CatalanNumber

Notations

Traditional name

Catalan number

Traditional notation

C_z

Mathematica StandardForm notation

`CatalanNumber[z]`

Primary definition

06.41.02.0001.01

$$C_z = \frac{2^{2z}\,\Gamma\left(z+\frac{1}{2}\right)}{\sqrt{\pi}\;\Gamma(z+2)}$$

Specific values

Specialized values

06.41.03.0001.01

$$C_{-n} = 0 \;/;\; n \in \mathbb{Z} \land n > 1$$

06.41.03.0002.01

$$C_n = \frac{2^n}{(n+1)!} \prod_{k=1}^{n} (2k-1) \;/;\; n \in \mathbb{N}$$

06.41.03.0003.01

$$C_n = \frac{(-1)^n\,(-2n)_n}{(n+1)!} \;/;\; n \in \mathbb{N}$$

06.41.03.0004.01

$$C_{n+\frac{1}{2}} = \frac{2^{3n+3}\,n!}{\pi \prod_{k=1}^{n+2} (2k-1)} \;/;\; n \in \mathbb{N}$$

06.41.03.0005.01

$$C_{-n-\frac{1}{2}} = \tilde{\infty} \;/;\; n \in \mathbb{N}$$

Values at fixed points

06.41.03.0006.01

$$C_{-10} = 0$$

06.41.03.0007.01

$$C_{-\frac{19}{2}} = \tilde{\infty}$$

06.41.03.0008.01

$$C_{-9} = 0$$

06.41.03.0009.01

$$C_{-\frac{17}{2}} = \tilde{\infty}$$

06.41.03.0010.01

$$C_{-8} = 0$$

06.41.03.0011.01

$$C_{-\frac{15}{2}} = \tilde{\infty}$$

06.41.03.0012.01

$$C_{-7} = 0$$

06.41.03.0013.01

$$C_{-\frac{13}{2}} = \tilde{\infty}$$

06.41.03.0014.01

$$C_{-6} = 0$$

06.41.03.0015.01

$$C_{-\frac{11}{2}} = \tilde{\infty}$$

06.41.03.0016.01

$$C_{-5} = 0$$

06.41.03.0017.01

$$C_{-\frac{9}{2}} = \tilde{\infty}$$

06.41.03.0018.01

$$C_{-4} = 0$$

06.41.03.0019.01

$$C_{-\frac{7}{2}} = \tilde{\infty}$$

06.41.03.0020.01

$$C_{-3} = 0$$

06.41.03.0021.01

$$C_{-\frac{5}{2}} = \tilde{\infty}$$

06.41.03.0022.01

$$C_{-2} = 0$$

06.41.03.0023.01

$$C_{-\frac{3}{2}} = \tilde{\infty}$$

06.41.03.0024.01
$$C_{-1} = -\frac{1}{2}$$

06.41.03.0025.01
$$C_{-\frac{1}{2}} = \tilde{\infty}$$

06.41.03.0026.01
$$C_0 = 1$$

06.41.03.0027.01
$$C_{\frac{1}{2}} = \frac{8}{3\pi}$$

06.41.03.0028.01
$$C_1 = 1$$

06.41.03.0029.01
$$C_{\frac{3}{2}} = \frac{64}{15\pi}$$

06.41.03.0030.01
$$C_2 = 2$$

06.41.03.0031.01
$$C_{\frac{5}{2}} = \frac{1024}{105\pi}$$

06.41.03.0032.01
$$C_3 = 5$$

06.41.03.0033.01
$$C_{\frac{7}{2}} = \frac{8192}{315\pi}$$

06.41.03.0034.01
$$C_4 = 14$$

06.41.03.0035.01
$$C_{\frac{9}{2}} = \frac{262\,144}{3465\pi}$$

06.41.03.0036.01
$$C_5 = 42$$

06.41.03.0037.01
$$C_{\frac{11}{2}} = \frac{2\,097\,152}{9009\pi}$$

06.41.03.0038.01
$$C_6 = 132$$

06.41.03.0039.01
$$C_{\frac{13}{2}} = \frac{33\,554\,432}{45\,045\pi}$$

06.41.03.0040.01
$$C_7 = 429$$

06.41.03.0041.01

$$C_{\frac{15}{2}} = \frac{268\,435\,456}{109\,395\,\pi}$$

06.41.03.0042.01

$$C_8 = 1430$$

06.41.03.0043.01

$$C_{\frac{17}{2}} = \frac{17\,179\,869\,184}{2\,078\,505\,\pi}$$

06.41.03.0044.01

$$C_9 = 4862$$

06.41.03.0045.01

$$C_{\frac{19}{2}} = \frac{137\,438\,953\,472}{4\,849\,845\,\pi}$$

06.41.03.0046.01

$$C_{10} = 16\,796$$

Values at infinities

06.41.03.0047.01

$$C_{\infty} = \infty$$

06.41.03.0048.01

$$C_{-\infty} = ¿$$

06.41.03.0049.01

$$C_{i\,\infty} = 0$$

06.41.03.0050.01

$$C_{-i\,\infty} = 0$$

06.41.03.0051.01

$$C_{\tilde{\infty}} = ¿$$

General characteristics

Domain and analyticity

C_z is an analytical function of z which is defined over the whole complex z-plane with the exception of countably many points $z = -k - 1/2 /; k \in \mathbb{N}$.

06.41.04.0001.01

$$z \longrightarrow C_z :: \mathbb{C} \longrightarrow \mathbb{C}$$

Symmetries and periodicities

Mirror symmetry

06.41.04.0002.01

$$C_{\bar{z}} = \overline{C_z}$$

Periodicity

No periodicity

Poles and essential singularities

The function C_z has an infinite set of singular points:

a) $z == -k - 1/2$ /; $k \in \mathbb{N}$ are the simple poles with residues $\frac{(-1)^k 2^{-2k-1}}{\sqrt{\pi}\, k!\, \Gamma(3/2-k)}$;

b) $z == \tilde{\infty}$ is the point of convergence of poles, which is an essential singular point.

06.41.04.0003.01

$$\mathcal{S}ing_z(C_z) == \left\{\left\{\left\{-k - \frac{1}{2}, 1\right\} /; k \in \mathbb{N}\right\}, \{\tilde{\infty}, \infty\}\right\}$$

06.41.04.0004.01

$$\text{res}_z(C_z)\left(-k - \frac{1}{2}\right) == \frac{(-1)^k\, 2^{-2k-1}}{\sqrt{\pi}\; k!\, \Gamma\left(\frac{3}{2} - k\right)} /; k \in \mathbb{N}$$

Branch points

The function C_z does not have branch points.

06.41.04.0005.01

$$\mathcal{BP}_z(C_z) == \{\}$$

Branch cuts

The function C_z does not have branch cuts.

06.41.04.0006.01

$$\mathcal{BC}_z(C_z) == \{\}$$

Series representations

Generalized power series

Expansions at $z == 0$

06.41.06.0001.01

$$C_z \propto 1 - z + \left(1 + \frac{\pi^2}{6}\right) z^2 - \left(2\,\zeta(3) + \frac{\pi^2}{6} + 1\right) z^3 + \ldots /; (z \to 0)$$

06.41.06.0002.01

$$C_z \propto 1 - z + \left(1 + \frac{\pi^2}{6}\right) z^2 - \left(2\,\zeta(3) + \frac{\pi^2}{6} + 1\right) z^3 + O\left(z^4\right)$$

06.41.06.0003.01

$$C_z = \frac{1}{\sqrt{\pi}} \sum_{k=0}^{\infty} \sum_{j=0}^{k} (j+1)\, d_{k-j} \sum_{r=0}^{j} \frac{(-1)^r}{r+1} \binom{j}{r} p_{r,j}\, z^k \;/;\; a_k = \frac{\Gamma^{(k)}\left(\frac{1}{2}\right)}{k!} \bigwedge b_k = \frac{\log^k(4)}{k!} \bigwedge c_k = \frac{\Gamma^{(k)}(2)}{k!} \bigwedge d_k = \sum_{n=0}^{k} a_n\, b_{k-n} \bigwedge p_{j,0} = 1 \bigwedge p_{j,k} = \frac{1}{k} \sum_{m=1}^{k} (-k + j\, m + m)\, c_m\, p_{j,k-m} \bigwedge k \in \mathbb{N} \bigwedge |z| < \frac{1}{2}$$

06.41.06.0004.01

$$C_z \propto 1 + O(z)$$

Expansions at $z == z_0$ /; $z_0 \neq -n - 1/2$

06.41.06.0005.01

$$C_z \propto C_{z_0} \left(1 + \left(\log(4) - \psi(z_0 + 2) + \psi\left(z_0 + \frac{1}{2}\right)\right)(z - z_0) + \frac{1}{2}\left(\left(\log(4) - \psi(z_0 + 2) + \psi\left(z_0 + \frac{1}{2}\right)\right)^2 - \psi^{(1)}(z_0 + 2) + \psi^{(1)}\left(z_0 + \frac{1}{2}\right)\right)(z - z_0)^2 + \ldots\right) /; (z \to z_0) \bigwedge -z_0 - \frac{1}{2} \notin \mathbb{N}$$

06.41.06.0006.01

$$C_z \propto C_{z_0} \left(1 + \left(\log(4) - \psi(z_0 + 2) + \psi\left(z_0 + \frac{1}{2}\right)\right)(z - z_0) + \frac{1}{2}\left(\left(\log(4) - \psi(z_0 + 2) + \psi\left(z_0 + \frac{1}{2}\right)\right)^2 - \psi^{(1)}(z_0 + 2) + \psi^{(1)}\left(z_0 + \frac{1}{2}\right)\right)(z - z_0)^2 + O\left((z - z_0)^3\right)\right) /; -z_0 - \frac{1}{2} \notin \mathbb{N}$$

06.41.06.0007.01

$$C_z = \frac{2^{2 z_0}}{\sqrt{\pi}\; \Gamma(z_0 + 2)} \sum_{k=0}^{\infty} \sum_{j=0}^{k} (j+1)\, d_{k-j} \sum_{r=0}^{j} \frac{(-1)^r \binom{j}{r}}{r+1} p_{r,j}\, (z - z_0)^k \;/;\; a_k = \frac{\Gamma^{(k)}\left(z_0 + \frac{1}{2}\right)}{k!} \bigwedge b_k = \frac{\log^k(4)}{k!} \bigwedge c_k = \frac{\Gamma^{(k)}(z_0 + 2)}{k!} \bigwedge d_k = \sum_{n=0}^{k} a_n\, b_{k-n} \bigwedge p_{j,0} = 1 \bigwedge p_{j,k} = \frac{1}{\Gamma(z_0 + 2)\, k} \sum_{m=1}^{k} (j\, m + m - k)\, c_m\, p_{j,k-m} \bigwedge k \in \mathbb{N} \bigwedge -z_0 - \frac{1}{2} \notin \mathbb{N}$$

06.41.06.0008.01

$$C_z \propto C_{z_0}\, (1 + O(z - z_0)) /; -z_0 - \frac{1}{2} \notin \mathbb{N}$$

Expansions at $z = -n - 1/2$

06.41.06.0009.01

$$C_z \propto \frac{(-1)^n\, 2^{-2n-1} \left(\log(4) - \psi\left(\frac{3}{2}-n\right) + \psi(n+1)\right)}{\sqrt{\pi}\, \Gamma\left(\frac{3}{2}-n\right) n!} + \frac{(-1)^n\, 2^{-2n-1}}{\sqrt{\pi}\, \Gamma\left(\frac{3}{2}-n\right) n! \left(n+z+\frac{1}{2}\right)} + \frac{(-1)^n\, 4^{-n-1}}{3\sqrt{\pi}\, \Gamma\left(\frac{3}{2}-n\right) n!} \left(3\log^2(4) + \pi^2 + 3\,\psi\left(\frac{3}{2}-n\right)^2 - 6\,\psi\left(\frac{3}{2}-n\right)(\log(4)+\psi(n+1)) + 3\,\psi(n+1)\,(\log(16)+\psi(n+1)) - 3\,\psi^{(1)}\left(\frac{3}{2}-n\right) - 3\,\psi^{(1)}(n+1)\right)\left(n+z+\frac{1}{2}\right) + \frac{(-1)^n\, 4^{-n-1}}{3\sqrt{\pi}\, \Gamma\left(\frac{3}{2}-n\right) n!} \left(\log^3(4) + \pi^2 \log(4) - 3\left(\psi^{(1)}\left(\frac{3}{2}-n\right) + \psi^{(1)}(n+1)\right)\log(4) - \psi\left(\frac{3}{2}-n\right)^3 + \psi(n+1)^3 + \log(64)\,\psi(n+1)^2 + 3\,\psi\left(\frac{3}{2}-n\right)^2 (\log(4)+\psi(n+1)) + \psi(n+1)\left(3\log^2(4) + \pi^2 - 3\,\psi^{(1)}\left(\frac{3}{2}-n\right) - 3\,\psi^{(1)}(n+1)\right) - \psi\left(\frac{3}{2}-n\right)\left(3\log^2(4) + \pi^2 + 3\,\psi(n+1)\,(\log(16)+\psi(n+1)) - 3\,\psi^{(1)}\left(\frac{3}{2}-n\right) - 3\,\psi^{(1)}(n+1)\right) - \psi^{(2)}\left(\frac{3}{2}-n\right) + \psi^{(2)}(n+1)\right)\left(n+z+\frac{1}{2}\right)^2 + \ldots \;/;\; \left(z \to -n-\frac{1}{2}\right) \bigwedge n \in \mathbb{N}$$

06.41.06.0010.01

$$C_z = \frac{(-1)^n\, 2^{-2n-1}}{e_0 \sqrt{\pi} \left(n+z+\frac{1}{2}\right)} \sum_{k=0}^{\infty} (k+1) \sum_{r=0}^{k} \frac{(-1)^r \binom{k}{r}}{r+1}\, p_{r,k} \left(n+z+\frac{1}{2}\right)^k \;/;$$

$$a_k = \frac{\pi^k\, i^k\, (1+(-1)^k)}{2\,(k+1)!} \bigwedge b_k = \frac{(-1)^k \log^k(4)}{k!} \bigwedge c_k = \frac{\Gamma^{(k)}\left(\frac{3}{2}-n\right)}{k!} \bigwedge d_k = \frac{(-1)^k\, \Gamma^{(k)}(n+1)}{k!} \bigwedge$$

$$e_k = \sum_{u=0}^{k} \sum_{v=0}^{u} \sum_{i=0}^{v} a_{u-v}\, b_{v-i}\, c_i\, d_{k-u} \bigwedge p_{j,0} = 1 \bigwedge p_{j,k} = \frac{1}{e_0\, k} \sum_{m=1}^{k} (-k + j\, m + m)\, e_m\, p_{j,k-m} \bigwedge k \in \mathbb{N} \bigwedge n \in \left(1 + O\left(n+z+\frac{1}{2}\right)\right)$$

06.41.06.0011.01

$$C_z \propto \frac{(-1)^n\, 2^{-2n-1}}{\sqrt{\pi}\, \Gamma\left(\frac{3}{2}-n\right) n! \left(n+z+\frac{1}{2}\right)} \left(1 + O\left(n+z+\frac{1}{2}\right)\right) \;/;\; n \in \mathbb{N}$$

Asymptotic series expansions

06.41.06.0012.01

$$C_x \propto \frac{4^x\, e^{-\frac{9}{8x}}}{\sqrt{\pi}\, x^{3/2}} \;/;\; (x \to \infty)$$

06.41.06.0013.01

$$C_z \propto \frac{2^{2z}}{\sqrt{\pi}\, z^{3/2}} \left(1 - \frac{9}{8z} + \frac{145}{128 z^2} - \frac{1155}{1024 z^3} + \frac{36939}{32768 z^4} - \frac{295911}{262144 z^5} + \frac{4735445}{4194304 z^6} - \frac{37844235}{33554432 z^7} + \frac{2421696563}{2147483648 z^8} - \frac{19402289907}{17179869184 z^9} + O\left(\frac{1}{z^{10}}\right)\right) \;/;\; \left|\arg\left(z+\frac{1}{2}\right)\right| < \pi \bigwedge (|z| \to \infty)$$

06.41.06.0014.01

$$C_z \propto \frac{2^{2z}}{\sqrt{\pi}\; z^{3/2}} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \left(\frac{3}{2}\right)_k B_k^{\left(-\frac{1}{2}\right)}\left(\frac{1}{2}\right) z^{-k} \;/;\; \left|\arg\left(z+\frac{1}{2}\right)\right| < \pi \bigwedge (|z| \to \infty)$$

06.41.06.0015.01

$$C_z \propto \frac{2^{2z}\tan(\pi z)}{\sqrt{\pi}\; (-z)^{3/2}} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \left(\frac{3}{2}\right)_k B_k^{\left(-\frac{1}{2}\right)}\left(\frac{1}{2}\right) z^{-k} \;/;\; \arg(z) = \pi \bigwedge z + \frac{1}{2} \notin \mathbb{Z} \bigwedge (|z| \to \infty)$$

06.41.06.0016.01

$$C_z \propto \frac{1}{\sqrt{\pi}\; z^{3/2}} \left(\frac{z^{3/2}\tan(\pi z)}{(-z)^{3/2}}\right)^{\left\lfloor \frac{\arg(z)+\pi}{2\pi} \right\rfloor} 2^{2z} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \left(\frac{3}{2}\right)_k B_k^{\left(-\frac{1}{2}\right)}\left(\frac{1}{2}\right) z^{-k} \;/;\; \frac{1}{2} - z \notin \mathbb{N}^+ \bigwedge (|z| \to \infty)$$

06.41.06.0017.01

$$C_z \propto \frac{2^{2z}}{\sqrt{\pi}\; z^{3/2}} \left(1 + O\left(\frac{1}{z}\right)\right) /;\; \left|\arg\left(z+\frac{1}{2}\right)\right| < \pi \bigwedge (|z| \to \infty)$$

Product representations

06.41.08.0001.01

$$C_z = \frac{2}{(z+1)!} \prod_{k=1}^{z-1} (4k+2)$$

06.41.08.0002.01

$$C_z = \frac{2^{2z+1} e^{\gamma(z+2)} (z+2)}{\sqrt{\pi}\; (2z+1)} \prod_{k=1}^{\infty} \frac{\left(1+\frac{1}{k}\right)^{z+\frac{1}{2}} \left(\frac{z+2}{k}+1\right) e^{-\frac{z+2}{k}}}{\frac{2z+1}{2k}+1} \;/;\; -z - \frac{1}{2} \mathbb{N}$$

Generating functions

06.41.11.0001.01

$$C_n = \left([t^n]\, \frac{1-\sqrt{1-4t}}{2t}\right) /;\; n \in \mathbb{N}$$

Transformations

Transformations and argument simplifications

Argument involving basic arithmetic operations

06.41.16.0001.01

$$C_{-z} = \frac{\tan(\pi z)}{\pi z \left(1-z^2\right) C_z}$$

06.41.16.0002.01

$$C_{z+1} = \frac{2(2z+1)}{z+2} C_z$$

06.41.16.0003.01

$$C_{n+z} = \frac{2^{2n} \left(z + \frac{1}{2}\right)_n}{(z+2)_n} C_z \;/;\; n \in \mathbb{Z}$$

06.41.16.0004.01

$$C_{z-1} = \frac{z+1}{2\,(2\,z-1)} C_z$$

06.41.16.0005.01

$$C_{z-n} = \frac{2^{-2n}\,(-z-1)_n}{\left(\frac{1}{2} - z\right)_n} C_z \;/;\; n \in \mathbb{Z}$$

Products, sums, and powers of the direct function

Products of the direct function

06.41.16.0006.01

$$C_z\, C_{-z} = \frac{\tan(\pi\, z)}{\pi\, z\left(1 - z^2\right)}$$

06.41.16.0007.01

$$C_z\, C_{1-z} = \frac{2\,(1-2\,z)\tan(\pi\, z)}{\pi\,(z-2)\,(z-1)\,z\,(z+1)}$$

06.41.16.0008.01

$$C_z\, C_{-n-z} = -\frac{4^{-n}\,(z+1)_{n-2}\tan(\pi\, z)}{\pi\,(z+1)\left(z + \frac{1}{2}\right)_n} \;/;\; n \in \mathbb{Z}$$

06.41.16.0009.01

$$\frac{C_{n+z}}{C_z} = 2^n \prod_{k=0}^{n-1} \frac{2\,k + 2\,z + 1}{k + z + 2} \;/;\; n \in \mathbb{Z}$$

06.41.16.0010.01

$$\frac{C_{z-n}}{C_z} = 2^{-n} \prod_{k=0}^{n-1} \frac{k - z - 1}{2\,k - 2\,z + 1} \;/;\; n \in \mathbb{N}^+$$

Identities

Recurrence identities

Consecutive neighbors

06.41.17.0001.01

$$C_z = \frac{2\,(2\,z-1)}{z+1} C_{z-1}$$

06.41.17.0002.01

$$C_z = \frac{z+2}{2\,(2\,z+1)} C_{z+1}$$

Distant neighbors

06.41.17.0003.01

$$C_z = \frac{(z+2)_n}{2^{2n}\left(z+\frac{1}{2}\right)_n} C_{n+z} \text{ /; } n \in \mathbb{Z}$$

06.41.17.0004.01

$$C_z = \frac{2^{2n}\left(\frac{1}{2}-z\right)_n}{(-z-1)_n} C_{z-n} \text{ /; } n \in \mathbb{Z}$$

Differentiation

Low-order differentiation

06.41.20.0001.01

$$\frac{\partial C_z}{\partial z} = C_z\left(\log(4) - \psi(z+2) + \psi\left(z+\frac{1}{2}\right)\right)$$

06.41.20.0002.01

$$\frac{\partial^2 C_z}{\partial z^2} = C_z\left(\left(\log(4) - \psi(z+2) + \psi\left(z+\frac{1}{2}\right)\right)^2 - \psi^{(1)}(z+2) + \psi^{(1)}\left(z+\frac{1}{2}\right)\right)$$

Summation

Infinite summation

06.41.23.0001.01

$$\sum_{n=0}^{\infty} C_n z^n = \frac{1-\sqrt{1-4z}}{2z} \text{ /; } |z| < \frac{1}{4}$$

Representations through equivalent functions

With related functions

06.41.27.0001.01

$$C_z = \frac{1}{z+1}\binom{2z}{z}$$

06.41.27.0002.01

$$C_z = \frac{(2z)!}{(z+1)!\, z!}$$

06.41.27.0003.01

$$C_z = \frac{1}{(z+1)!} 2^{z-\frac{1}{4}\cos(2\pi z)+\frac{1}{4}} \pi^{-\frac{1}{2}\sin^2(\pi z)} (2z-1)!!$$

06.41.27.0004.01

$$C_z = \frac{\Gamma(2z+1)}{\Gamma(z+1)\,\Gamma(z+2)}$$

06.41.27.0005.01

$$C_z = \frac{(z+1)_z}{\Gamma(z+2)}$$

06.41.27.0006.01

$$C_z = \frac{1}{z\,(z+1)\,\mathrm{B}(z,\, z+1)}$$

Zeros

06.41.30.0001.01

$$C_z = 0 \;/;\; z \in \mathbb{Z} \land z \leq -2$$

Copyright