ChebyshevU

Notations

Traditional name

Chebyshev polynomial of the second kind

Traditional notation

$U_n(z)$

Mathematica StandardForm notation

ChebyshevU[n, z]

Primary definition

05.05.02.0001.01

$$U_n(z) = \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{(-1)^k\,(n-k)!\,(2\,z)^{n-2\,k}}{k!\,(n-2\,k)!} \;/;\; n \in \mathbb{N}$$

05.05.02.0002.01

$$U_n(z) = -U_{-n-2}(z) \;/;\; n \in \mathbb{Z} \land n < 0$$

Specific values

Specialized values

For fixed n

05.05.03.0001.01

$$U_n(0) = \cos\left(\frac{\pi\,n}{2}\right)$$

05.05.03.0002.01

$$U_n(1) = 1 + n$$

05.05.03.0003.01

$$U_n(-1) = (-1)^n\,(1+n)$$

05.05.03.0016.01

$$U_n\left(-\frac{i}{2}\right) = (-i)^n\,F_{n+1}$$

For fixed z

05.05.03.0004.01

$U_0(z) == 1$

05.05.03.0005.01

$U_1(z) == 2\, z$

05.05.03.0006.01

$U_2(z) == 4\, z^2 - 1$

05.05.03.0007.01

$U_3(z) == 8\, z^3 - 4\, z$

05.05.03.0008.01

$U_4(z) == 16\, z^4 - 12\, z^2 + 1$

05.05.03.0009.01

$U_5(z) == 32\, z^5 - 32\, z^3 + 6\, z$

05.05.03.0010.01

$U_6(z) == 64\, z^6 - 80\, z^4 + 24\, z^2 - 1$

05.05.03.0011.01

$U_7(z) == 128\, z^7 - 192\, z^5 + 80\, z^3 - 8\, z$

05.05.03.0012.01

$U_8(z) == 256\, z^8 - 448\, z^6 + 240\, z^4 - 40\, z^2 + 1$

05.05.03.0013.01

$U_9(z) == 512\, z^9 - 1024\, z^7 + 672\, z^5 - 160\, z^3 + 10\, z$

05.05.03.0014.01

$U_{10}(z) == 1024\, z^{10} - 2304\, z^8 + 1792\, z^6 - 560\, z^4 + 60\, z^2 - 1$

Values at infinities

05.05.03.0017.01

$U_n(\infty) = \infty \;/;\; n > 0$

05.05.03.0018.01

$U_n(-\infty) = (-1)^n\, \infty \;/;\; n > 0$

General characteristics

Domain and analyticity

The function $U_n(z)$is defined over $\mathbb{N} \otimes \mathbb{C}$. For fixed n, the function $U_n(z)$ is a polynomial in z of degree n.

05.05.04.0001.01

$(n * z) \longrightarrow U_n(z) :: (\mathbb{N} \otimes \mathbb{C}) \longrightarrow \mathbb{C}$

Symmetries and periodicities

Parity

05.05.04.0002.01

$U_n(-z) == (-1)^n\, U_n(z)$

Mirror symmetry

05.05.04.0003.01

$U_n(\bar{z}) == \overline{U_n(z)}$

Periodicity

No periodicity

Poles and essential singularities

With respect to *z*

The function $U_n(z)$ is polynomial and has pole of order n at $z = \tilde{\infty}$.

05.05.04.0004.01

$\mathcal{Sing}_z(U_n(z)) == \{\{\tilde{\infty}, n\}\}$

Branch points

With respect to *z*

The function $U_n(z)$ does not have branch points.

05.05.04.0005.01

$\mathcal{BP}_z(U_n(z)) == \{\}$

Branch cuts

With respect to *z*

The function $U_n(z)$ does not have branch cuts.

05.05.04.0006.01

$\mathcal{BC}_z(U_n(z)) == \{\}$

Series representations

Generalized power series

Expansions at generic point $z == z_0$

For the function itself

05.05.06.0018.01

$$U_n(z) \propto U_n(z_0) + \frac{(n+1)\,T_{n+1}(z_0) - z_0\,U_n(z_0)}{z_0^2 - 1}(z - z_0) + \frac{1}{2\left(z_0^2-1\right)^2}\left(\left(\left(z_0^2-1\right)n^2 + 2\left(z_0^2-1\right)n + 3\,z_0^2\right)U_n(z_0) - 3\,(n+1)\,z_0\,T_{n+1}(z_0)\right)(z-z_0)^2 + \ldots\ /;\ (z \to z_0)$$

05.05.06.0019.01

$$U_n(z) \propto U_n(z_0) + \frac{(n+1)\, T_{n+1}(z_0) - z_0\, U_n(z_0)}{z_0^2 - 1} (z - z_0) + \frac{\left(\left(z_0^2 - 1\right) n^2 + 2\left(z_0^2 - 1\right) n + 3 z_0^2\right) U_n(z_0) - 3\,(n+1)\, z_0\, T_{n+1}(z_0)}{2\left(z_0^2 - 1\right)^2} (z - z_0)^2 + O\left((z - z_0)^3\right)$$

05.05.06.0020.01

$$U_n(z) = \sum_{k=0}^{n} 2^k\, C_{n-k}^{k+1}(z_0)\, (z - z_0)^k$$

05.05.06.0021.01

$$U_n(z) = \frac{\sqrt{\pi}\, (n+1)}{2} \sum_{k=0}^{n} \frac{(z_0 - 1)^{-k}}{k!}\, {}_3\tilde{F}_2\left(1, -n, n+2; \frac{3}{2}, 1-k; \frac{1-z_0}{2}\right) (z - z_0)^k$$

05.05.06.0022.01

$$U_n(z) = \sum_{k=0}^{n} 2^k \left(\sum_{i_1=0}^{n-k} \cdots \sum_{i_{k+1}=0}^{n-k} \delta_{\sum_{j=1}^{k+1} i_j, n-k} \prod_{j=1}^{k+1} U_{i_j}(z_0) \right) (z - z_0)^k$$

05.05.06.0023.01

$$U_n(z) \propto U_n(z_0)\, (1 + O(z - z_0))$$

Expansions at $z = 0$

For the function itself

05.05.06.0001.02

$$U_n(z) \propto \cos\left(\frac{\pi n}{2}\right) + (n+1) \sin\left(\frac{\pi n}{2}\right) z - n\,(n+2) \cos\left(\frac{\pi n}{2}\right) z^2 + \ldots /; (z \to 0)$$

05.05.06.0024.01

$$U_n(z) \propto \cos\left(\frac{\pi n}{2}\right) + (n+1) \sin\left(\frac{\pi n}{2}\right) z - n\,(n+2) \cos\left(\frac{\pi n}{2}\right) z^2 + O\left(z^3\right)$$

05.05.06.0002.01

$$U_n(z) = (n+1)\, F_{1\times 0\times 0}^{2\times 0\times 0}\left(\begin{matrix} -n, n+2;;; \\ \frac{3}{2};;; \end{matrix} \; \frac{1}{2}, -\frac{z}{2} \right)$$

05.05.06.0025.01

$$U_n(z) = \cos\left(\frac{\pi n}{2}\right) \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{\left(-\frac{n}{2}\right)_k \left(\frac{n}{2} + 1\right)_k}{\left(\frac{1}{2}\right)_k k!} z^{2k} + (n+1) \sin\left(\frac{\pi n}{2}\right) z \sum_{k=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \frac{\left(\frac{1-n}{2}\right)_k \left(\frac{n+3}{2}\right)_k}{\left(\frac{3}{2}\right)_k k!} z^{2k}$$

05.05.06.0026.01

$$U_n(z) = \cos\left(\frac{\pi n}{2}\right) {}_2F_1\left(-\frac{n}{2}, \frac{n}{2} + 1; \frac{1}{2}; z^2\right) + (n+1)\, \sin\left(\frac{\pi n}{2}\right) z\, {}_2F_1\left(\frac{1-n}{2}, \frac{n+3}{2}; \frac{3}{2}; z^2\right)$$

05.05.06.0027.01

$$U_n(z) = \frac{1}{\sqrt{1 - z^2}} \cos\left(\frac{\pi n}{2} - (n+1) \sin^{-1}(z)\right)$$

05.05.06.0028.01

$$U_n(z) = \cos\left(n\cos^{-1}(z)\right) + \frac{z}{\sqrt{1-z^2}} \sin\left(n\cos^{-1}(z)\right)$$

05.05.06.0029.01

$$U_n(z) = \frac{1}{\sqrt{1-z^2}} \left(\cos\left(\frac{\pi n}{2}\right) \cos\left((n+1)\sin^{-1}(z)\right) + \sin\left(\frac{\pi n}{2}\right) \sin\left((n+1)\sin^{-1}(z)\right)\right)$$

05.05.06.0030.01

$$U_n(z) = \cos\left(\frac{n}{2}\left(\pi - 2\sin^{-1}(z)\right)\right) + \frac{z}{\sqrt{1-z^2}} \sin\left(\frac{n}{2}\left(\pi - 2\sin^{-1}(z)\right)\right)$$

05.05.06.0003.02

$$U_n(z) \propto \cos\left(\frac{\pi n}{2}\right)(1 + O(z))$$

05.05.06.0004.01

$$U_n(z) = \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{(-1)^k (n-k)!\,(2z)^{n-2k}}{k!\,(n-2k)!}$$

05.05.06.0005.02

$$U_n(z) \propto (-1)^{\left\lfloor \frac{n}{2} \right\rfloor} ((n+1)z)^{n-2\left\lfloor \frac{n}{2} \right\rfloor} \left(1 + O\left(z^2\right)\right) /; n > 0$$

Expansions at $z = 1$

For the function itself

05.05.06.0006.02

$$U_n(z) \propto (n+1)\left(1 + \frac{n(2+n)}{3}(z-1) + \frac{-n(1-n)(2+n)(3+n)}{30}(z-1)^2 + \ldots\right) /; (z \to 1)$$

05.05.06.0031.01

$$U_n(z) \propto (n+1)\left(1 + \frac{n(2+n)}{3}(z-1) + \frac{-n(1-n)(2+n)(3+n)}{30}(z-1)^2 + O\left((z-1)^3\right)\right)$$

05.05.06.0007.01

$$U_n(z) = (n+1)\sum_{k=0}^{n} \frac{(-n)_k (n+2)_k}{\left(\frac{3}{2}\right)_k k!} \left(\frac{1-z}{2}\right)^k$$

05.05.06.0008.01

$$U_n(z) = (n+1)\,{}_2F_1\left(-n, n+2; \frac{3}{2}; \frac{1-z}{2}\right)$$

05.05.06.0032.01

$$U_n(z) = \frac{1}{\sqrt{1-z^2}} \sin\left(2(n+1)\sin^{-1}\left(\frac{\sqrt{1-z}}{\sqrt{2}}\right)\right)$$

05.05.06.0033.01

$$U_n(z) = \frac{1}{2(z-1)}\left({}_2F_1\left(-n-1, n+2; \frac{1}{2}; \frac{1-z}{2}\right) - {}_2F_1\left(-n, n+1; \frac{1}{2}; \frac{1-z}{2}\right)\right)$$

05.05.06.0034.01

$$U_n(z) = \frac{\sqrt{\pi}}{2\,(1-z)^{3/4}\,\sqrt[4]{z+1}} \left(P_n^{\frac{1}{2}}(z) - P_{n+1}^{\frac{1}{2}}(z)\right)$$

05.05.06.0035.01

$$U_n(z) = \frac{1}{\sqrt{2}\,(1-z)\,\sqrt{z+1}} \left({}_2F_1\left(\frac{1+2n}{4}, -\frac{1+2n}{4}; \frac{1}{2}; 1-z^2\right) - {}_2F_1\left(\frac{3+2n}{4}, -\frac{3+2n}{4}; \frac{1}{2}; 1-z^2\right)\right)$$

05.05.06.0036.01

$$U_n(z) = \frac{1}{\sqrt{2}\,(1-z)\,\sqrt{z+1}} \left(\cos\left(\frac{1}{2}\,(2n+1)\sin^{-1}\left(\sqrt{1-z^2}\right)\right) - \cos\left(\frac{1}{2}\,(2n+3)\sin^{-1}\left(\sqrt{1-z^2}\right)\right)\right)$$

05.05.06.0009.02

$$U_n(z) \propto (n+1)\,(1+O(z-1))$$

Expansions at $z == -1$

For the function itself

05.05.06.0010.02

$$U_n(z) \propto (-1)^n\,(n+1)\left(1 - \frac{n\,(2+n)}{3}\,(z+1) - \frac{n(1-n)\,(2+n)\,(3+n)}{30}\,(z+1)^2 - \ldots\right) /; (z \to -1)$$

05.05.06.0037.01

$$U_n(z) \propto (-1)^n\,(n+1)\left(1 - \frac{n\,(2+n)}{3}\,(z+1) - \frac{n(1-n)\,(2+n)\,(3+n)}{30}\,(z+1)^2 + O\left((z+1)^3\right)\right)$$

05.05.06.0011.01

$$U_n(z) = (-1)^n\,(1+n) \sum_{k=0}^{n} \frac{(-n)_k\,(n+2)_k}{\left(\frac{3}{2}\right)_k\,k!} \left(\frac{z+1}{2}\right)^k$$

05.05.06.0012.01

$$U_n(z) = (-1)^n\,(1+n)\,{}_2F_1\left(-n, n+2; \frac{3}{2}; \frac{z+1}{2}\right)$$

05.05.06.0038.01

$$U_n(z) = \frac{(-1)^n}{\sqrt{1-z^2}} \sin\left(2\,(n+1)\sin^{-1}\left(\frac{\sqrt{z+1}}{\sqrt{2}}\right)\right)$$

05.05.06.0039.01

$$U_n(z) = \cos\left(n\left(\pi - 2\sin^{-1}\left(\frac{\sqrt{z+1}}{\sqrt{2}}\right)\right)\right) + \frac{z}{\sqrt{1-z^2}} \sin\left(n\left(\pi - 2\sin^{-1}\left(\frac{\sqrt{z+1}}{\sqrt{2}}\right)\right)\right)$$

05.05.06.0040.01

$$U_n(z) = (-1)^n\left(\cos\left(2\,n\sin^{-1}\left(\frac{\sqrt{z+1}}{\sqrt{2}}\right)\right) - \frac{z}{\sqrt{1-z^2}} \sin\left(2\,n\sin^{-1}\left(\frac{\sqrt{z+1}}{\sqrt{2}}\right)\right)\right)$$

05.05.06.0013.02

$$U_n(z) \propto (-1)^n\,(1+n)\,(1+O(z+1))$$

Expansions at $z == \infty$

For the function itself

Expansions in $1/z$

05.05.06.0041.01

$$U_n(z) \propto 2^n z^n \left(1 + \frac{1-n}{4 z^2} + \frac{(2-n)(3-n)}{32 z^4} + \ldots\right) /; n > 0$$

05.05.06.0042.01

$$U_n(z) \propto 2^n z^n \left(1 + \frac{1-n}{4 z^2} + \frac{(2-n)(3-n)}{32 z^4} + O\left(\frac{1}{z^6}\right)\right) /; n > 0$$

05.05.06.0043.01

$$U_n(z) = 2^n z^n \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{\left(\frac{1-n}{2}\right)_k \left(-\frac{n}{2}\right)_k}{k!\,(-n)_k} z^{-2k}$$

05.05.06.0044.01

$$U_n(z) = 2^n z^n\, {}_2F_1\left(-\frac{n}{2}, \frac{1-n}{2}; -n; \frac{1}{z^2}\right) /; n > 0$$

05.05.06.0045.01

$$U_n(z) = \frac{1}{2\sqrt{z^2-1}} \left(\left(z + \sqrt{z^2-1}\right)^{n+1} - \left(z + \sqrt{z^2-1}\right)^{-n-1}\right)$$

Expansions in $1/(1-z)$

05.05.06.0014.01

$$U_n(z) \propto 2^n (z-1)^n \left(1 + \frac{n}{z-1} + \frac{(n-1)(2n-1)}{4(z-1)^2} + \ldots\right) /; (|z| \to \infty) \land n > 0$$

05.05.06.0046.01

$$U_n(z) \propto 2^n (z-1)^n \left(1 + \frac{n}{z-1} + \frac{(n-1)(2n-1)}{4(z-1)^2} + O\left(\frac{1}{n^3}\right)\right) /; n > 0$$

05.05.06.0015.01

$$U_n(z) = 2^n (z-1)^n \sum_{k=0}^{n} \frac{(-n)_k \left(-n-\frac{1}{2}\right)_k}{k!\,(-2n-1)_k} \left(\frac{2}{1-z}\right)^k$$

05.05.06.0047.01

$$T_n(z) = 2^{n-1} (z-1)^n\, {}_2F_1\left(-n, \frac{1}{2}-n; 1-2n; \frac{2}{1-z}\right) /; n \in \mathbb{N}$$

05.05.06.0048.01

$$U_n(z) = \frac{2^{-n-2}\sqrt{1-z}\,(z-1)^n}{\sqrt{-z-1}} \left(\sqrt{\frac{z+1}{z-1}} + 1\right)^{2n+2} - \frac{2^n \sqrt{1-z}\,(z-1)^{-n-2}}{\sqrt{-z-1}} \left(\sqrt{\frac{z+1}{z-1}} + 1\right)^{-2n-2}$$

05.05.06.0016.02

$$U_n(z) \propto 2^n z^n \left(1 + O\left(\frac{1}{z}\right)\right) /;\ n > 0$$

Expansions at $n == \infty$

05.05.06.0049.01

$$U_n(z) = \frac{\sin\left((n+1)\cos^{-1}(z)\right)}{\sqrt{1-z^2}}$$

05.05.06.0050.01

$$U_n(z) \propto \begin{cases} -\frac{i\, e^{i(n+1)\cos^{-1}(z)}}{2\sqrt{1-z^2}} & -\pi < \arg\left(\cos^{-1}(z)\right) < 0 \\ \frac{i\, e^{-i(n+1)\cos^{-1}(z)}}{2\sqrt{1-z^2}} & 0 < \arg\left(\cos^{-1}(z)\right) < \pi \\ \frac{\sin\left((n+1)\cos^{-1}(z)\right)}{\sqrt{1-z^2}} & \text{True} \end{cases} \;/;\; (n \to \infty)$$

Other series representations

05.05.06.0017.01

$$U_n(z) == \sum_{k=0}^{\left\lfloor \frac{n+1}{2} \right\rfloor} \binom{n+1}{2k} z^{n-2k} \left(z^2-1\right)^{k-1} \left(z^2 + \frac{2k}{n+1} - 1\right)$$

Integral representations

On the real axis

Of the direct function

05.05.07.0001.01

$$U_n(z) == \frac{n+1}{2} \int_0^\pi \left(z + \sqrt{z^2-1}\,\cos(t)\right)^n \sin(t)\, dt \;/;\; z \notin (-\infty, -1)$$

Integral representations of negative integer order

Rodrigues-type formula.

05.05.07.0002.01

$$U_n(z) == \frac{(-1)^n \sqrt{\pi}\,(n+1)}{2^{n+1}\,\Gamma\left(n+\frac{3}{2}\right)\sqrt{1-z^2}} \frac{\partial^n \left(1-z^2\right)^{n+\frac{1}{2}}}{\partial z^n}$$

Generating functions

05.05.11.0001.01

$$U_n(z) == \left([t^n]\, \frac{1}{t^2 - 2\,t\,z + 1}\right) /;\; -1 < z < 1$$

Differential equations

Ordinary linear differential equations and wronskians

For the direct function itself

05.05.13.0001.01

$$\left(1-z^2\right) w''(z) - 3\,z\,w'(z) + (n+2)\,n\,w(z) == 0 \;/;\; w(z) == c_1\,U_n(z) \;+\; c_2\,\frac{\cos\left((n+1)\cos^{-1}(z)\right)}{\sqrt{1-z^2}}$$

05.05.13.0002.01

$$W_z\left(U_n(z),\;\frac{\cos\left((n+1)\cos^{-1}(z)\right)}{\sqrt{1-z^2}}\right) == \frac{n+1}{\left(1-z^2\right)^{3/2}}$$

05.05.13.0003.01

$$\left(1-z^2\right) w''(z) - 3\,z\,w'(z) + (n+2)\,n\,w(z) == 0 \;/;\; w(z) == c_1\,U_n(z) \;+\; c_2\,\frac{\cosh\left((n+1)\cosh^{-1}(z)\right)}{\sqrt{1-z^2}}$$

05.05.13.0004.01

$$W_z\left(U_n(z),\;\frac{\cosh\left((n+1)\cosh^{-1}(z)\right)}{\sqrt{1-z^2}}\right) == \frac{n+1}{(z-1)^2\,(z+1)^{3/2}}\left(\sqrt{1-z}\;\cos\left((n+1)\cos^{-1}(z)\right)\cosh\left((n+1)\cosh^{-1}(z)\right) - \sqrt{z-1}\;\sin\left((n+1)\cos^{-1}(z)\right)\sinh\left((n+1)\cosh^{-1}(z)\right)\right)$$

05.05.13.0005.01

$$\left(1-z^2\right) w''(z) - 3\,z\,w'(z) + (n+2)\,n\,w(z) = 0 \;/;\; w(z) = c_1\,U_n(z) \;+ c_2\,\frac{1}{\sqrt{1-z^2}}\,T_{n+1}(z)$$

05.05.13.0006.01

$$W_z\left(U_n(z),\;\frac{1}{\sqrt{1-z^2}}\,T_{n+1}(z)\right) == \frac{n+1}{\left(1-z^2\right)^{3/2}}$$

05.05.13.0007.01

$$w''(z) - \left(\frac{3\,g(z)\,g'(z)}{1-g(z)^2} + \frac{g''(z)}{g'(z)}\right) w'(z) + \frac{n\,(n+2)\,g'(z)^2}{1-g(z)^2}\,w(z) = 0 \;/;\; w(z) = c_1\,U_n(g(z)) \;+ c_2\,\frac{1}{\sqrt{1-g(z)^2}}\,T_{n+1}(g(z))$$

05.05.13.0008.01

$$W_z\left(U_n(g(z)),\;\frac{1}{\sqrt{1-g(z)^2}}\,T_{n+1}(g(z))\right) == \frac{(n+1)\,g'(z)}{\left(1-g(z)^2\right)^{3/2}}$$

05.05.13.0009.01

$$w''(z) - \left(\frac{3\,g(z)\,g'(z)}{1-g(z)^2} + \frac{2\,h(z)\,h'(z)}{h(z)^2} + \frac{g''(z)}{g'(z)}\right) w'(z) + \left(\frac{n\,(n+2)\,g'(z)^2}{1-g(z)^2} + \frac{2\,h'(z)^2}{h(z)^2} + \frac{h'(z)}{h(z)}\left(\frac{3\,g(z)\,g'(z)}{1-g(z)^2} + \frac{g''(z)}{g'(z)}\right) - \frac{h''(z)}{h(z)}\right) w(z) = 0 \;/;\; w(z) = c_1\,h(z)\,U_n(g(z)) \;+ c_2\,h(z)\,\frac{1}{\sqrt{1-g(z)^2}}\,T_{n+1}(g(z))$$

05.05.13.0010.01

$$W_z\left(h(z)\, U_n(g(z)),\ \frac{h(z)}{\sqrt{1-g(z)^2}}\, T_{n+1}(g(z))\right) = \frac{(n+1)\, h(z)^2\, g'(z)}{\left(1-g(z)^2\right)^{3/2}}$$

05.05.13.0011.01

$$z^2\left(a^2 z^{2r}-1\right) w''(z) + \left(a^2 (2r-2s+1) z^{2r} + r + 2s - 1\right) z\, w'(z) + \left(a^2 z^{2r} (s+r n)(s-r(n+2)) - s(r+s)\right) w(z) = 0 /; w(z) = c_1 z^s U_n(a z^r) + c_2 \frac{z^s}{\sqrt{1-a^2 z^{2r}}} T_{n+1}(a z^r)$$

05.05.13.0012.01

$$W_z\left(z^s U_n(a z^r),\ \frac{z^s}{\sqrt{1-a^2 z^{2r}}} T_{n+1}(a z^r)\right) = \frac{a r z^{r+2s-1}(n+1)}{\left(1-a^2 z^{2r}\right)^{3/2}}$$

05.05.13.0013.01

$$\left(a^2 r^{2z}-1\right) w''(z) + \left(2a^2 (\log(r)-\log(s))\, r^{2z} + \log(r) + 2\log(s)\right) w'(z) + \left(-a^2 ((n+2)\log(r)-\log(s))(n\log(r)+\log(s))\, r^{2z} - \log(s)(\log(r)+\log(s))\right) w(z) = 0 /; w(z) = c_1 s^z U_n(a r^z) + c_2 \frac{s^z}{\sqrt{1-a^2 r^{2z}}} T_{n+1}(a r^z)$$

05.05.13.0014.01

$$W_z\left(s^z U_n(a r^z),\ \frac{s^z}{\sqrt{1-a^2 r^{2z}}} T_{n+1}(a r^z)\right) = \frac{a r^z s^{2z} (n+1)\log(r)}{\left(1-a^2 r^{2z}\right)^{3/2}}$$

Transformations

Transformations and argument simplifications

Argument involving basic arithmetic operations

05.05.16.0002.01

$$U_{-n}(z) = -U_{n-2}(z) /; n \in \mathbb{N}^+$$

05.05.16.0001.01

$$U_n(-z) = (-1)^n U_n(z)$$

Multiple arguments

05.05.16.0003.01

$$U_{2n}\left(\sqrt{\frac{z+1}{2}}\right) = U_{n-1}(z) + U_n(z)$$

05.05.16.0004.01

$$U_{2n}(z) = (-1)^n \left(U_n\left(1-2z^2\right) - U_{n-1}\left(1-2z^2\right)\right)$$

Identities

Recurrence identities

Consecutive neighbors

05.05.17.0001.01

$$U_n(z) == 2\,z\,U_{n+1}(z) - U_{n+2}(z)$$

05.05.17.0002.01

$$U_n(z) == 2\,z\,U_{n-1}(z) - U_{n-2}(z)$$

Distant neighbors

05.05.17.0008.01

$$U_n(z) == C_m(n, z)\,U_{n+m}(z) - C_{m-1}(n, z)\,U_{n+m+1}(z) \;/;\; C_0(n, z) == 1 \land C_1(n, z) == 2\,z \land C_m(n, z) == 2\,z\,C_{m-1}(n, z) - C_{m-2}(n, z) \land m > 0$$

05.05.17.0009.01

$$U_n(z) == C_m(n, z)\,U_{n-m}(z) - C_{m-1}(n, z)\,U_{n-m-1}(z) \;/;\; C_0(n, z) == 1 \land C_1(n, z) == 2\,z \land C_m(n, z) == 2\,z\,C_{m-1}(n, z) - C_{m-2}(n, z) \land m > 0$$

05.05.17.0003.01

$$U_n(z) == 2\,(-1)^{2\left\lfloor \frac{m}{2} \right\rfloor} z^{m-2\left\lfloor \frac{m}{2} \right\rfloor} \left(z^2\right)^{\frac{1-m}{2}+\left\lfloor \frac{m}{2} \right\rfloor} U_{\frac{m-1}{2}}\left(2\,z^2-1\right) U_{n+m}(z) - 2\,z^{1-m+2\left\lfloor \frac{m}{2} \right\rfloor} \left(z^2\right)^{\left\lfloor \frac{m+1}{2} \right\rfloor-\frac{m}{2}} U_{\frac{m}{2}-1}\left(2\,z^2-1\right) U_{n+m+1}(z) \;/;\; m > 0$$

05.05.17.0004.01

$$U_n(z) == 2\,(-1)^{2\left\lfloor \frac{m}{2} \right\rfloor} z^{m-2\left\lfloor \frac{m}{2} \right\rfloor} \left(z^2\right)^{\frac{1-m}{2}+\left\lfloor \frac{m}{2} \right\rfloor} U_{\frac{m-1}{2}}\left(2\,z^2-1\right) U_{n-m}(z) - 2\,z^{1-m+2\left\lfloor \frac{m}{2} \right\rfloor} \left(z^2\right)^{\left\lfloor \frac{m+1}{2} \right\rfloor-\frac{m}{2}} U_{\frac{m}{2}-1}\left(2\,z^2-1\right) U_{n-m-1}(z) \;/;\; m > 0$$

Functional identities

Relations between contiguous functions

Recurrence relations

05.05.17.0005.01

$$U_{n-1}(z) + U_{n+1}(z) == 2\,z\,U_n(z)$$

05.05.17.0006.01

$$U_n(z) == \frac{1}{2\,z}\left(U_{n-1}(z) + U_{n+1}(z)\right)$$

Normalized recurrence relation

05.05.17.0007.01

$$z\,p(n, z) == \frac{1}{4}\,p(n-1, z) + p(n+1, z) \;/;\; p(n, z) == 2^{-n}\,U_n(z)$$

Complex characteristics

Real part

05.05.19.0001.01

$$\operatorname{Re}(U_n(x + i\,y)) == \sum_{j=0}^{\left\lfloor \frac{n}{2} \right\rfloor} (-1)^j\,2^{2\,j}\,C_{n-2\,j}^{2\,j+1}(x)\,y^{2\,j} \;/;\; x \in \mathbb{R} \land y \in \mathbb{R}$$

Imaginary part

05.05.19.0002.01

$$\operatorname{Im}(U_n(x + i\, y)) == \sum_{j=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} (-1)^j\, 2^{2\,j+1}\, C_{-2\,j+n-1}^{2\,j+2}(x)\, y^{2\,j+1} \;/;\; x \in \mathbb{R} \bigwedge y \in \mathbb{R}$$

Argument

05.05.19.0003.01

$$\arg(U_n(x + i\, y)) == \tan^{-1}\left(\sum_{j=0}^{\left\lfloor \frac{n}{2} \right\rfloor} (-1)^j\, 2^{2\,j}\, C_{n-2\,j}^{2\,j+1}(x)\, y^{2\,j},\; \sum_{j=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} (-1)^j\, 2^{2\,j+1}\, C_{-2\,j+n-1}^{2\,j+2}(x)\, y^{2\,j+1}\right) /;\; x \in \mathbb{R} \bigwedge y \in \mathbb{R}$$

Conjugate value

05.05.19.0004.01

$$\overline{U_n(x + i\, y)} == \sum_{j=0}^{\left\lfloor \frac{n}{2} \right\rfloor} (-1)^j\, 2^{2\,j}\, C_{n-2\,j}^{2\,j+1}(x)\, y^{2\,j} - i \sum_{j=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} (-1)^j\, 2^{2\,j+1}\, C_{-2\,j+n-1}^{2\,j+2}(x)\, y^{2\,j+1} \;/;\; x \in \mathbb{R} \bigwedge y \in \mathbb{R}$$

Differentiation

Low-order differentiation

With respect to *z*

Forward shift operator:

05.05.20.0001.01

$$\frac{\partial U_n(z)}{\partial z} == \frac{(n+1)\, T_{n+1}(z) - z\, U_n(z)}{z^2 - 1}$$

05.05.20.0002.01

$$\frac{\partial^2 U_n(z)}{\partial z^2} == \frac{\left(z^2\left(n^2 + 2\,n + 3\right) - n\,(n+2)\right) U_n(z) - 3\, z\,(n+1)\, T_{n+1}(z)}{\left(z^2 - 1\right)^2}$$

Backward shift operator:

05.05.20.0003.01

$$\left(1 - z^2\right) \frac{\partial U_n(z)}{\partial z} - z\, U_n(z) == -(n+1)\, T_{n+1}(z)$$

05.05.20.0004.01

$$\frac{\partial\left(\sqrt{1 - z^2}\; U_n(z)\right)}{\partial z} == -(n+1)\left(1 - z^2\right)^{-\frac{1}{2}} T_{n+1}(z)$$

Symbolic differentiation

With respect to *z*

05.05.20.0005.02

$$\frac{\partial^m U_n(z)}{\partial z^m} = 2^m\, m!\, C_{n-m}^{m+1}(z) \;/;\; m \in \mathbb{N}$$

05.05.20.0006.02

$$\frac{\partial^m U_n(z)}{\partial z^m} = \frac{\sqrt{\pi}}{2}\,(n+1)\,(z-1)^{-m}\;{}_3\tilde{F}_2\!\left(1,\,-n,\,n+2;\,\frac{3}{2},\,1-m;\,\frac{1-z}{2}\right) /;\; m \in \mathbb{N}$$

05.05.20.0008.02

$$\frac{\partial^m U_n(z)}{\partial z^m} = 2^m\, m!\sum_{i_1=0}^{n-m}\cdots\sum_{i_{m+1}=0}^{n-m}\delta_{\sum_{j=1}^{m+1} i_j,\,n-m}\prod_{j=1}^{m+1} U_{i_j}(z) \;/;\; m \in \mathbb{N} \land n \in \mathbb{N}$$

Fractional integro-differentiation

With respect to *z*

05.05.20.0007.01

$$\frac{\partial^\alpha U_n(z)}{\partial z^\alpha} = \frac{\sqrt{\pi}}{2}\,(n+1)\,z^{-\alpha}\,\tilde{F}_{1\times1\times0}^{2\times1\times0}\!\left(\begin{matrix} -n,\,n+2;\,1;\,; \\ \frac{3}{2};\,1-\alpha;\,; \end{matrix}\;-\frac{z}{2},\,\frac{1}{2}\right)$$

Integration

Indefinite integration

Involving only one direct function

05.05.21.0001.01

$$\int U_n(a\,z)\,dz = \frac{1}{a(n+1)}\,T_{n+1}(a\,z)$$

05.05.21.0002.01

$$\int U_n(z)\,dz = \frac{1}{n+1}\,T_{n+1}(z)$$

05.05.21.0003.01

$$\int z^{\alpha-1}\,U_n(z)\,dz = \sum_{k=0}^{\left\lfloor\frac{n}{2}\right\rfloor}\frac{(-1)^k\,(n-k)!\,2^{n-2k}\,z^{n-2k+\alpha}}{k!\,(n-2k)!\,(n-2k+\alpha)}$$

Involving one direct function and elementary functions

Involving power function

05.05.21.0004.01

$$\int z^{\alpha-1}\,U_n(z)\,dz = \sum_{k=0}^{\left\lfloor\frac{n}{2}\right\rfloor}\frac{(-1)^k\,(n-k)!\,2^{n-2k}\,z^{n-2k+\alpha}}{k!\,(n-2k)!\,(n-2k+\alpha)}$$

Involving algebraic functions

05.05.21.0005.01

$$\int \left(1-z^2\right)^{\frac{1}{2}(-n-3)} U_n(z)\, dz = \frac{\left(1-z^2\right)^{\frac{1}{2}(-n-1)}}{n+2} U_{n+1}(z)$$

05.05.21.0006.01

$$\int \left(1-z^2\right)^{\frac{n-1}{2}} U_n(z)\, dz = -\frac{\left(1-z^2\right)^{\frac{n+1}{2}}}{n} U_{n-1}(z)$$

Definite integration

Involving the direct function

05.05.21.0007.01

$$\mathcal{P}\int_{-1}^{1} \frac{\sqrt{1-t^2}\; U_n(t)}{t-x}\, dt = -\pi\, T_{n+1}(x) \;/;\; -1 < x < 1$$

05.05.21.0008.01

$$-\frac{2}{\pi}\int_{-1}^{1} U_n(x)^2 \log\left(U_n(x)^2\right) \sqrt{1-x^2}\; dx = -\frac{2}{\pi}\left(\frac{1}{n+1} - 1\right) /;\; n \in \mathbb{N}$$

Entropy integral

Orthogonality:

05.05.21.0009.01

$$\int_{-1}^{1} \sqrt{1-t^2}\; U_m(t)\, U_n(t)\, dt = \frac{\pi}{2}\delta_{m,n}$$

Summation

Infinite summation

05.05.23.0001.01

$$\sum_{n=0}^{\infty} U_n(z)\, w^n = \frac{1}{w^2 - 2\, z\, w + 1} \;/;\; -1 < z < 1 \land |w| < 1$$

05.05.23.0002.01

$$\sum_{n=0}^{\infty} U_n(x)\, U_n(y) = \frac{\pi}{2} \frac{1}{\sqrt[4]{1-x^2}} \frac{1}{\sqrt[4]{1-x^2}} \delta(x-y) \;/;\; -1 < x < 1 \bigwedge -1 < y < 1$$

Operations

Orthogonality, completeness, and Fourier expansions

The set of functions $U_n(x)$, $n = 0, 1, \ldots$, forms a complete, orthogonal (with weight $\frac{2}{\pi}\sqrt{1-x^2}$) system on the interval $(-1, 1)$.

05.05.25.0001.01

$$\sum_{n=0}^{\infty}\left(\sqrt{\frac{2}{\pi}}\sqrt[4]{1-x^2}\,U_n(x)\right)\left(\sqrt{\frac{2}{\pi}}\sqrt[4]{1-y^2}\,U_n(y)\right)=\delta(x-y)\,/;\,-1<x<1\bigwedge -1<y<1$$

05.05.25.0002.01

$$\int_{-1}^{1}\left(\sqrt{\frac{2}{\pi}}\sqrt[4]{1-t^2}\,U_m(t)\right)\left(\sqrt{\frac{2}{\pi}}\sqrt[4]{1-t^2}\,U_n(t)\right)dt=\delta_{m,n}$$

Any sufficiently smooth function $f(x)$ can be expanded in the system $\{U_n(x)\}_{n=0,1,\ldots}$ as a generalized Fourier series, with its sum converging to $f(x)$ almost everywhere.

05.05.25.0003.01

$$f(x)=\sum_{n=0}^{\infty}c_n\,\psi_n(x)\,/;\,c_n=\int_{-1}^{1}\psi_n(t)\,f(t)\,dt\bigwedge\psi_n(x)=\sqrt{\frac{2}{\pi}}\sqrt[4]{1-x^2}\,U_n(x)\bigwedge -1<x<1$$

Representations through more general functions

Through hypergeometric functions

Involving ${}_2F_1$

05.05.26.0001.01

$$U_n(z)=(n+1)\,{}_2F_1\left(-n,n+2;\frac{3}{2};\frac{1-z}{2}\right)$$

05.05.26.0002.01

$$U_n(z)=(-1)^n\,(1+n)\,{}_2F_1\left(-n,n+2;\frac{3}{2};\frac{z+1}{2}\right)$$

Through hypergeometric functions of two variables

05.05.26.0003.01

$$U_n(z)=(n+1)\,F_{1\times 0\times 0}^{2\times 0\times 0}\left(\begin{matrix}-n,n+2;;;\\ \frac{3}{2};;;\end{matrix}\;\frac{1}{2},-\frac{z}{2}\right)$$

Through Meijer G

Classical cases for the direct function itself

05.05.26.0004.01

$$U_n(z)=-\frac{1}{2\sqrt{\pi}}\lim_{m\to n}\sin(\pi m)\,G_{2,2}^{1,2}\left(\frac{z-1}{2}\,\middle|\,\begin{matrix}m+1,-m-1\\ 0,-\frac{1}{2}\end{matrix}\right)$$

Classical cases involving algebraic functions

05.05.26.0005.01

$$(z+1)^{-n-2}\,U_n\left(\frac{1-z}{1+z}\right)=\frac{2^{2n}}{\Gamma(2n+2)}\,G_{2,2}^{1,2}\left(z\,\middle|\,\begin{matrix}-n-1,-n-\frac{1}{2}\\ 0,-\frac{1}{2}\end{matrix}\right)/;\,z\notin(-\infty,-1)$$

05.05.26.0006.01

$$(z+1)^{-n-2}\, U_n\left(\frac{z-1}{z+1}\right) = \frac{4^n}{\Gamma(2n+2)}\, G_{2,2}^{2,1}\left(z \,\middle|\, \begin{matrix} -n-1, -n-\frac{1}{2} \\ -\frac{1}{2}, 0 \end{matrix}\right) /;\, z \notin (-1, 0)$$

05.05.26.0007.01

$$(z+1)^{-\frac{n}{2}-1}\, U_n\left(\frac{1}{\sqrt{z+1}}\right) = \frac{2^n}{\Gamma(n+1)}\, G_{2,2}^{1,2}\left(z \,\middle|\, \begin{matrix} -\frac{n}{2}, -\frac{n+1}{2} \\ 0, -\frac{1}{2} \end{matrix}\right)$$

05.05.26.0008.01

$$(z+1)^{-\frac{n}{2}-1}\, U_n\left(\sqrt{\frac{z}{z+1}}\right) = \frac{2^n}{\Gamma(n+1)}\, G_{2,2}^{2,1}\left(z \,\middle|\, \begin{matrix} -\frac{n}{2}, \frac{1-n}{2} \\ 0, \frac{1}{2} \end{matrix}\right) /;\, z \notin (-1, 0)$$

05.05.26.0009.01

$$(z+1)^{-\frac{n}{2}-1}\, U_n\left(\frac{z+2}{2\sqrt{z+1}}\right) = \frac{1}{\Gamma(n+1)}\, G_{2,2}^{1,2}\left(z \,\middle|\, \begin{matrix} 0, -n-1 \\ 0, -1 \end{matrix}\right)$$

05.05.26.0010.01

$$(z+1)^{-\frac{n}{2}-1}\, U_n\left(\frac{2z+1}{2\sqrt{z}\,\sqrt{z+1}}\right) = G_{2,2}^{2,1}\left(z \,\middle|\, \begin{matrix} -\frac{n}{2}, 1-\frac{n}{2} \\ -\frac{n}{2}, \frac{n}{2}+1 \end{matrix}\right) /;\, z \notin (-1, 0)$$

05.05.26.0011.01

$$(z+1)^{-n-2}\,(z+2)\, U_n\left(\frac{z^2+2z+2}{2(z+1)}\right) = \frac{1}{\Gamma(2n+2)}\, G_{2,2}^{1,2}\left(z \,\middle|\, \begin{matrix} 0, -2n-2 \\ 0, -1 \end{matrix}\right) /;\, z \notin (-\infty, -1)$$

05.05.26.0012.01

$$(2z+1)\,(z+1)^{-n-2}\, U_n\left(\frac{2z^2+2z+1}{2z(z+1)}\right) = \frac{1}{\Gamma(2n+2)}\, G_{2,2}^{2,1}\left(z \,\middle|\, \begin{matrix} -n, 1-n \\ -n, n+2 \end{matrix}\right) /;\, z \notin (-1, 0)$$

Classical cases involving unit step θ

05.05.26.0013.01

$$\sqrt{1-z}\; \theta(1-|z|)\, U_n(2z-1) = \frac{\sqrt{\pi}\,(n+1)}{2}\, G_{2,2}^{2,0}\left(z \,\middle|\, \begin{matrix} -n-\frac{1}{2}, n+\frac{3}{2} \\ -\frac{1}{2}, 0 \end{matrix}\right) /;\, z \notin (-1, 0)$$

05.05.26.0014.01

$$\sqrt{z-1}\; \theta(|z|-1)\, U_n(2z-1) = \frac{1}{2}\sqrt{\pi}\,(n+1)\, G_{2,2}^{0,2}\left(z \,\middle|\, \begin{matrix} -n-\frac{1}{2}, n+\frac{3}{2} \\ -\frac{1}{2}, 0 \end{matrix}\right)$$

05.05.26.0015.01

$$\sqrt{1-z}\; \theta(1-|z|)\, U_n\left(\frac{2}{z}-1\right) = \frac{1}{2}\sqrt{\pi}\,(n+1)\, G_{2,2}^{2,0}\left(z \,\middle|\, \begin{matrix} \frac{3}{2}, 2 \\ -n, n+2 \end{matrix}\right)$$

05.05.26.0016.01

$$\sqrt{z-1}\; \theta(|z|-1)\, U_n\left(\frac{2}{z}-1\right) = \frac{1}{2}\sqrt{\pi}\,(n+1)\, G_{2,2}^{0,2}\left(z \,\middle|\, \begin{matrix} \frac{3}{2}, 2 \\ -n, n+2 \end{matrix}\right) /;\, z \notin (-\infty, -1)$$

05.05.26.0017.01

$$\left(1-z^2\right)\theta(1-|z|)\, U_n\left(\frac{z^2+1}{2z}\right) = 2(n+1)\, G_{2,2}^{2,0}\left(z \,\middle|\, \begin{matrix} 1-n, n+3 \\ -n, n+2 \end{matrix}\right)$$

05.05.26.0018.01

$$\left(z^2-1\right)\theta(|z|-1)\,U_n\!\left(\frac{z^2+1}{2\,z}\right)=2\,(n+1)\,G_{2,2}^{0,2}\!\left(z\,\middle|\,\begin{array}{c}1-n,\,n+3\\-n,\,n+2\end{array}\right)$$

05.05.26.0019.01

$$\sqrt{z-1}\,(2\,z-1)\,\theta(|z|-1)\,U_n\!\left(8\,z^2-8\,z+1\right)=\frac{1}{2}\sqrt{\pi}\,(n+1)\,G_{2,2}^{0,2}\!\left(z\,\middle|\,\begin{array}{c}-2\,n-\frac{3}{2},\,2\,n+\frac{5}{2}\\-\frac{1}{2},\,0\end{array}\right)/;\,\mathrm{Re}(z)>0$$

05.05.26.0020.01

$$\sqrt{1-z}\,(2-z)\,\theta(1-|z|)\,U_n\!\left(\frac{8}{z^2}-\frac{8}{z}+1\right)=\frac{1}{2}\sqrt{\pi}\,(n+1)\,G_{2,2}^{2,0}\!\left(z\,\middle|\,\begin{array}{c}\frac{5}{2},\,3\\-2\,n,\,2\,(n+2)\end{array}\right)/;\,\mathrm{Re}(z)>0$$

Generalized cases involving algebraic functions

05.05.26.0021.01

$$\left(z^2+1\right)^{-\frac{n}{2}-1}U_n\!\left(\frac{z}{\sqrt{z^2+1}}\right)=\frac{2^n}{\Gamma(n+1)}\,G_{2,2}^{2,1}\!\left(z,\,\frac{1}{2}\,\middle|\,\begin{array}{c}-\frac{n}{2},\,\frac{1-n}{2}\\0,\,\frac{1}{2}\end{array}\right)/;\,\mathrm{Re}(z)>0$$

05.05.26.0022.01

$$\left(z^2+1\right)^{-\frac{n}{2}-1}U_n\!\left(\frac{2\,z^2+1}{2\,z\,\sqrt{z^2+1}}\right)=\frac{1}{\Gamma(n+1)}\,G_{2,2}^{2,1}\!\left(z,\,\frac{1}{2}\,\middle|\,\begin{array}{c}-\frac{n}{2},\,1-\frac{n}{2}\\-\frac{n}{2},\,\frac{n}{2}+1\end{array}\right)/;\,\mathrm{Re}(z)>0$$

Through other functions

Involving some hypergeometric-type functions

05.05.26.0023.01

$$U_n(z)=\sqrt{\frac{\pi}{2}}\,\frac{n+1}{\sqrt[4]{z+1}}\,\sqrt[4]{\frac{1}{1-z}}\,P_{n+\frac{1}{2}}^{-\frac{1}{2}}(z)$$

05.05.26.0024.01

$$U_n(z)=\sqrt{\frac{\pi}{2}}\,\frac{n+1}{\sqrt{z+1}}\,\sqrt[4]{\frac{z+1}{z-1}}\,P_{n+\frac{1}{2}}^{-\frac{1}{2}}(z)$$

05.05.26.0025.01

$$U_n(z)=\frac{(n+1)!}{\left(\frac{3}{2}\right)_n}\,P_n^{\left(\frac{1}{2},\frac{1}{2}\right)}(z)$$

05.05.26.0026.01

$$U_n(z)=\frac{(n+1)\,P_n^{\left(\frac{1}{2},\frac{1}{2}\right)}(z)}{P_n^{\left(\frac{1}{2},\frac{1}{2}\right)}(1)}$$

05.05.26.0027.01

$$U_n(z)=C_n^1(z)$$

Involving spheroidal functions

05.05.26.0028.01

$$U_n(z) = -\frac{\sqrt{2}}{\sqrt{\pi}\,\sqrt[4]{1-z^2}}\,QS_{n+\frac{1}{2},\frac{1}{2}}(0,z)$$

Representations through equivalent functions

With related functions

05.05.27.0001.01

$$U_n(z) = \frac{z\,T_{n+1}(z) - T_{n+2}(z)}{1-z^2}$$

05.05.27.0002.01

$$U_n(z) = \frac{1}{n+1}\,\frac{\partial T_{n+1}(z)}{\partial z}$$

05.05.27.0003.01

$$U_n(z) = \frac{1}{2}\,\frac{\partial C_{n+1}^{(0)}(z)}{\partial z}$$

05.05.27.0004.01

$$U_n(x) = \frac{1}{\pi}\,\mathcal{P}\int_{-1}^{1}\frac{T_{n+1}(t)}{\sqrt{1-t^2}\,(t-x)}\,dt\,/;\,n>0\land -1<x<1$$

05.05.27.0007.01

$$U_{2n}(z) = \frac{(-1)^n}{\sqrt{1-z^2}}\,T_{2n+1}\left(\sqrt{1-z^2}\right)$$

With elementary functions

05.05.27.0005.01

$$U_n(z) = \frac{1}{2\sqrt{1-z^2}}\left(e^{\frac{i\pi n}{2}}\left(i\,z+\sqrt{1-z^2}\right)^{-n-1} + e^{-\frac{i\pi n}{2}}\left(i\,z+\sqrt{1-z^2}\right)^{n+1}\right)$$

05.05.27.0006.01

$$U_n(z) = \frac{\sin\left((n+1)\cos^{-1}(z)\right)}{\sqrt{1-z^2}}$$

05.05.27.0008.01

$$U_n(z) = \frac{(-1)^n}{\sqrt{1-z^2}}\,\sin\left(2\,(n+1)\sin^{-1}\left(\frac{\sqrt{z+1}}{\sqrt{2}}\right)\right)$$

05.05.27.0009.01

$$U_n(z) = (-i)^n\,F_{n+1}(2\,i\,z)$$

Theorems

Expansions in generalized Fourier series

$$f(x) = \sum_{k=0}^{\infty} c_k \, \psi_k(x) \, /; \quad c_k = \int_{-1}^{1} f(t) \, \psi_k(t) \, dt, \;\; \psi_k(x) = \sqrt{\frac{2}{\pi}} \left(1 - x^2\right)^{\frac{1}{4}} U_k(x), \;\; k \in \mathbb{N}.$$

One property of a unimodular matrix

If $\mathbb{A}$ is a unimodular matrix, then $\mathbb{A}^n = \mathbb{A} \, U_{n-1}\left(\frac{\text{Tr}(\mathbb{A})}{2}\right) - \mathbb{I} \, U_{n-2}\left(\frac{\text{Tr}(\mathbb{A})}{2}\right) /; n \in \mathbb{N}^+$.

The length of the hypotenuse of the *r*th Pythagorean triangle

The length of the hypotenuse of the r th Pythagorean triangle with consecutive integer legs is $U_r(3) - U_{r-1}(3)$.

History

– P. L. Chebyshev (1854,1855,1859)

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