Binomial

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Notations

Traditional name

Binomial coefficient

Traditional notation

$\binom{n}{k}$

Mathematica StandardForm notation

`Binomial[n, k]`

Primary definition

06.03.02.0001.01

$$\binom{n}{k} = \frac{\Gamma(n+1)}{\Gamma(k+1)\,\Gamma(n-k+1)} = \frac{n!}{k!\,(n-k)!} \;/;\; (\neg\,(n \in \mathbb{Z} \wedge k \in \mathbb{Z} \wedge k \leq n < 0))$$

For $\nu = n$, $\kappa == k$ negative integers with $k \leq n$, the binomial coefficient $\binom{\nu}{\kappa}$ cannot be uniquely defined by a limiting procedure based on the above definition because the two variables ν, κ can approach negative integers n, k with $k \leq n$ at different speeds. For negative integers with $k \leq n$ we define:

06.03.02.0002.01

$$\binom{n}{k} = 0 \;/;\; n \in \mathbb{Z} \wedge k \in \mathbb{Z} \wedge k \leq n < 0$$

Specific values

Specialized values

For fixed *n*

06.03.03.0001.01

$$\binom{n}{-k} = 0 \;/;\; k \in \mathbb{N}^+$$

06.03.03.0002.01

$$\binom{n}{0} = 1$$

06.03.03.0003.01

$$\binom{n}{1} = n$$

06.03.03.0004.01

$$\binom{n}{2} = \frac{(n-1)\,n}{2}$$

06.03.03.0005.01

$$\binom{n}{3} = \frac{(n-2)\,(n-1)\,n}{6}$$

06.03.03.0006.01

$$\binom{n}{4} = \frac{(n-3)\,(n-2)\,(n-1)\,n}{24}$$

06.03.03.0007.01

$$\binom{n}{n} = 1$$

06.03.03.0008.01

$$\binom{n}{n+1} = 0$$

06.03.03.0009.01

$$\binom{n}{k} = 0 \;/;\; k - n \in \mathbb{N}^+$$

06.03.03.0011.01

$$\binom{n}{k} = -\frac{n!\,\sin(\pi\,k)}{\pi\,(-k)_{n+1}} \;/;\; n \in \mathbb{N} \bigwedge \neg\, k \in \mathbb{Z}$$

Values at fixed points

06.03.03.0010.01

$$\binom{0}{0} = 1$$

General characteristics

Domain and analyticity

$\binom{n}{k}$ is an analytical function of n and k which is defined over $\mathbb{C}^2$.

06.03.04.0001.01

$$(n * k) \longrightarrow \binom{n}{k} :: (\mathbb{C} \otimes \mathbb{C}) \longrightarrow \mathbb{C}$$

Symmetries and periodicities

Mirror symmetry

06.03.04.0002.01

$$\binom{\overline{n}}{\overline{k}} = \overline{\binom{n}{k}}$$

Periodicity

No periodicity

Poles and essential singularities

With respect to *k*

For fixed n, the function $\binom{n}{k}$ has only one singular point at $k = \tilde{\infty}$. It is an essential singular point.

06.03.04.0003.01

$$\mathcal{S}ing_k\left(\binom{n}{k}\right) = \{\{\tilde{\infty}, \infty\}\}$$

With respect to *n*

For fixed k, the function $\binom{n}{k}$ has an infinite set of singular points:

a) $n = -j /; j \in \mathbb{N}^+$, are the simple poles with residues $\frac{(-1)^j}{j!\, k!\, (-j-k)!} /; k \notin \mathbb{Z}$;

b) $n = \tilde{\infty}$ is the point of convergence of poles, which is an essential singular point.

06.03.04.0004.01

$$\mathcal{S}ing_n\left(\binom{n}{k}\right) = \left\{\left\{\{-j, 1\} /; j \in \mathbb{N}^+\right\}, \{\tilde{\infty}, \infty\}\right\}$$

06.03.04.0005.01

$$\operatorname{res}_n\left(\binom{n}{k}\right)(-j) = \frac{(-1)^j}{j!\, k!\, (-j-k)!} /; j \in \mathbb{N}^+ \bigwedge k \notin \mathbb{Z}$$

Branch points

With respect to *k*

The function $\binom{n}{k}$ does not have branch points with respect to k.

06.03.04.0006.01

$$\mathcal{BP}_k\left(\binom{n}{k}\right) = \{\}$$

With respect to *n*

The function $\binom{n}{k}$ does not have branch points with respect to n.

06.03.04.0007.01

$$\mathcal{BP}_n\left(\binom{n}{k}\right) = \{\}$$

Branch cuts

With respect to k

The function $\binom{n}{k}$ does not have branch cuts with respect to k.

06.03.04.0008.01

$$\mathcal{BC}_k\left(\binom{n}{k}\right) == \{\}$$

With respect to n

The function $\binom{n}{k}$ does not have branch cuts with respect to n.

06.03.04.0009.01

$$\mathcal{BC}_n\left(\binom{n}{k}\right) == \{\}$$

Series representations

Generalized power series

Expansions at generic point $n == n_0$

For the function itself

06.03.06.0008.01

$$\binom{n}{k} \propto \binom{n_0}{k}\left(1 + \left(H_{n_0} - H_{n_0-k}\right)(n-n_0) + \frac{1}{2}\left(\left(H_{n_0} - H_{n_0-k}\right)^2 + \psi^{(1)}(n_0+1) - \psi^{(1)}(n_0-k+1)\right)(n-n_0)^2 + \ldots\right) /; (n \to n_0) \bigwedge k \notin \mathbb{N}^+$$

06.03.06.0009.01

$$\binom{n}{k} \propto \sum_{i=1}^{k} \frac{(-1)^{i+k} S_k^{(i)} (n_0-k+1)^i}{k!}\left(1 + \frac{i(n-n_0)}{n_0-k+1} + \frac{((i-1)i)(n-n_0)^2}{2(n_0-k+1)^2} + \ldots\right) /; (n \to n_0) \bigwedge k \in \mathbb{N}^+$$

06.03.06.0010.01

$$\binom{n}{k} \propto \binom{n_0}{k}\left(1 + \left(H_{n_0} - H_{n_0-k}\right)(n-n_0) + \frac{1}{2}\left(\left(H_{n_0} - H_{n_0-k}\right)^2 + \psi^{(1)}(n_0+1) - \psi^{(1)}(n_0-k+1)\right)(n-n_0)^2 + O\left((n-n_0)^3\right)\right) /; k \notin \mathbb{N}^+$$

06.03.06.0011.01

$$\binom{n}{k} \propto \sum_{i=1}^{k} \frac{(-1)^{i+k} S_k^{(i)} (n_0-k+1)^i}{k!}\left(1 + \frac{i(n-n_0)}{n_0-k+1} + \frac{((i-1)i)(n-n_0)^2}{2(n_0-k+1)^2} + O\left((n-n_0)^3\right)\right) /; k \in \mathbb{N}^+$$

06.03.06.0012.01

$$\binom{n}{k} == \frac{\sin(\pi k)}{\pi} \sum_{m=0}^{\infty} (-1)^{m-1}\, \Gamma(n_0+1)^{m+1}\, {}_{m+2}\tilde{F}_{m+1}(a_1, a_2, \ldots, a_{m+1}, k+1; a_1+1, a_2+1, \ldots, a_{m+1}+1; 1)\,(n-n_0)^m /;$$

$$a_1 == a_2 == \ldots == a_{m+1} == n_0 + 1 \bigwedge k \notin \mathbb{N}^+$$

06.03.06.0013.01

$$\binom{n}{k} = \sum_{m=0}^{\infty} \sum_{i=1}^{k} \frac{(-1)^{i+k} S_k^{(i)} (i-m+1)_m (n_0-k+1)^{i-m}}{m!\, k!} (n-n_0)^m \;/;\; k \in \mathbb{N}^+$$

06.03.06.0014.01

$$\binom{n}{k} \propto \binom{n_0}{k} (1 + O(n-n_0)) \;/;\; k \in \mathbb{N}^+$$

Expansions at generic point $k = k_0$

For the function itself

06.03.06.0015.01

$$\binom{n}{k} \propto \binom{n}{k_0} \Bigg(1 + (\cos(\pi(n-k_0))\,\Gamma(n-k_0+1) - 1)\left(H_{k_0} - H_{-n+k_0-1}\right)(k-k_0) - \frac{-2\,\Gamma(k_0-n)^2 - 2\,\pi\cot(\pi(n-k_0))\,\Gamma(k_0-n) + \pi^2}{4\,\Gamma(k_0-n)^2} \left(\left(H_{k_0} - H_{-n+k_0-1}\right)^2 - \psi^{(1)}(k_0+1) + \psi^{(1)}(k_0-n)\right)(k-k_0)^2 + \ldots\Bigg) \;/;\; (k \to k_0)$$

06.03.06.0016.01

$$\binom{n}{k} \propto \binom{n}{k_0} \Bigg(1 + (\cos(\pi(n-k_0))\,\Gamma(n-k_0+1) - 1)\left(H_{k_0} - H_{-n+k_0-1}\right)(k-k_0) - \frac{-2\,\Gamma(k_0-n)^2 - 2\,\pi\cot(\pi(n-k_0))\,\Gamma(k_0-n) + \pi^2}{4\,\Gamma(k_0-n)^2} \left(\left(H_{k_0} - H_{-n+k_0-1}\right)^2 - \psi^{(1)}(k_0+1) + \psi^{(1)}(k_0-n)\right)(k-k_0)^2 + O\left((k-k_0)^3\right)\Bigg)$$

06.03.06.0017.01

$$\binom{n}{k} = \Gamma(k_0-n) \sum_{m=0}^{\infty} \frac{\pi^{m-1}}{m!} \sum_{j=0}^{m} \binom{m}{j} \sin\left(\pi\left(\frac{m-j}{2} - n + k_0\right)\right) j!\; {}_{m+2}\tilde{F}_{m+1}(a_1, a_2, \ldots, a_{m+1}, -n; a_1+1, a_2+1, \ldots, a_{m+1}+1; 1) \left(-\frac{\Gamma(k_0-n)}{\pi}\right)^j (k-k_0)^m \;/;\; a_1 = a_2 = \ldots = a_{m+1} = k_0 - n$$

06.03.06.0018.01

$$\binom{n}{k} \propto \binom{n}{k_0} (1 + O(k-k_0))$$

Asymptotic series expansions

Expansions at $n = \infty$

06.03.06.0002.01

$$\binom{n}{k} \propto \frac{n^k}{\Gamma(k+1)} \sum_{j=0}^{\infty} \frac{(-1)^j (-k)_j\, B(j, k+1, 1)\, n^{-j}}{j!} \;/;\; (|n| \to \infty) \bigwedge B(n, \alpha, 1) = n! \left([t^n]\, \frac{t^\alpha e^t}{(e^t - 1)^\alpha}\right) \bigwedge |\arg(n+1)| < \pi$$

06.03.06.0003.01

$$\binom{n}{k} \propto \frac{n^k}{\Gamma(k+1)} \left(1 + \frac{(1-k)\,k}{2\,n} + O\left(\frac{1}{n^2}\right)\right) \;/;\; (|n| \to \infty) \land |\arg(n+1)| < \pi$$

Expansions at $k = \infty$

06.03.06.0004.01

$$\binom{n}{k} \propto \frac{\Gamma(n+1)\sin(\pi(k-n))\,k^{-n-1}}{\pi} \sum_{j=0}^{\infty} \frac{\left((-1)^j (n+1)_j\right) B(j, -n, -n)\, k^{-j}}{j!} \;/;$$

$$(|k| \to \infty) \bigwedge B(n, \alpha, z) == n! \left([t^n] \frac{t^{\alpha} e^{t z}}{(e^t - 1)^{\alpha}}\right) \bigwedge |\arg(k-n)| < \pi$$

06.03.06.0005.01

$$\binom{n}{k} \propto \frac{\Gamma(n+1)\sin(\pi(k-n))\,k^{-n-1}}{\pi} \left(1 + \frac{(n+1)\,n}{2\,k} + O\!\left(\frac{1}{k^2}\right)\right) /;\ (|k| \to \infty) \bigwedge |\arg(k-n)| < \pi$$

Expansions at $n == 2\,k == \infty$

06.03.06.0019.01

$$\binom{2\,k}{k} \propto \frac{4^k}{\sqrt{\pi\,k}} \left(1 - \frac{1}{8\,k} + \frac{1}{128\,k^2} + \frac{5}{1024\,k^3} - \frac{21}{32\,768\,k^4} + O\!\left(\frac{1}{k^5}\right)\right) /;\ (|k| \to \infty)$$

Residue representations

06.03.06.0006.01

$$\binom{n}{k} == \operatorname{res}_z\!\left(z^{-k-1} (z+1)^n\right)(0) /;\ k \in \mathbb{Z} \bigwedge n \in \mathbb{Z}$$

06.03.06.0007.01

$$\binom{n}{k} == \operatorname{res}_z\!\left(z^{k-n-1} (z+1)^n\right)(0) /;\ k \in \mathbb{Z} \bigwedge n \in \mathbb{Z}$$

Other series representations

06.03.06.0001.01

$$\binom{n}{k} == \frac{2^{n-1}}{n} \sum_{j=1}^{2\,n} \cos^n\!\left(\frac{j\,\pi}{n}\right) \cos\!\left(\frac{j\,\pi\,(n-2\,k)}{n}\right) /;\ n \in \mathbb{N}^+ \bigwedge k \in \mathbb{N}^+ \bigwedge k < n$$

Integral representations

On the real axis

06.03.07.0002.01

$$\binom{n}{k} == \frac{1}{2\,\pi} \int_{-\pi}^{\pi} e^{-i\,k\,t} \left(1 + e^{i\,t}\right)^n dt /;\ k \in \mathbb{R}\ k > -1 \bigwedge n \in \mathbb{R}$$

Contour integral representations

06.03.07.0001.01

$$\binom{n}{k} == \frac{1}{2\,\pi\,i} \int_{|z|==1} (z+1)^n\, z^{-k-1}\, dz$$

Identities

Recurrence identities

Consecutive neighbors

06.03.17.0010.01

$$\binom{n}{k} = \frac{n}{n-k}\binom{n-1}{k}$$

06.03.17.0011.01

$$\binom{n}{k} = \frac{n-k+1}{n+1}\binom{n+1}{k}$$

06.03.17.0012.01

$$\binom{n}{k} = \frac{n-k+1}{k}\binom{n}{k-1}$$

06.03.17.0013.01

$$\binom{n}{k} = \frac{k+1}{n-k}\binom{n}{k+1}$$

Distant neighbors

06.03.17.0014.01

$$\binom{n}{k} = \frac{(n-m+1)_m}{(n-m+1)_m}\binom{n-m}{k}$$

06.03.17.0015.01

$$\binom{n}{k} = \frac{(n-k+1)_m}{(n+1)_m}\binom{m+n}{k}$$

06.03.17.0016.01

$$\binom{n}{k} = \frac{(n-k+1)_m}{(k-m+1)_m}\binom{n}{k-m}$$

06.03.17.0017.01

$$\binom{n}{k} = \frac{(k+1)_m}{(n-m-k+1)_m}\binom{n}{k+m}$$

Functional identities

Consecutive neighbors

06.03.17.0018.01

$$\binom{n}{k} = \frac{k+1}{n-k}\binom{n}{k+1}$$

06.03.17.0019.01

$$\binom{n}{k} = \frac{n-k+1}{k}\binom{n}{k-1}$$

06.03.17.0020.01

$$\binom{n}{k} = \frac{n-k+1}{n+1}\binom{n+1}{k}$$

06.03.17.0021.01

$$\binom{n}{k} = \frac{n}{n-k}\binom{n-1}{k}$$

Distant neighbors

06.03.17.0022.01

$$\binom{n}{k} = \frac{(-1)^m (k+1)_m}{(k-n)_m} \binom{n}{k+m} /; m \in \mathbb{N}^+$$

06.03.17.0023.01

$$\binom{n}{k} = \frac{(n-k+1)_m}{(k-m+1)_m} \binom{n}{k-m} /; m \in \mathbb{N}^+$$

06.03.17.0024.01

$$\binom{n}{k} = \frac{(n-k+1)_m}{(n+1)_m} \binom{m+n}{k} /; m \in \mathbb{N}^+$$

06.03.17.0025.01

$$\binom{n}{k} = \frac{(n-m+1)_m}{(n-k-m+1)_m} \binom{n-m}{k} /; m \in \mathbb{N}^+$$

Additional relations between contiguous functions

06.03.17.0001.01

$$\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}$$

Relations of special kind

06.03.17.0002.01

$$\binom{n}{k} = \binom{n}{n-k}$$

06.03.17.0003.01

$$\binom{z_1 - z_2}{z_4 - z_2} \binom{z_1 - z_3}{z_4} \binom{z_1 - z_2 - z_3}{z_4 - z_3} = \binom{z_1 - z_2}{z_4} \binom{z_1 - z_3}{z_4 - z_3} \binom{z_1 - z_2 - z_3}{z_4 - z_2}$$

06.03.17.0004.01

$$\binom{n}{k} \binom{n}{p-1} = \sum_{j=0}^{k} \left(\binom{n+1}{p-j+k} \binom{n}{j} - \binom{n}{p-j+k} \binom{n+1}{j} \right) /; p \in \mathbb{N}^+ \land k \in \mathbb{N}^+ \land n \in \mathbb{N}^+ \land p \geq k$$

06.03.17.0005.01

$$\binom{2k}{k} = (-4)^k \binom{-\frac{1}{2}}{k} /; k \in \mathbb{N}$$

06.03.17.0006.01

$$\binom{n}{k} = (-1)^k \binom{k-n-1}{k} /; k \in \mathbb{Z}$$

06.03.17.0009.01

$$\binom{z}{k} = \frac{1}{z-k} \left((k+z+1) \sum_{j=0}^{k} (-1)^j \binom{j+w+z}{k-j} \binom{2j+w}{j} - \sum_{j=0}^{k} (-4)^j \binom{j+z}{k-j} \right) /; k \in \mathbb{Z}$$

Identities involving determinants

06.03.17.0007.01

$$\left| \left(\frac{1}{k!} \binom{a\,k+l+x}{a\,k-l+y} \right)_{\substack{0\le k\le n \\ 0\le l\le n}} \right| = a^{\binom{n+1}{2}} \prod_{k=0}^{n} \frac{(a\,k+x)!\,\prod_{l=1}^{k}((k+l-1)\,a+x+y+1)}{(a\,k+y)!\,(2\,k+x-y)!}$$

06.03.17.0008.01

$$\left| \left(\frac{1}{k! \binom{a\,k+l+x}{a\,k-l+y}} \right)_{\substack{0\le k\le n \\ 0\le l\le n}} \right| = (-a)^{\binom{n+1}{2}} \prod_{k=0}^{n} \frac{(2\,k+x-y)!\,(a\,k-n+y)!\,\prod_{l=1}^{k}((k+l-1)\,a+x+y+1)}{(a\,k+n+x)!}$$

Differentiation

Low-order differentiation

With respect to *n*

06.03.20.0001.01

$$\frac{\partial \binom{n}{k}}{\partial n} = \binom{n}{k} (\psi(n+1) - \psi(-k+n+1))$$

06.03.20.0002.01

$$\frac{\partial^2 \binom{n}{k}}{\partial n^2} = \binom{n}{k} \left((\psi(n+1) - \psi(1-k+n))^2 + \psi^{(1)}(n+1) - \psi^{(1)}(1-k+n) \right)$$

With respect to *k*

06.03.20.0003.01

$$\frac{\partial \binom{n}{k}}{\partial k} = \binom{n}{k} (\psi(1-k+n) - \psi(k+1))$$

06.03.20.0004.01

$$\frac{\partial^2 \binom{n}{k}}{\partial k^2} = \binom{n}{k} \left((\psi(k+1) - \psi(1-k+n))^2 - \psi^{(1)}(k+1) - \psi^{(1)}(1-k+n) \right)$$

Symbolic differentiation

With respect to *n*

06.03.20.0005.02

$$\frac{\partial^m \binom{n}{k}}{\partial n^m} = \frac{(-1)^{m-1}\,m!\,\sin(\pi\,k)}{\pi} \sum_{j=0}^{\infty} \frac{(k+1)_j}{j!\,(j+n+1)^{m+1}} \;/;\; m \in \mathbb{N} \bigwedge n \notin \mathbb{N}$$

06.03.20.0006.02

$$\frac{\partial^m \binom{n}{k}}{\partial n^m} == \frac{(-1)^{m-1} \sin(\pi k)\, m!}{\pi} \Gamma(n+1)^{m+1} {}_{m+2}\tilde{F}_{m+1}(a_1, a_2, \ldots, a_{m+1}, k+1; a_1+1, a_2+1, \ldots, a_{m+1}+1; 1) /;$$
$$a_1 == a_2 == \ldots == a_{m+1} == n+1 \bigwedge m \in \mathbb{N} \bigwedge k \notin \mathbb{N}^+$$

06.03.20.0007.02

$$\frac{\partial^m \binom{n}{k}}{\partial n^m} == \frac{1}{k!} \sum_{j=1}^{k} (-1)^{j+k} S_k^{(j)} (j-m+1)_m (1-k+n)^{j-m} /; m \in \mathbb{N} \bigwedge k \in \mathbb{N}^+$$

With respect to *k*

06.03.20.0008.02

$$\frac{\partial^m \binom{n}{k}}{\partial k^m} == \pi^{m-1} \Gamma(k-n)$$
$$\sum_{j=0}^{m} \binom{m}{j} \sin\left(\pi \left(\frac{m-j}{2} + k - n\right)\right) j!\, {}_{m+2}\tilde{F}_{m+1}(a_1, a_2, \ldots, a_{m+1}, -n; a_1+1, a_2+1, \ldots, a_{m+1}+1; 1) \left(-\frac{\Gamma(k-n)}{\pi}\right)^j /;$$
$$a_1 == a_2 == \ldots == a_{m+1} == k-n \bigwedge m \in \mathbb{N}$$

Summation

Finite summation

06.03.23.0001.01

$$\sum_{k=0}^{n} \binom{n}{k} == 2^n /; n \in \mathbb{N}^+$$

06.03.23.0002.01

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} == 0 /; n \in \mathbb{N}^+$$

06.03.23.0012.01

$$\sum_{j=k}^{n} \binom{n}{j} \binom{j}{k} == \binom{n}{k} 2^{n-k} /; n \in \mathbb{N} \bigwedge k \in \mathbb{N}$$

06.03.23.0013.01

$$\sum_{j=k}^{n} \binom{n}{j} (-1)^{j-k} \binom{j}{k} = \delta_{n,k} /; n \in \mathbb{N} \bigwedge k \in \mathbb{N}$$

06.03.23.0014.01

$$\sum_{k=0}^{n} \frac{(-1)^k}{k+1} \binom{j}{k} \binom{j-k-1}{n-k} == \frac{1}{j+1} \left(\binom{j}{n+1} + (-1)^n\right) /; n \in \mathbb{N} \bigwedge k \in \mathbb{N}$$

06.03.23.0015.01

$$\sum_{k=0}^{n-m} \frac{(-1)^k}{k+1} \binom{k+n}{2k+m} \binom{2k}{k} == \binom{n-1}{m-1} /; n \in \mathbb{N} \bigwedge k \in \mathbb{N}$$

06.03.23.0011.01

$$\sum_{k=0}^{n}\binom{n}{k}\binom{k}{\left\lfloor\frac{k}{2}\right\rfloor}\binom{n-k}{\left\lfloor\frac{n-k}{2}\right\rfloor}=\binom{n}{\left\lfloor\frac{n}{2}\right\rfloor}\binom{n+1}{\left\lfloor\frac{n+1}{2}\right\rfloor}\,/;\, n\in\mathbb{N}^+$$

06.03.23.0016.01

$$\sum_{k=0}^{n-r}(-2)^{-k}\binom{n}{k+r}\binom{k+n+r}{k}=(-1)^m\,2^{-2m}\binom{n}{m}\,/;\, n\in\mathbb{N}\bigwedge r\in\mathbb{N}\bigwedge \frac{n-r}{2}\in\mathbb{Z}\bigwedge m=\frac{n-r}{2}$$

Infinite summation

06.03.23.0017.01

$$\sum_{k=1}^{\infty}\frac{1}{\binom{2k}{k}}=\frac{1}{3}+\frac{2\pi}{9\sqrt{3}}$$

06.03.23.0018.01

$$\sum_{k=1}^{\infty}\frac{k}{\binom{2k}{k}}=\frac{2}{3}+\frac{2\pi}{9\sqrt{3}}$$

06.03.23.0019.01

$$\sum_{k=1}^{\infty}\frac{k^2}{\binom{2k}{k}}=\frac{4}{3}+\frac{10\pi}{27\sqrt{3}}$$

06.03.23.0020.01

$$\sum_{k=1}^{\infty}\frac{k^3}{\binom{2k}{k}}=\frac{10}{3}+\frac{74\pi}{81\sqrt{3}}$$

06.03.23.0021.01

$$\sum_{k=1}^{\infty}\frac{k^4}{\binom{2k}{k}}=\frac{32}{3}+\frac{238\pi}{81\sqrt{3}}$$

06.03.23.0022.01

$$\sum_{k=1}^{\infty}\frac{k^5}{\binom{2k}{k}}=42+\frac{938\pi}{81\sqrt{3}}$$

06.03.23.0023.01

$$\sum_{k=1}^{\infty}\frac{k^6}{\binom{2k}{k}}=196+\frac{13\,130\pi}{243\sqrt{3}}$$

06.03.23.0024.01

$$\sum_{k=1}^{\infty}\frac{k^7}{\binom{2k}{k}}=\frac{3170}{3}+\frac{23\,594\pi}{81\sqrt{3}}$$

06.03.23.0025.01

$$\sum_{k=1}^{\infty} \frac{k^8}{\binom{2k}{k}} == \frac{19384}{3} + \frac{1298462\,\pi}{729\sqrt{3}}$$

06.03.23.0026.01

$$\sum_{k=1}^{\infty} \frac{k^9}{\binom{2k}{k}} == \frac{132550}{3} + \frac{26637166\,\pi}{2187\sqrt{3}}$$

06.03.23.0027.01

$$\sum_{k=1}^{\infty} \frac{k^{10}}{\binom{2k}{k}} == \frac{1002212}{3} + \frac{201403930\,\pi}{2187\sqrt{3}}$$

06.03.23.0028.01

$$\sum_{k=1}^{\infty} \frac{k^{11}}{\binom{2k}{k}} == 2767310 + \frac{5005052234\,\pi}{6561\sqrt{3}}$$

06.03.23.0029.01

$$\sum_{k=1}^{\infty} \frac{k^{12}}{\binom{2k}{k}} == 24922352 + \frac{135226271914\,\pi}{19683\sqrt{3}}$$

06.03.23.0030.01

$$\sum_{k=1}^{\infty} \frac{2^{-k}}{\binom{2k}{k}} == \frac{1}{7} + \frac{8\cot^{-1}\left(\sqrt{7}\right)}{7\sqrt{7}}$$

06.03.23.0031.01

$$\sum_{k=1}^{\infty} \frac{2^{-k}\,k}{\binom{2k}{k}} == \frac{12}{49} + \frac{40\cot^{-1}\left(\sqrt{7}\right)}{49\sqrt{7}}$$

06.03.23.0032.01

$$\sum_{k=1}^{\infty} \frac{2^{-k}\,k^2}{\binom{2k}{k}} == \frac{128}{343} + \frac{296\cot^{-1}\left(\sqrt{7}\right)}{343\sqrt{7}}$$

06.03.23.0033.01

$$\sum_{k=1}^{\infty} \frac{2^{-k}\,k^3}{\binom{2k}{k}} == \frac{220}{343} + \frac{472\cot^{-1}\left(\sqrt{7}\right)}{343\sqrt{7}}$$

06.03.23.0034.01

$$\sum_{k=1}^{\infty} \frac{2^{-k}\,k^4}{\binom{2k}{k}} == \frac{3256}{2401} + \frac{7064\cot^{-1}\left(\sqrt{7}\right)}{2401\sqrt{7}}$$

06.03.23.0035.01

$$\sum_{k=1}^{\infty} \frac{2^{-k} k^5}{\binom{2k}{k}} = \frac{59348}{16807} + \frac{129880 \cot^{-1}(\sqrt{7})}{16807 \sqrt{7}}$$

06.03.23.0036.01

$$\sum_{k=1}^{\infty} \frac{2^{-k} k^6}{\binom{2k}{k}} = \frac{1288944}{117649} + \frac{2822168 \cot^{-1}(\sqrt{7})}{117649 \sqrt{7}}$$

06.03.23.0037.01

$$\sum_{k=1}^{\infty} \frac{2^{-k} k^7}{\binom{2k}{k}} = \frac{32385804}{823543} + \frac{70856152 \cot^{-1}(\sqrt{7})}{823543 \sqrt{7}}$$

06.03.23.0038.01

$$\sum_{k=1}^{\infty} \frac{2^{-k} k^8}{\binom{2k}{k}} = \frac{923030312}{5764801} + \frac{2019098264 \cot^{-1}(\sqrt{7})}{5764801 \sqrt{7}}$$

06.03.23.0039.01

$$\sum_{k=1}^{\infty} \frac{2^{-k} k^9}{\binom{2k}{k}} = \frac{29423171140}{40353607} + \frac{64364670040 \cot^{-1}(\sqrt{7})}{40353607 \sqrt{7}}$$

06.03.23.0040.01

$$\sum_{k=1}^{\infty} \frac{2^{-k} k^{10}}{\binom{2k}{k}} = \frac{148182746272}{40353607} + \frac{324165551144 \cot^{-1}(\sqrt{7})}{40353607 \sqrt{7}}$$

06.03.23.0041.01

$$\sum_{k=1}^{\infty} \frac{2^{-k} k^{11}}{\binom{2k}{k}} = \frac{5724192868004}{282475249} + \frac{12522296655976 \cot^{-1}(\sqrt{7})}{282475249 \sqrt{7}}$$

06.03.23.0042.01

$$\sum_{k=1}^{\infty} \frac{2^{-k} k^{12}}{\binom{2k}{k}} = \frac{240432226362024}{1977326743} + \frac{525970525309352 \cot^{-1}(\sqrt{7})}{1977326743 \sqrt{7}}$$

06.03.23.0043.01

$$\sum_{k=1}^{\infty} \frac{2^{-k}}{\binom{3k}{k}} = \frac{2}{25} + \frac{11 \pi}{250} - \frac{6 \log(2)}{125}$$

06.03.23.0044.01

$$\sum_{k=1}^{\infty} \frac{2^{-k} k}{\binom{3k}{k}} = \frac{81}{625} + \frac{79 \pi}{3125} - \frac{18 \log(2)}{3125}$$

06.03.23.0045.01

$$\sum_{k=1}^{\infty} \frac{2^{-k} k^2}{\binom{3k}{k}} = \frac{561}{3125} + \frac{673\pi}{31250} + \frac{42\log(2)}{15625}$$

06.03.23.0046.01

$$\sum_{k=1}^{\infty} \frac{2^{-k} k^3}{\binom{3k}{k}} = \frac{20517}{78125} + \frac{21331\pi}{781250} + \frac{774\log(2)}{390625}$$

06.03.23.0056.01

$$\sum_{k=1}^{\infty} \frac{10k^2 + 9k + 1}{k(2k-1)(2k+1)^2 \binom{2k}{k}} = 3 - \frac{\pi}{\sqrt{3}}$$

G.Huvent (2006)

Infinite summation

06.03.23.0003.01

$$\sum_{k=0}^{\infty} \binom{n}{k} (-1)^k = 0$$

06.03.23.0004.01

$$\sum_{k=0}^{\infty} \binom{n}{k} k = 2^{n-1} n$$

06.03.23.0005.01

$$\sum_{k=0}^{\infty} \binom{n}{k} \binom{m}{j-k} = \binom{m+n}{j}$$

06.03.23.0006.01

$$\sum_{k=0}^{\infty} \binom{n}{k} \binom{m}{j+k} = \binom{m+n}{m-j}$$

06.03.23.0007.01

$$\sum_{k=0}^{\infty} \binom{n}{k} \binom{k+m}{j} (-1)^k = \binom{m}{j-n} \csc(\pi(j-n)) \sin(\pi j) \;/;\; j \in \mathbb{N} \land m \in \mathbb{N} \land n \in \mathbb{N}$$

06.03.23.0008.01

$$\sum_{k=0}^{\infty} \binom{n}{k} \binom{m-k}{j} (-1)^k = \binom{m-n}{m-j} \sin(j\pi) \sin((m-n)\pi) \csc(m\pi) \csc((j-n)\pi) \;/;\; j \in \mathbb{N} \land m \in \mathbb{N} \land n \in \mathbb{N}$$

06.03.23.0009.01

$$\binom{\sum_{k=0}^{l} m_k p^k}{\sum_{k=0}^{l} n_k p^k} \bmod p = \left(\prod_{k=0}^{l} \binom{m_k}{n_k}\right) \bmod p \;/;\; p \in \mathbb{P} \land m_k \in \mathbb{N} \land n_k \in \mathbb{N} \land m_k < p \land n_k < p$$

06.03.23.0010.01

$$\max\left(\binom{n}{0}, \binom{n}{1}, \ldots, \binom{n}{n}\right) = \binom{n-l}{l} \;/;\; l = \left\lfloor \frac{1}{10}\left(5n - \sqrt{5n^2 + 10n + 9} + 7\right) \right\rfloor$$

Multiple sums

06.03.23.0047.01

$$\sum_{k=0}^{n}\sum_{j=0}^{k}\binom{n}{j} = (n+2)\,2^{n-1} \;/;\; n \in \mathbb{N}$$

06.03.23.0048.01

$$\sum_{k=0}^{n}\left(\sum_{j=0}^{k}\binom{n}{j}\right)^2 = (n+2)\,2^{2n-1} - \frac{1}{2}\,n\binom{2n}{n} \;/;\; n \in \mathbb{N}$$

06.03.23.0049.01

$$\sum_{k=0}^{n}\left(\sum_{j=0}^{k}\binom{n}{j}\right)^3 = (n+2)\,2^{3n-1} - 3\,2^{n-2}\,n\binom{2n}{n} \;/;\; n \in \mathbb{N}$$

06.03.23.0050.01

$$\sum_{m=0}^{k-1}\sum_{j=k}^{n}\frac{(-1)^{j+m-1}}{j-m}\binom{n}{m}\binom{n}{j} = \sum_{m=0}^{k-1}\binom{n}{m}^2 (H_{n-m} - H_m) \;/;\; n \in \mathbb{N}^+ \bigwedge k \in \mathbb{N} \bigwedge n \geq k+1$$

06.03.23.0051.01

$$\sum_{k=0}^{n}\sum_{l=0}^{m}(-1)^{k+l}\binom{n}{k}\binom{m}{l}\binom{k+l}{l} = \delta_{n,m} \;/;\; n \in \mathbb{N} \wedge m \in \mathbb{N}$$

06.03.23.0052.01

$$\sum_{k=0}^{n}\sum_{l=0}^{m}(-1)^{k+l}\binom{n}{k}\binom{n}{n-k}\binom{m}{l}\binom{m}{m-l}\mathrm{B}(-k-l+m+n+1,\, k+l+1) = \delta_{n,m}\,\frac{(2n)!}{n!^2} \;/;\; n \in \mathbb{N} \wedge m \in \mathbb{N}$$

06.03.23.0053.01

$$\sum_{k=0}^{n}\sum_{l=0}^{m}(-1)^{k+l}\binom{n+\alpha}{k}\binom{n+\beta}{n-k}\binom{m+\alpha}{l}\binom{m+\beta}{m-l}\mathrm{B}(-k-l+m+n+\alpha+1,\, k+l+\beta+1) = \delta_{n,m}\,\frac{1}{2n+\alpha+\beta+1}\,\frac{\mathrm{B}(n+\alpha+1,\, n+\beta+1)}{\mathrm{B}(n+1,\, n+\alpha+\beta+1)} \;/;\; n \in \mathbb{N} \wedge m \in \mathbb{N}$$

06.03.23.0054.01

$$\sum_{k=0}^{\frac{n}{2}}\sum_{l=0}^{\frac{m}{2}}\frac{(-1)^{k+l}}{m+n-2k-2l+1}\binom{n}{k}\binom{m}{l}\binom{2n-2k}{n}\binom{2m-2l}{m} = \delta_{n,m}\,\frac{2^{2n}}{2n+1} \;/;\; n \in \mathbb{N} \wedge m \in \mathbb{N}$$

06.03.23.0055.01

$$\sum_{k=1}^{\infty}\sum_{j=0}^{\infty}\frac{\binom{2k-1}{j}}{2^{2k-1}\,(2j-1)\left(j-k-n+\frac{1}{2}\right)} = \frac{2}{n}\left(H_{2n} - \frac{H_n}{2}\right) \;/;\; n \in \mathbb{N}^+$$

R. Lyons, P. Paule, A. Riese: A Computer Proof of a Series Evaluation in Terms of Harmonic Numbers Applicable Algebra in Engineering, Communication and Computing 13, 327-333 (2002)

Representations through more general functions

Through other functions

Involving some hypergeometric-type functions

06.03.26.0001.01

$$\binom{n}{k} == (n;\, n-k,\, k)$$

Representations through equivalent functions

With related functions

06.03.27.0001.01

$$\binom{n}{k} == \frac{n!}{(n-k)!\, k!}$$

06.03.27.0002.01

$$\binom{n}{k} == \frac{\Gamma(n+1)}{\Gamma(k+1)\, \Gamma(1-k+n)}$$

06.03.27.0003.01

$$\binom{n}{k} == \frac{(k+1)_{n-k}}{\Gamma(1-k+n)}$$

06.03.27.0004.01

$$\binom{n}{k} == \frac{(1-k+n)_k}{\Gamma(k+1)}$$

06.03.27.0005.01

$$\binom{n}{k} == \frac{(-1)^k\, (-n)_k}{k!} \;/;\; k \in \mathbb{Z}$$

06.03.27.0006.01

$$\binom{n}{k} = \frac{1}{k\, \mathrm{B}(k,\, n-k+1)}$$

06.03.27.0007.01

$$\binom{2z}{z} = (z+1)\, C_z$$

Inequalities

06.03.29.0001.01

$$\prod_{k=1}^{n} \left(k! \binom{n}{k}\right)^{\frac{k}{n^k}} \leq \left(\frac{n^n\, (n-1)^2}{n^n - n}\right)^{\frac{n-n^{2-n}}{(n-1)^2}} \;/;\; k \in \mathbb{N} \bigwedge n \in \mathbb{N} \bigwedge n \geq 2$$

06.03.29.0002.01

$$\binom{m\,n}{n} \geq \frac{m^{(n-1)\,m+1}}{(m-1)^{(m-1)\,(n-1)}}\, n^{-\frac{1}{2}} \;/;\; n \in \mathbb{N} \land n \in \mathbb{N} \land n \geq 1 \land m \geq 2$$

06.03.29.0003.01

$$d_n(\alpha+\beta+\lambda)\sum_{k=0}^{n}\frac{d_{n-k}(\lambda)}{d_k(\alpha+\beta)}\left|\sum_{j=0}^{k}a_j\,b_{k-j}\right|^2 \leq \left(\sum_{k=0}^{n}\frac{d_{n-k}(\beta+\lambda)}{d_k(\alpha)}\,|a_k|^2\right)\sum_{k=0}^{n}\frac{d_{n-k}\,(\alpha+\lambda)}{d_k(\beta)}\,|b_k|^2\ /;$$

$$d_k(\alpha)=\binom{k+\alpha-1}{k}\bigwedge\alpha\in\mathbb{R}^+\bigwedge\beta\in\mathbb{R}^+\bigwedge\lambda\in\mathbb{R}\bigwedge\lambda\geq 0\bigwedge\max(|\{a_1,\,a_2,\,\ldots,\,a_n\}|)>0\bigwedge\max(|\{b_1,\,b_2,\,\ldots,\,b_n\}|)>0$$

Generalized Cauchy–Schwarz inequality. $\lambda = 0$ gives the Cauchy–Schwarz inequality.

06.03.29.0004.01

$$2\,d_n(\gamma+\lambda+\mu+\nu)\sum_{k=0}^{n}\frac{d_{n-k}(\lambda)}{d_k(\gamma+\mu+\nu)}\left|\left(\sum_{j=0}^{k}u_j\,v_{k-j}\right)\sum_{j=0}^{k}x_j\,y_{k-j}\right|\leq$$

$$\left(\sum_{k=0}^{n}\frac{d_{n-k}(\lambda+\nu)}{d_k(\gamma+\mu)}\,|u_k|^2\right)\left(\sum_{k=0}^{n}\frac{d_{n-k}(\lambda+\mu)}{d_k(\gamma+\nu)}\,|y_k|^2\right)\left(\sum_{k=0}^{n}\frac{d_{n-k}(\gamma+\lambda+\nu)}{d_k(\mu)}\,|x_k|^2\right)\sum_{k=0}^{n}\frac{d_{n-k}(\gamma+\lambda+\mu)}{d_k(\nu)}\,|v_k|^2\ /;$$

$$d_k(\alpha)=\binom{k+\alpha-1}{k}\bigwedge\mu\in\mathbb{R}^+\bigwedge\nu\in\mathbb{R}^+\,0\bigwedge\gamma\in\mathbb{R}\bigwedge\gamma\geq 0\bigwedge\lambda\in\mathbb{R}\bigwedge\lambda\geq 0\bigwedge\max(|\{u_1,\,u_2,\,\ldots,\,u_n\}|)>0\bigwedge$$
$$\max(|\{v_1,\,v_2,\,\ldots,\,v_n\}|)>0\bigwedge\max(|\{x_1,\,x_2,\,\ldots,\,x_n\}|)>0\bigwedge\max(|\{y_1,\,y_2,\,\ldots,\,y_n\}|)>0$$

06.03.29.0005.01

$$d_n(\alpha+\beta+\gamma)\,d_k(\alpha)\,d_j(\alpha)\,d_{-j-k+n}(\gamma)\leq d_{j+k}(\alpha+\beta)\,d_{n-j}(\alpha+\gamma)\,d_{n-k}(\beta+\gamma)\ /;$$

$$d_k(\alpha)=\binom{k+\alpha-1}{k}\bigwedge\alpha\in\mathbb{R}^+\bigwedge\beta\in\mathbb{R}^+\bigwedge\gamma\in\mathbb{R}^+\bigwedge j\in\mathbb{N}\bigwedge k\in\mathbb{N}\bigwedge n\in\mathbb{N}^+\bigwedge j+k\leq n$$

Zeros

06.03.30.0001.01

$$\binom{n}{k}=0\ /;\ k=-j\land j\in\mathbb{N}^+\land -n\notin\mathbb{N}^+$$

06.03.30.0002.01

$$\binom{n}{k}=0\ /;\ n=k-j\land j\in\mathbb{N}^+\land -n\notin\mathbb{N}^+$$

Theorems

The binomial expansions

$$(a+b)^n=\sum_{k=0}^{n}\binom{n}{k}a^k\,b^{n-k};$$

$$(a+b)^n=\sum_{k=0}^{n}\binom{n}{k}a\,(a-k\,z)^{k-1}\,(b+k\,z)^{n-k}\ /;\ n\in\mathbb{N}^+,\ a\neq 0$$

The derivative of products

$$\frac{\partial^n(f(z)\,g(z))}{\partial z^n}=\sum_{k=0}^{n}\binom{n}{k}\frac{\partial^k f(z)}{\partial z^k}\,\frac{\partial^{n-k}g(z)}{\partial z^{n-k}}.$$

Representations of polynomials in Bezier form

Every polynomial $p_n(x)$ can be represented in Bezier form, that is, as $p_n(x) = \sum_{k=0}^{n} \beta_k B_k^n(x)$,

where the Bernstein polynomials $B_k^n(x)$ are given as $B_k^n(x) = \binom{n}{k} x^k (1-x)^{n-k}$.

Inversion of the Hilbert matrix

The inverse $\boldsymbol{A}^{-1}$ of the Hilbert matrix $\boldsymbol{A} = \{a_{i\,j}\}_{1 \le i, j \le n}$ with entries $a_{i\,j} = \frac{1}{i+j-1}$ has the entries

$$(-1)^{i+j} (i+j-1) \binom{n+i-1}{n-j} \binom{n+j-1}{n-i} \binom{i+j-2}{i-1}^2.$$

Generalized binomial theorem

$$A_n(x, y, p, q) = \sum_{k=0}^{n} \binom{n}{k} k! \, (k+x) \, A_{n-k}(k+x, y, p-1, q) \,/;$$

$$A_n(x, y, p, q) = \sum_{k=0}^{n} \binom{n}{k} (k+x)^{k+p} (n-k+y)^{n-k+q} \bigwedge n \in \mathbb{N} \bigwedge p, q \in \mathbb{Z}$$

Weyl ordering

The Weyl ordering of the operator product $(\hat{q}^n \, \hat{p}^m)_W$ of the operators $\hat{q}$ and $\hat{p}$ is given by

$$(\hat{q}^n \, \hat{p}^m)_W = 2^{-n} \sum_{k=0}^{n} \binom{n}{k} \hat{q}^{n-k} \, \hat{p}^m \, \hat{q}^k.$$

History

– Chia Hsien(1050), al-Karaji (about 1100), Omar al-Khayyami (1080), Bhaskara Acharya (1150), al-Samaw'al (1175), Yang Hui (1261), Tshu shi Kih (1303), Shih-Chieh Chu (1303), M. Stifel (1544), Cardano (1545), Scheubel (1545), Peletier (1549), Tartaglia (1556), Cardan (1570), Stevin (1585), Faulhaber (1615), Girard (1629), Oughtred (1631), Briggs (1633), Mersenne (1636), Fermat (1636), Wallis (1656), Montmort (1708), De Moivre (1730)

- B. Pascal (1653) gave a recursion relation

– I. Newton (1676) studied cases with fractional arguments

– G.W. Leibniz (1695)

– L. Euler (1774, 1781) used notations with parentheses

– C. F. Gauss (1812)

– A. von Ettinghausen (1826) introduced the binomial symbol

– Förstemann (1835) gave combinatorical interpretation of binomial coefficient

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