrow q(m) :: \mathbb{C} \longrightarrow \mathbb{C}$$

Symmetries and periodicities

Mirror symmetry

09.53.04.0002.01

$q(\overline{m}) = \overline{q(m)} \;/;\; m \notin (1, \infty)$

Periodicity

No periodicity

Poles and essential singularities

The function $q(m)$ does not have poles and essential singularities.

09.53.04.0003.01

$\mathcal{Sing}_m(q(m)) = \{\}$

Branch points

The function $q(m)$ has two branch points: $m = 1$, $m = \tilde{\infty}$.

09.53.04.0004.01

$\mathcal{BP}_m(q(m)) = \{1, \tilde{\infty}\}$

09.53.04.0005.01

$\mathcal{R}_m(q(m), 1) = \log$

09.53.04.0006.01

$\mathcal{R}_m(q(m), \tilde{\infty}) = \log$

Branch cuts

The function $q(m)$ is a single-valued function on the m-plane cut along the interval $(1, \infty)$ where it is continuous from below.

09.53.04.0007.01

$\mathcal{BC}_m(q(m)) = \{\{(1, \infty), i\}\}$

09.53.04.0008.01

$$\lim_{\epsilon \to +0} q(x - i\,\epsilon) = q(x) \;/;\; x > 1$$

09.53.04.0009.01

$$\lim_{\epsilon \to +0} q(x + i\,\epsilon) = \exp\left(-\frac{\pi\, K(1-x)}{2\, i\, K(1-x) + K(x)}\right) \;/;\; x > 1$$

Series representations

Generalized power series

Expansions at generic point $m = m_0$

09.53.06.0008.01

$$q(m) \propto q(m_0) - \frac{\pi^2 q(m_0)}{4 K(m_0)^2 (m_0-1) m_0} (m-m_0) + \frac{\pi^2 q(m_0) \left(4 m_0 K(m_0)^2 - 4 E(m_0) K(m_0) + \pi^2\right)}{32 K(m_0)^4 (m_0-1)^2 m_0^2} (m-m_0)^2 - \frac{\pi^2 q(m_0)}{384 K(m_0)^6 (m_0-1)^3 m_0^3} \left(4 K(m_0) \left(\left(8 m_0^2 - 2 m_0 + 2\right) K(m_0)^3 + 6 E(m_0)^2 K(m_0) + 3 \pi^2 m_0 K(m_0) - 3 E(m_0) \left(4 m_0 K(m_0)^2 + \pi^2\right)\right) + \pi^4\right) (m-m_0)^3 + \frac{\pi^2 q(m_0)}{6144 K(m_0)^8 (m_0-1)^4 m_0^4} \left(8 K(m_0) \left(4 (4 m_0 - 1)(m_0 (3 m_0 - 1) + 2) K(m_0)^5 + 2 \pi^2 (m_0 (11 m_0 - 2) + 2) K(m_0)^3 - 24 E(m_0)^3 K(m_0)^2 + 3 \pi^4 m_0 K(m_0) + 18 E(m_0)^2 \left(4 m_0 K(m_0)^2 + \pi^2\right) K(m_0) + E(m_0) \left(-8 (m_0 (11 m_0 - 2) + 2) K(m_0)^4 - 36 \pi^2 m_0 K(m_0)^2 - 3 \pi^4\right)\right) + \pi^6\right) (m-m_0)^4 - \frac{\pi^2 q(m_0)}{122880 K(m_0)^{10} (m_0-1)^5 m_0^5} \left(8 K(m_0) \left(32 (m_0 (m_0 (m_0 (24 m_0 - 23) + 37) - 18) + 4) K(m_0)^7 + 20 \pi^2 (5 m_0 - 1)(m_0 (4 m_0 - 1) + 2) K(m_0)^5 + 240 E(m_0)^4 K(m_0)^3 + 10 \pi^4 (m_0 (7 m_0 - 1) + 1) K(m_0)^3 - 240 E(m_0)^3 \left(4 m_0 K(m_0)^2 + \pi^2\right) K(m_0)^2 + 5 \pi^6 m_0 K(m_0) + 60 E(m_0)^2 \left(4 (m_0 (7 m_0 - 1) + 1) K(m_0)^4 + 12 \pi^2 m_0 K(m_0)^2 + \pi^4\right) K(m_0) + 5 E(m_0) \left(-16 (5 m_0 - 1)(m_0 (4 m_0 - 1) + 2) K(m_0)^6 - 24 \pi^2 (m_0 (7 m_0 - 1) + 1) K(m_0)^4 - 24 \pi^4 m_0 K(m_0)^2 - \pi^6\right)\right) + \pi^8\right) (m-m_0)^5 + \ldots \;/;\; (m \to m_0)$$

09.53.06.0009.01

$$q(m) = \sum_{k=0}^{\infty} \frac{1}{k!} q(m_0) \sum_{p=0}^{k} \frac{\pi^p}{p!} \left(\sum_{j=0}^{p} (-1)^{j+p} \binom{p}{j} \left(\frac{K(1-m_0)}{K(m_0)}\right)^j \text{Function}\left[u, \left(\frac{K(1-u)}{K(u)}\right)^{p-j}\right]^{(k)} (m_0) \right) (m-m_0)^k$$

09.53.06.0010.01

$$q(m) \propto q(m_0) (1 + O(m-m_0))$$

Expansions at $m == 0$

09.53.06.0001.01

$$q(m) \propto \frac{m}{16} + \frac{m^2}{32} + \frac{21 m^3}{1024} + \frac{31 m^4}{2048} + \frac{6257 m^5}{524288} + \frac{10293 m^6}{1048576} + \frac{279025 m^7}{33554432} + \frac{483127 m^8}{67108864} + \frac{435506703 m^9}{68719476736} + \frac{776957575 m^{10}}{137438953472} + \frac{22417045555 m^{11}}{4398046511104} + \frac{40784671953 m^{12}}{8796093022208} + \frac{9569130097211 m^{13}}{2251799813685248} + \frac{17652604545791 m^{14}}{4503599627370496} + \frac{523910972020563 m^{15}}{144115188075855872} + \frac{976501268709949 m^{16}}{288230376151711744} + \frac{935823746406530603 m^{17}}{295147905179352825856} + \frac{1758220447807291611 m^{18}}{590295810358705651712} + \frac{53030538453624441751 m^{19}}{18889465931478580854784} + \frac{100268465197007602645 m^{20}}{37778931862957161709568} + O\left(m^{21}\right) \;/;\; (m \to 0)$$

09.53.06.0011.01

$$q(m) = \exp\left(\frac{\log(m) \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k^2 m^k}{(k!)^2} - 2 \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k^2 \left(\psi(k+1) - \psi\left(k+\frac{1}{2}\right)\right) m^k}{(k!)^2}}{\sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k \left(\frac{1}{2}\right)_k m^k}{(k!)^2}} \right)$$

09.53.06.0002.01

$$q(m) \propto \frac{m}{16} (1 + O(m)) \;/;\; (m \to 0)$$

Expansions at $m == 1$

09.53.06.0003.01

$$q(m) == e^{\frac{\pi^2}{\log\left(\frac{1-m}{16}\right)}} \left(1 + \frac{\pi^2 (m-1)}{2 \log^2\left(\frac{1-m}{16}\right)} - \frac{13 \pi^2 (m-1)^2}{64 \log^2\left(\frac{1-m}{16}\right)} + \frac{\pi^2 (m-1)^2}{4 \log^3\left(\frac{1-m}{16}\right)} + \frac{23 \pi^2 (m-1)^3}{192 \log^2\left(\frac{1-m}{16}\right)} - \frac{13 \pi^2 (m-1)^3}{64 \log^3\left(\frac{1-m}{16}\right)} + \ldots\right)$$

09.53.06.0004.01

$$q(m) == e^{\pi^2 w} \sum_{k=0}^{\infty} \frac{\pi^{2k} w^k}{k!} \left(\frac{A}{B} - 1\right)^k /;$$

$$(m \to 1) \bigwedge A == \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k \left(\frac{1}{2}\right)_k (1-m)^k}{k!^2} \bigwedge B == \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k^2 (1-m)^k}{k!^2} \left(1 + 2 w \left(\log(4) - \psi(k+1) + \psi\left(k + \frac{1}{2}\right)\right)\right) \bigwedge w == \frac{1}{\log\left(\frac{1-m}{16}\right)}$$

09.53.06.0005.01

$$q(m) \propto \exp\left(\frac{\pi^2}{\log\left(\frac{1-m}{16}\right)}\right) \left(1 + O\left(\frac{m-1}{\log^2\left(\frac{1-m}{16}\right)}\right)\right) /; (m \to 1)$$

Expansions at $m == \infty$

09.53.06.0006.01

$$q(m) \propto \exp\left(-\frac{\sqrt{-m}\, \pi \log(16\, m)}{\sqrt{m}\, \log(-16\, m)}\right) \left(1 + \frac{\sqrt{-m^2}\, \pi (\log(-m) - \log(m))}{2 \log^2(-16\, m)\, m^2} + \frac{\pi \sqrt{-m^2}\, (\log(-m) - \log(m))}{64 \log^4(-16\, m)\, m^{7/2}} \right.$$

$$\left. \left(13 \sqrt{m}\, \log^2(-16\, m) + 8 \left(\pi \sqrt{-m} + 2 \sqrt{m}\right) \log(-16\, m) - 8 \sqrt{-m}\, \pi \log(16\, m)\right) + \ldots\right) /; (|m| \to \infty)$$

09.53.06.0007.01

$$q(m) \propto (16\, m)^{\frac{\pi \sqrt{m}}{\sqrt{-m}\, \log(-16\, m)}} \left(1 + O\left(\frac{\sqrt{-m^2}\, (\log(-m) - \log(m))}{\log^2(-16\, m)\, m^2}\right)\right) /; (|m| \to \infty)$$

Differential equations

Ordinary nonlinear differential equations

09.53.13.0001.01

$$(m-1)^2 m^2 w'(m)^4 + w(m)^2 \left(-3 (m-1)^2 w''(m)^2 m^2 + 2 (m-1)^2 w'(m)\, w^{(3)}(m)\, m^2 - \left(m^2 - m + 1\right) w'(m)^2\right) == 0 /; w(m) == q(m)$$

Differentiation

Low-order differentiation

09.53.20.0001.02

$$\frac{\partial q(m)}{\partial m} == -\frac{\pi^2 q(m)}{4 (m-1)\, m\, K(m)^2}$$

09.53.20.0002.02

$$\frac{\partial^2 q(m)}{\partial m^2} = \frac{\pi^2 \left(4\, m\, K(m)^2 - 4\, E(m)\, K(m) + \pi^2\right) q(m)}{16\, (m-1)^2\, m^2\, K(m)^4}$$

09.53.20.0004.01

$$\frac{\partial^3 q(m)}{\partial m^3} = -\frac{1}{64\, (m-1)^3\, m^3\, K(m)^6} \left(\pi^2 \left(4\, K(m) \left(\left(8\, m^2 - 2\, m + 2\right) K(m)^3 + 3\, m\, \pi^2\, K(m) + 6\, E(m)^2\, K(m) - 3\, E(m) \left(4\, m\, K(m)^2 + \pi^2\right)\right) + \pi^4\right) q(m)\right)$$

09.53.20.0005.01

$$\frac{\partial^4 q(m)}{\partial m^4} = \frac{1}{256\, (m-1)^4\, m^4\, K(m)^8} \left(\pi^2 \left(8\, K(m) \left(4\, (4\, m-1)\, (m\, (3\, m-1)+2)\, K(m)^5 + 2\, (m\, (11\, m-2)+2)\, \pi^2\, K(m)^3 - 24\, E(m)^3\, K(m)^2 + 3\, m\, \pi^4\, K(m) + 18\, E(m)^2 \left(4\, m\, K(m)^2 + \pi^2\right) K(m) + E(m) \left(-8\, (m\, (11\, m-2)+2)\, K(m)^4 - 36\, m\, \pi^2\, K(m)^2 - 3\, \pi^4\right)\right) + \pi^6\right) q(m)\right)$$

09.53.20.0006.01

$$\frac{\partial^5 q(m)}{\partial m^5} = -\frac{1}{1024\, (m-1)^5\, m^5\, K(m)^{10}} \left(\pi^2 \left(8\, K(m) \left(32\, (m\, (m\, (m\, (24\, m-23)+37)-18)+4)\, K(m)^7 + 20\, (5\, m-1)\, (m\, (4\, m-1)+2)\, \pi^2\, K(m)^5 + 10\, (m\, (7\, m-1)+1)\, \pi^4\, K(m)^3 + 240\, E(m)^4\, K(m)^3 - 240\, E(m)^3 \left(4\, m\, K(m)^2 + \pi^2\right) K(m)^2 + 5\, m\, \pi^6\, K(m) + 60\, E(m)^2 \left(4\, (m\, (7\, m-1)+1)\, K(m)^4 + 12\, m\, \pi^2\, K(m)^2 + \pi^4\right) K(m) + 5\, E(m) \left(-16\, (5\, m-1)\, (m\, (4\, m-1)+2)\, K(m)^6 - 24\, (m\, (7\, m-1)+1)\, \pi^2\, K(m)^4 - 24\, m\, \pi^4\, K(m)^2 - \pi^6\right)\right) + \pi^8\right) q(m)\right)$$

Symbolic differentiation

09.53.20.0003.02

$$\frac{\partial^n q(m)}{\partial m^n} = q(m) \sum_{k=0}^{n} \frac{\pi^k}{k!} \sum_{j=0}^{k} (-1)^{j+k} \binom{k}{j} \left(\frac{K(1-m)}{K(m)}\right)^j \frac{\partial^n \left(\frac{K(1-m)}{K(m)}\right)^{k-j}}{\partial m^n} \;/;\; n \in \mathbb{N}$$

Summation

Infinite summation

09.53.23.0001.01

$$\sum_{k=1}^{\infty} \frac{q(m)^k}{q(m)^{2k}+1} = \frac{K(m)}{2\pi} - \frac{1}{4}$$

09.53.23.0002.01

$$\sum_{k=1}^{\infty} \frac{k^{2n}\, q(m)^k}{q(m)^{2k}+1} = \frac{\left((-1)^n\, K(m)^{2n+1}\right) \mathrm{JacobiAmplitude}^{(2n+1,0)}(0,\, m)}{2\, \pi^{2n+1}} \;/;\; n \in \mathbb{N}^+$$

Operations

Limit operation

09.53.25.0001.01

$$\lim_{m\to 0} \frac{1}{m^4}\left(q(m) - \left(\frac{31\, m^4}{2048} + \frac{21\, m^3}{1024} + \frac{m^2}{32} + \frac{m}{16}\right)\right) == 0$$

Representations through more general functions

Through other functions

Involving some hypergeometric-type functions

09.53.26.0001.01

$$q(m) == \exp\left(-\frac{\pi\, K(1-m)}{K(m)}\right)$$

Representations through equivalent functions

With inverse function

09.53.27.0001.01

$$q^{-1}(q(m)) == m$$

The function $q(m)$ is the inverse function of the (more basic) function $q^{-1}(m)$. While $q^{-1}(q(m)) == m$ holds for all m, the relation $q\left(q^{-1}(m)\right) == m$ holds only for m from a restricted domain near the negative real axis.

History

– C. G. J. Jacobi (1828) for real arguments
– K. Weierstrass (1883) for complex arguments

Applications include q -series for elliptic functions.

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