BesselK

View the online version at
● functions.wolfram.com
Download the
● PDF File

Notations

Traditional name

Modified Bessel function of the second kind

Traditional notation

$K_\nu(z)$

Mathematica StandardForm notation

`BesselK[ν, z]`

Primary definition

03.04.02.0001.01

$$K_\nu(z) = \frac{\pi \csc(\pi \nu)}{2} (I_{-\nu}(z) - I_\nu(z)) /; \nu \notin \mathbb{Z}$$

03.04.02.0002.01

$$K_\nu(z) = \lim_{\mu \to \nu} K_\mu(z) /; \nu \in \mathbb{Z}$$

Specific values

Specialized values

For fixed ν

03.04.03.0001.01

$$K_\nu(0) = \tilde{\infty} /; \operatorname{Re}(\nu) \neq 0$$

03.04.03.0002.01

$$K_\nu(0) = ¿ /; \operatorname{Re}(\nu) = 0 \land \nu \neq 0$$

For fixed z

Explicit rational ν

03.04.03.0005.01

$$K_{\frac{1}{3}}(z) = \frac{\sqrt[3]{2} \sqrt[6]{3} \pi}{\sqrt[3]{z}} \operatorname{Ai}\left(\left(\frac{3}{2}\right)^{2/3} z^{2/3}\right)$$

03.04.03.0003.01

$$K_{\frac{1}{2}}(z) = \sqrt{\frac{\pi}{2}}\,\frac{e^{-z}}{\sqrt{z}}$$

03.04.03.0006.01

$$K_{\frac{2}{3}}(z) = -\frac{2^{2/3}\,\pi}{\sqrt[6]{3}\,z^{2/3}}\,\mathrm{Ai}'\!\left(\left(\frac{3}{2}\right)^{2/3} z^{2/3}\right)$$

03.04.03.0011.01

$$K_{\frac{4}{3}}(z) = \frac{2\sqrt[3]{2}\,\pi\,\mathrm{Ai}\!\left(\left(\frac{3}{2}\right)^{2/3} z^{2/3}\right) - 6^{2/3}\,\pi\,z^{2/3}\,\mathrm{Ai}'\!\left(\left(\frac{3}{2}\right)^{2/3} z^{2/3}\right)}{3^{5/6}\,z^{4/3}}$$

03.04.03.0012.01

$$K_{\frac{3}{2}}(z) = \sqrt{\frac{\pi}{2}}\,\frac{e^{-z}\,(z+1)}{z^{3/2}}$$

03.04.03.0013.01

$$K_{\frac{5}{3}}(z) = \frac{9\sqrt[3]{2}\,\pi\,z^{4/3}\,\mathrm{Ai}\!\left(\left(\frac{3}{2}\right)^{2/3} z^{2/3}\right) - 4\,6^{2/3}\,\pi\,\mathrm{Ai}'\!\left(\left(\frac{3}{2}\right)^{2/3} z^{2/3}\right)}{3\,3^{5/6}\,z^{5/3}}$$

03.04.03.0014.01

$$K_{\frac{7}{3}}(z) = \frac{\pi}{3\sqrt[3]{2}\,3^{5/6}\,z^{7/3}}\left(2^{2/3}\,(9z^2+16)\,\mathrm{Ai}\!\left(\left(\frac{3}{2}\right)^{2/3} z^{2/3}\right) - 16\,3^{2/3}\,z^{2/3}\,\mathrm{Ai}'\!\left(\left(\frac{3}{2}\right)^{2/3} z^{2/3}\right)\right)$$

03.04.03.0015.01

$$K_{\frac{5}{2}}(z) = \sqrt{\frac{\pi}{2}}\,\frac{e^{-z}\,(z^2+3z+3)}{z^{5/2}}$$

03.04.03.0016.01

$$K_{\frac{8}{3}}(z) = \frac{1}{9\,3^{5/6}\,z^{8/3}}\left(90\sqrt[3]{2}\,\pi\,z^{4/3}\,\mathrm{Ai}\!\left(\left(\frac{3}{2}\right)^{2/3} z^{2/3}\right) - 6^{2/3}\,\pi\,(9z^2+40)\,\mathrm{Ai}'\!\left(\left(\frac{3}{2}\right)^{2/3} z^{2/3}\right)\right)$$

03.04.03.0017.01

$$K_{\frac{10}{3}}(z) = \frac{16\sqrt[3]{2}\,\pi\,(9z^2+14)\,\mathrm{Ai}\!\left(\left(\frac{3}{2}\right)^{2/3} z^{2/3}\right) - 6^{2/3}\,\pi\,z^{2/3}\,(9z^2+112)\,\mathrm{Ai}'\!\left(\left(\frac{3}{2}\right)^{2/3} z^{2/3}\right)}{9\,3^{5/6}\,z^{10/3}}$$

03.04.03.0018.01

$$K_{\frac{7}{2}}(z) = \sqrt{\frac{\pi}{2}}\,\frac{e^{-z}\,(z^3+6z^2+15z+15)}{z^{7/2}}$$

03.04.03.0019.01

$$K_{\frac{11}{3}}(z) = \frac{1}{27\,3^{5/6}\,z^{11/3}}\left(9\sqrt[3]{2}\,\pi\,z^{4/3}\,(9z^2+160)\,\mathrm{Ai}\!\left(\left(\frac{3}{2}\right)^{2/3} z^{2/3}\right) - 20\,6^{2/3}\,\pi\,(9z^2+32)\,\mathrm{Ai}'\!\left(\left(\frac{3}{2}\right)^{2/3} z^{2/3}\right)\right)$$

03.04.03.0020.01

$$K_{\frac{13}{3}}(z) = \frac{\pi}{27\sqrt[3]{2}\,3^{5/6}\,z^{13/3}}\left(2^{2/3}\,(81z^4+3024z^2+4480)\,\mathrm{Ai}\!\left(\left(\frac{3}{2}\right)^{2/3} z^{2/3}\right) - 56\,3^{2/3}\,z^{2/3}\,(9z^2+80)\,\mathrm{Ai}'\!\left(\left(\frac{3}{2}\right)^{2/3} z^{2/3}\right)\right)$$

03.04.03.0021.01

$$K_{\frac{9}{2}}(z) = \sqrt{\frac{\pi}{2}} \frac{e^{-z}\left(z^4+10z^3+45z^2+105z+105\right)}{z^{9/2}}$$

03.04.03.0022.01

$$K_{\frac{14}{3}}(z) = \frac{288\sqrt[3]{2}\,\pi z^{4/3}\left(9z^2+110\right)\operatorname{Ai}\left(\left(\frac{3}{2}\right)^{2/3} z^{2/3}\right) - 6^{2/3}\pi\left(81z^4+4320z^2+14080\right)\operatorname{Ai}'\left(\left(\frac{3}{2}\right)^{2/3} z^{2/3}\right)}{81\,3^{5/6} z^{14/3}}$$

Symbolic rational ν

03.04.03.0004.01

$$K_\nu(z) = \sqrt{\frac{\pi}{2}} \frac{e^{-z}}{\sqrt{z}} \sum_{j=0}^{\left\lfloor |\nu|-\frac{1}{2}\right\rfloor} \frac{\left(j+|\nu|-\frac{1}{2}\right)!}{j!\left(-j+|\nu|-\frac{1}{2}\right)!}(2z)^{-j} \;/;\; \nu-\frac{1}{2}\in\mathbb{Z}$$

03.04.03.0007.01

$$K_\nu(z) = \frac{(-1)^{|\nu|-\frac{1}{3}} 2^{|\nu|-\frac{2}{3}} \pi z^{-|\nu|}\Gamma\left(-\frac{1}{3}\right)}{3^{5/6}\Gamma(1-|\nu|)} \left(3^{2/3} z^{2/3}\operatorname{Ai}'\left(\left(\frac{3}{2}\right)^{2/3} z^{2/3}\right)\sum_{k=0}^{|\nu|-\frac{4}{3}} \frac{\left(|\nu|-k-\frac{4}{3}\right)!}{k!\left(|\nu|-2k-\frac{4}{3}\right)!\left(\frac{4}{3}\right)_k(1-|\nu|)_k}\left(-\frac{z^2}{4}\right)^k - \right.$$

$$\left. 2^{2/3}\operatorname{Ai}\left(\left(\frac{3}{2}\right)^{2/3} z^{2/3}\right)\sum_{k=0}^{|\nu|-\frac{1}{3}} \frac{\left(|\nu|-k-\frac{1}{3}\right)!}{k!\left(|\nu|-2k-\frac{1}{3}\right)!\left(\frac{1}{3}\right)_k(1-|\nu|)_k}\left(-\frac{z^2}{4}\right)^k\right) \;/;\; |\nu|-\frac{1}{3}\in\mathbb{Z}$$

03.04.03.0008.01

$$K_\nu(z) = \frac{(-1)^{|\nu|+\frac{1}{3}} 2^{|\nu|-\frac{4}{3}} \pi z^{-|\nu|}\Gamma\left(-\frac{2}{3}\right)}{3\,3^{5/6}\Gamma(1-|\nu|)} \left(9z^{4/3}\operatorname{Ai}\left(\left(\frac{3}{2}\right)^{2/3} z^{2/3}\right)\sum_{k=0}^{|\nu|-\frac{5}{3}} \frac{\left(|\nu|-k-\frac{5}{3}\right)!}{k!\left(|\nu|-2k-\frac{5}{3}\right)!\left(\frac{5}{3}\right)_k(1-|\nu|)_k}\left(-\frac{z^2}{4}\right)^k - \right.$$

$$\left. 4\sqrt[3]{2}\,3^{2/3}\operatorname{Ai}'\left(\left(\frac{3}{2}\right)^{2/3} z^{2/3}\right)\sum_{k=0}^{|\nu|-\frac{2}{3}} \frac{\left(|\nu|-k-\frac{2}{3}\right)!}{k!\left(|\nu|-2k-\frac{2}{3}\right)!\left(\frac{2}{3}\right)_k(1-|\nu|)_k}\left(-\frac{z^2}{4}\right)^k\right) \;/;\; |\nu|-\frac{2}{3}\in\mathbb{Z}$$

Values at fixed points

03.04.03.0009.01

$$K_0(0) = \infty$$

Values at infinities

03.04.03.0010.01

$$\lim_{x\to\infty} K_\nu(x) = 0$$

03.04.03.0023.01

$$K_\nu\left(e^{i\lambda}\infty\right) = \begin{cases} 0 & -\frac{\pi}{2}\le\lambda\le\frac{\pi}{2} \\ \tilde{\infty} & \text{True} \end{cases} \;/;\; \operatorname{Im}(\lambda)=0$$

03.04.03.0024.01

$$K_\nu(i\infty) = 0$$

03.04.03.0025.01

$K_\nu(-i\,\infty) = 0$

General characteristics

Domain and analyticity

$K_\nu(z)$ is an analytical function of ν and z which is defined in $\mathbb{C}^2$.

03.04.04.0001.01

$(\nu * z) \longrightarrow K_\nu(z) :: (\mathbb{C} \otimes \mathbb{C}) \longrightarrow \mathbb{C}$

Symmetries and periodicities

Parity

$K_\nu(z)$ is an even function with respect to its parameter.

03.04.04.0002.01

$K_{-\nu}(z) = K_\nu(z)$

Mirror symmetry

03.04.04.0004.01

$K_{\bar{\nu}}(\bar{z}) = \overline{K_\nu(z)} \;/;\; z \notin (-\infty, 0)$

Periodicity

No periodicity

Poles and essential singularities

With respect to z

For fixed ν, the function $K_\nu(z)$ has an essential singularity at $z = \tilde{\infty}$. At the same time, the point $z = \tilde{\infty}$ is a branch point.

03.04.04.0005.01

$\mathcal{S}ing_z(K_\nu(z)) = \{\{\tilde{\infty}, \infty\}\}$

With respect to ν

For fixed z, the function $K_\nu(z)$ has only one singular point at $\nu = \tilde{\infty}$. It is an essential singular point.

03.04.04.0006.01

$\mathcal{S}ing_\nu(K_\nu(z)) = \{\{\tilde{\infty}, \infty\}\}$

Branch points

With respect to z

For fixed ν, the function $K_\nu(z)$ has two branch points: $z = 0$, $z = \tilde{\infty}$. At the same time, the point $z = \tilde{\infty}$ is an essential singularity.

03.04.04.0007.01

$\mathcal{BP}_z(K_\nu(z)) = \{0, \tilde{\infty}\}$

03.04.04.0008.01

$\mathcal{R}_z(K_\nu(z), 0) = \log /; \nu \notin \mathbb{Q}$

03.04.04.0009.01

$\mathcal{R}_z\left(K_{\frac{p}{q}}(z), 0\right) = q /; p \in \mathbb{Z} \bigwedge q - 1 \in \mathbb{N}^+ \bigwedge \gcd(p, q) = 1$

03.04.04.0010.01

$\mathcal{R}_z(K_\nu(z), \tilde{\infty}) = \log /; \nu \notin \mathbb{Q}$

03.04.04.0011.01

$\mathcal{R}_z\left(K_{\frac{p}{q}}(z), \tilde{\infty}\right) = q /; p \in \mathbb{Z} \bigwedge q - 1 \in \mathbb{N}^+ \bigwedge \gcd(p, q) = 1$

With respect to ν

For fixed z, the function $K_\nu(z)$ does not have branch points.

03.04.04.0012.01

$\mathcal{BP}_\nu(K_\nu(z)) = \{\}$

Branch cuts

With respect to z

For fixed ν, the function $K_\nu(z)$ has one infinitely long branch cut. For fixed ν, the function $K_\nu(z)$ is a single-valued function on the z-plane cut along the interval $(-\infty, 0)$, where it is continuous from above.

03.04.04.0013.01

$\mathcal{BC}_z(K_\nu(z)) = \{\{(-\infty, 0), -i\}\}$

03.04.04.0014.01

$\lim_{\epsilon \to +0} K_\nu(x + i\,\epsilon) = K_\nu(x) /; x \in \mathbb{R} \wedge x < 0$

03.04.04.0015.01

$\lim_{\epsilon \to +0} K_\nu(x - i\,\epsilon) = \frac{\pi \csc(\nu\,\pi)}{2} \left(e^{2\pi i \nu} I_{-\nu}(x) - e^{-2\pi i \nu} I_\nu(x)\right) /; \nu \notin \mathbb{Z} \bigwedge x \in \mathbb{R} \bigwedge x < 0$

03.04.04.0018.01

$\lim_{\epsilon \to +0} K_\nu(x - i\,\epsilon) = e^{2 i \pi \nu} K_\nu(x) + 2\,i\,\pi\, I_\nu(x) \cos(\pi\,\nu) /; x \in \mathbb{R} \wedge x < 0$

03.04.04.0016.01

$\lim_{\epsilon \to +0} K_\nu(x - i\,\epsilon) = (-1)^\nu\, 2\,i\,\pi\, I_\nu(x) + K_\nu(x) /; \nu \in \mathbb{Z} \wedge x \in \mathbb{R} \wedge x < 0$

With respect to ν

For fixed z, the function $K_\nu(z)$ is an entire function of ν and does not have branch cuts.

03.04.04.0017.01

$\mathcal{BC}_\nu(K_\nu(z)) = \{\}$

Series representations

Generalized power series

Expansions at $\nu == \pm n$

03.04.06.0015.01

$$K_\nu(z) \propto K_n(z) + \left(\frac{1}{2}\operatorname{sgn}(n)\,|n|!\left(\frac{z}{2}\right)^{-|n|}\sum_{k=0}^{|n|-1}\frac{1}{(|n|-k)\,k!}K_k(z)\left(\frac{z}{2}\right)^k\right)(\nu-n)+\ldots\;/;\;(\nu\to n)\land n\in\mathbb{Z}$$

Expansions at generic point $z == z_0$

For the function itself

03.04.06.0016.01

$$K_\nu(z) \propto K_\nu(z_0)\left(\frac{1}{z_0}\right)^{-\nu\left\lfloor\frac{\arg(z-z_0)}{2\pi}\right\rfloor} z_0^{-\nu\left\lfloor\frac{\arg(z-z_0)}{2\pi}\right\rfloor} - 2\,i\,\pi\,I_\nu(z_0)\cos(\pi\,\nu)\left\lfloor\frac{\arg(z-z_0)}{2\pi}\right\rfloor\left\lfloor\frac{\arg(z_0)+\pi}{2\pi}\right\rfloor + \frac{1}{2}\left(-(K_{\nu-1}(z_0)+K_{\nu+1}(z_0))\left(\frac{1}{z_0}\right)^{-\nu\left\lfloor\frac{\arg(z-z_0)}{2\pi}\right\rfloor} z_0^{-\nu\left\lfloor\frac{\arg(z-z_0)}{2\pi}\right\rfloor} - 2\,i\,\pi\,(I_{\nu-1}(z_0)+I_{\nu+1}(z_0))\cos(\pi\,\nu)\left\lfloor\frac{\arg(z-z_0)}{2\pi}\right\rfloor\left\lfloor\frac{\arg(z_0)+\pi}{2\pi}\right\rfloor\right)(z-z_0) + \frac{1}{8}\left((K_{\nu-2}(z_0)+2\,K_\nu(z_0)+K_{\nu+2}(z_0))\left(\frac{1}{z_0}\right)^{-\nu\left\lfloor\frac{\arg(z-z_0)}{2\pi}\right\rfloor} z_0^{-\nu\left\lfloor\frac{\arg(z-z_0)}{2\pi}\right\rfloor} - 2\,i\,\pi\,(I_{\nu-2}(z_0)+2\,I_\nu(z_0)+I_{\nu+2}(z_0))\cos(\pi\,\nu)\left\lfloor\frac{\arg(z-z_0)}{2\pi}\right\rfloor\left\lfloor\frac{\arg(z_0)+\pi}{2\pi}\right\rfloor\right)(z-z_0)^2 + \ldots\;/;\;(z\to z_0)$$

03.04.06.0017.01

$$K_\nu(z) \propto K_\nu(z_0)\left(\frac{1}{z_0}\right)^{-\nu\left\lfloor\frac{\arg(z-z_0)}{2\pi}\right\rfloor} z_0^{-\nu\left\lfloor\frac{\arg(z-z_0)}{2\pi}\right\rfloor} - 2\,i\,\pi\,I_\nu(z_0)\cos(\pi\,\nu)\left\lfloor\frac{\arg(z-z_0)}{2\pi}\right\rfloor\left\lfloor\frac{\arg(z_0)+\pi}{2\pi}\right\rfloor + \frac{1}{2}\left(-(K_{\nu-1}(z_0)+K_{\nu+1}(z_0))\left(\frac{1}{z_0}\right)^{-\nu\left\lfloor\frac{\arg(z-z_0)}{2\pi}\right\rfloor} z_0^{-\nu\left\lfloor\frac{\arg(z-z_0)}{2\pi}\right\rfloor} - 2\,i\,\pi\,(I_{\nu-1}(z_0)+I_{\nu+1}(z_0))\cos(\pi\,\nu)\left\lfloor\frac{\arg(z-z_0)}{2\pi}\right\rfloor\left\lfloor\frac{\arg(z_0)+\pi}{2\pi}\right\rfloor\right)(z-z_0) + \frac{1}{8}\left((K_{\nu-2}(z_0)+2\,K_\nu(z_0)+K_{\nu+2}(z_0))\left(\frac{1}{z_0}\right)^{-\nu\left\lfloor\frac{\arg(z-z_0)}{2\pi}\right\rfloor} z_0^{-\nu\left\lfloor\frac{\arg(z-z_0)}{2\pi}\right\rfloor} - 2\,i\,\pi\,(I_{\nu-2}(z_0)+2\,I_\nu(z_0)+I_{\nu+2}(z_0))\cos(\pi\,\nu)\left\lfloor\frac{\arg(z-z_0)}{2\pi}\right\rfloor\left\lfloor\frac{\arg(z_0)+\pi}{2\pi}\right\rfloor\right)(z-z_0)^2 + O\left((z-z_0)^3\right)$$

03.04.06.0018.01

$$K_\nu(z) = \sum_{k=0}^{\infty}\frac{K_\nu^{(0,k)}(z_0)}{k!}(z-z_0)^k\;/;\;|\arg(z_0)|<\pi$$

03.04.06.0019.01

$$K_\nu(z) = \sum_{k=0}^{\infty} \frac{1}{2\,k!} G_{2,4}^{2,2}\left(\frac{z_0}{2}, \frac{1}{2} \middle| \begin{matrix} \frac{1-k}{2}, -\frac{k}{2} \\ \frac{\nu-k}{2}, -\frac{1}{2}(k+\nu), \frac{1}{2}, 0 \end{matrix}\right) (z-z_0)^k \text{ /; } |\arg(z_0)| < \pi$$

03.04.06.0020.01

$$K_\nu(z) = 2^{-2\nu-1} \pi^{3/2} z_0^{-\nu} \csc(\pi\nu)$$
$$\sum_{k=0}^{\infty} \frac{2^k z_0^{-k}}{k!} \left(16^\nu \left(\frac{1}{z_0}\right)^{-\nu \left\lfloor \frac{\arg(z-z_0)}{2\pi} \right\rfloor} z_0^{-\nu \left\lfloor \frac{\arg(z-z_0)}{2\pi} \right\rfloor} \Gamma(1-\nu)\, {}_2\tilde{F}_3\left(\frac{1-\nu}{2}, 1-\frac{\nu}{2}; \frac{1}{2}(-k-\nu+1), \frac{1}{2}(-k-\nu+2), 1-\nu; \frac{z_0^2}{4}\right) - \right.$$
$$\left. z_0^{2\nu} \left(\frac{1}{z_0}\right)^{\nu \left\lfloor \frac{\arg(z-z_0)}{2\pi} \right\rfloor} z_0^{\nu \left\lfloor \frac{\arg(z-z_0)}{2\pi} \right\rfloor} \Gamma(\nu+1)\, {}_2\tilde{F}_3\left(\frac{\nu+1}{2}, \frac{\nu+2}{2}; \frac{1}{2}(-k+\nu+1), \frac{1}{2}(-k+\nu+2), \nu+1; \frac{z_0^2}{4}\right)\right) (z-z_0)^k \text{ /; } \nu \notin \mathbb{Z}$$

03.04.06.0021.01

$$K_\nu(z) = \sum_{k=0}^{\infty} \frac{1}{k!} \left(2^{-k} \right.$$
$$\left. \sum_{j=0}^{k} \binom{k}{j} \left((-1)^k \left(\frac{1}{z_0}\right)^{-\nu \left\lfloor \frac{\arg(z-z_0)}{2\pi} \right\rfloor} z_0^{-\nu \left\lfloor \frac{\arg(z-z_0)}{2\pi} \right\rfloor} K_{-2j+k+\nu}(z_0) - 2\pi i \cos(\pi\nu) \left\lfloor \frac{\arg(z-z_0)}{2\pi} \right\rfloor \left\lfloor \frac{\arg(z_0)+\pi}{2\pi} \right\rfloor I_{-2j+k+\nu}(z_0)\right) (z-z_0)^k \right)$$

03.04.06.0022.01

$$K_\nu(z) = \sum_{k=0}^{\infty} \frac{z_0^{-k}}{k!}$$
$$\sum_{m=0}^{k} (-1)^{k+m} \binom{k}{m} (-\nu)_{k-m} \sum_{i=0}^{m} \frac{(-1)^i\, 2^{2i-m} (-m)_{2(m-i)} (\nu)_i}{(m-i)!} \left(\frac{z_0}{2} \sum_{j=0}^{i-1} \frac{(i-j-1)!}{j!\,(i-2j-1)!\,(-i-\nu+1)_j\,(\nu)_{j+1}} \left(-\frac{z_0^2}{4}\right)^j K_{\nu-1}(z_0) + \right.$$
$$\left. \sum_{j=0}^{i} \frac{(i-j)!}{j!\,(i-2j)!\,(-i-\nu+1)_j\,(\nu)_j} \left(-\frac{z_0^2}{4}\right)^j K_\nu(z_0)\right) (z-z_0)^k \text{ /; } |\arg(z_0)| < \pi$$

03.04.06.0023.01

$$K_\nu(z) \propto K_\nu(z_0) \left(\frac{1}{z_0}\right)^{-\nu \left\lfloor \frac{\arg(z-z_0)}{2\pi} \right\rfloor} z_0^{-\nu \left\lfloor \frac{\arg(z-z_0)}{2\pi} \right\rfloor} - 2 i \pi I_\nu(z_0) \cos(\pi\nu) \left\lfloor \frac{\arg(z-z_0)}{2\pi} \right\rfloor \left\lfloor \frac{\arg(z_0)+\pi}{2\pi} \right\rfloor (1 + O(z-z_0))$$

Expansions on branch cuts

For the function itself

03.04.06.0024.01

$$K_\nu(z) \propto e^{-2 i \pi \nu \left\lfloor \frac{\arg(z-x)}{2\pi} \right\rfloor} K_\nu(x) - 2 i \pi I_\nu(x) \cos(\pi \nu) \left\lfloor \frac{\arg(z-x)}{2\pi} \right\rfloor - \frac{1}{2} \left(e^{-2 i \pi \nu \left\lfloor \frac{\arg(z-x)}{2\pi} \right\rfloor} (K_{\nu-1}(x) + K_{\nu+1}(x)) + 2 i \pi (I_{\nu-1}(x) + I_{\nu+1}(x)) \cos(\pi \nu) \left\lfloor \frac{\arg(z-x)}{2\pi} \right\rfloor \right) (z-x) + \frac{1}{8} \left(e^{-2 i \pi \nu \left\lfloor \frac{\arg(z-x)}{2\pi} \right\rfloor} (K_{\nu-2}(x) + 2 K_\nu(x) + K_{\nu+2}(x)) - 2 i \pi (I_{\nu-2}(x) + 2 I_\nu(x) + I_{\nu+2}(x)) \cos(\pi \nu) \left\lfloor \frac{\arg(z-x)}{2\pi} \right\rfloor \right) (z-x)^2 + \ldots /; (z \to x) \land x \in \mathbb{R} \land x < 0$$

03.04.06.0025.01

$$K_\nu(z) \propto e^{-2 i \pi \nu \left\lfloor \frac{\arg(z-x)}{2\pi} \right\rfloor} K_\nu(x) - 2 i \pi I_\nu(x) \cos(\pi \nu) \left\lfloor \frac{\arg(z-x)}{2\pi} \right\rfloor - \frac{1}{2} \left(e^{-2 i \pi \nu \left\lfloor \frac{\arg(z-x)}{2\pi} \right\rfloor} (K_{\nu-1}(x) + K_{\nu+1}(x)) + 2 i \pi (I_{\nu-1}(x) + I_{\nu+1}(x)) \cos(\pi \nu) \left\lfloor \frac{\arg(z-x)}{2\pi} \right\rfloor \right) (z-x) + \frac{1}{8} \left(e^{-2 i \pi \nu \left\lfloor \frac{\arg(z-x)}{2\pi} \right\rfloor} (K_{\nu-2}(x) + 2 K_\nu(x) + K_{\nu+2}(x)) - 2 i \pi (I_{\nu-2}(x) + 2 I_\nu(x) + I_{\nu+2}(x)) \cos(\pi \nu) \left\lfloor \frac{\arg(z-x)}{2\pi} \right\rfloor \right) (z-x)^2 + O\left((z-x)^3\right) /; x \in \mathbb{R} \land x < 0$$

03.04.06.0026.01

$$K_\nu(z) = 2^{-2\nu-1} \pi^{3/2} x^{-\nu} \csc(\pi \nu) \sum_{k=0}^{\infty} \frac{2^k x^{-k}}{k!} \left(16^\nu e^{-2 i \pi \nu \left\lfloor \frac{\arg(z-x)}{2\pi} \right\rfloor} \Gamma(1-\nu) \, {}_2\tilde{F}_3\left(\frac{1-\nu}{2}, 1 - \frac{\nu}{2}; \frac{1}{2}(-k-\nu+1), \frac{1}{2}(-k-\nu+2), 1-\nu; \frac{x^2}{4} \right) - x^{2\nu} e^{2 i \pi \nu \left\lfloor \frac{\arg(z-x)}{2\pi} \right\rfloor} \Gamma(\nu+1) \, {}_2\tilde{F}_3\left(\frac{\nu+1}{2}, \frac{\nu+2}{2}; \frac{1}{2}(-k+\nu+1), \frac{1}{2}(-k+\nu+2), \nu+1; \frac{x^2}{4} \right) \right) (z-x)^k /; \nu \notin \mathbb{Z}$$

03.04.06.0027.01

$$K_\nu(z) = \sum_{k=0}^{\infty} \frac{2^{-k}}{k!} \sum_{j=0}^{k} \binom{k}{j} \left((-1)^k e^{-2 \nu \pi i \left\lfloor \frac{\arg(z-x)}{2\pi} \right\rfloor} K_{-2j+k+\nu}(x) - 2 \pi i \cos(\pi \nu) \left\lfloor \frac{\arg(z-x)}{2\pi} \right\rfloor I_{-2j+k+\nu}(x) \right) (z-x)^k /; x \in \mathbb{R} \land x < 0$$

03.04.06.0028.01

$$K_\nu(z) \propto e^{-2 i \pi \nu \left\lfloor \frac{\arg(z-x)}{2\pi} \right\rfloor} K_\nu(x) - 2 i \pi I_\nu(x) \cos(\pi \nu) \left\lfloor \frac{\arg(z-x)}{2\pi} \right\rfloor (1 + O(z-x)) /; x \in \mathbb{R} \land x < 0$$

Expansions at $z == 0$

For the function itself

General case

03.04.06.0001.02

$$K_\nu(z) \propto \frac{1}{2} \left(\Gamma(\nu) \left(\frac{z}{2}\right)^{-\nu} \left(1 + \frac{z^2}{4(1-\nu)} + \frac{z^4}{32(1-\nu)(2-\nu)} + \ldots \right) + \Gamma(-\nu) \left(\frac{z}{2}\right)^{\nu} \left(1 + \frac{z^2}{4(\nu+1)} + \frac{z^4}{32(\nu+1)(\nu+2)} + \ldots \right) \right) /; (z \to 0) \bigwedge \nu \notin \mathbb{Z}$$

03.04.06.0029.01

$$K_\nu(z) \propto \frac{1}{2}\left(\Gamma(\nu)\left(\frac{z}{2}\right)^{-\nu}\left(1+\frac{z^2}{4(1-\nu)}+\frac{z^4}{32(1-\nu)(2-\nu)}+O\left(z^6\right)\right)+\Gamma(-\nu)\left(\frac{z}{2}\right)^{\nu}\left(1+\frac{z^2}{4(\nu+1)}+\frac{z^4}{32(\nu+1)(\nu+2)}+O\left(z^6\right)\right)\right)/;\ \nu\notin\mathbb{Z}$$

03.04.06.0030.01

$$K_\nu(z) == \frac{1}{2}\left(\Gamma(\nu)\left(\frac{z}{2}\right)^{-\nu}\left(\sum_{k=0}^{\infty}\frac{\left(\frac{z}{2}\right)^{2k}}{(1-\nu)_k\,k!}\right)+\Gamma(-\nu)\left(\frac{z}{2}\right)^{\nu}\left(\sum_{k=0}^{\infty}\frac{\left(\frac{z}{2}\right)^{2k}}{(\nu+1)_k\,k!}\right)\right)/;\ \nu\notin\mathbb{Z}$$

03.04.06.0002.01

$$K_\nu(z) == \frac{\pi\csc(\pi\nu)}{2}\left(\sum_{k=0}^{\infty}\frac{1}{\Gamma(k-\nu+1)\,k!}\left(\frac{z}{2}\right)^{2k-\nu}-\sum_{k=0}^{\infty}\frac{1}{\Gamma(k+\nu+1)\,k!}\left(\frac{z}{2}\right)^{2k+\nu}\right)/;\ \nu\notin\mathbb{Z}$$

03.04.06.0031.01

$$K_\nu(z) == \frac{1}{2}\left(\Gamma(\nu)\left(\frac{z}{2}\right)^{-\nu}{}_0F_1\left(;1-\nu;\frac{z^2}{4}\right)+\Gamma(-\nu)\left(\frac{z}{2}\right)^{\nu}{}_0F_1\left(;\nu+1;\frac{z^2}{4}\right)\right)/;\ \nu\notin\mathbb{Z}$$

03.04.06.0003.01

$$K_\nu(z) == 2^{\nu-1}\pi z^{-\nu}\csc(\pi\nu)\,{}_0\tilde{F}_1\left(;1-\nu;\frac{z^2}{4}\right)-2^{-\nu-1}\pi z^{\nu}\csc(\pi\nu)\,{}_0\tilde{F}_1\left(;\nu+1;\frac{z^2}{4}\right)/;\ \nu\notin\mathbb{Z}$$

03.04.06.0007.01

$$K_\nu(z) \propto 2^{-\nu-1}\Gamma(-\nu)z^{\nu}\left(1+O\left(z^2\right)\right)+2^{\nu-1}\Gamma(\nu)z^{-\nu}\left(1+O\left(z^2\right)\right)/;\ \nu\notin\mathbb{Z}$$

03.04.06.0032.01

$$K_\nu(z) == F_\infty(z,\nu)\ /;$$

$$\left(\left(F_m(z,\nu) == \frac{1}{2}\left(\Gamma(\nu)\left(\frac{z}{2}\right)^{-\nu}\sum_{k=0}^{m}\frac{\left(\frac{z}{2}\right)^{2k}}{(1-\nu)_k\,k!}+\Gamma(-\nu)\left(\frac{z}{2}\right)^{\nu}\sum_{k=0}^{m}\frac{\left(\frac{z}{2}\right)^{2k}}{(\nu+1)_k\,k!}\right) == K_\nu(z)+\frac{\pi}{\sin(\nu\pi)(m+1)!}\left(\frac{2^{-2m-\nu-3}z^{2m+\nu+2}}{\Gamma(m+\nu+2)}\right.\right.\right.$$

$$\left.\left.\left.{}_1F_2\left(1;m+2,m+\nu+2;\frac{z^2}{4}\right)-\frac{2^{-2m+\nu-3}z^{2m-\nu+2}}{\Gamma(m-\nu+2)}\,{}_1F_2\left(1;m+2,m-\nu+2;\frac{z^2}{4}\right)\right)\right)\bigwedge m\in\mathbb{N}\right)/;\ \nu\notin\mathbb{Z}$$

Summed form of the truncated series expansion.

Logarithmic cases

03.04.06.0033.01

$$K_0(z) \propto \left(-\gamma+\frac{1}{4}(1-\gamma)z^2+\frac{1}{128}(3-2\gamma)z^4+\ldots\right)-\log\left(\frac{z}{2}\right)\left(1+\frac{z^2}{4}+\frac{z^4}{64}+\ldots\right)/;\ (z\to 0)$$

03.04.06.0034.01

$$K_1(z) \propto \frac{1}{z}+\frac{z}{4}\left(2\gamma-1+\frac{1}{8}\left(2\gamma-\frac{5}{2}\right)z^2+\frac{1}{192}\left(2\gamma-\frac{10}{3}\right)z^4+\ldots\right)+\frac{z}{2}\log\left(\frac{z}{2}\right)\left(1+\frac{z^2}{8}+\frac{z^4}{192}+\ldots\right)/;\ (z\to 0)$$

03.04.06.0035.01

$$K_2(z) \propto \frac{2}{z^2}-\frac{1}{2}-\frac{z^2}{8}\log\left(\frac{z}{2}\right)\left(1+\frac{z^2}{12}+\frac{z^4}{384}+\ldots\right)+\frac{z^2}{16}\left(\frac{3}{2}-2\gamma+\frac{1}{12}\left(\frac{17}{6}-2\gamma\right)z^2+\frac{1}{384}\left(\frac{43}{12}-2\gamma\right)z^4+\ldots\right)/;\ (z\to 0)$$

03.04.06.0036.01

$$K_n(z) \propto \frac{(n-1)!}{2}\left(\frac{z}{2}\right)^{-n}\left(1-\frac{z^2}{4(n-1)}+\frac{z^4}{32(n-1)(n-2)}+\ldots\right)+\frac{(-1)^{n-1}}{n!}\log\left(\frac{z}{2}\right)\left(\frac{z}{2}\right)^n\left(1+\frac{z^2}{4(n+1)}+\frac{z^4}{32(n+1)(n+2)}+\ldots\right)+ \frac{(-1)^n 2^{-n-1} z^n}{n!}\left(\psi(n+1)-\gamma+\frac{(\psi(n+2)-\gamma+1)z^2}{4(n+1)}+\frac{\left(\psi(n+3)+\frac{3}{2}-\gamma\right)z^4}{32(n+1)(n+2)}+\ldots\right) /; (z\to 0)\land n-3\in\mathbb{N}$$

03.04.06.0037.01

$$K_\nu(z) = \frac{1}{2}\left(\frac{z}{2}\right)^{-|\nu|}\sum_{k=0}^{|\nu|-1}\frac{(-1)^k(|\nu|-k-1)!}{k!}\left(\frac{z}{2}\right)^{2k}+ (-1)^{\nu-1}\log\left(\frac{z}{2}\right)\left(\frac{z}{2}\right)^{|\nu|}\sum_{k=0}^{\infty}\frac{\left(\frac{z}{2}\right)^{2k}}{k!(k+|\nu|)!}+\frac{(-1)^\nu}{2}\left(\frac{z}{2}\right)^{|\nu|}\sum_{k=0}^{\infty}\frac{\psi(k+1)+\psi(k+|\nu|+1)}{k!(k+|\nu|)!}\left(\frac{z}{2}\right)^{2k} /; \nu\in\mathbb{Z}$$

03.04.06.0004.01

$$K_\nu(z) = (-1)^{\nu-1}\,{}_0\tilde{F}_1\left(;|\nu|+1;\frac{z^2}{4}\right)\log\left(\frac{z}{2}\right)\left(\frac{z}{2}\right)^{|\nu|}+ \frac{1}{2}\sum_{k=0}^{|\nu|-1}\frac{(-1)^k(|\nu|-k-1)!}{k!}\left(\frac{z}{2}\right)^{2k-|\nu|}+\frac{(-1)^\nu}{2}\sum_{k=0}^{\infty}\frac{\psi(k+1)+\psi(k+|\nu|+1)}{k!(k+|\nu|)!}\left(\frac{z}{2}\right)^{2k+|\nu|} /; \nu\in\mathbb{Z}$$

03.04.06.0038.01

$$K_n(z) = (-1)^{n-1}\log\left(\frac{z}{2}\right)\left(\frac{z}{2}\right)^n\sum_{k=0}^{\infty}\frac{\left(\frac{z}{2}\right)^{2k}}{k!(k+n)!}+ \frac{1}{2}\left(\frac{z}{2}\right)^{-n}\sum_{k=0}^{n-1}\frac{(-1)^k(n-k-1)!}{k!}\left(\frac{z}{2}\right)^{2k}+\frac{(-1)^n}{2}\left(\frac{z}{2}\right)^n\sum_{k=0}^{\infty}\frac{\psi(k+1)+\psi(k+n+1)}{k!(k+n)!}\left(\frac{z}{2}\right)^{2k} /; n\in\mathbb{N}$$

03.04.06.0005.01

$$K_n(z) = (-1)^{n-1}\log\left(\frac{z}{2}\right)I_n(z)+\frac{1}{2}\sum_{k=0}^{n-1}\frac{(-1)^k(n-k-1)!}{k!}\left(\frac{z}{2}\right)^{2k-n}+\frac{(-1)^n}{2}\sum_{k=0}^{\infty}\frac{\psi(k+1)+\psi(k+n+1)}{k!(k+n)!}\left(\frac{z}{2}\right)^{2k+n} /; n\in\mathbb{N}$$

03.04.06.0006.01

$$K_n(z) = \frac{(-1)^n}{8}\left(-\frac{2^{4-n}z^{n-2}}{(n-1)!}\,{}_3F_0\left(1,1,1-n;;\frac{4}{z^2}\right)+ \frac{2^{-n}z^{2+n}}{(n+1)(n+1)!}\left((n+1)F_{2\times 0\times 1}^{0\times 1\times 2}\left(\begin{matrix};1;1,1;\\2,n+2;;2;\end{matrix}\frac{z^2}{4},\frac{z^2}{4}\right)+F_{2\times 0\times 1}^{0\times 1\times 2}\left(\begin{matrix};1;1,n+1;\\2,n+2;;n+2;\end{matrix}\frac{z^2}{4},\frac{z^2}{4}\right)\right)- 4I_n(z)\left(2\log\left(\frac{z}{2}\right)-\psi(n+1)+\gamma\right)\right) /; n\in\mathbb{N}$$

03.04.06.0008.01

$$K_\nu(z) \propto (-1)^{\nu-1}\delta_{|\nu|}\log\left(\frac{z}{2}\right)\left(1+O\left(z^2\right)\right)+\frac{1}{2}\sum_{k=0}^{|\nu|-1}\frac{(-1)^k(-k+|\nu|-1)!}{k!}\left(\frac{z}{2}\right)^{2k-|\nu|}\left(1+O\left(z^2\right)\right) /; \nu\in\mathbb{Z}$$

03.04.06.0039.01

$$K_0(z) \propto -\log\left(\frac{z}{2}\right)\left(1+O\left(z^2\right)\right)-\gamma\left(1+O\left(z^2\right)\right)$$

03.04.06.0040.01

$$K_1(z) \propto \frac{1}{z}\left(1+O\left(z^2\right)\right)+\frac{z}{2}\log\left(\frac{z}{2}\right)\left(1+O\left(z^2\right)\right)$$

03.04.06.0041.01

$$K_2(z) \propto \frac{2}{z^2}\left(1+O\left(z^2\right)\right)-\frac{z^2}{8}\log\left(\frac{z}{2}\right)\left(1+O\left(z^2\right)\right)$$

03.04.06.0042.01

$$K_n(z) \propto \frac{(n-1)!}{2}\left(\frac{z}{2}\right)^{-n}\left(1+O\left(z^2\right)\right)+\frac{(-1)^{n-1}}{n!}\log\left(\frac{z}{2}\right)\left(\frac{z}{2}\right)^n\left(1+O\left(z^2\right)\right)\;/;\;n-3\in\mathbb{N}$$

03.04.06.0043.01

$$K_n(z) = F_\infty(z, n)\;/;$$
$$\Bigg(\Bigg(F_m(z,n) = \frac{(-1)^n}{2}\left(\frac{z}{2}\right)^n\sum_{k=0}^{m}\frac{\psi(k+1)+\psi(k+n+1)}{k!\,(k+n)!}\left(\frac{z}{2}\right)^{2k}+(-1)^{n-1}\log\left(\frac{z}{2}\right)I_n(z)+\frac{1}{2}\sum_{k=0}^{n-1}\frac{(-1)^k\,(-k+n-1)!}{k!}\left(\frac{z}{2}\right)^{2k-n} =$$
$$z^n\left(z^2\right)^{-\frac{n}{2}}K_n\left(\sqrt{z^2}\right)-\log\left(\frac{z^2}{4}\right)\frac{(-1)^n\,2^{-2m-n-3}\,z^{2(m+1)+n}}{(m+1)!\,(m+n+1)!}\,{}_1F_2\left(1;m+2,m+n+2;\frac{z^2}{4}\right)+$$
$$(-1)^{n-1}I_n(z)\log\left(\frac{z}{2}\right)+\frac{1}{2}(-1)^n\log\left(\frac{z^2}{4}\right)I_n(z)-\frac{(-1)^n}{2}\left(\frac{z}{2}\right)^n G_{2,4}^{2,2}\left(\frac{z^2}{4}\,\middle|\,\begin{matrix}m+1,m+1\\m+1,m+1,0,-n\end{matrix}\right)\Bigg)\bigwedge n\in\mathbb{N}\Bigg)$$

Summed form of the truncated series expansion.

Generic formulas for main term

03.04.06.0044.01

$$K_\nu(z) \propto \begin{cases}-\log\left(\frac{z}{2}\right)-\gamma & \nu=0\\ \frac{1}{2}\left(|\nu|-1\right)!\left(\frac{z}{2}\right)^{-|\nu|} & \nu\in\mathbb{Z}\bigwedge\nu\ne 0\\ \frac{1}{2}\Gamma(\nu)\left(\frac{z}{2}\right)^{-\nu}+\frac{1}{2}\Gamma(-\nu)\left(\frac{z}{2}\right)^{\nu} & \text{True}\end{cases}\;/;\;(z\to 0)$$

Asymptotic series expansions

Expansions for any z in exponential form

Using exponential function with branch cut-containing arguments

03.04.06.0045.01

$$K_\nu(z) \propto \frac{1}{2}\sqrt{\frac{\pi}{2}}\csc(\pi\nu)\left(-z^2\right)^{-\frac{1}{4}(2\nu+1)}\left(e^{i\sqrt{-z^2}+\frac{1}{4}i\pi(3-2\nu)}\left(z^\nu-e^{i\pi\nu}(-z)^\nu\right)\left(1+\frac{i\left(-1+4\nu^2\right)}{8\sqrt{-z^2}}+\frac{9-40\nu^2+16\nu^4}{128z^2}+\ldots\right)+\right.$$
$$\left.e^{-i\sqrt{-z^2}+\frac{1}{4}i\pi(1-2\nu)}\left((-z)^\nu-e^{i\pi\nu}z^\nu\right)\left(1-\frac{i\left(-1+4\nu^2\right)}{8\sqrt{-z^2}}+\frac{9-40\nu^2+16\nu^4}{128z^2}+\ldots\right)\right)\;/;\;(|z|\to\infty)$$

03.04.06.0046.01

$$K_\nu(z) \propto \frac{1}{2}\sqrt{\frac{\pi}{2}}\csc(\pi\nu)\left(-z^2\right)^{-\frac{1}{4}(2\nu+1)}\left(e^{-i\sqrt{-z^2}+\frac{1}{4}i\pi(1-2\nu)}\left((-z)^\nu - e^{i\pi\nu}z^\nu\right)\left(\sum_{k=0}^{n}\frac{\left(\nu+\frac{1}{2}\right)_k\left(\frac{1}{2}-\nu\right)_k}{k!}\left(\frac{i}{2\sqrt{-z^2}}\right)^k + O\left(\frac{1}{z^{n+1}}\right)\right) + e^{i\sqrt{-z^2}+\frac{1}{4}i\pi(3-2\nu)}\left(z^\nu - e^{i\pi\nu}(-z)^\nu\right)\left(\sum_{k=0}^{n}\frac{\left(\nu+\frac{1}{2}\right)_k\left(\frac{1}{2}-\nu\right)_k}{k!}\left(-\frac{i}{2\sqrt{-z^2}}\right)^k + O\left(\frac{1}{z^{n+1}}\right)\right)\right) /; (|z| \to \infty)$$

03.04.06.0011.01

$$K_\nu(z) \propto \frac{1}{2}\sqrt{\frac{\pi}{2}}\, z^{-\nu}\left(-z^2\right)^{-\frac{2\nu+1}{4}}\csc(\pi\nu)\, e^{-\frac{i\pi\nu}{2}}\left(\exp\left(\frac{i\pi}{4} - i\sqrt{-z^2}\right)\left(\left(-z^2\right)^\nu - e^{i\pi\nu}z^{2\nu}\right){}_2F_0\left(\frac{1}{2}-\nu, \nu+\frac{1}{2};;\frac{i}{2\sqrt{-z^2}}\right) + \exp\left(-\frac{i\pi}{4} + i\sqrt{-z^2}\right)\left(e^{i\pi\nu}\left(-z^2\right)^\nu - z^{2\nu}\right){}_2F_0\left(\frac{1}{2}-\nu, \nu+\frac{1}{2};;-\frac{i}{2\sqrt{-z^2}}\right)\right) /; (|z| \to \infty)$$

03.04.06.0047.01

$$K_\nu(z) \propto \frac{1}{2}\sqrt{\frac{\pi}{2}}\, z^{-\nu}\left(-z^2\right)^{-\frac{1}{4}(2\nu+1)}\csc(\pi\nu)\, e^{-\frac{1}{2}(i\pi\nu)}\left(e^{-\frac{1}{4}(i\pi)+i\sqrt{-z^2}}\left(e^{i\pi\nu}\left(-z^2\right)^\nu - z^{2\nu}\right)\left(1+O\left(\frac{1}{z}\right)\right) + e^{\frac{i\pi}{4}-i\sqrt{-z^2}}\left(\left(-z^2\right)^\nu - e^{i\pi\nu}z^{2\nu}\right)\left(1+O\left(\frac{1}{z}\right)\right)\right) /; (|z| \to \infty)$$

Using exponential function with branch cut-free arguments

03.04.06.0048.01

$$K_\nu(z) \propto \frac{1}{\sqrt{z}}\sqrt{\frac{\pi}{2}}\, e^{-z}\left(1 + \frac{4\nu^2-1}{8z} + \frac{16\nu^4-40\nu^2+9}{128z^2} + \ldots\right) /; (|z| \to \infty)$$

03.04.06.0049.01

$$K_\nu(z) \propto \sqrt{\frac{\pi}{2}}\frac{1}{\sqrt{z}}\, e^{-z}\left(\sum_{k=0}^{n}\frac{\left(\nu+\frac{1}{2}\right)_k\left(\frac{1}{2}-\nu\right)_k}{k!}\left(-\frac{1}{2z}\right)^k + O\left(\frac{1}{z^{n+1}}\right)\right) /; (|z| \to \infty)$$

03.04.06.0009.01

$$K_\nu(z) \propto \sqrt{\frac{\pi}{2}}\frac{e^{-z}}{\sqrt{z}}\,{}_2F_0\left(\nu+\frac{1}{2}, \frac{1}{2}-\nu;;-\frac{1}{2z}\right) /; (|z| \to \infty)$$

03.04.06.0010.01

$$K_\nu(z) \propto \sqrt{\frac{\pi}{2}}\frac{e^{-z}}{\sqrt{z}}\left(1+O\left(\frac{1}{z}\right)\right) /; (|z| \to \infty)$$

03.04.06.0050.01

$$K_\nu(z) \propto \sqrt{\frac{\pi}{2}}\frac{e^{-z}}{\sqrt{z}} /; (|z| \to \infty)$$

Expansions at $\nu == i\infty$

03.04.06.0051.01

$$K_{i\tau}(x) \propto \sqrt{\frac{2\pi}{\tau}}\, e^{-\frac{\pi\tau}{2}} \sin\left(\frac{x^2}{4\tau} - \tau + \tau\log\left(\frac{2\tau}{x}\right) + \frac{\pi}{4}\right)\left(1 + O\left(\frac{1}{\tau}\right)\right) /; (\tau \to \infty) \land x \in \mathbb{R} \land x > 0$$

Residue representations

03.04.06.0012.02

$$K_\nu(z) = \frac{\pi\csc(\pi\nu)}{2}\sum_{j=0}^{\infty}\operatorname{res}_s\left(\Gamma(s)\cos(\pi s)\left(\frac{\left(\frac{z}{2}\right)^{-\nu}}{\Gamma(1-\nu-s)} - \frac{\left(\frac{z}{2}\right)^{\nu}}{\Gamma(1+\nu-s)}\right)\left(\frac{z^2}{4}\right)^{-s}\right)(-j) /; \nu \notin \mathbb{Z}$$

03.04.06.0013.01

$$K_\nu(z) = \frac{1}{2}\left(\sum_{j=0}^{\infty}\operatorname{res}_s\left(\left(\Gamma\left(s+\frac{\nu}{2}\right)\left(\frac{z}{2}\right)^{-2s}\right)\Gamma\left(s-\frac{\nu}{2}\right)\right)\left(\frac{\nu}{2}-j\right) + \sum_{j=0}^{\infty}\operatorname{res}_s\left(\left(\Gamma\left(s-\frac{\nu}{2}\right)\left(\frac{z}{2}\right)^{-2s}\right)\Gamma\left(s+\frac{\nu}{2}\right)\right)\left(-\frac{\nu}{2}-j\right)\right) /; \nu \notin \mathbb{Z}$$

03.04.06.0014.02

$$K_n(z) = \frac{1}{2}\sum_{j=0}^{|n|-1}\operatorname{res}_s\left(\left(\Gamma\left(s+\frac{|n|}{2}\right)\left(\frac{z}{2}\right)^{-2s}\right)\Gamma\left(s-\frac{|n|}{2}\right)\right)\left(\frac{|n|}{2}-j\right) + \frac{1}{2}\sum_{j=0}^{\infty}\operatorname{res}_s\left(\left(\frac{z}{2}\right)^{-2s}\Gamma\left(s+\frac{|n|}{2}\right)\Gamma\left(s-\frac{|n|}{2}\right)\right)\left(-\frac{|n|}{2}-j\right) /; n \in \mathbb{Z}$$

Integral representations

On the real axis

Of the direct function

03.04.07.0001.01

$$K_\nu(z) = \frac{\sqrt{\pi}\, z^\nu}{2^\nu\,\Gamma\left(\nu+\frac{1}{2}\right)}\int_1^\infty e^{-zt}\left(t^2-1\right)^{\nu-\frac{1}{2}}\,dt /; \operatorname{Re}(\nu) > -\frac{1}{2} \bigwedge \operatorname{Re}(z) > 0$$

03.04.07.0002.01

$$K_\nu(z) = \int_0^\infty e^{-z\cosh(t)}\cosh(\nu t)\,dt /; \operatorname{Re}(z) > 0$$

03.04.07.0003.01

$$K_\nu(z) = \frac{\sqrt{\pi}}{\Gamma\left(\nu+\frac{1}{2}\right)}\left(\frac{z}{2}\right)^\nu\int_0^\infty e^{-z\cosh(t)}\sinh^{2\nu}(t)\,dt /; \operatorname{Re}(\nu) > -\frac{1}{2} \bigwedge \operatorname{Re}(z) > 0$$

03.04.07.0004.01

$$K_\nu(x) = \sec\left(\frac{\pi\nu}{2}\right)\int_0^\infty \cos(x\sinh(t))\cosh(\nu t)\,dt /; |\operatorname{Re}(\nu)| < 1 \land x > 0$$

03.04.07.0005.01

$$K_\nu(x) = \csc\left(\frac{\pi\nu}{2}\right)\int_0^\infty \sin(x\sinh(t))\sinh(\nu t)\,dt /; |\operatorname{Re}(\nu)| < 1 \land x > 0$$

03.04.07.0006.01

$$K_\nu(x) = \frac{2^\nu}{\sqrt{\pi}\, x^\nu}\,\Gamma\left(\nu+\frac{1}{2}\right)\int_0^\infty \cos(xt)\left(t^2+1\right)^{-\nu-\frac{1}{2}}\,dt /; \operatorname{Re}(\nu) > -\frac{1}{2} \bigwedge x > 0$$

03.04.07.0007.01

$$K_0(x) = \int_0^\infty \frac{\cos(x\,t)}{\sqrt{t^2+1}}\,dt \,/;\, x > 0$$

03.04.07.0008.01

$$K_0(x) = \int_0^\infty \cos(x \sinh(t))\,dt \,/;\, x > 0$$

03.04.07.0009.01

$$K_0(z) = -\frac{1}{\pi}\int_0^\pi e^{z\cos(t)}\left(\log\left(2\,z\sin^2(t)\right)+\gamma\right)dt$$

03.04.07.0010.01

$$K_0(z) = -\frac{1}{\pi}\int_0^\pi e^{-z\cos(t)}\left(\log\left(2\,z\sin^2(t)\right)+\gamma\right)dt$$

Contour integral representations

03.04.07.0011.01

$$K_\nu(z) = \frac{1}{4\,\pi\,i}\int_{\gamma-i\,\infty}^{\gamma+i\,\infty}\Gamma(s)\,\Gamma(s-\nu)\left(\frac{z}{2}\right)^{\nu-2\,s}ds \,/;\, \gamma > \max(\mathrm{Re}(\nu),\,0)$$

03.04.07.0012.01

$$K_\nu(z) = z^{-\nu}\left(z^2\right)^{\nu/2}\frac{1}{4\,\pi\,i}\int_{\mathcal{L}}\Gamma\left(s+\frac{\nu}{2}\right)\Gamma\left(s-\frac{\nu}{2}\right)\left(\frac{z^2}{4}\right)^{-s}ds - \pi\,(-z)^{-\frac{\nu}{2}}\,z^{-\frac{3\nu}{2}}\left(z^{2\,\nu}-\left(z^2\right)^\nu\right)\csc(\pi\,\nu)\,\frac{1}{4\,\pi\,i}\int_{\mathcal{L}}\frac{\Gamma\left(s+\frac{\nu}{2}\right)}{\Gamma\left(1+\frac{\nu}{2}-s\right)}\left(-\frac{z^2}{4}\right)^{-s}ds \,/;\, \nu \notin \mathbb{Z}$$

Limit representations

03.04.09.0001.01

$$K_\nu(z) = \lim_{\lambda\to\infty}\lambda^{-\nu}\,e^{-\nu\,\pi\,i}\,Q_\lambda^\nu\left(\cosh\left(\frac{z}{\lambda}\right)\right)$$

Differential equations

Ordinary linear differential equations and wronskians

For the direct function itself

03.04.13.0001.01

$$z^2\,w''(z) + z\,w'(z) - \left(z^2+\nu^2\right)w(z) = 0 \,/;\, w(z) = c_1\,I_\nu(z) + c_2\,K_\nu(z)$$

03.04.13.0002.01

$$W_z(I_\nu(z),\,K_\nu(z)) = -\frac{1}{z}$$

03.04.13.0003.01

$$w''(z) - a\,z^n\,w(z) = 0 \,/;\, w(z) = \sqrt{z}\left(c_1\,I_{\frac{1}{n+2}}\left(\frac{2\sqrt{a}\,z^{\frac{n+2}{2}}}{n+2}\right) + c_2\,K_{\frac{1}{n+2}}\left(\frac{2\sqrt{a}\,z^{\frac{n+2}{2}}}{n+2}\right)\right)$$

03.04.13.0004.01

$$W_z\left(\sqrt{z}\, I_{\frac{1}{n+2}}\left(\frac{2\sqrt{a}\, z^{\frac{n+2}{2}}}{n+2}\right), \sqrt{z}\, K_{\frac{1}{n+2}}\left(\frac{2\sqrt{a}\, z^{\frac{n+2}{2}}}{n+2}\right)\right) = -\frac{n}{2} - 1$$

03.04.13.0005.01

$$w''(z) - \left(m^2 + \frac{1}{z^2}\left(\nu^2 - \frac{1}{4}\right)\right) w(z) = 0 \;/;\; w(z) = c_1 \sqrt{z}\, I_\nu\left(\sqrt{m^2}\, z\right) + c_2 \sqrt{z}\, K_\nu\left(\sqrt{m^2}\, z\right)$$

03.04.13.0006.01

$$W_z\left(\sqrt{z}\, I_\nu\left(\sqrt{m^2}\, z\right), \sqrt{z}\, K_\nu\left(\sqrt{m^2}\, z\right)\right) = -1$$

03.04.13.0007.01

$$w''(z) - \left(\frac{m^2}{4z} + \frac{\nu^2 - 1}{4z^2}\right) w(z) = 0 \;/;\; w(z) = c_1 \sqrt{z}\, I_\nu\left(\sqrt{m^2}\, \sqrt{z}\right) + c_2 \sqrt{z}\, K_\nu\left(\sqrt{m^2}\, \sqrt{z}\right)$$

03.04.13.0008.01

$$W_z\left(\sqrt{z}\, I_\nu\left(\sqrt{m^2}\, \sqrt{z}\right), \sqrt{z}\, K_\nu\left(\sqrt{m^2}\, \sqrt{z}\right)\right) = -\frac{1}{2}$$

03.04.13.0009.01

$$w''(z) - \frac{2\nu - 1}{z} w'(z) - w(z)\, m^2 = 0 \;/;\; w(z) = c_1 z^\nu I_\nu(m z) + c_2 z^\nu K_\nu(m z)$$

03.04.13.0010.01

$$W_z(z^\nu I_\nu(m z), z^\nu K_\nu(m z)) = -z^{2\nu-1}$$

03.04.13.0011.01

$$z^2 w''(z) + (2z + 1)\, z\, w'(z) + \left(z - \nu^2\right) w(z) = 0 \;/;\; w(z) = c_1 e^{-z} I_\nu(z) + c_2 e^{-z} K_\nu(z)$$

03.04.13.0012.01

$$W_z(e^{-z} I_\nu(z), e^{-z} K_\nu(z)) = -\frac{e^{-2z}}{z}$$

03.04.13.0013.01

$$z^2 w''(z) + (1 - 2z)\, z\, w'(z) - \left(\nu^2 + z\right) w(z) = 0 \;/;\; w(z) = c_1 e^z I_\nu(z) + c_2 e^z K_\nu(z)$$

03.04.13.0014.01

$$W_z(e^z I_\nu(z), e^z K_\nu(z)) = -\frac{e^{2z}}{z}$$

03.04.13.0015.01

$$\left(z^2 + \nu^2\right) w''(z)\, z^2 + \left(z^2 + 3\nu^2\right) w'(z)\, z - \left(z^2 - \nu^2 + \left(z^2 + \nu^2\right)^2\right) w(z) = 0 \;/;\; w(z) = c_1 \frac{\partial I_\nu(z)}{\partial z} + c_2 \frac{\partial K_\nu(z)}{\partial z}$$

03.04.13.0019.01

$$w''(z) - \left(\frac{g''(z)}{g'(z)} - \frac{g'(z)}{g(z)}\right) w'(z) - \left(\frac{\nu^2}{g(z)^2} + 1\right) g'(z)^2\, w(z) = 0 \;/;\; w(z) = c_1 I_\nu(g(z)) + c_2 K_\nu(g(z))$$

03.04.13.0020.01

$$W_z(I_\nu(g(z)), K_\nu(g(z))) = -\frac{g'(z)}{g(z)}$$

03.04.13.0021.01

$$w''(z) - \left(-\frac{g'(z)}{g(z)} + \frac{2\,h'(z)}{h(z)} + \frac{g''(z)}{g'(z)}\right) w'(z) - \left(\left(\frac{\nu^2}{g(z)^2} + 1\right) g'(z)^2 + \frac{h'(z)\,g'(z)}{g(z)\,h(z)} + \frac{h(z)\,h''(z) - 2\,h'(z)^2}{h(z)^2} - \frac{h'(z)\,g''(z)}{h(z)\,g'(z)}\right) w(z) = 0 \;/;\; w(z) = c_1\,h(z)\,I_\nu(g(z)) + c_2\,h(z)\,K_\nu(g(z))$$

03.04.13.0022.01

$$W_z(h(z)\,I_\nu(g(z)),\,h(z)\,K_\nu(g(z))) == -\frac{h(z)^2\,g'(z)}{g(z)}$$

03.04.13.0023.01

$$z^2\,w''(z) + z\,(1-2\,s)\,w'(z) + \left(s^2 - r^2\left(a^2\,z^{2\,r} + \nu^2\right)\right) w(z) = 0 \;/;\; w(z) = c_1\,z^s\,I_\nu(a\,z^r) + c_2\,z^s\,K_\nu(a\,z^r)$$

03.04.13.0024.01

$$W_z(z^s\,I_\nu(a\,z^r),\,z^s\,K_\nu(a\,z^r)) == -r\,z^{2\,s-1}$$

03.04.13.0025.01

$$w''(z) - 2\log(s)\,w'(z) - \left(\left(a^2\,r^{2\,z} + \nu^2\right)\log^2(r) - \log^2(s)\right) w(z) = 0 \;/;\; w(z) = c_1\,s^z\,I_\nu(a\,r^z) + c_2\,s^z\,K_\nu(a\,r^z)$$

03.04.13.0026.01

$$W_z(s^z\,I_\nu(a\,r^z),\,s^z\,K_\nu(a\,r^z)) == -s^{2\,z}\log(r)$$

Involving related functions

03.04.13.0016.01

$$\left(\prod_{k=1}^{4}\left(z\,\frac{d}{d\,z}\right)\right) w(z) - 2\left(\mu^2 + \nu^2\right)\left(\prod_{k=1}^{2}\left(z\,\frac{d}{d\,z}\right)\right) w(z) + \left(\nu^2 - \mu^2\right)^2 w(z) - 4\,z^2\left(\left(\prod_{k=1}^{2}\left(z\,\frac{d}{d\,z}\right)\right) w(z) + 2\,w(z) + 3\,z\,w'(z)\right) == 0 \;/;\; w(z) == c_1\,I_\mu(z)\,I_\nu(z) + c_2\,I_\nu(z)\,K_\mu(z) + c_3\,I_\mu(z)\,K_\nu(z) + c_4\,K_\mu(z)\,K_\nu(z)$$

03.04.13.0017.01

$$\left(\prod_{k=1}^{3}\left(z\,\frac{d}{d\,z}\right)\right) w(z) - 4\left(z^2 + \nu^2\right) z\,\frac{\partial w(z)}{\partial z} - 4\,z^2\,w(z) == 0 \;/;\; w(z) == c_1\,I_\nu(z)^2 + c_2\,K_\nu(z)\,I_\nu(z) + c_3\,K_\nu(z)^2$$

03.04.13.0018.01

$$z^3\,w^{(3)}(z) - z\left(4\,z^2 + 4\,\nu^2 - 1\right) w'(z) + \left(4\,\nu^2 - 1\right) w(z) == 0 \;/;\; w(z) == c_1\,z\,I_\nu(z)^2 + c_2\,z\,K_\nu(z)\,I_\nu(z) + c_3\,z\,K_\nu(z)^2$$

Transformations

Transformations and argument simplifications

Argument involving basic arithmetic operations

03.04.16.0001.01

$$K_\nu(-z) == z^\nu\,K_\nu(z)\,(-z)^{-\nu} + \frac{\pi}{2}\left((-z)^{-\nu}\,z^\nu - (-z)^\nu\,z^{-\nu}\right) I_\nu(z)\csc(\pi\,\nu) \;/;\; \nu \notin \mathbb{Z}$$

03.04.16.0002.01

$$K_\nu(-z) == (-1)^\nu\,K_\nu(z) + (\log(z) - \log(-z))\,I_\nu(z) \;/;\; \nu \in \mathbb{Z}$$

03.04.16.0003.01

$$K_\nu(i\,z) == \frac{\pi}{2}\left(\frac{z^\nu\cos(\pi\,\nu)}{(i\,z)^\nu} - \frac{(i\,z)^\nu}{z^\nu}\right)\csc(\pi\,\nu)\,J_\nu(z) - \frac{\pi\,z^\nu}{2\,(i\,z)^\nu}\,Y_\nu(z) \;/;\; \nu \notin \mathbb{Z}$$

03.04.16.0004.01

$$K_\nu(i z) = \frac{1}{i^\nu} (\log(z) - \log(i z)) J_\nu(z) - \frac{\pi}{2 i^\nu} Y_\nu(z) /; \nu \in \mathbb{Z}$$

03.04.16.0005.01

$$K_\nu(-i z) = \frac{\pi}{2} \left(\frac{z^\nu \cos(\pi \nu)}{(-i z)^\nu} - \frac{(-i z)^\nu}{z^\nu} \right) \csc(\pi \nu) J_\nu(z) - \frac{\pi z^\nu}{2 (-i z)^\nu} Y_\nu(z) /; \nu \notin \mathbb{Z}$$

03.04.16.0006.01

$$K_\nu(-i z) = \frac{1}{(-i)^\nu} (\log(z) - \log(-i z)) J_\nu(z) - \frac{\pi}{2 (-i)^\nu} Y_\nu(z) /; \nu \in \mathbb{Z}$$

03.04.16.0007.01

$$K_\nu(c (d z^n)^m) = \frac{(c d^m z^{m n})^\nu}{(c (d z^n)^m)^\nu} K_\nu(c d^m z^{m n}) - \frac{\pi}{2} \csc(\pi \nu) \left(\frac{(c (d z^n)^m)^\nu}{(c d^m z^{m n})^\nu} - \frac{(c d^m z^{m n})^\nu}{(c (d z^n)^m)^\nu} \right) I_\nu(c d^m z^{m n}) /; 2 m \in \mathbb{Z} \bigwedge \nu \notin \mathbb{Z}$$

03.04.16.0008.01

$$K_\nu(c (d z^n)^m) = \left(\frac{(d z^n)^m}{d^m z^{m n}} \right)^\nu (K_\nu(c d^m z^{m n}) - (-1)^\nu I_\nu(c d^m z^{m n}) (\log(c (d z^n)^m) - \log(c d^m z^{m n}))) /; 2 m \in \mathbb{Z} \land \nu \in \mathbb{Z}$$

03.04.16.0013.01

$$K_\nu\left(\sqrt{z^2}\right) = z^\nu \left(z^2\right)^{-\frac{\nu}{2}} K_\nu(z) - \frac{\pi \csc(\pi \nu)}{2} \left(z^{-\nu} \left(z^2\right)^{\nu/2} - z^\nu \left(z^2\right)^{-\frac{\nu}{2}} \right) I_\nu(z) /; \nu \notin \mathbb{Z}$$

03.04.16.0014.01

$$K_\nu\left(\sqrt{z^2}\right) = \left(\frac{\sqrt{z^2}}{z} \right)^\nu \left(K_\nu(z) - (-1)^\nu \left(\log\left(\sqrt{z^2}\right) - \log(z) \right) I_\nu(z) \right) /; \nu \in \mathbb{Z}$$

Addition formulas

03.04.16.0009.01

$$K_\nu(z_1 - z_2) = \sum_{k=-\infty}^{\infty} K_{k+\nu}(z_1) I_k(z_2) /; \left| \frac{z_2}{z_1} \right| < 1$$

03.04.16.0010.01

$$K_\nu(z_1 + z_2) = \sum_{k=-\infty}^{\infty} (-1)^k K_{\nu-k}(z_1) I_k(z_2) /; \left| \frac{z_2}{z_1} \right| < 1$$

Multiple arguments

03.04.16.0011.01

$$K_\nu(z_1 z_2) = z_1^\nu \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \left(z_1^2 - 1\right)^k K_{k+\nu}(z_2) \left(\frac{z_2}{2} \right)^k /; \left|z_1^2 - 1\right| < 1$$

03.04.16.0012.01

$$K_\nu(z_1 z_2) = z_1^{-\nu} \sum_{k=0}^{\infty} \frac{(-1)^k \left(z_1^2 - 1\right)^k}{k!} K_{\nu-k}(z_2) \left(\frac{z_2}{2} \right)^k /; \left|z_1^2 - 1\right| < 1 \bigvee \nu \in \mathbb{Z}$$

Identities

Recurrence identities

Consecutive neighbors

03.04.17.0001.01

$$K_\nu(z) = K_{\nu+2}(z) - \frac{2(\nu+1)}{z} K_{\nu+1}(z)$$

03.04.17.0002.01

$$K_\nu(z) = K_{\nu-2}(z) + \frac{2(\nu-1)}{z} K_{\nu-1}(z)$$

Distant neighbors

Increasing

03.04.17.0003.01

$$K_\nu(z) = (-1)^n 2^{n-1} z^{-n} (\nu+1)_{n-1} \left(2(n+\nu) \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{(n-k)!}{k!\,(n-2k)!\,(-n-\nu)_k\,(\nu+1)_k} \left(-\frac{z^2}{4}\right)^k K_{n+\nu}(z) - z \sum_{k=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \frac{(n-k-1)!}{k!\,(n-2k-1)!\,(-n-\nu+1)_k\,(\nu+1)_k} \left(-\frac{z^2}{4}\right)^k K_{n+\nu+1}(z)\right) /;\ n \in \mathbb{N}$$

03.04.17.0014.01

$$K_\nu(z) = (-1)^n 2^{n-1} z^{-n} (\nu+1)_{n-1} \left(2(n+\nu)\, {}_3F_4\!\left(1, \frac{1-n}{2}, -\frac{n}{2}; 1, -n, -n-\nu, \nu+1; z^2\right) K_{n+\nu}(z) - z\, {}_3F_4\!\left(1, \frac{1-n}{2}, 1-\frac{n}{2}; 1, 1-n, -n-\nu+1, \nu+1; z^2\right) K_{n+\nu+1}(z)\right) /;\ n \in \mathbb{N}$$

03.04.17.0006.01

$$K_\nu(z) = \frac{\left(z^2 + 4(\nu+1)(\nu+2)\right) K_{\nu+2}(z) - 2z(\nu+1) K_{\nu+3}(z)}{z^2}$$

03.04.17.0007.01

$$K_\nu(z) = \frac{z\left(z^2 + 4(\nu+1)(\nu+2)\right) K_{\nu+4}(z) - 4(\nu+2)\left(z^2 + 2(\nu+1)(\nu+3)\right) K_{\nu+3}(z)}{z^3}$$

03.04.17.0008.01

$$K_\nu(z) = \frac{1}{z^4} \left(\left(z^4 + 12(\nu+2)(\nu+3) z^2 + 16(\nu+1)(\nu+2)(\nu+3)(\nu+4)\right) K_{\nu+4}(z) - 4z(\nu+2)\left(z^2 + 2(\nu+1)(\nu+3)\right) K_{\nu+5}(z)\right)$$

03.04.17.0009.01

$$K_\nu(z) = -\frac{1}{z^5} \left(2(\nu+3)\left(3z^4 + 16(\nu+2)(\nu+4) z^2 + 16(\nu+1)(\nu+2)(\nu+4)(\nu+5)\right) K_{\nu+5}(z) - z\left(z^4 + 12(\nu+2)(\nu+3) z^2 + 16(\nu+1)(\nu+2)(\nu+3)(\nu+4)\right) K_{\nu+6}(z)\right)$$

03.04.17.0015.01

$$K_\nu(z) = C_n(\nu, z) K_{\nu+n}(z) + C_{n-1}(\nu, z) K_{\nu+n+1}(z) \ /;$$
$$C_0(\nu, z) = 1 \bigwedge C_1(\nu, z) = -\frac{2(\nu+1)}{z} \bigwedge C_n(\nu, z) = -\frac{2(n+\nu)}{z} C_{n-1}(\nu, z) + C_{n-2}(\nu, z) \bigwedge n \in \mathbb{N}^+$$

03.04.17.0016.01

$$K_\nu(z) = \mathcal{C}_n(\nu, z)\, K_{\nu+n}(z) + \mathcal{C}_{n-1}(\nu, z)\, K_{\nu+n+1}(z) \;/;\; \mathcal{C}_n(\nu, z) = (-2)^n z^{-n} (\nu+1)_n \,{}_2F_3\left(\frac{1-n}{2}, -\frac{n}{2}; \nu+1, -n, -n-\nu; z^2\right) \bigwedge n \in \mathbb{N}^+$$

Decreasing

03.04.17.0004.01

$$K_\nu(z) = (-1)^n 2^{n-1} z^{-n} (1-\nu)_{n-1} \left(2(n-\nu) \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{(n-k)!}{k!\,(n-2k)!\,(1-\nu)_k\,(\nu-n)_k} \left(-\frac{z^2}{4}\right)^k K_{\nu-n}(z) - z \sum_{k=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \frac{(n-k-1)!}{k!\,(n-2k-1)!\,(1-\nu)_k\,(\nu-n+1)_k} \left(-\frac{z^2}{4}\right)^k K_{\nu-n-1}(z)\right) /;\; n \in \mathbb{N}$$

03.04.17.0017.01

$$K_\nu(z) = (-1)^n 2^{n-1} z^{-n} (1-\nu)_{n-1} \left(2(n-\nu)\,{}_3F_4\left(1, \frac{1-n}{2}, -\frac{n}{2}; 1, -n, 1-\nu, \nu-n; z^2\right) K_{\nu-n}(z) - z\,{}_3F_4\left(1, \frac{1-n}{2}, 1-\frac{n}{2}; 1, 1-n, 1-\nu, \nu-n+1; z^2\right) K_{\nu-n-1}(z)\right) /;\; n \in \mathbb{N}$$

03.04.17.0010.01

$$K_\nu(z) = \frac{2z(\nu-1) K_{\nu-3}(z) + \left(z^2 + 4(\nu-2)(\nu-1)\right) K_{\nu-2}(z)}{z^2}$$

03.04.17.0011.01

$$K_\nu(z) = \frac{z\left(z^2 + 4(\nu-2)(\nu-1)\right) K_{\nu-4}(z) + 4\left(z^2 + 2(\nu-3)(\nu-1)\right)(\nu-2) K_{\nu-3}(z)}{z^3}$$

03.04.17.0012.01

$$K_\nu(z) = \frac{1}{z^4}\left(4z\left(z^2 + 2(\nu-3)(\nu-1)\right)(\nu-2) K_{\nu-5}(z) + \left(z^4 + 12(\nu-3)(\nu-2) z^2 + 16(\nu-4)(\nu-3)(\nu-2)(\nu-1)\right) K_{\nu-4}(z)\right)$$

03.04.17.0013.01

$$K_\nu(z) = \frac{1}{z^5}\left(z\left(z^4 + 12(\nu-3)(\nu-2) z^2 + 16(\nu-4)(\nu-3)(\nu-2)(\nu-1)\right) K_{\nu-6}(z) + 2\left(3z^4 + 16(\nu-4)(\nu-2) z^2 + 16(\nu-5)(\nu-4)(\nu-2)(\nu-1)\right)(\nu-3) K_{\nu-5}(z)\right)$$

03.04.17.0018.01

$$K_\nu(z) = \mathcal{C}_n(\nu, z)\, K_{\nu-n}(z) + \mathcal{C}_{n-1}(\nu, z)\, K_{\nu-n-1}(z) \;/;\; \mathcal{C}_0(\nu, z) = 1 \bigwedge \mathcal{C}_1(\nu, z) = \frac{2(\nu-1)}{z} \bigwedge \mathcal{C}_n(\nu, z) = \frac{2(-n+\nu)}{z}\, \mathcal{C}_{n-1}(\nu, z) + \mathcal{C}_{n-2}(\nu, z) \bigwedge n \in \mathbb{N}^+$$

03.04.17.0019.01

$$K_\nu(z) = \mathcal{C}_n(\nu, z)\, K_{\nu-n}(z) + \mathcal{C}_{n-1}(\nu, z)\, K_{\nu-n-1}(z) \;/;\; \mathcal{C}_n(\nu, z) = (-2)^n z^{-n} (1-\nu)_n \,{}_2F_3\left(\frac{1-n}{2}, -\frac{n}{2}; 1-\nu, -n, \nu-n; z^2\right) \bigwedge n \in \mathbb{N}^+$$

Functional identities

Relations between contiguous functions

03.04.17.0005.01

$$K_\nu(z) = \frac{z}{2\,\nu}\,(K_{\nu+1}(z) - K_{\nu-1}(z))$$

Differentiation

Low-order differentiation

With respect to ν

03.04.20.0001.01

$$K_\nu^{(1,0)}(z) = \frac{\pi\csc(\pi\,\nu)}{2}\left(-2\cos(\pi\,\nu)\,K_\nu(z) - \log\!\left(\frac{z}{2}\right)(I_{-\nu}(z) + I_\nu(z)) + \sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{\psi(k-\nu+1)}{\Gamma(k-\nu+1)}\left(\frac{z}{2}\right)^{2k-\nu} + \frac{\psi(k+\nu+1)}{\Gamma(k+\nu+1)}\left(\frac{z}{2}\right)^{2k+\nu}\right)\right)\;/;\;\nu\notin\mathbb{Z}$$

03.04.20.0002.01

$$K_\nu^{(1,0)}(z) = \frac{\pi\csc(\pi\,\nu)}{2}\left((I_{-\nu}(z) + I_\nu(z))\,(\log(2) - \log(z) + \psi(\nu)) + \frac{\pi\,\nu\cot(\pi\,\nu) + 1}{\nu}\,I_\nu(z)\right) + \frac{2^{\nu-3}\,z^{2-\nu}\,\Gamma(\nu-1)}{\nu-1}\,F_{2\times0\times1}^{0\times1\times2}\!\left(\begin{matrix};1;1,1-\nu;\\2,2-\nu;;2-\nu;\end{matrix}\;\frac{z^2}{4},\frac{z^2}{4}\right) + \frac{2^{-\nu-3}\,z^{\nu+2}\,\Gamma(-\nu-1)}{\nu+1}\,F_{2\times0\times1}^{0\times1\times2}\!\left(\begin{matrix};1;1,1+\nu;\\2,2+\nu;;2+\nu;\end{matrix}\;\frac{z^2}{4},\frac{z^2}{4}\right)\;/;\;\nu\notin\mathbb{Z}$$

03.04.20.0017.01

$$K_0^{(1,0)}(z) = 0$$

03.04.20.0003.01

$$K_n^{(1,0)}(z) = \frac{1}{2}\,n!\left(\frac{z}{2}\right)^{-n}\sum_{k=0}^{n-1}\frac{1}{(n-k)\,k!}\,K_k(z)\left(\frac{z}{2}\right)^k\;/;\;n\in\mathbb{N}$$

03.04.20.0018.01

$$K_n^{(1,0)}(z) = \frac{1}{2}\,\mathrm{sgn}(n)\,|n|!\left(\frac{z}{2}\right)^{-|n|}\sum_{k=0}^{|n|-1}\frac{1}{(|n|-k)\,k!}\,K_k(z)\left(\frac{z}{2}\right)^k\;/;\;n\in\mathbb{Z}$$

03.04.20.0019.01

$$K_{n+\frac{1}{2}}^{(1,0)}(z) = \frac{(-1)^{n+1}\,\pi}{2}\,(\mathrm{Chi}(2\,z) - \mathrm{Shi}(2\,z))\left(I_{n+\frac{1}{2}}(z) + I_{-n-\frac{1}{2}}(z)\right) + \frac{1}{2}\,(n+1)!\sum_{k=0}^{n}\frac{1}{k!\,(n-k+1)}\left(\frac{z}{2}\right)^{k-n-1}K_{k-\frac{1}{2}}(z) + \frac{(-1)^n\sqrt{\pi\,z}\,(n+1)!}{2}\sum_{k=1}^{n+1}\frac{1}{(n-k+1)!\,k}\left(-\frac{2}{z}\right)^k\left(I_{n-k+\frac{1}{2}}(z) + I_{k-n-\frac{1}{2}}(z)\right)\sum_{p=0}^{k-1}\frac{z^p}{p!}\,K_{p-\frac{1}{2}}(2\,z)\;/;\;n\in\mathbb{N}$$

Brychkov Yu.A. (2005)

03.04.20.0020.01

$$K_{-n-\frac{1}{2}}^{(1,0)}(z) = \frac{(-1)^n\,\pi}{2}\left(I_{n+\frac{1}{2}}(z) + I_{-n-\frac{1}{2}}(z)\right)(\mathrm{Chi}(2\,z) - \mathrm{Shi}(2\,z)) - \frac{(n+1)!}{2}\sum_{k=0}^{n}\frac{1}{k!\,(n-k+1)}\left(\frac{z}{2}\right)^{k-n-1}K_{k-\frac{1}{2}}(z) - \frac{(-1)^n\sqrt{\pi\,z}\,(n+1)!}{2}\sum_{k=1}^{n+1}\frac{1}{(n-k+1)!\,k}\left(-\frac{2}{z}\right)^k\left(I_{n-k+\frac{1}{2}}(z) + I_{k-n-\frac{1}{2}}(z)\right)\sum_{p=0}^{k-1}\frac{z^p}{p!}\,K_{p-\frac{1}{2}}(2\,z)\;/;\;n\in\mathbb{N}$$

Brychkov Yu.A. (2005)

With respect to z

03.04.20.0004.01

$$\frac{\partial K_\nu(z)}{\partial z} = -K_{\nu-1}(z) - \frac{\nu}{z} K_\nu(z)$$

03.04.20.0005.01

$$\frac{\partial K_\nu(z)}{\partial z} = \frac{\nu}{z} K_\nu(z) - K_{\nu+1}(z)$$

03.04.20.0006.01

$$\frac{\partial K_\nu(z)}{\partial z} = -\frac{1}{2} (K_{\nu-1}(z) + K_{\nu+1}(z))$$

03.04.20.0007.01

$$\frac{\partial K_0(z)}{\partial z} = -K_1(z)$$

03.04.20.0008.01

$$\frac{\partial (z^\nu K_\nu(z))}{\partial z} = -z^\nu K_{\nu-1}(z)$$

03.04.20.0009.01

$$\frac{\partial (z^{-\nu} K_\nu(z))}{\partial z} = -z^{-\nu} K_{\nu+1}(z)$$

03.04.20.0010.01

$$\frac{\partial^2 K_\nu(z)}{\partial z^2} = \frac{1}{4} (K_{\nu-2}(z) + 2 K_\nu(z) + K_{\nu+2}(z))$$

Symbolic differentiation

With respect to ν

03.04.20.0011.02

$$K_\nu^{(m,0)}(z) = \frac{\pi}{2} \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{z}{2}\right)^{2k} \frac{\partial^m}{\partial \nu^m} \left(\csc(\pi \nu) \left(\frac{1}{\Gamma(k-\nu+1)} \left(\frac{z}{2}\right)^{-\nu} - \frac{1}{\Gamma(k+\nu+1)} \left(\frac{z}{2}\right)^{\nu}\right)\right) /; m \in \mathbb{N} \bigwedge \nu \notin \mathbb{Z}$$

With respect to z

03.04.20.0021.01

$$\frac{\partial^n K_\nu(z)}{\partial z^n} = z^{-n} \sum_{m=0}^{n} (-1)^{m+n} \binom{n}{m} (-\nu)_{n-m} \sum_{k=0}^{m} \frac{(-1)^k 2^{2k-m} (-m)_{2(m-k)} (\nu)_k}{(m-k)!}$$
$$\left(\frac{z}{2} \sum_{j=0}^{k-1} \frac{(k-j-1)!}{j! (k-2j-1)! (1-k-\nu)_j (\nu)_{j+1}} \left(-\frac{z^2}{4}\right)^j K_{\nu-1}(z) + \sum_{j=0}^{k} \frac{(k-j)!}{j! (k-2j)! (1-k-\nu)_j (\nu)_j} \left(-\frac{z^2}{4}\right)^j K_\nu(z)\right) /; n \in \mathbb{N}$$

03.04.20.0012.02

$$\frac{\partial^n K_\nu(z)}{\partial z^n} = 2^{n-2\nu-1} \pi^{3/2} z^{-n-\nu} \csc(\pi \nu) \left(16^\nu \Gamma(1-\nu) \, {}_2\tilde{F}_3\left(\frac{1-\nu}{2}, 1-\frac{\nu}{2}; \frac{1-\nu-n}{2}, \frac{2-\nu-n}{2}, 1-\nu; \frac{z^2}{4}\right) -\right.$$
$$\left. z^{2\nu} \Gamma(\nu+1) \, {}_2\tilde{F}_3\left(\frac{\nu+1}{2}, \frac{\nu+2}{2}; \frac{\nu-n+1}{2}, \frac{\nu-n+2}{2}, \nu+1; \frac{z^2}{4}\right)\right) /; \nu \notin \mathbb{Z} \bigwedge n \in \mathbb{N}$$

03.04.20.0013.02

$$\frac{\partial^n K_\nu(z)}{\partial z^n} = \frac{1}{2} G_{2,4}^{2,2}\left(\frac{z}{2}, \frac{1}{2} \middle| \begin{array}{c} \frac{1-n}{2}, -\frac{n}{2} \\ \frac{\nu-n}{2}, -\frac{1}{2}(n+\nu), \frac{1}{2}, 0 \end{array}\right) /; n \in \mathbb{N}$$

03.04.20.0014.02

$$\frac{\partial^n K_\nu(z)}{\partial z^n} = (-1)^n 2^{-n} \sum_{k=0}^{n} \binom{n}{k} K_{2k-n+\nu}(z) /; n \in \mathbb{N}$$

Fractional integro-differentiation

With respect to z

03.04.20.0015.01

$$\frac{\partial^\alpha K_\nu(z)}{\partial z^\alpha} = 2^{\alpha-2\nu-1} \pi^{3/2} z^{-\alpha-\nu} \csc(\pi \nu) \left(16^\nu \Gamma(1-\nu) \, {}_2\tilde{F}_3\left(\frac{1-\nu}{2}, 1-\frac{\nu}{2}; 1-\nu, \frac{1-\alpha-\nu}{2}, 1-\frac{\alpha+\nu}{2}; \frac{z^2}{4}\right) - z^{2\nu} \Gamma(\nu+1) \, {}_2\tilde{F}_3\left(\frac{\nu+1}{2}, \frac{\nu}{2}+1; \nu+1, \frac{1-\alpha+\nu}{2}, 1-\frac{\alpha-\nu}{2}; \frac{z^2}{4}\right)\right) /; \nu \notin \mathbb{Z}$$

03.04.20.0022.01

$$\frac{\partial^\alpha K_\nu(z)}{\partial z^\alpha} = 2^{|\nu|-1} z^{-\alpha-|\nu|} \sum_{k=\left\lfloor \frac{|\nu|-1}{2} \right\rfloor+1}^{|\nu|-1} \frac{(-1)^k (|\nu|-k-1)! \, \Gamma(2k-|\nu|+1)}{k! \, \Gamma(2k-\alpha-|\nu|+1)} \left(\frac{z}{2}\right)^{2k} +$$
$$(-1)^{|\nu|-1} 2^{|\nu|-1} z^{-\alpha-|\nu|} \sum_{k=0}^{\left\lfloor \frac{|\nu|-1}{2} \right\rfloor} \frac{(-1)^k (|\nu|-k-1)!}{k! \, (|\nu|-2k-1)! \, \Gamma(2k-\alpha-|\nu|+1)} (\log(z) - \psi(2k-\alpha-|\nu|+1) + \psi(|\nu|-2k)) \left(\frac{z}{2}\right)^{2k} +$$
$$(-1)^{\nu-1} 2^{-|\nu|} z^{|\nu|-\alpha} \sum_{k=0}^{\infty} \frac{\mathcal{FC}_{\log}^{(\alpha)}(z, 2k+|\nu|)}{k! \, (k+|\nu|)!} \left(\frac{z}{2}\right)^{2k} - (-1)^{\nu-1} \log(2) \, 2^{-|\nu|} z^{|\nu|-\alpha} \sum_{k=0}^{\infty} \frac{\Gamma(2k+|\nu|+1)}{k! \, (k+|\nu|)! \, \Gamma(2k-\alpha+|\nu|+1)} \left(\frac{z}{2}\right)^{2k} +$$
$$(-1)^\nu 2^{-|\nu|-1} z^{|\nu|-\alpha} \sum_{k=0}^{\infty} \frac{\Gamma(2k+|\nu|+1) (\psi(k+1) + \psi(k+|\nu|+1))}{k! \, (k+|\nu|)! \, \Gamma(2k-\alpha+|\nu|+1)} \left(\frac{z}{2}\right)^{2k} /; \nu \in \mathbb{Z}$$

03.04.20.0016.01

$$\frac{\partial^\alpha K_\nu(z)}{\partial z^\alpha} = (-1)^{\nu-1} 2^{\alpha-2\nu} \sqrt{\pi} \, \Gamma(\nu+1) \, {}_2\tilde{F}_3\left(\frac{\nu+1}{2}, \frac{\nu+2}{2}; \frac{1-\alpha+\nu}{2}, \frac{2-\alpha+\nu}{2}, \nu+1; \frac{z^2}{4}\right) \log\left(\frac{z}{2}\right) z^{\nu-\alpha} +$$
$$\frac{1}{2} z^{-\alpha} \sum_{k=\left\lfloor \frac{\nu+1}{2} \right\rfloor}^{\nu-1} \frac{(-1)^k (-k+\nu-1)! \, (2k-\nu)!}{k! \, \Gamma(2k-\alpha-\nu+1)} \left(\frac{z}{2}\right)^{2k-\nu} + \sum_{k=0}^{\left\lfloor \frac{\nu-1}{2} \right\rfloor} \frac{(-1)^k (\nu-k-1)!}{2^{2k-\nu+1} k!} \mathcal{FC}_{\exp}^{(\alpha)}(z, 2k-\nu) z^{2k-\alpha-\nu} +$$
$$\frac{(-1)^\nu}{2} z^{-\alpha} \sum_{k=0}^{\infty} \frac{(2k+\nu)! \, (\psi(k+1) + \psi(k+\nu+1) - 2\psi(2k+\nu+1) + 2\psi(2k-\alpha+\nu+1))}{k! \, (k+\nu)! \, \Gamma(2k-\alpha+\nu+1)} \left(\frac{z}{2}\right)^{2k+\nu} /; \nu \in \mathbb{N}$$

Integration

Indefinite integration

Involving only one direct function

03.04.21.0001.01

$$\int K_\nu(a\,z)\,dz = 2^{-\nu-2}\,\pi\,z\,(a\,z)^{-\nu}\csc(\pi\,\nu)\left(4^\nu\,\Gamma\!\left(\frac{1}{2}-\frac{\nu}{2}\right){}_1\tilde{F}_2\!\left(\frac{1}{2}-\frac{\nu}{2};1-\nu,\frac{3}{2}-\frac{\nu}{2};\frac{a^2\,z^2}{4}\right)-(a\,z)^{2\,\nu}\,\Gamma\!\left(\frac{\nu+1}{2}\right){}_1\tilde{F}_2\!\left(\frac{\nu+1}{2};\nu+1,\frac{\nu+3}{2};\frac{a^2\,z^2}{4}\right)\right)\,/;\,\nu\notin\mathbb{Z}$$

03.04.21.0002.01

$$\int K_\nu(z)\,dz = \frac{2^{\nu-1}\,\pi\,z^{1-\nu}\csc(\pi\,\nu)}{\Gamma(2-\nu)}\,{}_1F_2\!\left(\frac{1-\nu}{2};1-\nu,\frac{3-\nu}{2};\frac{z^2}{4}\right)-\frac{2^{-\nu-1}\,\pi\,z^{\nu+1}\csc(\pi\,\nu)}{\Gamma(\nu+2)}\,{}_1F_2\!\left(\frac{\nu+1}{2};\frac{\nu+3}{2},\nu+1;\frac{z^2}{4}\right)\,/;\,\nu\notin\mathbb{Z}$$

03.04.21.0003.01

$$\int K_\nu(z)\,dz = \frac{1}{2}\,G_{1,3}^{2,1}\!\left(\frac{z}{2},\frac{1}{2}\,\middle|\,\begin{matrix}1\\ \frac{1-\nu}{2},\frac{\nu+1}{2},0\end{matrix}\right)$$

03.04.21.0004.01

$$\int K_0(z)\,dz = \frac{\pi\,z}{2}\,(K_0(z)\,\boldsymbol{L}_{-1}(z)+K_1(z)\,\boldsymbol{L}_0(z))$$

03.04.21.0005.01

$$\int K_1(z)\,dz = -K_0(z)$$

03.04.21.0006.01

$$\int K_2(z)\,dz = \frac{1}{2}\,G_{1,3}^{2,1}\!\left(\frac{z}{2},\frac{1}{2}\,\middle|\,\begin{matrix}1\\ -\frac{1}{2},\frac{3}{2},0\end{matrix}\right)$$

03.04.21.0120.01

$$\int K_2(z)\,dz = -2\,K_1(z)-\frac{1}{2}\,\pi\,z\,(K_0(z)\,\boldsymbol{L}_{-1}(z)+K_1(z)\,\boldsymbol{L}_0(z))$$

03.04.21.0121.01

$$\int K_3(z)\,dz = K_0(z)-2\,K_2(z)$$

03.04.21.0122.01

$$\int K_4(z)\,dz = 2\,(K_1(z)-K_3(z))+\frac{1}{2}\,\pi\,z\,(K_0(z)\,\boldsymbol{L}_{-1}(z)+K_1(z)\,\boldsymbol{L}_0(z))$$

03.04.21.0123.01

$$\int K_5(z)\,dz = 2\,(K_2(z)-K_4(z))-K_0(z)$$

03.04.21.0124.01

$$\int K_{2\,n}(z)\,dz = \frac{1}{2}\,((-1)^n\,\pi\,z)\,(K_0(z)\,\boldsymbol{L}_{-1}(z)+K_1(z)\,\boldsymbol{L}_0(z))+2\sum_{k=0}^{n-1}(-1)^{k+n}\,K_{2\,k+1}(z)\,/;\,n\in\mathbb{N}$$

03.04.21.0125.01

$$\int K_{2\,n+1}(z)\,dz = 2\sum_{k=1}^{n}(-1)^{k+n-1}\,K_{2\,k}(z)-(-1)^n\,K_0(z)\,/;\,n\in\mathbb{N}$$

Involving one direct function and elementary functions

Involving power function

Involving power

Linear arguments

03.04.21.0007.01

$$\int z^{\alpha-1} K_\nu(a z)\, dz = 2^{-\nu-2} \pi z^\alpha (a z)^{-\nu} \csc(\pi \nu) \left(4^\nu \Gamma\left(\frac{\alpha-\nu}{2}\right) {}_1\tilde{F}_2\left(\frac{\alpha-\nu}{2}; 1-\nu, \frac{1}{2}(\alpha-\nu+2); \frac{a^2 z^2}{4}\right) - (a z)^{2\nu} \Gamma\left(\frac{\alpha+\nu}{2}\right) {}_1\tilde{F}_2\left(\frac{\alpha+\nu}{2}; \nu+1, \frac{1}{2}(\alpha+\nu+2); \frac{a^2 z^2}{4}\right)\right) /; \nu \notin \mathbb{Z}$$

03.04.21.0008.01

$$\int z^{\alpha-1} K_\nu(z)\, dz = -\frac{2^{\nu-1} \pi z^{\alpha-\nu} \csc(\pi \nu)}{(\nu-\alpha)\Gamma(1-\nu)} {}_1F_2\left(\frac{\alpha-\nu}{2}; 1-\nu, \frac{\alpha-\nu}{2}+1; \frac{z^2}{4}\right) - \frac{2^{-\nu-1} \pi z^{\alpha+\nu} \csc(\pi \nu)}{(\alpha+\nu)\Gamma(\nu+1)} {}_1F_2\left(\frac{\alpha+\nu}{2}; \nu+1, \frac{\alpha+\nu}{2}+1; \frac{z^2}{4}\right) /; \nu \notin \mathbb{Z}$$

03.04.21.0009.01

$$\int z^{\alpha-1} K_0(z)\, dz = \frac{z^\alpha}{\alpha}\left(K_0(z)\, {}_1F_2\left(1; \frac{\alpha}{2}+1, \frac{\alpha}{2}; \frac{z^2}{4}\right) + \frac{z}{\alpha} K_1(z)\, {}_1F_2\left(1; \frac{\alpha}{2}+1, \frac{\alpha}{2}+1; \frac{z^2}{4}\right)\right)$$

03.04.21.0010.01

$$\int z^{1-\nu} K_\nu(z)\, dz = -z^{1-\nu} K_{\nu-1}(z)$$

03.04.21.0011.01

$$\int z^{-\nu} K_\nu(z)\, dz = 2^{-\nu-1} \pi z \csc(\pi \nu)\left(-\frac{4^\nu z^{-2\nu}}{(2\nu-1)\Gamma(1-\nu)} {}_1F_2\left(\frac{1}{2}-\nu; 1-\nu, \frac{3}{2}-\nu; \frac{z^2}{4}\right) - \frac{1}{\Gamma(\nu+1)} {}_1F_2\left(\frac{1}{2}; \frac{3}{2}, \nu+1; \frac{z^2}{4}\right)\right)$$

03.04.21.0012.01

$$\int z^{\nu+3} K_\nu(z)\, dz = -\frac{1}{2\Gamma(-\nu)}\left(\pi \csc(\pi \nu)\left(2 I_{-\nu-2}(z)\Gamma(-\nu) z^{\nu+2} + 2\nu I_{\nu+2}(z)\Gamma(-\nu) z^{\nu+2} + 2 I_{\nu+2}(z)\Gamma(-\nu) z^{\nu+2} - I_{-\nu-1}(z)\Gamma(-\nu) z^{\nu+3} + I_{\nu+3}(z)\Gamma(-\nu) z^{\nu+3} + 2^{\nu+3} + 2^{\nu+3}\nu\right)\right)$$

03.04.21.0013.01

$$\int z^{\nu+1} K_\nu(z)\, dz = -z^{\nu+1} K_{-\nu-1}(z)$$

03.04.21.0014.01

$$\int z^\nu K_\nu(z)\, dz = 2^{-\nu-2} \pi z \csc(\pi \nu)\left(4^\nu \sqrt{\pi}\; {}_1\tilde{F}_2\left(\frac{1}{2}; 1-\nu, \frac{3}{2}; \frac{z^2}{4}\right) - z^{2\nu} \Gamma\left(\nu+\frac{1}{2}\right) {}_1\tilde{F}_2\left(\nu+\frac{1}{2}; \nu+1, \nu+\frac{3}{2}; \frac{z^2}{4}\right)\right)$$

03.04.21.0015.01

$$\int z K_0(z)\, dz = -z K_1(z)$$

03.04.21.0126.01

$$\int \frac{K_{2n}(z)}{z}\, dz = -\frac{1}{2n}\left((-1)^n K_0(z) + K_{2n}(z) + 2\sum_{k=1}^{n-1} (-1)^{k+n} K_{2k}(z)\right) /; n \in \mathbb{N}^+$$

03.04.21.0127.01

$$\int \frac{K_{2n+1}(z)}{z}\, dz = -\frac{1}{2n+1}\left(K_{2n+1}(z) + \frac{1}{2}((-1)^n \pi z)(K_0(z)\,\boldsymbol{L}_{-1}(z) + K_1(z)\,\boldsymbol{L}_0(z)) + 2\sum_{k=0}^{n-1} (-1)^{k+n} K_{2k+1}(z)\right) /; n \in \mathbb{N}$$

03.04.21.0128.01

$$\int \frac{K_{2n}(z)}{z^2}\,dz = -\frac{K_{2n}(z)}{z} + \frac{1}{2(2n+1)}\left(K_{2n+1}(z) + \frac{1}{2}\left((-1)^n \pi z\right)\left(K_0(z)\,\boldsymbol{L}_{-1}(z) + K_1(z)\,\boldsymbol{L}_0(z)\right) + 2\sum_{k=0}^{n-1}(-1)^{k+n} K_{2k+1}(z)\right) + \frac{1}{2(2n-1)}\left(K_{2n-1}(z) - \frac{1}{2}\left((-1)^n \pi z\right)\left(K_0(z)\,\boldsymbol{L}_{-1}(z) + K_1(z)\,\boldsymbol{L}_0(z)\right) - 2\sum_{k=0}^{n-2}(-1)^{k+n} K_{2k+1}(z)\right) /;\ n \in \mathbb{N}^+$$

03.04.21.0129.01

$$\int \frac{K_{2n+1}(z)}{z^2}\,dz = \frac{(-1)^n K_0(z) + K_{2n}(z)}{4n} + \frac{K_{2n+2}(z) - (-1)^n K_0(z)}{4(n+1)} - \frac{K_{2n+1}(z)}{z} + \frac{1}{2(n+1)}\sum_{k=1}^{n}(-1)^{k+n-1} K_{2k}(z) + \frac{1}{2n}\sum_{k=1}^{n-1}(-1)^{k+n} K_{2k}(z) /;\ n \in \mathbb{N}^+$$

03.04.21.0130.01

$$\int \frac{K_\nu(z)}{z^m}\,dz = -\frac{K_\nu(z)}{(m-1)\,z^{m-1}} - \frac{1}{2(m-1)}\int \frac{1}{z^{m-1}}\left(K_{\nu-1}(z) + K_{\nu+1}(z)\right)dz /;\ m \in \mathbb{Z} \land m > 1$$

Power arguments

03.04.21.0016.01

$$\int z^{\alpha-1} K_\nu(a z^r)\,dz = \frac{1}{r}\left(2^{-\nu-2}\pi z^\alpha (a z^r)^{-\nu}\csc(\pi\nu)\left(4^\nu\,\Gamma\!\left(\frac{\alpha - r\nu}{2r}\right) {}_1\tilde{F}_2\!\left(\frac{\alpha - r\nu}{2r}; 1-\nu, \frac{1}{2}\left(\frac{\alpha}{r} - \nu + 2\right); \frac{1}{4} a^2 z^{2r}\right) - (a z^r)^{2\nu}\,\Gamma\!\left(\frac{\alpha + r\nu}{2r}\right) {}_1\tilde{F}_2\!\left(\frac{\alpha + r\nu}{2r}; \nu + 1, \frac{\alpha + r(\nu+2)}{2r}; \frac{1}{4} a^2 z^{2r}\right)\right)\right)$$

Involving exponential function

Involving exp

Linear arguments

03.04.21.0017.01

$$\int e^{-az} K_\nu(a z)\,dz = -2^{-\nu}\pi z (a z)^{-\nu}\nu\csc(\pi\nu) \left(\Gamma(2\nu)\,{}_2\tilde{F}_2\!\left(\nu + \frac{1}{2}, \nu+1; \nu+2, 2\nu+1; -2az\right)(az)^{2\nu} + 4^\nu\,\Gamma(-2\nu)\,{}_2\tilde{F}_2\!\left(\frac{1}{2} - \nu, 1-\nu; 1-2\nu, 2-\nu; -2az\right)\right)$$

03.04.21.0018.01

$$\int e^{az} K_\nu(a z)\,dz = -2^{-\nu}\pi z (a z)^{-\nu}\nu\csc(\pi\nu) \left(\Gamma(2\nu)\,{}_2\tilde{F}_2\!\left(\nu + \frac{1}{2}, \nu+1; \nu+2, 2\nu+1; 2az\right)(az)^{2\nu} + 4^\nu\,\Gamma(-2\nu)\,{}_2\tilde{F}_2\!\left(\frac{1}{2} - \nu, 1-\nu; 1-2\nu, 2-\nu; 2az\right)\right)$$

Power arguments

03.04.21.0019.01

$$\int e^{-a z^r} K_\nu(a z^r)\, dz = -\frac{1}{r^2 \nu^2 - 1}\left(2^{-\nu-1} z\, (a z^r)^{-\nu}\left((r\,\Gamma(1-\nu) + \Gamma(-\nu))\, {}_2F_2\left(\nu + \frac{1}{2}, \nu + \frac{1}{r}; \nu + \frac{1}{r} + 1, 2\nu + 1; -2 a z^r\right)(a z^r)^{2\nu} + 4^\nu (\Gamma(\nu) + r\,\Gamma(\nu+1))\, {}_2F_2\left(\frac{1}{2} - \nu, \frac{1}{r} - \nu; 1 - 2\nu, -\nu + \frac{1}{r} + 1; -2 a z^r\right)\right)\right)$$

03.04.21.0020.01

$$\int e^{a z^r} K_\nu(a z^r)\, dz = -\frac{1}{r^2 \nu^2 - 1}\left(2^{-\nu-1} z\, (a z^r)^{-\nu}\left((r\,\Gamma(1-\nu) + \Gamma(-\nu))\, {}_2F_2\left(\nu + \frac{1}{2}, \nu + \frac{1}{r}; \nu + \frac{1}{r} + 1, 2\nu + 1; 2 a z^r\right)(a z^r)^{2\nu} + 4^\nu (\Gamma(\nu) + r\,\Gamma(\nu+1))\, {}_2F_2\left(\frac{1}{2} - \nu, \frac{1}{r} - \nu; 1 - 2\nu, -\nu + \frac{1}{r} + 1; 2 a z^r\right)\right)\right)$$

Involving exponential function and a power function

Involving exp and power

Linear arguments

03.04.21.0021.01

$$\int z^{\alpha-1} e^{-a z} K_\nu(a z)\, dz = 2^{-\nu-1}\sqrt{\pi}\, z^\alpha (a z)^{-\nu} \csc(\pi \nu)\left(\Gamma\left(\frac{1}{2} - \nu\right)\Gamma(\alpha - \nu)\, {}_2\tilde{F}_2\left(\frac{1}{2} - \nu, \alpha - \nu; 1 - 2\nu, \alpha - \nu + 1; -2 a z\right) - 4^\nu (a z)^{2\nu}\,\Gamma\left(\nu + \frac{1}{2}\right)\Gamma(\alpha + \nu)\, {}_2\tilde{F}_2\left(\nu + \frac{1}{2}, \alpha + \nu; \alpha + \nu + 1, 2\nu + 1; -2 a z\right)\right)$$

03.04.21.0022.01

$$\int z^{-\nu} e^{-a z} K_\nu(a z)\, dz = \frac{2^{-\nu} e^{-a z} z^{-\nu}}{a (2\nu - 1)\,\Gamma(\nu)}\left(2^\nu a z\,(K_{1-\nu}(a z) - K_\nu(a z))\,\Gamma(\nu) - e^{a z} \pi (a z)^\nu \csc(\pi \nu)\right)$$

03.04.21.0023.01

$$\int z^\nu e^{-a z} K_\nu(a z)\, dz = \frac{z^\nu}{2\nu + 1}\left(e^{-a z} z\,(K_\nu(a z) - K_{-\nu-1}(a z)) - \frac{2^\nu \pi (a z)^{-\nu} \csc(\pi \nu)}{a\,\Gamma(-\nu)}\right)$$

03.04.21.0024.01

$$\int z^{\alpha-1} e^{a z} K_\nu(a z)\, dz = 2^{-\nu-1}\sqrt{\pi}\, z^\alpha (a z)^{-\nu} \csc(\pi \nu)\left(\Gamma\left(\frac{1}{2} - \nu\right)\Gamma(\alpha - \nu)\, {}_2\tilde{F}_2\left(\frac{1}{2} - \nu, \alpha - \nu; 1 - 2\nu, \alpha - \nu + 1; 2 a z\right) - 4^\nu (a z)^{2\nu}\,\Gamma\left(\nu + \frac{1}{2}\right)\Gamma(\alpha + \nu)\, {}_2\tilde{F}_2\left(\nu + \frac{1}{2}, \alpha + \nu; \alpha + \nu + 1, 2\nu + 1; 2 a z\right)\right)$$

03.04.21.0025.01

$$\int z^{-\nu} e^{a z} K_\nu(a z)\, dz = \frac{2^{-\nu} z^{-\nu}}{a (2\nu - 1)\,\Gamma(\nu)}\left(\pi (a z)^\nu \csc(\pi \nu) - 2^\nu a\, e^{a z} z\,(K_{1-\nu}(a z) + K_\nu(a z))\,\Gamma(\nu)\right)$$

03.04.21.0026.01

$$\int z^\nu e^{a z} K_\nu(a z)\, dz = \frac{z^\nu (a z)^{-\nu}}{a (2\nu + 1)\,\Gamma(-\nu)}\left(a\, e^{a z} z\,(K_{-\nu-1}(a z) + K_\nu(a z))\,\Gamma(-\nu)\,(a z)^\nu + 2^\nu \pi \csc(\pi \nu)\right)$$

Power arguments

03.04.21.0027.01

$$\int z^{\alpha-1} e^{-a z^r} K_\nu(a z^r)\, dz = 2^{-\nu-1} z^\alpha (a z^r)^{-\nu} \left(\frac{\Gamma(-\nu) (a z^r)^{2\nu}}{\alpha + r\nu} {}_2F_2\left(\nu + \frac{1}{2}, \frac{\alpha}{r} + \nu; \frac{\alpha}{r} + \nu + 1, 2\nu + 1; -2 a z^r\right) + \frac{4^\nu \Gamma(\nu)}{\alpha - r\nu} {}_2F_2\left(\frac{1}{2} - \nu, \frac{\alpha}{r} - \nu; 1 - 2\nu, \frac{\alpha}{r} - \nu + 1; -2 a z^r\right)\right)$$

03.04.21.0028.01

$$\int z^{\alpha-1} e^{a z^r} K_\nu(a z^r)\, dz = 2^{-\nu-1} z^\alpha (a z^r)^{-\nu} \left(\frac{\Gamma(-\nu) (a z^r)^{2\nu}}{\alpha + r\nu} {}_2F_2\left(\nu + \frac{1}{2}, \frac{\alpha}{r} + \nu; \frac{\alpha}{r} + \nu + 1, 2\nu + 1; 2 a z^r\right) + \frac{4^\nu \Gamma(\nu)}{\alpha - r\nu} {}_2F_2\left(\frac{1}{2} - \nu, \frac{\alpha}{r} - \nu; 1 - 2\nu, \frac{\alpha}{r} - \nu + 1; 2 a z^r\right)\right)$$

Involving hyperbolic functions

Involving sinh

Linear arguments

03.04.21.0029.01

$$\int \sinh(a z) K_\nu(a z)\, dz = -\frac{1}{\nu^2 - 4} \Bigg(2^{-\nu-1} z (a z)^{1-\nu} \Bigg((\Gamma(1-\nu) + 2\Gamma(-\nu))\, {}_3F_4\left(\frac{\nu}{2} + \frac{3}{4}, \frac{\nu}{2} + 1, \frac{\nu}{2} + \frac{5}{4}; \frac{3}{2}, \frac{\nu}{2} + 2, \nu + 1, \nu + \frac{3}{2}; a^2 z^2\right) (a z)^{2\nu} + \left(2^{2\nu+1} \Gamma(\nu) + 4^\nu \Gamma(\nu+1)\right) {}_3F_4\left(\frac{3}{4} - \frac{\nu}{2}, 1 - \frac{\nu}{2}, \frac{5}{4} - \frac{\nu}{2}; \frac{3}{2}, 1 - \nu, \frac{3}{2} - \nu, 2 - \frac{\nu}{2}; a^2 z^2\right)\Bigg)\Bigg)$$

03.04.21.0030.01

$$\int \sinh(b + a z) K_\nu(a z)\, dz = 2^{-\nu-1} \pi z (a z)^{-\nu} \csc(\pi\nu) \Bigg(-\frac{a z (a z)^{2\nu} \cosh(b)}{(\nu+2)\Gamma(\nu+1)} {}_3F_4\left(\frac{\nu}{2} + \frac{3}{4}, \frac{\nu}{2} + 1, \frac{\nu}{2} + \frac{5}{4}; \frac{3}{2}, \frac{\nu}{2} + 2, \nu + 1, \nu + \frac{3}{2}; a^2 z^2\right) - \frac{4^\nu a z \cosh(b)}{(\nu-2)\Gamma(1-\nu)} {}_3F_4\left(\frac{3}{4} - \frac{\nu}{2}, 1 - \frac{\nu}{2}, \frac{5}{4} - \frac{\nu}{2}; \frac{3}{2}, 1 - \nu, \frac{3}{2} - \nu, 2 - \frac{\nu}{2}; a^2 z^2\right) + \left(\frac{4^\nu}{\Gamma(2-\nu)} {}_3F_4\left(\frac{1}{4} - \frac{\nu}{2}, \frac{1}{2} - \frac{\nu}{2}, \frac{3}{4} - \frac{\nu}{2}; \frac{1}{2}, \frac{1}{2} - \nu, 1 - \nu, \frac{3}{2} - \frac{\nu}{2}; a^2 z^2\right) - \frac{(a z)^{2\nu}}{\Gamma(\nu+2)} {}_3F_4\left(\frac{\nu}{2} + \frac{1}{4}, \frac{\nu}{2} + \frac{1}{2}, \frac{\nu}{2} + \frac{3}{4}; \frac{1}{2}, \frac{\nu}{2} + \frac{3}{2}, \nu + \frac{1}{2}, \nu + 1; a^2 z^2\right)\right) \sinh(b)\Bigg)$$

Power arguments

03.04.21.0031.01

$$\int \sinh(a z^r)\, K_\nu(a z^r)\, dz = -\frac{1}{(r(\nu-1)-1)(\nu r + r + 1)} \left(2^{-\nu-1} z (a z^r)^{1-\nu} \left((r\,\Gamma(1-\nu) + (r+1)\,\Gamma(-\nu))\, {}_3F_4\left(\frac{\nu}{2}+\frac{3}{4}, \frac{\nu}{2}+\frac{5}{4}, \frac{\nu}{2}+\frac{1}{2}+\frac{1}{2r}; \frac{3}{2}, \frac{\nu}{2}+\frac{3}{2}+\frac{1}{2r}, \nu+1, \nu+\frac{3}{2}; a^2 z^{2r}\right) (a z^r)^{2\nu} + 4^\nu ((r+1)\,\Gamma(\nu) + r\,\Gamma(\nu+1))\, {}_3F_4\left(\frac{3}{4}-\frac{\nu}{2}, \frac{5}{4}-\frac{\nu}{2}, -\frac{\nu}{2}+\frac{1}{2}+\frac{1}{2r}; \frac{3}{2}, 1-\nu, \frac{3}{2}-\nu, -\frac{\nu}{2}+\frac{3}{2}+\frac{1}{2r}; a^2 z^{2r}\right)\right)\right)$$

03.04.21.0032.01

$$\int \sinh(a z^r + b)\, K_\nu(a z^r)\, dz = 2^{-\nu-1} \pi z (a z^r)^{-\nu} \csc(\pi\nu) \left(-\frac{4^\nu \sinh(b)}{(r\nu-1)\,\Gamma(1-\nu)}\, {}_3F_4\left(\frac{1}{4}-\frac{\nu}{2}, \frac{3}{4}-\frac{\nu}{2}, \frac{1}{2r}-\frac{\nu}{2}; \frac{1}{2}, \frac{1}{2}-\nu, 1-\nu, -\frac{\nu}{2}+\frac{1}{2r}+1; a^2 z^{2r}\right) + \frac{1}{\Gamma(\nu+1)} \left((a z^r)^{2\nu} \left(-\frac{a z^r \cosh(b)}{\nu r + r + 1}\, {}_3F_4\left(\frac{\nu}{2}+\frac{3}{4}, \frac{\nu}{2}+\frac{5}{4}, \frac{\nu}{2}+\frac{1}{2}+\frac{1}{2r}; \frac{3}{2}, \frac{\nu}{2}+\frac{3}{2}+\frac{1}{2r}, \nu+1, \nu+\frac{3}{2}; a^2 z^{2r}\right) - \frac{\sinh(b)}{r\nu+1}\, {}_3F_4\left(\frac{\nu}{2}+\frac{1}{4}, \frac{\nu}{2}+\frac{3}{4}, \frac{\nu}{2}+\frac{1}{2r}; \frac{1}{2}, \frac{\nu}{2}+\frac{1}{2r}+1, \nu+\frac{1}{2}, \nu+1; a^2 z^{2r}\right)\right)\right) - \frac{4^\nu a z^r \cosh(b)}{(r(\nu-1)-1)\,\Gamma(1-\nu)}\, {}_3F_4\left(\frac{3}{4}-\frac{\nu}{2}, \frac{5}{4}-\frac{\nu}{2}, -\frac{\nu}{2}+\frac{1}{2}+\frac{1}{2r}; \frac{3}{2}, 1-\nu, \frac{3}{2}-\nu, -\frac{\nu}{2}+\frac{3}{2}+\frac{1}{2r}; a^2 z^{2r}\right)\right)$$

Involving cosh

Linear arguments

03.04.21.0033.01

$$\int \cosh(a z)\, K_\nu(a z)\, dz = 2^{-\nu-1} \pi z (a z)^{-\nu} \csc(\pi\nu) \left(\frac{4^\nu}{\Gamma(2-\nu)}\, {}_3F_4\left(\frac{1}{4}-\frac{\nu}{2}, \frac{1}{2}-\frac{\nu}{2}, \frac{3}{4}-\frac{\nu}{2}; \frac{1}{2}, \frac{1}{2}-\nu, 1-\nu, \frac{3}{2}-\frac{\nu}{2}; a^2 z^2\right) - \frac{(a z)^{2\nu}}{\Gamma(\nu+2)}\, {}_3F_4\left(\frac{\nu}{2}+\frac{1}{4}, \frac{\nu}{2}+\frac{1}{2}, \frac{\nu}{2}+\frac{3}{4}; \frac{1}{2}, \frac{\nu}{2}+\frac{3}{2}, \nu+\frac{1}{2}, \nu+1; a^2 z^2\right)\right)$$

03.04.21.0034.01

$$\int \cosh(b + a z)\, K_\nu(a z)\, dz = 2^{-\nu-1} \pi z (a z)^{-\nu} \csc(\pi\nu) \left(-\frac{(a z)^{2\nu} \cosh(b)}{\Gamma(\nu+2)}\, {}_3F_4\left(\frac{\nu}{2}+\frac{1}{4}, \frac{\nu}{2}+\frac{1}{2}, \frac{\nu}{2}+\frac{3}{4}; \frac{1}{2}, \frac{\nu}{2}+\frac{3}{2}, \nu+\frac{1}{2}, \nu+1; a^2 z^2\right) + \frac{4^\nu \cosh(b)}{\Gamma(2-\nu)}\, {}_3F_4\left(\frac{1}{4}-\frac{\nu}{2}, \frac{1}{2}-\frac{\nu}{2}, \frac{3}{4}-\frac{\nu}{2}; \frac{1}{2}, \frac{1}{2}-\nu, 1-\nu, \frac{3}{2}-\frac{\nu}{2}; a^2 z^2\right) + a z \left(-\frac{(a z)^{2\nu}}{(\nu+2)\,\Gamma(\nu+1)}\, {}_3F_4\left(\frac{\nu}{2}+\frac{3}{4}, \frac{\nu}{2}+1, \frac{\nu}{2}+\frac{5}{4}; \frac{3}{2}, \frac{\nu}{2}+2, \nu+1, \nu+\frac{3}{2}; a^2 z^2\right) - \frac{4^\nu}{(\nu-2)\,\Gamma(1-\nu)}\, {}_3F_4\left(\frac{3}{4}-\frac{\nu}{2}, 1-\frac{\nu}{2}, \frac{5}{4}-\frac{\nu}{2}; \frac{3}{2}, 1-\nu, \frac{3}{2}-\nu, 2-\frac{\nu}{2}; a^2 z^2\right)\right) \sinh(b)\right)$$

Power arguments

03.04.21.0035.01

$$\int \cosh(a z^r)\, K_\nu(a z^r)\, dz = -\frac{1}{r^2 \nu^2 - 1} \left(2^{-\nu-1} z\, (a z^r)^{-\nu} \left((r\, \Gamma(1-\nu) + \Gamma(-\nu))\, {}_3F_4\!\left(\frac{\nu}{2}+\frac{1}{4}, \frac{\nu}{2}+\frac{3}{4}, \frac{\nu}{2}+\frac{1}{2r}; \frac{1}{2}, \frac{\nu}{2}+\frac{1}{2r}+1, \nu+\frac{1}{2}, \nu+1; a^2 z^{2r}\right)(a z^r)^{2\nu} + 4^\nu\, (\Gamma(\nu) + r\, \Gamma(\nu+1))\, {}_3F_4\!\left(\frac{1}{4}-\frac{\nu}{2}, \frac{3}{4}-\frac{\nu}{2}, \frac{1}{2r}-\frac{\nu}{2}; \frac{1}{2}, \frac{1}{2}-\nu, 1-\nu, -\frac{\nu}{2}+\frac{1}{2r}+1; a^2 z^{2r}\right)\right)\right)$$

03.04.21.0036.01

$$\int \cosh(a z^r + b)\, K_\nu(a z^r)\, dz = 2^{-\nu-1} \pi\, z\, (a z^r)^{-\nu} \csc(\pi \nu) \left(-\frac{4^\nu \cosh(b)}{(r \nu - 1)\, \Gamma(1-\nu)}\, {}_3F_4\!\left(\frac{1}{4}-\frac{\nu}{2}, \frac{3}{4}-\frac{\nu}{2}, \frac{1}{2r}-\frac{\nu}{2}; \frac{1}{2}, \frac{1}{2}-\nu, 1-\nu, -\frac{\nu}{2}+\frac{1}{2r}+1; a^2 z^{2r}\right) + \frac{1}{\Gamma(\nu+1)} \left((a z^r)^{2\nu} \left(-\frac{\cosh(b)}{r \nu + 1}\, {}_3F_4\!\left(\frac{\nu}{2}+\frac{1}{4}, \frac{\nu}{2}+\frac{3}{4}, \frac{\nu}{2}+\frac{1}{2r}; \frac{1}{2}, \frac{\nu}{2}+\frac{1}{2r}+1, \nu+\frac{1}{2}, \nu+1; a^2 z^{2r}\right) - \frac{a z^r \sinh(b)}{\nu r + r + 1}\, {}_3F_4\!\left(\frac{\nu}{2}+\frac{3}{4}, \frac{\nu}{2}+\frac{5}{4}, \frac{\nu}{2}+\frac{1}{2}+\frac{1}{2r}; \frac{3}{2}, \frac{\nu}{2}+\frac{3}{2}+\frac{1}{2r}, \nu+1, \nu+\frac{3}{2}; a^2 z^{2r}\right)\right)\right) - \frac{4^\nu a z^r \sinh(b)}{(r(\nu-1)-1)\, \Gamma(1-\nu)}\, {}_3F_4\!\left(\frac{3}{4}-\frac{\nu}{2}, \frac{5}{4}-\frac{\nu}{2}, -\frac{\nu}{2}+\frac{1}{2}+\frac{1}{2r}; \frac{3}{2}, 1-\nu, \frac{3}{2}-\nu, -\frac{\nu}{2}+\frac{3}{2}+\frac{1}{2r}; a^2 z^{2r}\right)\right)$$

Involving hyperbolic functions and a power function

Involving sinh and power

Linear arguments

03.04.21.0037.01

$$\int z^{\alpha-1} \sinh(a z)\, K_\nu(a z)\, dz = \frac{1}{(\alpha-\nu+1)(\alpha+\nu+1)} \left(2^{-\nu-1} z^\alpha\, (a z)^{1-\nu} \left((\Gamma(1-\nu) + (\alpha+1)\, \Gamma(-\nu))\, {}_3F_4\!\left(\frac{\nu}{2}+\frac{3}{4}, \frac{\nu}{2}+\frac{5}{4}, \frac{\alpha}{2}+\frac{\nu}{2}+\frac{1}{2}; \frac{3}{2}, \frac{\alpha}{2}+\frac{\nu}{2}+\frac{3}{2}, \nu+1, \nu+\frac{3}{2}; a^2 z^2\right)(a z)^{2\nu} + 4^\nu\, ((\alpha+1)\, \Gamma(\nu) + \Gamma(\nu+1))\, {}_3F_4\!\left(\frac{3}{4}-\frac{\nu}{2}, \frac{5}{4}-\frac{\nu}{2}, \frac{\alpha}{2}-\frac{\nu}{2}+\frac{1}{2}; \frac{3}{2}, 1-\nu, \frac{3}{2}-\nu, \frac{\alpha}{2}-\frac{\nu}{2}+\frac{3}{2}; a^2 z^2\right)\right)\right)$$

03.04.21.0038.01

$$\int z^{\alpha-1} \sinh(b + a z)\, K_\nu(a z)\, dz = 2^{-\nu-3} \pi^{3/2} z^\alpha (a z)^{-\nu} \csc(\pi \nu) \left(-\left(a z \cosh(b)\, \Gamma\!\left(\nu+\frac{3}{2}\right) \Gamma\!\left(\frac{1}{2}(\alpha+\nu+1)\right) {}_3\tilde{F}_4\!\left(\frac{1}{4}(2\nu+3), \frac{1}{4}(2\nu+5), \frac{1}{2}(\alpha+\nu+1); \frac{3}{2}, \frac{1}{2}(\alpha+\nu+3), \nu+1, \nu+\frac{3}{2}; a^2 z^2\right) + 2\, \Gamma\!\left(\frac{\alpha+\nu}{2}\right) \Gamma\!\left(\nu+\frac{1}{2}\right) {}_3\tilde{F}_4\!\left(\frac{1}{4}(2\nu+1), \frac{1}{4}(2\nu+3), \frac{\alpha+\nu}{2}; \frac{1}{2}, \frac{1}{2}(\alpha+\nu+2), \nu+\frac{1}{2}, \nu+1; a^2 z^2\right) \sinh(b)\right)(a z)^{2\nu} + 4^\nu a z \cosh(b)\, \Gamma\!\left(\frac{3}{2}-\nu\right) \Gamma\!\left(\frac{1}{2}(\alpha-\nu+1)\right) {}_3\tilde{F}_4\!\left(\frac{1}{4}(3-2\nu), \frac{1}{4}(5-2\nu), \frac{1}{2}(\alpha-\nu+1); \frac{3}{2}, 1-\nu, \frac{3}{2}-\nu, \frac{1}{2}(\alpha-\nu+3); a^2 z^2\right) + 2^{2\nu+1}\, \Gamma\!\left(\frac{\alpha-\nu}{2}\right) \Gamma\!\left(\frac{1}{2}-\nu\right) {}_3\tilde{F}_4\!\left(\frac{1}{4}(1-2\nu), \frac{1}{4}(3-2\nu), \frac{\alpha-\nu}{2}; \frac{1}{2}, \frac{1}{2}-\nu, 1-\nu, \frac{1}{2}(\alpha-\nu+2); a^2 z^2\right) \sinh(b)\right)$$

Power arguments

03.04.21.0039.01

$$\int z^{\alpha-1}\sinh(a z^r)\,K_\nu(a z^r)\,dz = 2^{-\nu-1}\pi z^\alpha (a z^r)^{1-\nu}\csc(\pi\nu)\left(\frac{4^\nu}{(-\nu r+r+\alpha)\,\Gamma(1-\nu)}\,{}_3F_4\!\left(\frac{3}{4}-\frac{\nu}{2},\frac{5}{4}-\frac{\nu}{2},\frac{\alpha}{2r}-\frac{\nu}{2}+\frac{1}{2};\frac{3}{2},1-\nu,\frac{3}{2}-\nu,\frac{\alpha}{2r}-\frac{\nu}{2}+\frac{3}{2};a^2 z^{2r}\right)-\frac{(a z^r)^{2\nu}}{(\nu r+r+\alpha)\,\Gamma(\nu+1)}\,{}_3F_4\!\left(\frac{\nu}{2}+\frac{3}{4},\frac{\nu}{2}+\frac{5}{4},\frac{\alpha}{2r}+\frac{\nu}{2}+\frac{1}{2};\frac{3}{2},\frac{\alpha}{2r}+\frac{\nu}{2}+\frac{3}{2},\nu+1,\nu+\frac{3}{2};a^2 z^{2r}\right)\right)$$

03.04.21.0040.01

$$\int z^{\alpha-1}\sinh(a z^r+b)\,K_\nu(a z^r)\,dz = 2^{-\nu-1}\pi z^\alpha (a z^r)^{-\nu}\csc(\pi\nu)\left(\frac{4^\nu a z^r\cosh(b)}{(-\nu r+r+\alpha)\,\Gamma(1-\nu)}\,{}_3F_4\!\left(\frac{3}{4}-\frac{\nu}{2},\frac{5}{4}-\frac{\nu}{2},\frac{\alpha}{2r}-\frac{\nu}{2}+\frac{1}{2};\frac{3}{2},1-\nu,\frac{3}{2}-\nu,\frac{\alpha}{2r}-\frac{\nu}{2}+\frac{3}{2};a^2 z^{2r}\right)+\frac{4^\nu\sinh(b)}{(\alpha-r\nu)\,\Gamma(1-\nu)}\,{}_3F_4\!\left(\frac{1}{4}-\frac{\nu}{2},\frac{3}{4}-\frac{\nu}{2},\frac{\alpha}{2r}-\frac{\nu}{2};\frac{1}{2},\frac{1}{2}-\nu,1-\nu,\frac{\alpha}{2r}-\frac{\nu}{2}+1;a^2 z^{2r}\right)+\frac{1}{\Gamma(\nu+1)}\left((a z^r)^{2\nu}\left(-\frac{a z^r\cosh(b)}{\nu r+r+\alpha}\,{}_3F_4\!\left(\frac{\nu}{2}+\frac{3}{4},\frac{\nu}{2}+\frac{5}{4},\frac{\alpha}{2r}+\frac{\nu}{2}+\frac{1}{2};\frac{3}{2},\frac{\alpha}{2r}+\frac{\nu}{2}+\frac{3}{2},\nu+1,\nu+\frac{3}{2};a^2 z^{2r}\right)-\frac{\sinh(b)}{\alpha+r\nu}\,{}_3F_4\!\left(\frac{\nu}{2}+\frac{1}{4},\frac{\nu}{2}+\frac{3}{4},\frac{\alpha}{2r}+\frac{\nu}{2};\frac{1}{2},\frac{\alpha}{2r}+\frac{\nu}{2}+1,\nu+\frac{1}{2},\nu+1;a^2 z^{2r}\right)\right)\right)\right)$$

Involving cosh and power

Linear arguments

03.04.21.0041.01

$$\int z^{\alpha-1}\cosh(a z)\,K_\nu(a z)\,dz = -\frac{1}{(\nu-\alpha)(\alpha+\nu)}\left(2^{-\nu-1} z^\alpha (a z)^{-\nu}\left((\Gamma(1-\nu)+\alpha\,\Gamma(-\nu))\,{}_3F_4\!\left(\frac{\nu}{2}+\frac{1}{4},\frac{\nu}{2}+\frac{3}{4},\frac{\alpha}{2}+\frac{\nu}{2};\frac{1}{2},\frac{\alpha}{2}+\frac{\nu}{2}+1,\nu+\frac{1}{2},\nu+1;a^2 z^2\right)(a z)^{2\nu}+4^\nu(\alpha\,\Gamma(\nu)+\Gamma(\nu+1))\,{}_3F_4\!\left(\frac{1}{4}-\frac{\nu}{2},\frac{3}{4}-\frac{\nu}{2},\frac{\alpha}{2}-\frac{\nu}{2};\frac{1}{2},\frac{1}{2}-\nu,1-\nu,\frac{\alpha}{2}-\frac{\nu}{2}+1;a^2 z^2\right)\right)\right)$$

03.04.21.0042.01

$$\int z^{\alpha-1}\cosh(b+a z)\,K_\nu(a z)\,dz = 2^{-\nu-3}\pi^{3/2} z^\alpha (a z)^{-\nu}\csc(\pi\nu)\left(-\left(2\cosh(b)\,\Gamma\!\left(\frac{\alpha+\nu}{2}\right)\Gamma\!\left(\nu+\frac{1}{2}\right){}_3\tilde{F}_4\!\left(\frac{1}{4}(2\nu+1),\frac{1}{4}(2\nu+3),\frac{\alpha+\nu}{2};\frac{1}{2},\frac{1}{2}(\alpha+\nu+2),\nu+\frac{1}{2},\nu+1;a^2 z^2\right)+a z\,\Gamma\!\left(\nu+\frac{3}{2}\right)\Gamma\!\left(\frac{1}{2}(\alpha+\nu+1)\right){}_3\tilde{F}_4\!\left(\frac{1}{4}(2\nu+3),\frac{1}{4}(2\nu+5),\frac{1}{2}(\alpha+\nu+1);\frac{3}{2},\frac{1}{2}(\alpha+\nu+3),\nu+1,\nu+\frac{3}{2};a^2 z^2\right)\sinh(b)\right)(a z)^{2\nu}+2^{2\nu+1}\cosh(b)\,\Gamma\!\left(\frac{\alpha-\nu}{2}\right)\Gamma\!\left(\frac{1}{2}-\nu\right){}_3\tilde{F}_4\!\left(\frac{1}{4}(1-2\nu),\frac{1}{4}(3-2\nu),\frac{\alpha-\nu}{2};\frac{1}{2},\frac{1}{2}-\nu,1-\nu,\frac{1}{2}(\alpha-\nu+2);a^2 z^2\right)+4^\nu a z\,\Gamma\!\left(\frac{3}{2}-\nu\right)\Gamma\!\left(\frac{1}{2}(\alpha-\nu+1)\right){}_3\tilde{F}_4\!\left(\frac{1}{4}(3-2\nu),\frac{1}{4}(5-2\nu),\frac{1}{2}(\alpha-\nu+1);\frac{3}{2},1-\nu,\frac{3}{2}-\nu,\frac{1}{2}(\alpha-\nu+3);a^2 z^2\right)\sinh(b)\right)$$

Power arguments

03.04.21.0043.01

$$\int z^{\alpha-1}\cosh(a z^r)\,K_\nu(a z^r)\,dz = 2^{-\nu-1} z^\alpha (a z^r)^{-\nu}\left(\frac{(a z^r)^{2\nu}\,\Gamma(-\nu)}{\alpha+r\nu}\,{}_3F_4\left(\frac{\nu}{2}+\frac{1}{4},\frac{\nu}{2}+\frac{3}{4},\frac{\alpha}{2r}+\frac{\nu}{2};\frac{1}{2},\frac{\alpha}{2r}+\frac{\nu}{2}+1,\nu+\frac{1}{2},\nu+1;a^2 z^{2r}\right)+\frac{4^\nu\,\Gamma(\nu)}{\alpha-r\nu}\,{}_3F_4\left(\frac{1}{4}-\frac{\nu}{2},\frac{3}{4}-\frac{\nu}{2},\frac{\alpha}{2r}-\frac{\nu}{2};\frac{1}{2},\frac{1}{2}-\nu,1-\nu,\frac{\alpha}{2r}-\frac{\nu}{2}+1;a^2 z^{2r}\right)\right)$$

03.04.21.0044.01

$$\int z^{\alpha-1}\cosh(a z^r+b)\,K_\nu(a z^r)\,dz = 2^{-\nu-1}\pi z^\alpha (a z^r)^{-\nu}\csc(\pi\nu)\left(\frac{4^\nu\cosh(b)}{(\alpha-r\nu)\,\Gamma(1-\nu)}\,{}_3F_4\left(\frac{1}{4}-\frac{\nu}{2},\frac{3}{4}-\frac{\nu}{2},\frac{\alpha}{2r}-\frac{\nu}{2};\frac{1}{2},\frac{1}{2}-\nu,1-\nu,\frac{\alpha}{2r}-\frac{\nu}{2}+1;a^2 z^{2r}\right)+\frac{4^\nu a z^r\sinh(b)}{(-\nu r+r+\alpha)\,\Gamma(1-\nu)}\,{}_3F_4\left(\frac{3}{4}-\frac{\nu}{2},\frac{5}{4}-\frac{\nu}{2},\frac{\alpha}{2r}-\frac{\nu}{2}+\frac{1}{2};\frac{3}{2},1-\nu,\frac{3}{2}-\nu,\frac{\alpha}{2r}-\frac{\nu}{2}+\frac{3}{2};a^2 z^{2r}\right)+\frac{1}{\Gamma(\nu+1)}\left((a z^r)^{2\nu}\left(-\frac{\cosh(b)}{\alpha+r\nu}\,{}_3F_4\left(\frac{\nu}{2}+\frac{1}{4},\frac{\nu}{2}+\frac{3}{4},\frac{\alpha}{2r}+\frac{\nu}{2};\frac{1}{2},\frac{\alpha}{2r}+\frac{\nu}{2}+1,\nu+\frac{1}{2},\nu+1;a^2 z^{2r}\right)-\frac{a z^r\sinh(b)}{\nu r+r+\alpha}\,{}_3F_4\left(\frac{\nu}{2}+\frac{3}{4},\frac{\nu}{2}+\frac{5}{4},\frac{\alpha}{2r}+\frac{\nu}{2}+\frac{1}{2};\frac{3}{2},\frac{\alpha}{2r}+\frac{\nu}{2}+\frac{3}{2},\nu+1,\nu+\frac{3}{2};a^2 z^{2r}\right)\right)\right)\right)$$

Involving functions of the direct function

Involving elementary functions of the direct function

Involving powers of the direct function

Linear arguments

03.04.21.0045.01

$$\int K_\nu(a z)^2\,dz = -\frac{1}{\left(4\nu^2-1\right)\Gamma(1-\nu)^2\,\Gamma(\nu+1)^2}\left(4^{-\nu-1}\pi^2 z\,(a z)^{-2\nu}\csc^2(\pi\nu)\left(2^{2\nu+1}\left(4\nu^2-1\right)\Gamma(1-\nu)\,\Gamma(\nu+1)\,{}_2F_3\left(\frac{1}{2},\frac{1}{2};\frac{3}{2},1-\nu,\nu+1;a^2 z^2\right)(a z)^{2\nu}-(2\nu-1)\,\Gamma(1-\nu)^2\,{}_2F_3\left(\nu+\frac{1}{2},\nu+\frac{1}{2};\nu+1,\nu+\frac{3}{2},2\nu+1;a^2 z^2\right)(a z)^{4\nu}+\left(2^{4\nu+1}\nu+16^\nu\right)\Gamma(\nu+1)^2\,{}_2F_3\left(\frac{1}{2}-\nu,\frac{1}{2}-\nu;1-2\nu,1-\nu,\frac{3}{2}-\nu;a^2 z^2\right)\right)\right)$$

03.04.21.0046.01

$$\int K_\nu(z)^2\, dz = -\frac{4^{-\nu-1}\pi^2 z^{1-2\nu}\csc^2(\pi\nu)}{(4\nu^2-1)\,\Gamma(1-\nu)^2\,\Gamma(\nu+1)^2}\Big(2^{2\nu+1}(4\nu^2-1)\,\Gamma(1-\nu)\,\Gamma(\nu+1)\,z^{2\nu}\,{}_2F_3\Big(\frac{1}{2},\frac{1}{2};\frac{3}{2},1-\nu,\nu+1;z^2\Big)-(2\nu-1)\,\Gamma(1-\nu)^2 z^{4\nu}\,{}_2F_3\Big(\nu+\frac{1}{2},\nu+\frac{1}{2};\nu+1,\nu+\frac{3}{2},2\nu+1;z^2\Big)+2^{4\nu}(1+2\nu)\,\Gamma(\nu+1)^2\,{}_2F_3\Big(\frac{1}{2}-\nu,\frac{1}{2}-\nu;1-2\nu,1-\nu,\frac{3}{2}-\nu;z^2\Big)\Big)$$

Power arguments

03.04.21.0047.01

$$\int K_\nu(a z^r)^2\, dz = -\Big(4^{-\nu-1}\pi^2 z\,(a z^r)^{-2\nu}\csc^2(\pi\nu)\Big(2^{2\nu+1}(4r^2\nu^2-1)\,\Gamma(1-\nu)\,\Gamma(\nu+1)\,{}_2F_3\Big(\frac{1}{2},\frac{1}{2r};1+\frac{1}{2r},1-\nu,\nu+1;a^2z^{2r}\Big)(a z^r)^{2\nu}-(2r\nu-1)\,\Gamma(1-\nu)^2\,{}_2F_3\Big(\nu+\frac{1}{2},\nu+\frac{1}{2r};\nu+1,\nu+\frac{1}{2r}+1,2\nu+1;a^2z^{2r}\Big)(a z^r)^{4\nu}+(2^{4\nu+1}r\nu+16^\nu)\,\Gamma(\nu+1)^2\,{}_2F_3\Big(\frac{1}{2}-\nu,\frac{1}{2r}-\nu;1-2\nu,1-\nu,-\nu+\frac{1}{2r}+1;a^2z^{2r}\Big)\Big)\Big)\Big/\Big((4r^2\nu^2-1)\,\Gamma(1-\nu)^2\,\Gamma(\nu+1)^2\Big)$$

Involving products of the direct function

Linear arguments

03.04.21.0048.01

$$\int K_\mu(a z)\,K_\nu(a z)\, dz = 2^{-\mu-\nu-2}\pi^2 z\,(a z)^{-\mu-\nu}\csc(\pi\mu)\csc(\pi\nu)\Bigg(\frac{1}{\Gamma(\nu+1)}\Bigg((a z)^{2\nu}\Bigg(\frac{(a z)^{2\mu}}{(\mu+\nu+1)\,\Gamma(\mu+1)}\,{}_3F_4\Big(\frac{\mu}{2}+\frac{\nu}{2}+\frac{1}{2},\frac{\mu}{2}+\frac{\nu}{2}+\frac{1}{2},\frac{\mu}{2}+\frac{\nu}{2}+1;\mu+1,\frac{\mu}{2}+\frac{\nu}{2}+\frac{3}{2},\nu+1,\mu+\nu+1;a^2z^2\Big)+\frac{4^\mu}{(\mu-\nu-1)\,\Gamma(1-\mu)}\,{}_3F_4\Big(-\frac{\mu}{2}+\frac{\nu}{2}+\frac{1}{2},-\frac{\mu}{2}+\frac{\nu}{2}+\frac{1}{2},-\frac{\mu}{2}+\frac{\nu}{2}+1;1-\mu,-\frac{\mu}{2}+\frac{\nu}{2}+\frac{3}{2},\nu+1,-\mu+\nu+1;a^2z^2\Big)\Bigg)\Bigg)+\frac{4^\nu (a z)^{2\mu}}{(-\mu+\nu-1)\,\Gamma(\mu+1)\,\Gamma(1-\nu)}\,{}_3F_4\Big(\frac{\mu}{2}-\frac{\nu}{2}+\frac{1}{2},\frac{\mu}{2}-\frac{\nu}{2}+\frac{1}{2},\frac{\mu}{2}-\frac{\nu}{2}+1;\mu+1,1-\nu,\mu-\nu+1,\frac{\mu}{2}-\frac{\nu}{2}+\frac{3}{2};a^2z^2\Big)-\frac{4^{\mu+\nu}}{(\mu+\nu-1)\,\Gamma(1-\mu)\,\Gamma(1-\nu)}\,{}_3F_4\Big(-\frac{\mu}{2}-\frac{\nu}{2}+\frac{1}{2},-\frac{\mu}{2}-\frac{\nu}{2}+\frac{1}{2},-\frac{\mu}{2}-\frac{\nu}{2}+1;1-\mu,1-\nu,-\mu-\nu+1,-\frac{\mu}{2}-\frac{\nu}{2}+\frac{3}{2};a^2z^2\Big)\Bigg)$$

03.04.21.0049.01

$$\int K_\nu(a z)\,K_{\nu+1}(a z)\,dz = 2^{-\mu-\nu-4}\,\pi^2\,z\,(a z)^{-\mu-\nu-1}\csc(\pi\mu)\csc(\pi(\nu+1))\left(2\left(\frac{1}{\Gamma(\nu+2)}\left((a z)^{2(\nu+1)}\left(\frac{(a z)^{2\mu}}{(\mu+\nu+2)\,\Gamma(\mu+1)}\,{}_3F_4\!\left(\frac{\mu}{2}+\frac{\nu}{2}+1,\frac{\mu}{2}+\frac{\nu}{2}+1,\frac{\mu}{2}+\frac{\nu}{2}+\frac{3}{2};\mu+1,\frac{\mu}{2}+\frac{\nu}{2}+2,\nu+2,\mu+\nu+2;a^2z^2\right)+\frac{4^\mu}{(\mu-\nu-2)\,\Gamma(1-\mu)}\,{}_3F_4\!\left(-\frac{\mu}{2}+\frac{\nu}{2}+1,-\frac{\mu}{2}+\frac{\nu}{2}+1,-\frac{\mu}{2}+\frac{\nu}{2}+\frac{3}{2};1-\mu,-\frac{\mu}{2}+\frac{\nu}{2}+2,\nu+2,-\mu+\nu+2;a^2z^2\right)\right)\right)+\frac{4^{\nu+1}(a z)^{2\mu}}{(\nu-\mu)\,\Gamma(\mu+1)\,\Gamma(-\nu)}\,{}_3F_4\!\left(\frac{\mu}{2}-\frac{\nu}{2},\frac{\mu}{2}-\frac{\nu}{2},\frac{\mu}{2}-\frac{\nu}{2}+\frac{1}{2};\mu+1,\mu-\nu,\frac{\mu}{2}-\frac{\nu}{2}+1,-\nu;a^2z^2\right)\right)-\frac{2^{2\mu+2\nu+3}}{(\mu+\nu)\,\Gamma(1-\mu)\,\Gamma(-\nu)}\,{}_3F_4\!\left(-\frac{\mu}{2}-\frac{\nu}{2},-\frac{\mu}{2}-\frac{\nu}{2},-\frac{\mu}{2}-\frac{\nu}{2}+\frac{1}{2};1-\mu,-\mu-\nu,-\frac{\mu}{2}-\frac{\nu}{2}+1,-\nu;a^2z^2\right)\right)$$

03.04.21.0050.01

$$\int K_0(a z)\,K_1(a z)\,dz = -\frac{K_0(a z)^2}{2a}$$

Power arguments

03.04.21.0051.01

$$\int K_\mu(a z^r)\,K_\nu(a z^r)\,dz = 2^{-\mu-\nu-2}\,\pi^2\,z\,(a z^r)^{-\mu-\nu}\csc(\pi\mu)\csc(\pi\nu)\left(\frac{1}{\Gamma(\nu+1)}\left((a z^r)^{2\nu}\left(\frac{(a z^r)^{2\mu}}{(r(\mu+\nu)+1)\,\Gamma(\mu+1)}\,{}_3F_4\!\left(\frac{\mu}{2}+\frac{\nu}{2}+\frac{1}{2},\frac{\mu}{2}+\frac{\nu}{2}+1,\frac{\mu}{2}+\frac{\nu}{2}+\frac{1}{2r};\mu+1,\frac{\mu}{2}+\frac{\nu}{2}+\frac{1}{2r}+1,\nu+1,\mu+\nu+1;a^2z^{2r}\right)+\frac{1}{(r\mu-r\nu-1)\,\Gamma(1-\mu)}\left(4^\mu\,{}_3F_4\!\left(-\frac{\mu}{2}+\frac{\nu}{2}+\frac{1}{2},-\frac{\mu}{2}+\frac{\nu}{2}+1,-\frac{\mu}{2}+\frac{\nu}{2}+\frac{1}{2r};1-\mu,-\frac{\mu}{2}+\frac{\nu}{2}+\frac{1}{2r}+1,\nu+1,-\mu+\nu+1;a^2z^{2r}\right)\right)\right)\right)+\frac{4^\nu(a z^r)^{2\mu}}{(-r\mu+r\nu-1)\,\Gamma(\mu+1)\,\Gamma(1-\nu)}\,{}_3F_4\!\left(\frac{\mu}{2}-\frac{\nu}{2}+\frac{1}{2},\frac{\mu}{2}-\frac{\nu}{2}+1,\frac{\mu}{2}-\frac{\nu}{2}+\frac{1}{2r};\mu+1,1-\nu,\mu-\nu+1,\frac{\mu}{2}-\frac{\nu}{2}+\frac{1}{2r}+1;a^2z^{2r}\right)-\frac{4^{\mu+\nu}}{(r(\mu+\nu)-1)\,\Gamma(1-\mu)\,\Gamma(1-\nu)}\,{}_3F_4\!\left(-\frac{\mu}{2}-\frac{\nu}{2}+\frac{1}{2},-\frac{\mu}{2}-\frac{\nu}{2}+1,-\frac{\mu}{2}-\frac{\nu}{2}+\frac{1}{2r};1-\mu,1-\nu,-\mu-\nu+1,-\frac{\mu}{2}-\frac{\nu}{2}+\frac{1}{2r}+1;a^2z^{2r}\right)\right)$$

03.04.21.0052.01

$$\int K_\nu\!\left(a\sqrt{z}\right)K_\nu\!\left(b\sqrt{z}\right)dz = \frac{2\sqrt{z}}{a^2-b^2}\left(b\,K_{\nu-1}\!\left(b\sqrt{z}\right)K_\nu\!\left(a\sqrt{z}\right)-a\,K_{\nu-1}\!\left(a\sqrt{z}\right)K_\nu\!\left(b\sqrt{z}\right)\right)$$

Involving functions of the direct function and elementary functions

Involving elementary functions of the direct function and elementary functions

Involving powers of the direct function and a power function

Linear arguments

03.04.21.0053.01

$$\int z^{\alpha-1} K_\nu(a z)^2 \, dz = \frac{1}{4} \pi z^\alpha \csc(\pi \nu) \left(\frac{4^\nu \Gamma(\nu) (a z)^{-2\nu}}{(\alpha - 2\nu) \Gamma(1-\nu)} {}_2F_3\left(\frac{1}{2} - \nu, \frac{\alpha}{2} - \nu; 1 - 2\nu, 1 - \nu, \frac{\alpha}{2} - \nu + 1; a^2 z^2\right) + 4^{-\nu} \Gamma(1-\nu) \Gamma(2\nu) \Gamma\left(\frac{\alpha}{2} + \nu\right) {}_2\tilde{F}_3\left(\nu + \frac{1}{2}, \frac{\alpha}{2} + \nu; \nu + 1, \frac{\alpha}{2} + \nu + 1, 2\nu + 1; a^2 z^2\right) (a z)^{2\nu} - \frac{2}{\alpha \nu} {}_2F_3\left(\frac{1}{2}, \frac{\alpha}{2}; \frac{\alpha}{2} + 1, 1 - \nu, \nu + 1; a^2 z^2\right) \right)$$

03.04.21.0054.01

$$\int z^{1-2\nu} K_\nu(a z)^2 \, dz = \frac{1}{a^2 (2\nu - 1) \Gamma(\nu)^2} \left(2^{-2\nu-3} \pi^2 z^{-2\nu} \csc^2(\pi \nu) \left(a^2 z^2 \left(2^{2\nu+1} (I_{-\nu}(a z) I_\nu(a z) - I_{1-\nu}(a z) I_{\nu-1}(a z)) + 4^\nu \left(I_{1-\nu}(a z)^2 + I_{\nu-1}(a z)^2 - I_{-\nu}(a z)^2 - I_\nu(a z)^2\right)\right) \Gamma(\nu)^2 - 4 (a z)^{2\nu}\right)\right)$$

03.04.21.0055.01

$$\int z^{2\nu+1} K_\nu(a z)^2 \, dz = \frac{1}{8 a^2 (2\nu + 1) \Gamma(-\nu)^2} \left(\pi^2 z^{2\nu} (a z)^{-2\nu} \csc^2(\pi \nu) \left(2^{2(\nu+1)} - a z (a z)^{2\nu} \Gamma(-\nu)^2 \left(a z I_{\nu+1}(a z)^2 - (a z (I_{-\nu-1}(a z) + I_{1-\nu}(a z)) - 2\nu I_{-\nu}(a z)) I_{\nu+1}(a z) + a \left(I_{-\nu-1}(a z)^2 - \frac{4 K_\nu(a z)^2 \sin^2(\pi \nu)}{\pi^2}\right) z\right)\right)\right)$$

03.04.21.0056.01

$$\int z K_\nu(a z)^2 \, dz = \frac{1}{2} z^2 \left(K_\nu(a z)^2 - K_{\nu-1}(a z) K_{\nu+1}(a z)\right)$$

03.04.21.0057.01

$$\int z K_0(a z)^2 \, dz = \frac{1}{2} z^2 \left(K_0(a z)^2 - K_1(a z)^2\right)$$

03.04.21.0058.01

$$\int \frac{1}{z K_\nu(a z)^2} \, dz = \frac{I_\nu(a z)}{K_\nu(a z)}$$

03.04.21.0059.01

$$\int \frac{K_\nu(a z)^2}{z} \, dz = \frac{1}{8} \pi \csc(\pi \nu) \left(4^{-\nu} \pi (a z)^{-2\nu} \csc(\pi \nu) \left(2 \Gamma(2\nu) {}_2\tilde{F}_3\left(\nu, \nu + \frac{1}{2}; \nu + 1, \nu + 1, 2\nu + 1; a^2 z^2\right) (a z)^{4\nu} - 4^\nu {}_3\tilde{F}_4\left(1, 1, \frac{3}{2}; 2, 2, 2 - \nu, \nu + 2; a^2 z^2\right) (a z)^{2(\nu+1)} + 2^{4\nu+1} \Gamma(-2\nu) {}_2\tilde{F}_3\left(\frac{1}{2} - \nu, -\nu; 1 - 2\nu, 1 - \nu, 1 - \nu; a^2 z^2\right)\right) - \frac{4 \log(z)}{\nu}\right)$$

03.04.21.0060.01

$$\int \frac{K_\nu(a\,z)^2}{z^2}\,dz = \frac{1}{4\,z\left(4\,\nu^2-1\right)}\left(\pi\,\csc(\pi\,\nu)\left(4\,a\,z\,(a\,z\,I_{\nu-2}(a\,z)+I_{\nu-1}(a\,z))\,K_\nu(a\,z)+\pi\left(2\,a^2\,I_{-\nu-1}(a\,z)^2\,z^2+2\,a^2\,I_{\nu-1}(a\,z)^2\,z^2-2\,a^2\,I_{-\nu-2}(a\,z)\,I_{-\nu}(a\,z)\,z^2-2\,a\,I_{-\nu-1}(a\,z)\,I_{-\nu}(a\,z)\,z+2\,a\,I_{1-\nu}(a\,z)\,(I_\nu(a\,z)-2\,a\,z\,I_{\nu-1}(a\,z))\,z+2\,\nu\,I_{-\nu}(a\,z)^2+I_{-\nu}(a\,z)^2-(2\,\nu-1)\,I_\nu(a\,z)^2+2\left(a^2\,I_{2-\nu}(a\,z)\,z^2+(2\,\nu-1)\,I_{-\nu}(a\,z)\right)I_\nu(a\,z)\right)\csc(\pi\,\nu)\right)\right)$$

Power arguments

03.04.21.0061.01

$$\int z^{\alpha-1}\,K_\nu(a\,z^r)^2\,dz = \left(4^{-\nu-1}\,\pi^2\,z^\alpha\,(a\,z^r)^{-2\,\nu}\,\csc^2(\pi\,\nu)\left(2^{2\,\nu+1}\left(4\,r^2\,\nu^2-\alpha^2\right)\Gamma(1-\nu)\,\Gamma(\nu+1)\,{}_2F_3\!\left(\frac{1}{2},\frac{\alpha}{2\,r};\frac{\alpha}{2\,r}+1,1-\nu,\nu+1;a^2\,z^{2\,r}\right)(a\,z^r)^{2\,\nu}+\alpha\left(\left(16^\nu\,\alpha+2^{4\,\nu+1}\,r\,\nu\right)\Gamma(\nu+1)^2\,{}_2F_3\!\left(\frac{1}{2}-\nu,\frac{\alpha}{2\,r}-\nu;1-2\,\nu,1-\nu,\frac{\alpha}{2\,r}-\nu+1;a^2\,z^{2\,r}\right)-(a\,z^r)^{4\,\nu}\,(2\,r\,\nu-\alpha)\,\Gamma(1-\nu)^2\,{}_2F_3\!\left(\nu+\frac{1}{2},\frac{\alpha}{2\,r}+\nu;\nu+1,\frac{\alpha}{2\,r}+\nu+1,2\,\nu+1;a^2\,z^{2\,r}\right)\right)\right)\right)\Big/\left(\left(\alpha^3-4\,r^2\,\alpha\,\nu^2\right)\Gamma(1-\nu)^2\,\Gamma(\nu+1)^2\right)$$

Involving products of the direct function and a power function

Linear arguments

03.04.21.0062.01

$$\int z^{\alpha-1}\,K_\mu(a\,z)\,K_\nu(a\,z)\,dz = 2^{-\mu-\nu-2}\,\pi^2\,z^\alpha\,(a\,z)^{-\mu-\nu}\,\csc(\pi\,\mu)\,\csc(\pi\,\nu)\left(\frac{1}{\Gamma(\nu+1)}\left((a\,z)^{2\,\nu}\left(\frac{(a\,z)^{2\,\mu}}{(\alpha+\mu+\nu)\,\Gamma(\mu+1)}\,{}_3F_4\!\left(\frac{\mu}{2}+\frac{\nu}{2}+\frac{1}{2},\frac{\mu}{2}+\frac{\nu}{2}+1,\frac{\alpha}{2}+\frac{\mu}{2}+\frac{\nu}{2};\mu+1,\frac{\alpha}{2}+\frac{\mu}{2}+\frac{\nu}{2}+1,\nu+1,\mu+\nu+1;a^2\,z^2\right)-\frac{4^\mu}{(\alpha-\mu+\nu)\,\Gamma(1-\mu)}\,{}_3F_4\!\left(-\frac{\mu}{2}+\frac{\nu}{2}+\frac{1}{2},-\frac{\mu}{2}+\frac{\nu}{2}+1,\frac{\alpha}{2}+\frac{\nu}{2}-\frac{\mu}{2};1-\mu,\frac{\alpha}{2}+\frac{\nu}{2}-\frac{\mu}{2}+1,\nu+1,-\mu+\nu+1;a^2\,z^2\right)\right)\right)-\frac{4^{\mu+\nu}}{(-\alpha+\mu+\nu)\,\Gamma(1-\mu)\,\Gamma(1-\nu)}\,{}_3F_4\!\left(-\frac{\mu}{2}-\frac{\nu}{2}+\frac{1}{2},-\frac{\mu}{2}-\frac{\nu}{2}+1,\frac{\alpha}{2}-\frac{\mu}{2}-\frac{\nu}{2};1-\mu,1-\nu,-\mu-\nu+1,\frac{\alpha}{2}-\frac{\mu}{2}-\frac{\nu}{2}+1;a^2\,z^2\right)-\frac{4^\nu\,(a\,z)^{2\,\mu}}{(\alpha+\mu-\nu)\,\Gamma(\mu+1)\,\Gamma(1-\nu)}\,{}_3F_4\!\left(\frac{\mu}{2}-\frac{\nu}{2}+\frac{1}{2},\frac{\mu}{2}-\frac{\nu}{2}+1,\frac{\alpha}{2}+\frac{\mu}{2}-\frac{\nu}{2};\mu+1,1-\nu,\mu-\nu+1,\frac{\alpha}{2}+\frac{\mu}{2}-\frac{\nu}{2}+1;a^2\,z^2\right)\right)$$

03.04.21.0063.01

$$\int z^{1-\mu-\nu} K_\mu(a z) K_\nu(a z)\, dz = 2^{-\mu-\nu-3} \pi^2 z^{-\mu-\nu} (a z)^{-\mu-\nu} \csc(\pi \mu) \csc(\pi \nu) \left(\frac{4^\nu z^2 (a z)^{2\mu}}{(\nu-1)\Gamma(\mu+1)\Gamma(1-\nu)} {}_2F_3\left(\frac{\mu}{2}-\frac{\nu}{2}+\frac{1}{2}, \frac{\mu}{2}-\frac{\nu}{2}+1; \mu+1, 2-\nu, \mu-\nu+1; a^2 z^2\right) + \frac{(a z)^{2\nu}}{\Gamma(\nu+1)} \left(\frac{4 \mu \nu (a z)^{2\mu}}{a^2 (\mu+\nu-1)\Gamma(\mu+1)} \left({}_2F_3\left(\frac{\mu}{2}+\frac{\nu}{2}-\frac{1}{2}, \frac{\mu}{2}+\frac{\nu}{2}; \mu, \nu, \mu+\nu; a^2 z^2\right)-1\right) + \frac{4^\mu z^2}{(\mu-1)\Gamma(1-\mu)} {}_2F_3\left(-\frac{\mu}{2}+\frac{\nu}{2}+\frac{1}{2}, -\frac{\mu}{2}+\frac{\nu}{2}+1; 2-\mu, \nu+1, -\mu+\nu+1; a^2 z^2\right)\right) - \frac{4^{\mu+\nu} z^2}{(\mu+\nu-1)\Gamma(1-\mu)\Gamma(1-\nu)} {}_2F_3\left(-\frac{\mu}{2}-\frac{\nu}{2}+\frac{1}{2}, -\frac{\mu}{2}-\frac{\nu}{2}+1; 1-\mu, 1-\nu, -\mu-\nu+2; a^2 z^2\right)\right)$$

03.04.21.0064.01

$$\int z^{\mu+\nu+1} K_\mu(a z) K_\nu(a z)\, dz = 2^{-\mu-\nu-3} \pi^2 z^{\mu+\nu} (a z)^{-\mu-\nu} \csc(\pi \mu) \csc(\pi \nu) \left(-\frac{4^\nu z^2 (a z)^{2\mu}}{(\mu+1)\Gamma(\mu+1)\Gamma(1-\nu)} {}_2F_3\left(\frac{\mu}{2}-\frac{\nu}{2}+\frac{1}{2}, \frac{\mu}{2}-\frac{\nu}{2}+1; \mu+2, 1-\nu, \mu-\nu+1; a^2 z^2\right) - \frac{4^\mu z^2 (a z)^{2\nu}}{(\nu+1)\Gamma(1-\mu)\Gamma(\nu+1)} {}_2F_3\left(-\frac{\mu}{2}+\frac{\nu}{2}+\frac{1}{2}, -\frac{\mu}{2}+\frac{\nu}{2}+1; 1-\mu, \nu+2, -\mu+\nu+1; a^2 z^2\right) + \frac{z^2 (a z)^{2(\mu+\nu)}}{(\mu+\nu+1)\Gamma(\mu+1)\Gamma(\nu+1)} {}_2F_3\left(\frac{\mu}{2}+\frac{\nu}{2}+\frac{1}{2}, \frac{\mu}{2}+\frac{\nu}{2}+1; \mu+1, \nu+1, \mu+\nu+2; a^2 z^2\right) - \frac{4^{\mu+\nu+1} \mu \nu}{a^2 (\mu+\nu+1)\Gamma(1-\mu)\Gamma(1-\nu)} \left({}_2F_3\left(-\frac{\mu}{2}-\frac{\nu}{2}-\frac{1}{2}, -\frac{\mu}{2}-\frac{\nu}{2}; -\mu, -\mu-\nu, -\nu; a^2 z^2\right)-1\right)\right)$$

03.04.21.0065.01

$$\int z K_\nu(a z) K_\nu(b z)\, dz = \frac{z}{a^2-b^2} \left(b K_{\nu-1}(b z) K_\nu(a z) - a K_{\nu-1}(a z) K_\nu(b z)\right)$$

03.04.21.0066.01

$$\int \frac{K_\mu(a z) K_\nu(a z)}{z}\, dz = 2^{-\mu-\nu-2} \pi^2 (a z)^{-\mu-\nu} \csc(\pi \mu) \csc(\pi \nu) \left(-4^\nu \Gamma(\mu-\nu) {}_2\tilde{F}_3\left(\frac{\mu-\nu}{2}, \frac{1}{2}(\mu-\nu+1); \mu+1, 1-\nu, \mu-\nu+1; a^2 z^2\right)(a z)^{2\mu} + \left((a z)^{2\mu} \Gamma(\mu+\nu) {}_2\tilde{F}_3\left(\frac{\mu+\nu}{2}, \frac{1}{2}(\mu+\nu+1); \mu+1, \nu+1, \mu+\nu+1; a^2 z^2\right) - 4^\mu \Gamma(\nu-\mu) {}_2\tilde{F}_3\left(\frac{\nu-\mu}{2}, \frac{1}{2}(-\mu+\nu+1); 1-\mu, \nu+1, -\mu+\nu+1; a^2 z^2\right)\right)(a z)^{2\nu} + 4^{\mu+\nu} \Gamma(-\mu-\nu) {}_2\tilde{F}_3\left(\frac{1}{2}(-\mu-\nu), \frac{1}{2}(-\mu-\nu+1); 1-\mu, 1-\nu, -\mu-\nu+1; a^2 z^2\right)\right)$$

03.04.21.0067.01

$$\int \frac{K_\mu(a z)\, K_\nu(a z)}{z^2}\, dz = 2^{-\mu-\nu-2}\, a\, \pi^2\, (a z)^{-\mu-\nu-1} \csc(\pi \mu) \csc(\pi \nu) \left(\left(\frac{4^\mu\, \Gamma(-\mu+\nu+1)}{\mu-\nu+1}\, {}_2\tilde{F}_3\!\left(\frac{1}{2}(-\mu+\nu-1), \frac{1}{2}(-\mu+\nu+2); 1-\mu, \nu+1, -\mu+\nu+1; a^2 z^2\right) + \frac{(a z)^{2\mu}\, \Gamma(\mu+\nu+1)}{\mu+\nu-1}\, {}_2\tilde{F}_3\!\left(\frac{1}{2}(\mu+\nu-1), \frac{1}{2}(\mu+\nu+2); \mu+1, \nu+1, \mu+\nu+1; a^2 z^2\right) \right) (a z)^{2\nu} - \frac{4^{\mu+\nu}\, \Gamma(-\mu-\nu+1)}{\mu+\nu+1}\, {}_2\tilde{F}_3\!\left(\frac{1}{2}(-\mu-\nu-1), \frac{1}{2}(-\mu-\nu+2); 1-\mu, 1-\nu, -\mu-\nu+1; a^2 z^2\right) + \frac{4^\nu\, (a z)^{2\mu}\, \Gamma(\mu-\nu+1)}{-\mu+\nu+1}\, {}_2\tilde{F}_3\!\left(\frac{1}{2}(\mu-\nu-1), \frac{1}{2}(\mu-\nu+2); \mu+1, 1-\nu, \mu-\nu+1; a^2 z^2\right) \right)$$

Power arguments

03.04.21.0068.01

$$\int z^{\alpha-1}\, K_\mu(a z^r)\, K_\nu(a z^r)\, dz = 2^{-\mu-\nu-2}\, \pi^2\, z^\alpha\, (a z^r)^{-\mu-\nu} \csc(\pi \mu) \csc(\pi \nu) \left(\frac{1}{\Gamma(\nu+1)} \left((a z^r)^{2\nu} \left(\frac{1}{(\alpha + r(\mu+\nu))\, \Gamma(\mu+1)} \left((a z^r)^{2\mu}\, {}_3F_4\!\left(\frac{\mu}{2}+\frac{\nu}{2}+\frac{1}{2}, \frac{\mu}{2}+\frac{\nu}{2}+1, \frac{\alpha}{2r}+\frac{\mu}{2}+\frac{\nu}{2}; \mu+1, \frac{\alpha}{2r}+\frac{\mu}{2}+\frac{\nu}{2}+1, \nu+1, \mu+\nu+1; a^2 z^{2r}\right)\right) - \frac{1}{(\alpha - r\mu + r\nu)\, \Gamma(1-\mu)} \left(4^\mu\, {}_3F_4\!\left(-\frac{\mu}{2}+\frac{\nu}{2}+\frac{1}{2}, -\frac{\mu}{2}+\frac{\nu}{2}+1, \frac{\alpha}{2r}+\frac{\nu}{2}-\frac{\mu}{2}; 1-\mu, \frac{\alpha}{2r}+\frac{\nu}{2}-\frac{\mu}{2}+1, \nu+1, -\mu+\nu+1; a^2 z^{2r}\right)\right)\right)\right) + \frac{4^{\mu+\nu}}{(\alpha - r(\mu+\nu))\, \Gamma(1-\mu)\, \Gamma(1-\nu)}\, {}_3F_4\!\left(-\frac{\mu}{2}-\frac{\nu}{2}+\frac{1}{2}, -\frac{\mu}{2}-\frac{\nu}{2}+1, \frac{\alpha}{2r}-\frac{\mu}{2}-\frac{\nu}{2}; 1-\mu, 1-\nu, -\mu-\nu+1, \frac{\alpha}{2r}-\frac{\mu}{2}-\frac{\nu}{2}+1; a^2 z^{2r}\right) - \frac{4^\nu\, (a z^r)^{2\mu}}{(\alpha + r\mu - r\nu)\, \Gamma(\mu+1)\, \Gamma(1-\nu)}\, {}_3F_4\!\left(\frac{\mu}{2}-\frac{\nu}{2}+\frac{1}{2}, \frac{\mu}{2}-\frac{\nu}{2}+1, \frac{\alpha}{2r}+\frac{\mu}{2}-\frac{\nu}{2}; \mu+1, 1-\nu, \mu-\nu+1, \frac{\alpha}{2r}+\frac{\mu}{2}-\frac{\nu}{2}+1; a^2 z^{2r}\right) \right)$$

03.04.21.0069.01

$$\int z^{\alpha-1}\, K_{\nu-1}(a z^r)\, K_\nu(a z^r)\, dz = -\frac{1}{16}\, \pi^2\, z^\alpha \csc^2(\pi \nu) \left(\frac{4 \sin(\pi \nu)\, z^{-r}}{a\, \pi\, (r-\alpha)} \left({}_2F_3\!\left(\frac{1}{2}, \frac{\alpha}{2r}-\frac{1}{2}; \frac{\alpha}{2r}+\frac{1}{2}, 1-\nu, \nu; a^2 z^{2r}\right) + 1 \right) + \frac{2^{2\nu+1}\, (a z^r)^{1-2\nu}}{(-2\nu r + r + \alpha)\, \Gamma(1-\nu)\, \Gamma(2-\nu)}\, {}_2F_3\!\left(\frac{3}{2}-\nu, \frac{\alpha}{2r}-\nu+\frac{1}{2}; 2-2\nu, 2-\nu, \frac{\alpha}{2r}-\nu+\frac{3}{2}; a^2 z^{2r}\right) + \frac{2^{3-2\nu}\, (a z^r)^{2\nu-1}}{(\alpha + r(2\nu-1))\, \Gamma(\nu)\, \Gamma(\nu+1)}\, {}_2F_3\!\left(\nu+\frac{1}{2}, \frac{\alpha}{2r}+\nu-\frac{1}{2}; 2\nu, \nu+1, \frac{\alpha}{2r}+\nu+\frac{1}{2}; a^2 z^{2r}\right) - \frac{4 \csc(\pi \nu)}{(r-\alpha)(r+\alpha)\, \Gamma(2-\nu)\, \Gamma(\nu+1)} \left(a\, r \sin(\pi \nu)\, z^r\, {}_2F_3\!\left(\frac{3}{2}, \frac{\alpha}{2r}+\frac{1}{2}; \frac{\alpha}{2r}+\frac{3}{2}, 2-\nu, \nu+1; a^2 z^{2r}\right) + \pi (r+\alpha)(\nu-1)\, \nu\, I_{1-\nu}(a z^r)\, I_\nu(a z^r) \right) \right)$$

Involving direct function and Bessel-type functions

Involving Bessel functions

Involving Bessel J

Linear arguments

03.04.21.0070.01

$$\int J_\nu(a z) K_\nu(a z) \, dz = \frac{z}{16 \sqrt{\pi}} G_{1,5}^{3,1}\left(\frac{a z}{2 \sqrt{2}}, \frac{1}{4} \,\middle|\, \begin{matrix} \frac{3}{4} \\ 0, \frac{1}{2}, \frac{\nu}{2}, -\frac{1}{4}, -\frac{\nu}{2} \end{matrix} \right)$$

03.04.21.0071.01

$$\int J_{-\nu}(a z) K_\nu(a z) \, dz = \frac{z}{16 \sqrt{\pi}} G_{1,5}^{3,1}\left(\frac{a z}{2 \sqrt{2}}, \frac{1}{4} \,\middle|\, \begin{matrix} \frac{3}{4} \\ 0, \frac{1}{2}, -\frac{\nu}{2}, -\frac{1}{4}, \frac{\nu}{2} \end{matrix} \right)$$

Power arguments

03.04.21.0072.01

$$\int J_\nu(a z^r) K_\nu(a z^r) \, dz = \frac{z}{16 \sqrt{\pi} \, r} G_{1,5}^{3,1}\left(\frac{a z^r}{2 \sqrt{2}}, \frac{1}{4} \,\middle|\, \begin{matrix} 1 - \frac{1}{4 r} \\ 0, \frac{1}{2}, \frac{\nu}{2}, -\frac{1}{4 r}, -\frac{\nu}{2} \end{matrix} \right)$$

03.04.21.0073.01

$$\int J_{-\nu}(a z^r) K_\nu(a z^r) \, dz = \frac{z}{16 \sqrt{\pi} \, r} G_{1,5}^{3,1}\left(\frac{a z^r}{2 \sqrt{2}}, \frac{1}{4} \,\middle|\, \begin{matrix} 1 - \frac{1}{4 r} \\ 0, \frac{1}{2}, -\frac{\nu}{2}, -\frac{1}{4 r}, \frac{\nu}{2} \end{matrix} \right)$$

03.04.21.0074.01

$$\int J_\nu\left(a \sqrt{z}\right) K_\nu\left(b \sqrt{z}\right) dz = \frac{2 \sqrt{z}}{a^2 + b^2} \left(a J_{\nu+1}\left(a \sqrt{z}\right) K_\nu\left(b \sqrt{z}\right) - b J_\nu\left(a \sqrt{z}\right) K_{\nu+1}\left(b \sqrt{z}\right) \right)$$

03.04.21.0075.01

$$\int J_{-\nu}\left(a \sqrt{z}\right) K_\nu\left(b \sqrt{z}\right) dz = \frac{2 \sqrt{z}}{a^2 + b^2} \left(a J_{1-\nu}\left(a \sqrt{z}\right) K_\nu\left(b \sqrt{z}\right) - b J_{-\nu}\left(a \sqrt{z}\right) K_{1-\nu}\left(b \sqrt{z}\right) \right)$$

Involving Bessel J and power

Linear arguments

03.04.21.0076.01

$$\int z^{\alpha-1} J_\nu(a z) K_\nu(a z) \, dz = \frac{z^\alpha}{16 \sqrt{\pi}} G_{1,5}^{3,1}\left(\frac{a z}{2 \sqrt{2}}, \frac{1}{4} \,\middle|\, \begin{matrix} 1 - \frac{\alpha}{4} \\ 0, \frac{1}{2}, \frac{\nu}{2}, -\frac{\alpha}{4}, -\frac{\nu}{2} \end{matrix} \right)$$

03.04.21.0077.01

$$\int z^{\alpha-1} J_{-\nu}(a z) K_\nu(a z) \, dz = \frac{z^\alpha}{16 \sqrt{\pi}} G_{1,5}^{3,1}\left(\frac{a z}{2 \sqrt{2}}, \frac{1}{4} \,\middle|\, \begin{matrix} 1 - \frac{\alpha}{4} \\ 0, \frac{1}{2}, -\frac{\nu}{2}, -\frac{\alpha}{4}, \frac{\nu}{2} \end{matrix} \right)$$

03.04.21.0078.01

$$\int z\, J_\nu(a\, z)\, K_\nu(b\, z)\, dz = \frac{z}{a^2+b^2}\,(a\, J_{\nu+1}(a\, z)\, K_\nu(b\, z) - b\, J_\nu(a\, z)\, K_{\nu+1}(b\, z))$$

03.04.21.0079.01

$$\int z\, J_{-\nu}(a\, z)\, K_\nu(b\, z)\, dz = \frac{z}{a^2+b^2}\,(a\, J_{1-\nu}(a\, z)\, K_\nu(b\, z) - b\, J_{-\nu}(a\, z)\, K_{1-\nu}(b\, z))$$

Power arguments

03.04.21.0080.01

$$\int z^{\alpha-1}\, J_\nu(a\, z^r)\, K_\nu(a\, z^r)\, dz = \frac{z^\alpha}{16\sqrt{\pi}\, r}\, G_{1,5}^{3,1}\left(\frac{a\, z^r}{2\sqrt{2}}, \frac{1}{4} \,\middle|\, \begin{matrix} 1-\frac{\alpha}{4r} \\ 0, \frac{1}{2}, \frac{\nu}{2}, -\frac{\alpha}{4r}, -\frac{\nu}{2} \end{matrix}\right)$$

03.04.21.0081.01

$$\int z^{\alpha-1}\, J_{-\nu}(a\, z^r)\, K_\nu(a\, z^r)\, dz = \frac{z^\alpha}{16\sqrt{\pi}\, r}\, G_{1,5}^{3,1}\left(\frac{a\, z^r}{2\sqrt{2}}, \frac{1}{4} \,\middle|\, \begin{matrix} 1-\frac{\alpha}{4r} \\ 0, \frac{1}{2}, -\frac{\nu}{2}, -\frac{\alpha}{4r}, \frac{\nu}{2} \end{matrix}\right)$$

Involving Bessel I

Linear arguments

03.04.21.0082.01

$$\int I_\mu(a\, z)\, K_\nu(a\, z)\, dz = \frac{1}{\Gamma(\mu+1)}\Bigg(2^{-\mu-\nu-2}\,\pi\, z\,(a\, z)^{\mu-\nu}\csc(\pi\,\nu)$$
$$\Bigg(-\frac{2\,(a\, z)^{2\nu}}{(\mu+\nu+1)\,\Gamma(\nu+1)}\, {}_3F_4\left(\frac{\mu}{2}+\frac{\nu}{2}+\frac{1}{2}, \frac{\mu}{2}+\frac{\nu}{2}+\frac{1}{2}, \frac{\mu}{2}+\frac{\nu}{2}+1; \mu+1, \frac{\mu}{2}+\frac{\nu}{2}+\frac{3}{2}, \nu+1, \mu+\nu+1; a^2 z^2\right) -$$
$$\frac{2^{2\nu+1}}{(-\mu+\nu-1)\,\Gamma(1-\nu)}\, {}_3F_4\left(\frac{\mu}{2}-\frac{\nu}{2}+\frac{1}{2}, \frac{\mu}{2}-\frac{\nu}{2}+\frac{1}{2}, \frac{\mu}{2}-\frac{\nu}{2}+1; \mu+1, 1-\nu, \mu-\nu+1, \frac{\mu}{2}-\frac{\nu}{2}+\frac{3}{2}; a^2 z^2\right)\Bigg)\Bigg)$$

03.04.21.0083.01

$$\int I_\nu(a\, z)\, K_\nu(a\, z)\, dz =$$
$$\frac{\pi\, z\csc(\pi\,\nu)}{4\,\Gamma(\nu+1)^2}\left(\frac{2\,\Gamma(\nu+1)}{\Gamma(1-\nu)}\, {}_2F_3\left(\frac{1}{2}, \frac{1}{2}; \frac{3}{2}, 1-\nu, \nu+1; a^2 z^2\right) - \frac{4^{-\nu}\,(a\, z)^{2\nu}}{\nu+\frac{1}{2}}\, {}_2F_3\left(\nu+\frac{1}{2}, \nu+\frac{1}{2}; \nu+1, \nu+\frac{3}{2}, 2\nu+1; a^2 z^2\right)\right)$$

03.04.21.0084.01

$$\int I_{-\nu}(a\, z)\, K_\nu(a\, z)\, dz =$$
$$\frac{\pi\, z\csc(\pi\,\nu)}{2\,\Gamma(1-\nu)^2}\left(\frac{4^\nu\,(a\, z)^{-2\nu}}{1-2\nu}\, {}_2F_3\left(\frac{1}{2}-\nu, \frac{1}{2}-\nu; 1-2\nu, 1-\nu, \frac{3}{2}-\nu; a^2 z^2\right) - \frac{\Gamma(1-\nu)}{\Gamma(\nu+1)}\, {}_2F_3\left(\frac{1}{2}, \frac{1}{2}; \frac{3}{2}, 1-\nu, \nu+1; a^2 z^2\right)\right)$$

03.04.21.0085.01

$$\int I_0(a\, z)\, K_0(a\, z)\, dz = \frac{z}{4\sqrt{\pi}}\, G_{2,4}^{2,2}\left(a\, z, \frac{1}{2} \,\middle|\, \begin{matrix} \frac{1}{2}, \frac{1}{2} \\ 0, 0, -\frac{1}{2}, 0 \end{matrix}\right)$$

Power arguments

03.04.21.0086.01

$$\int I_\mu(a z^r) K_\nu(a z^r) \, dz = \frac{1}{\Gamma(\mu+1)} \left(2^{-\mu-\nu-1} \pi z (a z^r)^{\mu-\nu} \csc(\pi\nu) \left(-\frac{4^\nu}{(r(\nu-\mu)-1)\Gamma(1-\nu)} {}_3F_4\left(\frac{\mu}{2}-\frac{\nu}{2}+\frac{1}{2}, \frac{\mu}{2}-\frac{\nu}{2}+1, \frac{\mu}{2}-\frac{\nu}{2}+\frac{1}{2r}; \mu+1, 1-\nu, \mu-\nu+1, \frac{\mu}{2}-\frac{\nu}{2}+\frac{1}{2r}+1; a^2 z^{2r} \right) - \frac{(a z^r)^{2\nu}}{(r(\mu+\nu)+1)\Gamma(\nu+1)} {}_3F_4\left(\frac{\mu}{2}+\frac{\nu}{2}+\frac{1}{2}, \frac{\mu}{2}+\frac{\nu}{2}+1, \frac{\mu}{2}+\frac{\nu}{2}+\frac{1}{2r}; \mu+1, \frac{\mu}{2}+\frac{\nu}{2}+\frac{1}{2r}+1, \nu+1, \mu+\nu+1; a^2 z^{2r} \right) \right) \right)$$

03.04.21.0087.01

$$\int I_\nu(a z^r) K_\nu(a z^r) \, dz = \frac{\pi z \csc(\pi\nu)}{2\Gamma(\nu+1)^2} \left(\frac{4^{-\nu} (a z^r)^{2\nu}}{-2r\nu-1} {}_2F_3\left(\nu+\frac{1}{2}, \nu+\frac{1}{2r}; \nu+1, \nu+\frac{1}{2r}+1, 2\nu+1; a^2 z^{2r} \right) + \frac{\Gamma(\nu+1)}{\Gamma(1-\nu)} {}_2F_3\left(\frac{1}{2}, \frac{1}{2r}; 1+\frac{1}{2r}, 1-\nu, \nu+1; a^2 z^{2r} \right) \right)$$

03.04.21.0088.01

$$\int I_{-\nu}(a z^r) K_\nu(a z^r) \, dz = \frac{\pi z \csc(\pi\nu)}{2\Gamma(1-\nu)^2} \left(\frac{4^\nu (a z^r)^{-2\nu}}{1-2r\nu} {}_2F_3\left(\frac{1}{2}-\nu, \frac{1}{2r}-\nu; 1-2\nu, 1-\nu, -\nu+\frac{1}{2r}+1; a^2 z^{2r} \right) - \frac{\Gamma(1-\nu)}{\Gamma(\nu+1)} {}_2F_3\left(\frac{1}{2}, \frac{1}{2r}; 1+\frac{1}{2r}, 1-\nu, \nu+1; a^2 z^{2r} \right) \right)$$

03.04.21.0089.01

$$\int I_\nu(a\sqrt{z}) K_\nu(b\sqrt{z}) \, dz = \frac{2\sqrt{z}}{a^2-b^2} \left(a I_{\nu+1}(a\sqrt{z}) K_\nu(b\sqrt{z}) + b I_\nu(a\sqrt{z}) K_{\nu+1}(b\sqrt{z}) \right)$$

03.04.21.0090.01

$$\int I_{-\nu}(a\sqrt{z}) K_\nu(b\sqrt{z}) \, dz = \frac{2\sqrt{z}}{a^2-b^2} \left(a I_{1-\nu}(a\sqrt{z}) K_\nu(b\sqrt{z}) + b I_{-\nu}(a\sqrt{z}) K_{1-\nu}(b\sqrt{z}) \right)$$

Involving Bessel *I* and power

Linear arguments

03.04.21.0091.01

$$\int z^{\alpha-1} I_\mu(a z) K_\nu(a z) \, dz = \frac{1}{\Gamma(\mu+1)} \left(2^{-\mu-\nu-2} \pi z^\alpha (a z)^{\mu-\nu} \csc(\pi\nu) \left(\frac{2^{2\nu+1}}{(\alpha+\mu-\nu)\Gamma(1-\nu)} {}_3F_4\left(\frac{\mu}{2}-\frac{\nu}{2}+\frac{1}{2}, \frac{\mu}{2}-\frac{\nu}{2}+1, \frac{\alpha}{2}+\frac{\mu}{2}-\frac{\nu}{2}; \mu+1, 1-\nu, \mu-\nu+1, \frac{\alpha}{2}+\frac{\mu}{2}-\frac{\nu}{2}+1; a^2 z^2 \right) - \frac{2(a z)^{2\nu}}{(\alpha+\mu+\nu)\Gamma(\nu+1)} {}_3F_4\left(\frac{\mu}{2}+\frac{\nu}{2}+\frac{1}{2}, \frac{\mu}{2}+\frac{\nu}{2}+1, \frac{\alpha}{2}+\frac{\mu}{2}+\frac{\nu}{2}; \mu+1, \frac{\alpha}{2}+\frac{\mu}{2}+\frac{\nu}{2}+1, \nu+1, \mu+\nu+1; a^2 z^2 \right) \right) \right)$$

03.04.21.0092.01

$$\int z^{\alpha-1} I_\nu(a z) K_\nu(a z) \, dz = \frac{\pi z^\alpha \csc(\pi \nu)}{2 \Gamma(\nu+1)^2} \left(\frac{\Gamma(\nu+1)}{\alpha \Gamma(1-\nu)} {}_2F_3\left(\frac{1}{2}, \frac{\alpha}{2}; \frac{\alpha}{2}+1, 1-\nu, \nu+1; a^2 z^2\right) - \frac{4^{-\nu} (a z)^{2\nu}}{\alpha+2\nu} {}_2F_3\left(\nu+\frac{1}{2}, \frac{\alpha}{2}+\nu; \nu+1, \frac{\alpha}{2}+\nu+1, 2\nu+1; a^2 z^2\right)\right)$$

03.04.21.0093.01

$$\int z^{\alpha-1} I_{-\nu}(a z) K_\nu(a z) \, dz = \frac{\pi z^\alpha \csc(\pi \nu)}{4 \Gamma(1-\nu)^2} \left(\frac{2^{2\nu+1} (a z)^{-2\nu}}{\alpha-2\nu} {}_2F_3\left(\frac{1}{2}-\nu, \frac{\alpha}{2}-\nu; 1-2\nu, 1-\nu, \frac{\alpha}{2}-\nu+1; a^2 z^2\right) - \frac{2 \Gamma(1-\nu)}{\alpha \Gamma(\nu+1)} {}_2F_3\left(\frac{1}{2}, \frac{\alpha}{2}; \frac{\alpha}{2}+1, 1-\nu, \nu+1; a^2 z^2\right)\right)$$

03.04.21.0094.01

$$\int z^{\alpha-1} I_0(a z) K_0(a z) \, dz = \frac{z^\alpha}{4\sqrt{\pi}} G_{2,4}^{2,2}\left(a z, \frac{1}{2} \middle| \begin{matrix} \frac{1}{2}, 1-\frac{\alpha}{2} \\ 0, 0, 0, -\frac{\alpha}{2} \end{matrix}\right)$$

03.04.21.0095.01

$$\int z I_\nu(a z) K_\nu(b z) \, dz = \frac{z}{a^2-b^2} (a I_{\nu+1}(a z) K_\nu(b z) + b I_\nu(a z) K_{\nu+1}(b z))$$

03.04.21.0096.01

$$\int z I_{-\nu}(a z) K_\nu(b z) \, dz = \frac{z}{a^2-b^2} (b I_{-\nu}(a z) K_{1-\nu}(b z) + a I_{1-\nu}(a z) K_\nu(b z))$$

03.04.21.0097.01

$$\int z I_\nu(a z) K_\nu(a z) \, dz = -\frac{1}{8 a^2} \left(\csc(\pi \nu) \left(2 a^2 \pi I_\nu(a z)^2 z^2 + a^2 \pi I_{-\nu-1}(a z) I_{\nu-1}(a z) z^2 - 2 a^2 \pi I_{-\nu}(a z) I_\nu(a z) z^2 + a^2 \pi I_{1-\nu}(a z) I_{\nu+1}(a z) z^2 - 2 a^2 \pi I_{\nu-1}(a z) I_{\nu+1}(a z) z^2 + 4 \nu \sin(\pi \nu)\right)\right)$$

03.04.21.0098.01

$$\int \frac{I_\mu(a z) K_\nu(a z)}{z} \, dz = 2^{-\mu-\nu-1} \pi (a z)^{\mu-\nu} \csc(\pi \nu) \left(4^\nu \Gamma(\mu-\nu) {}_2\tilde{F}_3\left(\frac{\mu-\nu}{2}, \frac{1}{2}(\mu-\nu+1); \mu+1, 1-\nu, \mu-\nu+1; a^2 z^2\right) - (a z)^{2\nu} \Gamma(\mu+\nu) {}_2\tilde{F}_3\left(\frac{\mu+\nu}{2}, \frac{1}{2}(\mu+\nu+1); \mu+1, \nu+1, \mu+\nu+1; a^2 z^2\right)\right)$$

03.04.21.0099.01

$$\int \frac{1}{z I_\nu(a z) K_\nu(a z)} \, dz = \log\left(\frac{I_\nu(a z)}{K_\nu(a z)}\right)$$

Power arguments

03.04.21.0100.01

$$\int z^{\alpha-1} I_\mu(a z^r) K_\nu(a z^r) \, dz = \frac{1}{\Gamma(\mu+1)} \left(2^{-\mu-\nu-1} \pi z^\alpha (a z^r)^{\mu-\nu} \csc(\pi\nu) \left(\frac{4^\nu}{(\alpha + r(\mu-\nu)) \Gamma(1-\nu)} \, {}_3F_4\left(\frac{\mu}{2} - \frac{\nu}{2} + \frac{1}{2}, \frac{\mu}{2} - \frac{\nu}{2} + 1, \frac{\alpha}{2r} + \frac{\mu}{2} - \frac{\nu}{2}; \mu+1, 1-\nu, \mu-\nu+1, \frac{\alpha}{2r} + \frac{\mu}{2} - \frac{\nu}{2} + 1; a^2 z^{2r} \right) - \frac{(a z^r)^{2\nu}}{(\alpha + r(\mu+\nu)) \Gamma(\nu+1)} \, {}_3F_4\left(\frac{\mu}{2} + \frac{\nu}{2} + \frac{1}{2}, \frac{\mu}{2} + \frac{\nu}{2} + 1, \frac{\alpha}{2r} + \frac{\mu}{2} + \frac{\nu}{2}; \mu+1, \frac{\alpha}{2r} + \frac{\mu}{2} + \frac{\nu}{2} + 1, \nu+1, \mu+\nu+1; a^2 z^{2r} \right) \right) \right)$$

03.04.21.0101.01

$$\int z^{\alpha-1} I_\nu(a z^r) K_\nu(a z^r) \, dz = \frac{\pi z^\alpha \csc(\pi\nu)}{2 \Gamma(\nu+1)^2} \left(\frac{\Gamma(\nu+1)}{\alpha \Gamma(1-\nu)} \, {}_2F_3\left(\frac{1}{2}, \frac{\alpha}{2r}; \frac{\alpha}{2r} + 1, 1-\nu, \nu+1; a^2 z^{2r} \right) - \frac{4^{-\nu} (a z^r)^{2\nu}}{\alpha + 2 r \nu} \, {}_2F_3\left(\nu + \frac{1}{2}, \frac{\alpha}{2r} + \nu; \nu+1, \frac{\alpha}{2r} + \nu + 1, 2\nu+1; a^2 z^{2r} \right) \right)$$

03.04.21.0102.01

$$\int z^{\alpha-1} I_{-\nu}(a z^r) K_\nu(a z^r) \, dz = \frac{\pi z^\alpha \csc(\pi\nu)}{4 \Gamma(1-\nu)^2} \left(\frac{2^{2\nu+1} (a z^r)^{-2\nu}}{\alpha - 2 r \nu} \, {}_2F_3\left(\frac{1}{2} - \nu, \frac{\alpha}{2r} - \nu; 1 - 2\nu, 1-\nu, \frac{\alpha}{2r} - \nu + 1; a^2 z^{2r} \right) - \frac{2 \Gamma(1-\nu)}{\alpha \Gamma(\nu+1)} \, {}_2F_3\left(\frac{1}{2}, \frac{\alpha}{2r}; \frac{\alpha}{2r} + 1, 1-\nu, \nu+1; a^2 z^{2r} \right) \right)$$

Involving Bessel Y

Linear arguments

03.04.21.0103.01

$$\int Y_\nu(a z) K_\nu(a z) \, dz = -\frac{z}{16\sqrt{\pi}} G_{2,6}^{4,1}\left(\frac{a z}{2\sqrt{2}}, \frac{1}{4} \,\middle|\, \begin{matrix} \frac{3}{4}, \frac{1-\nu}{2} \\ 0, \frac{1}{2}, -\frac{\nu}{2}, \frac{\nu}{2}, -\frac{1}{4}, \frac{1-\nu}{2} \end{matrix} \right)$$

03.04.21.0104.01

$$\int Y_{-\nu}(a z) K_\nu(a z) \, dz = -\frac{z}{16\sqrt{\pi}} G_{2,6}^{4,1}\left(\frac{a z}{2\sqrt{2}}, \frac{1}{4} \,\middle|\, \begin{matrix} \frac{3}{4}, \frac{\nu+1}{2} \\ 0, \frac{1}{2}, -\frac{\nu}{2}, \frac{\nu}{2}, -\frac{1}{4}, \frac{\nu+1}{2} \end{matrix} \right)$$

Power arguments

03.04.21.0105.01

$$\int Y_\nu(a z^r) K_\nu(a z^r) \, dz = -\frac{z}{16\sqrt{\pi}\, r} G_{2,6}^{4,1}\left(\frac{a z^r}{2\sqrt{2}}, \frac{1}{4} \,\middle|\, \begin{matrix} 1 - \frac{1}{4r}, \frac{1-\nu}{2} \\ 0, \frac{1}{2}, -\frac{\nu}{2}, \frac{\nu}{2}, -\frac{1}{4r}, \frac{1-\nu}{2} \end{matrix} \right)$$

03.04.21.0106.01

$$\int Y_{-\nu}(a z^r) K_\nu(a z^r) \, dz = -\frac{z}{16\sqrt{\pi}\, r} G_{2,6}^{4,1}\left(\frac{a z^r}{2\sqrt{2}}, \frac{1}{4} \,\middle|\, \begin{matrix} 1 - \frac{1}{4r}, \frac{\nu+1}{2} \\ 0, \frac{1}{2}, -\frac{\nu}{2}, \frac{\nu}{2}, -\frac{1}{4r}, \frac{\nu+1}{2} \end{matrix} \right)$$

03.04.21.0107.01

$$\int Y_\nu\left(a\sqrt{z}\right)K_\nu\left(b\sqrt{z}\right)dz = \frac{2\sqrt{z}}{a^2+b^2}\left(a\,K_\nu\left(b\sqrt{z}\right)Y_{\nu+1}\left(a\sqrt{z}\right)-b\,K_{\nu+1}\left(b\sqrt{z}\right)Y_\nu\left(a\sqrt{z}\right)\right)$$

03.04.21.0108.01

$$\int Y_{-\nu}\left(a\sqrt{z}\right)K_\nu\left(b\sqrt{z}\right)dz = \frac{2\sqrt{z}}{a^2+b^2}\left(a\,K_\nu\left(b\sqrt{z}\right)Y_{1-\nu}\left(a\sqrt{z}\right)-b\,K_{1-\nu}\left(b\sqrt{z}\right)Y_{-\nu}\left(a\sqrt{z}\right)\right)$$

Involving Bessel Y and power

Linear arguments

03.04.21.0109.01

$$\int z^{\alpha-1}\,Y_\nu(a\,z)\,K_\nu(a\,z)\,dz = -\frac{z^\alpha}{16\sqrt{\pi}}\,G_{2,6}^{4,1}\left(\frac{a\,z}{2\sqrt{2}},\frac{1}{4}\,\middle|\,\begin{array}{c}1-\frac{\alpha}{4},\frac{1-\nu}{2}\\0,\frac{1}{2},-\frac{\nu}{2},\frac{\nu}{2},-\frac{\alpha}{4},\frac{1-\nu}{2}\end{array}\right)$$

03.04.21.0110.01

$$\int z^{\alpha-1}\,Y_{-\nu}(a\,z)\,K_\nu(a\,z)\,dz = -\frac{z^\alpha}{16\sqrt{\pi}}\,G_{2,6}^{4,1}\left(\frac{a\,z}{2\sqrt{2}},\frac{1}{4}\,\middle|\,\begin{array}{c}1-\frac{\alpha}{4},\frac{\nu+1}{2}\\0,\frac{1}{2},-\frac{\nu}{2},\frac{\nu}{2},-\frac{\alpha}{4},\frac{\nu+1}{2}\end{array}\right)$$

03.04.21.0111.01

$$\int z\,Y_\nu(a\,z)\,K_\nu(b\,z)\,dz = \frac{z}{a^2+b^2}\left(a\,K_\nu(b\,z)\,Y_{\nu+1}(a\,z)-b\,K_{\nu+1}(b\,z)\,Y_\nu(a\,z)\right)$$

03.04.21.0112.01

$$\int z\,Y_{-\nu}(a\,z)\,K_\nu(b\,z)\,dz = \frac{z}{a^2+b^2}\left(a\,K_\nu(b\,z)\,Y_{1-\nu}(a\,z)-b\,K_{1-\nu}(b\,z)\,Y_{-\nu}(a\,z)\right)$$

Power arguments

03.04.21.0113.01

$$\int z^{\alpha-1}\,Y_\nu(a\,z^r)\,K_\nu(a\,z^r)\,dz = -\frac{z^\alpha}{16\sqrt{\pi}\,r}\,G_{2,6}^{4,1}\left(\frac{a\,z^r}{2\sqrt{2}},\frac{1}{4}\,\middle|\,\begin{array}{c}1-\frac{\alpha}{4r},\frac{1-\nu}{2}\\0,\frac{1}{2},-\frac{\nu}{2},\frac{\nu}{2},-\frac{\alpha}{4r},\frac{1-\nu}{2}\end{array}\right)$$

03.04.21.0114.01

$$\int z^{\alpha-1}\,Y_{-\nu}(a\,z^r)\,K_\nu(a\,z^r)\,dz = -\frac{z^\alpha}{16\sqrt{\pi}\,r}\,G_{2,6}^{4,1}\left(\frac{a\,z^r}{2\sqrt{2}},\frac{1}{4}\,\middle|\,\begin{array}{c}1-\frac{\alpha}{4r},\frac{\nu+1}{2}\\0,\frac{1}{2},-\frac{\nu}{2},\frac{\nu}{2},-\frac{\alpha}{4r},\frac{\nu+1}{2}\end{array}\right)$$

Definite integration

For the direct function itself

03.04.21.0115.01

$$\int_0^\infty K_\nu(t)\,dt = \frac{\pi}{2}\sec\left(\frac{\pi\,\nu}{2}\right)\;/;\;|\mathrm{Re}(\nu)|<1$$

03.04.21.0116.01

$$\int_0^\infty t^{\alpha-1}\,K_\nu(t)\,dt = 2^{\alpha-2}\,\Gamma\left(\frac{\alpha-\nu}{2}\right)\Gamma\left(\frac{\alpha+\nu}{2}\right)\;/;\;\mathrm{Re}(\alpha)>|\mathrm{Re}(\nu)|$$

Involving the direct function

03.04.21.0117.01

$$\int_0^\infty K_\nu(t)^2 \,dt = \frac{1}{4}\pi^2 \sec(\pi\nu) \;/;\; |\mathrm{Re}(\nu)| < \frac{1}{2}$$

03.04.21.0118.01

$$\int_0^\infty t^{\alpha-1} K_\nu(t)^2 \,dt = \frac{\sqrt{\pi}}{4\,\Gamma\left(\frac{\alpha+1}{2}\right)} \Gamma\left(\frac{\alpha}{2}\right) \Gamma\left(\frac{\alpha}{2}-\nu\right) \Gamma\left(\frac{\alpha}{2}+\nu\right) \;/;\; \mathrm{Re}(\alpha) > 2\,|\mathrm{Re}(\nu)|$$

03.04.21.0119.01

$$\int_0^\infty \frac{1}{\sqrt{x}} e^{-a x} I_0\left(\frac{b x}{2}\right) K_0\left(\frac{b x}{2}\right) dx = 2\sqrt{\frac{2}{\pi}} \frac{\sqrt{a-\sqrt{a^2-b^2}}}{b} \operatorname{sech}^2(\alpha)\, K\left(\operatorname{sech}^2(\alpha)\right) K\left(\tanh^2(\alpha)\right) /;$$

$$\mathrm{Re}(a) \geq \mathrm{Re}(b) > 0 \bigwedge \cosh^{-1}\left(\frac{\sqrt{b+\sqrt{2a^2-2a\sqrt{a^2-b^2}}}}{\sqrt{2b}}\right)$$

Integral transforms

Fourier cos transforms

03.04.22.0001.01

$$\mathcal{F}c_t[K_\nu(t)](z) = \sqrt{\frac{\pi}{2}} \frac{\cosh\left(\nu \sinh^{-1}(z)\right)}{\sqrt{z^2+1}} \sec\left(\frac{\pi\nu}{2}\right) /;\; |\mathrm{Re}(\nu)| < 1 \land z > 0$$

Fourier sin transforms

03.04.22.0002.01

$$\mathcal{F}s_t[K_\nu(t)](z) = \sqrt{\frac{\pi}{2}} \frac{\sinh\left(\nu \sinh^{-1}(z)\right)}{\sqrt{z^2+1}} \csc\left(\frac{\pi\nu}{2}\right) /;\; |\mathrm{Re}(\nu)| < 2 \land z > 0$$

Laplace transforms

03.04.22.0003.01

$$\mathcal{L}_t[K_\nu(t)](z) = 2^{-\nu-1} \pi z^{-\nu-1} \csc(\pi\nu) \left(4^\nu z^{2\nu} {}_2F_1\left(\frac{1}{2}-\frac{\nu}{2}, 1-\frac{\nu}{2}; 1-\nu; \frac{1}{z^2}\right) - {}_2F_1\left(\frac{\nu+1}{2}, \frac{\nu+2}{2}; \nu+1; \frac{1}{z^2}\right)\right) /;$$

$$|\mathrm{Re}(\nu)| < 1 \land \mathrm{Re}(z) > 0$$

Mellin transforms

03.04.22.0004.01

$$\mathcal{M}_t[K_\nu(t)](z) = 2^{z-2}\, \Gamma\left(\frac{z-\nu}{2}\right) \Gamma\left(\frac{z+\nu}{2}\right) /;\; \mathrm{Re}(z) > |\mathrm{Re}(\nu)|$$

Hankel transforms

03.04.22.0005.01

$$\mathcal{H}_{t;\mu}[K_\nu(t)](z) = \frac{z^{\mu+\frac{1}{2}}}{\sqrt{2}\,\Gamma(\mu+1)} \Gamma\left(\frac{1}{4}(2\mu-2\nu+3)\right) \Gamma\left(\frac{1}{4}(2\mu+2\nu+3)\right) {}_2F_1\left(\frac{1}{4}(2\mu-2\nu+3), \frac{1}{4}(2\mu+2\nu+3); \mu+1; -z^2\right) /; \operatorname{Re}(\mu-\nu) > -\frac{3}{2} \bigwedge \operatorname{Re}(\mu+\nu) > -\frac{3}{2} \bigwedge z > 0$$

Representations through more general functions

Through hypergeometric functions

Involving ${}_0\tilde{F}_1$

03.04.26.0001.01

$$K_\nu(z) = \pi \csc(\nu\pi) \left(2^{\nu-1} z^{-\nu} {}_0\tilde{F}_1\left(; 1-\nu; \frac{z^2}{4}\right) - 2^{-\nu-1} z^\nu {}_0\tilde{F}_1\left(; \nu+1; \frac{z^2}{4}\right)\right) /; \nu \notin \mathbb{Z}$$

Involving ${}_0F_1$

03.04.26.0002.01

$$K_\nu(z) = 2^{\nu-1} \Gamma(\nu) z^{-\nu} {}_0F_1\left(; 1-\nu; \frac{z^2}{4}\right) + 2^{-\nu-1} \Gamma(-\nu) z^\nu {}_0F_1\left(; \nu+1; \frac{z^2}{4}\right) /; \nu \notin \mathbb{Z}$$

Involving hypergeometric U

03.04.26.0003.01

$$K_\nu(z) = \sqrt{\pi}\, (2z)^\nu e^{-z} U\left(\nu + \frac{1}{2}, 2\nu+1, 2z\right)$$

03.04.26.0004.01

$$K_\nu(z) = \sqrt{\pi}\, (2z)^{-\nu} e^{-z} U\left(\frac{1}{2} - \nu, 1-2\nu, 2z\right)$$

Involving ${}_1F_1$

03.04.26.0005.01

$$K_\nu(z) = 2^{\nu-1}\, \Gamma(\nu) z^{-\nu} e^{-z} {}_1F_1\left(\frac{1}{2} - \nu; 1-2\nu; 2z\right) + 2^{-\nu-1} \Gamma(-\nu) z^\nu e^{-z} {}_1F_1\left(\frac{1}{2} + \nu; 1+2\nu; 2z\right) /; \nu \notin \mathbb{Z}$$

Through Meijer G

Classical cases for the direct function itself

03.04.26.0006.01

$$K_\nu\left(\sqrt{z^2}\right) = \frac{1}{2} G_{0,2}^{2,0}\left(\frac{z^2}{4} \,\middle|\, \frac{\nu}{2}, -\frac{\nu}{2}\right)$$

03.04.26.0007.01

$$K_\nu(z) = \frac{1}{2} z^{-\nu} \left(z^2\right)^{\nu/2} G_{0,2}^{2,0}\left(\frac{z^2}{4} \,\middle|\, \frac{\nu}{2}, -\frac{\nu}{2}\right) - \frac{\pi}{2} (-z)^{-\frac{\nu}{2}} z^{-\frac{3\nu}{2}} \left(z^{2\nu} - \left(z^2\right)^\nu\right) \csc(\pi\nu)\, G_{0,2}^{1,0}\left(-\frac{z^2}{4} \,\middle|\, \frac{\nu}{2}, -\frac{\nu}{2}\right) /; \nu \notin \mathbb{Z}$$

03.04.26.0008.01

$$K_\nu(z) = \frac{1}{2} G_{0,2}^{2,0}\left(\frac{z^2}{4} \,\middle|\, \frac{\nu}{2}, -\frac{\nu}{2}\right) /; \operatorname{Re}(z) > 0$$

03.04.26.0009.01

$$K_\nu\left(\sqrt{z}\right) = \frac{1}{2} G_{0,2}^{2,0}\left(\frac{z}{4} \,\middle|\, \frac{\nu}{2}, -\frac{\nu}{2}\right)$$

Classical cases involving exp

03.04.26.0010.01

$$e^{-z} K_\nu(z) = \sqrt{\pi}\; G_{1,2}^{2,0}\left(2z \,\middle|\, \begin{matrix} \frac{1}{2} \\ -\nu, \nu \end{matrix}\right)$$

03.04.26.0011.01

$$e^{z} K_\nu(z) = \frac{\cos(\pi\nu)}{\sqrt{\pi}} G_{1,2}^{2,1}\left(2z \,\middle|\, \begin{matrix} \frac{1}{2} \\ -\nu, \nu \end{matrix}\right)$$

Classical cases involving cosh

03.04.26.0012.01

$$\cosh\left(\sqrt{z}\right) K_\nu\left(\sqrt{z}\right) = \frac{1}{2\sqrt{2}} G_{2,4}^{2,2}\left(z \,\middle|\, \begin{matrix} \frac{1}{4}, \frac{3}{4} \\ -\frac{\nu}{2}, \frac{\nu}{2}, \frac{1-\nu}{2}, \frac{\nu+1}{2} \end{matrix}\right)$$

Classical cases involving sinh

03.04.26.0013.01

$$\sinh\left(\sqrt{z}\right) K_\nu\left(\sqrt{z}\right) = \frac{1}{2\sqrt{2}} G_{2,4}^{2,2}\left(z \,\middle|\, \begin{matrix} \frac{1}{4}, \frac{3}{4} \\ \frac{1-\nu}{2}, \frac{\nu+1}{2}, -\frac{\nu}{2}, \frac{\nu}{2} \end{matrix}\right)$$

Classical cases for powers of Bessel *K*

03.04.26.0014.01

$$K_\nu\left(\sqrt{z}\right)^2 = \frac{\sqrt{\pi}}{2} G_{1,3}^{3,0}\left(z \,\middle|\, \begin{matrix} \frac{1}{2} \\ 0, -\nu, \nu \end{matrix}\right)$$

Classical cases for products of Bessel *K*

03.04.26.0015.01

$$K_{\nu-1}\left(\sqrt{z}\right) K_\nu\left(\sqrt{z}\right) = \frac{\sqrt{\pi}}{2}\; G_{1,3}^{3,0}\left(z \,\middle|\, \begin{matrix} 0 \\ -\frac{1}{2}, \nu-\frac{1}{2}, \frac{1}{2}-\nu \end{matrix}\right)$$

03.04.26.0016.01

$$K_\mu\left(\sqrt{z}\right) K_\nu\left(\sqrt{z}\right) = \frac{\sqrt{\pi}}{2} G_{2,4}^{4,0}\left(z \,\middle|\, \begin{matrix} 0, \frac{1}{2} \\ \frac{\mu+\nu}{2}, \frac{\mu-\nu}{2}, \frac{\nu-\mu}{2}, -\frac{\mu+\nu}{2} \end{matrix}\right)$$

03.04.26.0017.01

$$K_\nu\left(-\sqrt{-z}\right) K_\nu\left(\sqrt{-z}\right) = \frac{\cos(\nu\pi)}{2\sqrt{\pi}} G_{1,3}^{3,1}\left(z \,\middle|\, \begin{matrix} \frac{1}{2} \\ 0, -\nu, \nu \end{matrix}\right)$$

03.04.26.0018.01

$$K_\nu\left(\sqrt[4]{-z}\right)K_\nu\left(-\frac{(-z)^{3/4}}{\sqrt{z}}\right)=\frac{1}{8\sqrt{\pi}}\,G_{0,4}^{4,0}\!\left(\frac{z}{64}\,\middle|\,0,\frac{1}{2},-\frac{\nu}{2},\frac{\nu}{2}\right)$$

Classical cases involving Bessel J

03.04.26.0019.01

$$J_\nu\left(\sqrt[4]{z}\right)K_\nu\left(\sqrt[4]{z}\right)=\frac{1}{4\sqrt{\pi}}\,G_{0,4}^{3,0}\!\left(\frac{z}{64}\,\middle|\,0,\frac{1}{2},\frac{\nu}{2},-\frac{\nu}{2}\right)$$

03.04.26.0068.01

$$J_{-\nu}\left(\sqrt[4]{z}\right)K_\nu\left(\sqrt[4]{z}\right)=\frac{1}{4\sqrt{\pi}}\,G_{0,4}^{3,0}\!\left(\frac{z}{64}\,\middle|\,0,\frac{1}{2},-\frac{\nu}{2},\frac{\nu}{2}\right)$$

03.04.26.0020.01

$$\left(J_{-\nu}\left(\sqrt[4]{z}\right)+J_\nu\left(\sqrt[4]{z}\right)\right)K_\nu\left(\sqrt[4]{z}\right)=\frac{1}{2\sqrt{\pi}}\cos\left(\frac{\pi\,\nu}{2}\right)G_{0,4}^{3,0}\!\left(\frac{z}{64}\,\middle|\,\frac{1}{2},-\frac{\nu}{2},\frac{\nu}{2},0\right)$$

03.04.26.0021.01

$$\left(J_{-\nu}\left(\sqrt[4]{z}\right)-J_\nu\left(\sqrt[4]{z}\right)\right)K_\nu\left(\sqrt[4]{z}\right)=\frac{1}{2\sqrt{\pi}}\sin\left(\frac{\pi\,\nu}{2}\right)G_{0,4}^{3,0}\!\left(\frac{z}{64}\,\middle|\,0,-\frac{\nu}{2},\frac{\nu}{2},\frac{1}{2}\right)$$

Classical cases involving Bessel I

03.04.26.0022.01

$$I_\nu\left(\sqrt{z}\right)K_\nu\left(\sqrt{z}\right)=\frac{1}{2\sqrt{\pi}}\,G_{1,3}^{2,1}\!\left(z\,\middle|\,\begin{matrix}\frac{1}{2}\\0,\nu,-\nu\end{matrix}\right)$$

03.04.26.0069.01

$$I_{-\nu}\left(\sqrt{z}\right)K_\nu\left(\sqrt{z}\right)=\frac{1}{2\sqrt{\pi}}\,G_{1,3}^{2,1}\!\left(z\,\middle|\,\begin{matrix}\frac{1}{2}\\0,-\nu,\nu\end{matrix}\right)$$

03.04.26.0023.01

$$I_\mu\left(\sqrt{z}\right)K_\nu\left(\sqrt{z}\right)=\frac{1}{2\sqrt{\pi}}\,G_{2,4}^{2,2}\!\left(z\,\middle|\,\begin{matrix}0,\frac{1}{2}\\\frac{\mu-\nu}{2},\frac{\mu+\nu}{2},-\frac{\mu+\nu}{2},\frac{\nu-\mu}{2}\end{matrix}\right)/;\,-\mu-\nu-1\notin\mathbb{N}\land\nu-\mu-1\notin\mathbb{N}$$

03.04.26.0070.01

$$I_\nu\left(\sqrt{z}\right)K_{n+\nu+1}\left(\sqrt{z}\right)=\frac{(-1)^n\,\pi^{3/2}\csc\left(\frac{1}{4}\pi\,(4\,\nu+(-1)^n)\right)}{\sqrt{2}}\,G_{4,6}^{2,2}\!\left(z\,\middle|\,\begin{matrix}0,\frac{1}{2},\nu+\frac{3}{4},\frac{1}{4}\\\frac{n+1}{2},\frac{n+1}{2}+\nu,\nu+\frac{3}{4},\frac{1}{4},-\frac{1}{2}(n+1),-\frac{1}{2}(n+1)-\nu\end{matrix}\right)-$$

$$(-1)^n\sqrt{\pi}\,\csc(\nu\,\pi)\sum_{k=0}^{\left\lfloor\frac{n}{2}\right\rfloor}\frac{(-1)^{\left\lfloor\frac{n+1}{2}\right\rfloor}z^{\frac{1}{2}(2k-n-1)}\,\Gamma\left(k-n+\left\lfloor\frac{n}{2}\right\rfloor+\frac{1}{2}\right)\left(1-k+\left\lfloor\frac{n}{2}\right\rfloor\right)_{n-\left\lfloor\frac{n}{2}\right\rfloor}}{k!\,\Gamma(k-n-\nu)\,\Gamma(k+\nu+1)}\,/;\,n\in\mathbb{N}$$

03.04.26.0071.01

$$I_\nu\left(\sqrt{z}\right)K_{-n-\nu-1}\left(\sqrt{z}\right)=\frac{(-1)^n\,\pi^{3/2}\csc\left(\frac{1}{4}\pi\,(4\,\nu+(-1)^n)\right)}{\sqrt{2}}\,G_{4,6}^{2,2}\!\left(z\,\middle|\,\begin{matrix}0,\frac{1}{2},\nu+\frac{3}{4},\frac{1}{4}\\\frac{n+1}{2},\frac{n+1}{2}+\nu,\nu+\frac{3}{4},\frac{1}{4},-\frac{1}{2}(n+1),-\frac{1}{2}(n+1)-\nu\end{matrix}\right)-$$

$$(-1)^n\sqrt{\pi}\,\csc(\nu\,\pi)\sum_{k=0}^{\left\lfloor\frac{n}{2}\right\rfloor}\frac{(-1)^{\left\lfloor\frac{n+1}{2}\right\rfloor}z^{\frac{1}{2}(2k-n-1)}\,\Gamma\left(k-n+\left\lfloor\frac{n}{2}\right\rfloor+\frac{1}{2}\right)\left(1-k+\left\lfloor\frac{n}{2}\right\rfloor\right)_{n-\left\lfloor\frac{n}{2}\right\rfloor}}{k!\,\Gamma(k-n-\nu)\,\Gamma(k+\nu+1)}\,/;\,n\in\mathbb{N}\,/;\,n\in\mathbb{N}$$

03.04.26.0072.01

$$I_\nu(\sqrt{z})K_{\nu+1}(\sqrt{z}) = \frac{\pi^{3/2}}{\cos(\pi\nu)+\sin(\pi\nu)} G_{4,6}^{2,2}\left(z \,\middle|\, \begin{matrix} 0, \frac{1}{2}, \nu+\frac{3}{4}, \frac{1}{4} \\ \frac{1}{2}, \nu+\frac{1}{2}, \nu+\frac{3}{4}, \frac{1}{4}, -\frac{1}{2}, -\nu-\frac{1}{2} \end{matrix}\right) + \frac{1}{\sqrt{z}}$$

03.04.26.0073.01

$$I_\nu(\sqrt{z})K_{\nu+2}(\sqrt{z}) = \frac{2(\nu+1)}{z} + \frac{\pi^{3/2}\csc(\pi\nu)}{\cot(\pi\nu)-1} G_{4,6}^{2,2}\left(z \,\middle|\, \begin{matrix} 0, \frac{1}{2}, \nu+\frac{3}{4}, \frac{1}{4} \\ 1, \nu+1, \nu+\frac{3}{4}, \frac{1}{4}, -1, -\nu-1 \end{matrix}\right)$$

03.04.26.0024.01

$$\left(I_{-\nu}(\sqrt{z}) + I_\nu(\sqrt{z})\right)K_\nu(\sqrt{z}) == \frac{\cos(\pi\nu)}{\sqrt{\pi}} G_{1,3}^{2,1}\left(z \,\middle|\, \begin{matrix} \frac{1}{2} \\ -\nu, \nu, 0 \end{matrix}\right)$$

03.04.26.0074.01

$$\left(I_{-\nu}(\sqrt{z}) - I_\nu(\sqrt{z})\right)K_\nu(\sqrt{z}) = \frac{\sin(\pi\nu)}{\sqrt{\pi}} G_{1,3}^{3,0}\left(z \,\middle|\, \begin{matrix} \frac{1}{2} \\ 0, \nu, -\nu \end{matrix}\right)$$

03.04.26.0025.01

$$I_\nu(\sqrt{z})K_\mu(\sqrt{z}) + I_\mu(\sqrt{z})K_\nu(\sqrt{z}) == \frac{1}{\sqrt{\pi}}\cos\left(\frac{1}{2}\pi(\mu-\nu)\right) G_{2,4}^{3,1}\left(z \,\middle|\, \begin{matrix} \frac{1}{2}, 0 \\ \frac{\mu+\nu}{2}, \frac{\mu-\nu}{2}, \frac{\nu-\mu}{2}, -\frac{\mu+\nu}{2} \end{matrix}\right)$$

03.04.26.0026.01

$$I_\nu(\sqrt{z})K_\mu(\sqrt{z}) - I_\mu(\sqrt{z})K_\nu(\sqrt{z}) == \frac{1}{\sqrt{\pi}}\sin\left(\frac{1}{2}\pi(\mu-\nu)\right) G_{2,4}^{3,1}\left(z \,\middle|\, \begin{matrix} 0, \frac{1}{2} \\ \frac{\mu+\nu}{2}, \frac{\mu-\nu}{2}, \frac{\nu-\mu}{2}, -\frac{\mu+\nu}{2} \end{matrix}\right)$$

03.04.26.0075.01

$$I_\nu(\sqrt{z})^2 - \frac{1}{\pi^2}K_\nu(\sqrt{z})^2 = -\sec(\pi\nu)\sqrt{\pi}\, G_{3,5}^{3,0}\left(z \,\middle|\, \begin{matrix} \frac{1}{2}, \frac{1}{4}(1-2\nu), \frac{1}{4}(3-2\nu) \\ 0, -\nu, \nu, \frac{1}{4}(1-2\nu), \frac{1}{4}(3-2\nu) \end{matrix}\right)$$

Classical cases involving Bessel *Y*

03.04.26.0027.01

$$K_\nu(\sqrt[4]{z})Y_\nu(\sqrt[4]{z}) == -\frac{1}{4\sqrt{\pi}} G_{1,5}^{4,0}\left(\frac{z}{64} \,\middle|\, \begin{matrix} \frac{1-\nu}{2} \\ 0, \frac{1}{2}, \frac{\nu}{2}, -\frac{\nu}{2}, \frac{1-\nu}{2} \end{matrix}\right)$$

03.04.26.0076.01

$$K_\nu(\sqrt[4]{z})Y_{-\nu}(\sqrt[4]{z}) = -\frac{1}{4\sqrt{\pi}} G_{1,5}^{4,0}\left(\frac{z}{64} \,\middle|\, \begin{matrix} \frac{\nu+1}{2} \\ 0, \frac{1}{2}, \frac{\nu}{2}, -\frac{\nu}{2}, \frac{\nu+1}{2} \end{matrix}\right)$$

03.04.26.0028.01

$$K_0(\sqrt[4]{z}) - \frac{\pi}{2}Y_0(\sqrt[4]{z}) == \frac{\pi}{2} G_{0,4}^{2,0}\left(\frac{z}{256} \,\middle|\, 0, 0, \frac{1}{2}, \frac{1}{2}\right)$$

03.04.26.0029.01

$$K_n(\sqrt[4]{z}) + \frac{\pi}{2}Y_n(\sqrt[4]{z}) == \frac{\pi}{2} G_{1,5}^{3,0}\left(\frac{z}{256} \,\middle|\, \begin{matrix} -\frac{n}{4} \\ \frac{2-n}{4}, \frac{n}{4}, \frac{n+2}{4}, -\frac{n}{4}, -\frac{n}{4} \end{matrix}\right) /; n \in \mathbb{N}$$

03.04.26.0030.01

$$K_n\left(\sqrt[4]{z}\right)-\frac{\pi}{2}Y_n\left(\sqrt[4]{z}\right)=\frac{\pi}{2}G_{1,5}^{3,0}\left(\frac{z}{256}\,\middle|\,\begin{array}{c}\frac{2-n}{4}\\ -\frac{n}{4},\frac{n}{4},\frac{n+2}{4},\frac{2-n}{4},\frac{2-n}{4}\end{array}\right)/;n\in\mathbb{N}$$

Classical cases involving ${}_0F_1$

03.04.26.0031.01

$$K_\nu(z)\,{}_0F_1\left(;b;\frac{z^2}{4}\right)=\frac{2^{b-2}\,\Gamma(b)}{\sqrt{\pi}}G_{2,4}^{2,2}\left(z^2\,\middle|\,\begin{array}{c}\frac{1-b}{2},1-\frac{b}{2}\\ -\frac{\nu}{2},\frac{\nu}{2},1-b-\frac{\nu}{2},1-b+\frac{\nu}{2}\end{array}\right)/;-b-\nu\notin\mathbb{N}\bigwedge\nu-b\notin\mathbb{N}\bigwedge-\frac{\pi}{2}<\arg(z)\le\frac{\pi}{2}$$

03.04.26.0077.01

$$K_\nu(z)\,{}_0F_1\left(;-n-\nu;\frac{z^2}{4}\right)=2^{-n-\nu-\frac{3}{2}}\,\pi\,\Gamma(-n-\nu)$$

$$\left(\sqrt{\pi}\,\csc\left(\frac{1}{4}\pi\left(4\nu-(-1)^n\right)\right)G_{4,6}^{2,2}\left(z^2\,\middle|\,\begin{array}{c}\frac{1}{2}(n+\nu+1),\frac{1}{2}(n+\nu+2),\frac{1}{4}(-2n-2\nu+1),\frac{1}{4}(2n+2\nu+3)\\ n+\frac{\nu}{2}+1,-\frac{\nu}{2},\frac{1}{4}(-2n-2\nu+1),\frac{1}{4}(2n+2\nu+3),\frac{\nu}{2},n+\frac{3\nu}{2}+1\end{array}\right)-\right.$$

$$\left.\sqrt{\frac{2}{\pi}}\csc(\nu\pi)\sum_{k=0}^{\left\lfloor\frac{n}{2}\right\rfloor}\frac{(-1)^{\left\lfloor\frac{n+1}{2}\right\rfloor}z^{2k+\nu}\,\Gamma\left(k-n+\left\lfloor\frac{n}{2}\right\rfloor+\frac{1}{2}\right)\left(1-k+\left\lfloor\frac{n}{2}\right\rfloor\right)_{n-\left\lfloor\frac{n}{2}\right\rfloor}}{k!\,\Gamma(k-n-\nu)\,\Gamma(k+\nu+1)}\right)/;n\in\mathbb{N}\bigwedge-\frac{\pi}{2}<\arg(z)\le\frac{\pi}{2}$$

03.04.26.0078.01

$$K_\nu(z)\,{}_0F_1\left(;\nu-n;\frac{z^2}{4}\right)=$$

$$2^{-n+\nu-\frac{3}{2}}\,\pi\,\Gamma(\nu-n)\left(\sqrt{\frac{2}{\pi}}\csc(\nu\pi)\sum_{k=0}^{\left\lfloor\frac{n}{2}\right\rfloor}\frac{(-1)^{\left\lfloor\frac{n+1}{2}\right\rfloor}z^{2k-\nu}\,\Gamma\left(k-n+\left\lfloor\frac{n}{2}\right\rfloor+\frac{1}{2}\right)\left(1-k+\left\lfloor\frac{n}{2}\right\rfloor\right)_{n-\left\lfloor\frac{n}{2}\right\rfloor}}{k!\,\Gamma(k-n+\nu)\,\Gamma(k-\nu+1)}-\sqrt{\pi}\,\csc\left(\frac{1}{4}\pi\left(4\nu+(-1)^n\right)\right)\right.$$

$$\left.G_{4,6}^{2,2}\left(z^2\,\middle|\,\begin{array}{c}\frac{1}{2}(n-\nu+1),\frac{1}{2}(n-\nu+2),\frac{1}{4}(-2n+2\nu+1),\frac{1}{4}(2n-2\nu+3)\\ n-\frac{\nu}{2}+1,\frac{\nu}{2},\frac{1}{4}(-2n+2\nu+1),\frac{1}{4}(2n-2\nu+3),-\frac{\nu}{2},n-\frac{3\nu}{2}+1\end{array}\right)\right)/;n\in\mathbb{N}\bigwedge-\frac{\pi}{2}<\arg(z)\le\frac{\pi}{2}$$

03.04.26.0079.01

$$K_\nu(z)\,{}_0F_1\left(;\nu;\frac{z^2}{4}\right)=2^{\nu-1}\,\Gamma(\nu)\left(z^{-\nu}-\frac{\pi^{3/2}\csc\left(\frac{1}{4}(4\nu+1)\pi\right)}{\sqrt{2}}G_{4,6}^{2,2}\left(z^2\,\middle|\,\begin{array}{c}\frac{1-\nu}{2},1-\frac{\nu}{2},\frac{1}{4}(3-2\nu),\frac{1}{4}(2\nu+1)\\ 1-\frac{\nu}{2},\frac{\nu}{2},\frac{1}{4}(3-2\nu),1-\frac{3\nu}{2},-\frac{\nu}{2},\frac{1}{4}(2\nu+1)\end{array}\right)\right)/;$$

$$-\frac{\pi}{2}<\arg(z)\le\frac{\pi}{2}$$

03.04.26.0080.01

$$K_\nu(z)\,{}_0F_1\left(;\nu-1;\frac{z^2}{4}\right)=$$

$$2^{\nu-2}\,\Gamma(\nu-1)\left(2(\nu-1)z^{-\nu}+\frac{\pi^{3/2}\csc\left(\pi\left(\nu-\frac{5}{4}\right)\right)}{\sqrt{2}}G_{4,6}^{2,2}\left(z^2\,\middle|\,\begin{array}{c}\frac{3-\nu}{2},1-\frac{\nu}{2},\frac{1}{4}(5-2\nu),\frac{1}{4}(2\nu-1)\\ 2-\frac{\nu}{2},\frac{\nu}{2},\frac{1}{4}(5-2\nu),2-\frac{3\nu}{2},-\frac{\nu}{2},\frac{1}{4}(2\nu-1)\end{array}\right)\right)/;-\frac{\pi}{2}<\arg(z)\le\frac{\pi}{2}$$

03.04.26.0081.01

$$K_\nu(z)\,{}_0F_1\left(;-\nu;\frac{z^2}{4}\right)=2^{-\nu-1}\,\Gamma(-\nu)\left(z^\nu-\frac{\pi^{3/2}\csc\left(\frac{1}{4}\pi(1-4\nu)\right)}{\sqrt{2}}\,G_{4,6}^{2,2}\left(z^2\,\middle|\,\begin{array}{c}\frac{\nu+1}{2},\frac{\nu+2}{2},\frac{1}{4}(1-2\nu),\frac{1}{4}(2\nu+3)\\-\frac{\nu}{2},\frac{\nu+2}{2},\frac{1}{4}(1-2\nu),\frac{\nu}{2},\frac{3\nu}{2}+1,\frac{1}{4}(2\nu+3)\end{array}\right)\right)/;\;-\frac{\pi}{2}<\arg(z)\leq\frac{\pi}{2}$$

03.04.26.0082.01

$$K_\nu(z)\,{}_0F_1\left(;-\nu-1;\frac{z^2}{4}\right)=2^{-\nu-2}\,\Gamma(-\nu-1)\left(\frac{\pi^{3/2}\csc\left(\pi\left(\nu+\frac{1}{4}\right)\right)}{\sqrt{2}}\,G_{4,6}^{2,2}\left(z^2\,\middle|\,\begin{array}{c}\frac{\nu+2}{2},\frac{\nu+3}{2},\frac{1}{4}(-2\nu-1),\frac{1}{4}(2\nu+5)\\-\frac{\nu}{2},\frac{\nu+4}{2},\frac{1}{4}(-2\nu-1),\frac{\nu}{2},\frac{3\nu}{2}+2,\frac{1}{4}(2\nu+5)\end{array}\right)-2z^\nu(\nu+1)\right)/;\;-\frac{\pi}{2}<\arg(z)\leq\frac{\pi}{2}$$

03.04.26.0032.01

$$K_\nu(z)\,{}_0F_1\left(;\nu+1;-\frac{z^2}{4}\right)=\frac{2^{-\frac{\nu}{2}-2}\,\Gamma(\nu+1)}{\sqrt{\pi}}\,G_{0,4}^{3,0}\left(\frac{z^4}{64}\,\middle|\,\frac{2-\nu}{4},-\frac{\nu}{4},\frac{\nu}{4},-\frac{3\nu}{4}\right)/;\;-\frac{\pi}{2}<\arg(z)\leq\frac{\pi}{2}$$

03.04.26.0033.01

$$K_\nu(z)\,{}_0F_1\left(;1-\nu;-\frac{z^2}{4}\right)=\frac{2^{\frac{\nu}{2}-2}\,\Gamma(1-\nu)}{\sqrt{\pi}}\,G_{0,4}^{3,0}\left(\frac{z^4}{64}\,\middle|\,\frac{\nu+2}{4},-\frac{\nu}{4},\frac{\nu}{4},\frac{3\nu}{4}\right)/;\;-\frac{\pi}{2}<\arg(z)\leq\frac{\pi}{2}$$

03.04.26.0083.01

$$K_\nu\left(2\sqrt{z}\right)\,{}_0F_1(;b;z)=\frac{2^{b-2}\,\Gamma(b)}{\sqrt{\pi}}\,G_{2,4}^{2,2}\left(4z\,\middle|\,\begin{array}{c}\frac{1-b}{2},1-\frac{b}{2}\\-\frac{\nu}{2},\frac{\nu}{2},-b-\frac{\nu}{2}+1,-b+\frac{\nu}{2}+1\end{array}\right)/;\;-b-\nu\notin\mathbb{N}\land-b+\nu\notin\mathbb{N}$$

03.04.26.0084.01

$$K_\nu\left(2\sqrt{z}\right)\,{}_0F_1(;\nu-n;z)=2^{-n-\frac{3}{2}}\sqrt{\pi}\;z^{-\frac{\nu}{2}}\,\Gamma(\nu-n)\left(\sqrt{2}\,\csc(\nu\pi)\sum_{k=0}^{\left\lfloor\frac{n}{2}\right\rfloor}\frac{(-1)^{\left\lfloor\frac{n+1}{2}\right\rfloor}4^k z^k\,\Gamma\left(k-n+\left\lfloor\frac{n}{2}\right\rfloor+\frac{1}{2}\right)\left(1-k+\left\lfloor\frac{n}{2}\right\rfloor\right)_{n-\left\lfloor\frac{n}{2}\right\rfloor}}{k!\,\Gamma(k-\nu+1)\,\Gamma(k-n+\nu)}-2^\nu\pi z^{\nu/2}\csc\left(\frac{1}{4}\pi\left(4\nu+(-1)^n\right)\right)G_{4,6}^{2,2}\left(4z\,\middle|\,\begin{array}{c}\frac{1}{2}(n-\nu+1),\frac{1}{2}(n-\nu+2),\frac{1}{4}(-2n+2\nu+1),\frac{1}{4}(2n-2\nu+3)\\n-\frac{\nu}{2}+1,\frac{\nu}{2},\frac{1}{4}(-2n+2\nu+1),\frac{1}{4}(2n-2\nu+3),-\frac{\nu}{2},n-\frac{3\nu}{2}+1\end{array}\right)\right)/;\;n\in\mathbb{N}$$

03.04.26.0085.01

$$K_\nu\left(2\sqrt{z}\right)\,{}_0F_1(;-n-\nu;z)=2^{-n-\frac{3}{2}}\sqrt{\pi}\;z^{\nu/2}\,\Gamma(-n-\nu)\left(2^{-\nu}\pi z^{-\frac{\nu}{2}}\csc\left(\frac{1}{4}\pi\left(4\nu-(-1)^n\right)\right)G_{4,6}^{2,2}\left(4z\,\middle|\,\begin{array}{c}\frac{1}{2}(n+\nu+1),\frac{1}{2}(n+\nu+2),\frac{1}{4}(-2n-2\nu+1),\frac{1}{4}(2n+2\nu+3)\\n+\frac{\nu}{2}+1,-\frac{\nu}{2},\frac{1}{4}(-2n-2\nu+1),\frac{1}{4}(2n+2\nu+3),\frac{\nu}{2},n+\frac{3\nu}{2}+1\end{array}\right)-\sqrt{2}\,\csc(\nu\pi)\sum_{k=0}^{\left\lfloor\frac{n}{2}\right\rfloor}\frac{(-1)^{\left\lfloor\frac{n+1}{2}\right\rfloor}4^k z^k\,\Gamma\left(k-n+\left\lfloor\frac{n}{2}\right\rfloor+\frac{1}{2}\right)\left(1-k+\left\lfloor\frac{n}{2}\right\rfloor\right)_{n-\left\lfloor\frac{n}{2}\right\rfloor}}{k!\,\Gamma(k+\nu+1)\,\Gamma(k-n-\nu)}\right)/;\;n\in\mathbb{N}$$

03.04.26.0086.01

$$K_\nu\left(2\sqrt{z}\right){}_0F_1(;\nu;z) = \frac{1}{2}\Gamma(\nu)\left(z^{-\frac{\nu}{2}} - 2^{\nu-\frac{1}{2}}\pi^{3/2}\csc\left(\frac{1}{4}(4\nu+1)\pi\right)G_{4,6}^{2,2}\left(4z\,\middle|\,\begin{matrix}\frac{1-\nu}{2}, 1-\frac{\nu}{2}, \frac{1}{4}(3-2\nu), \frac{1}{4}(2\nu+1)\\ 1-\frac{\nu}{2}, \frac{\nu}{2}, \frac{1}{4}(3-2\nu), 1-\frac{3\nu}{2}, -\frac{\nu}{2}, \frac{1}{4}(2\nu+1)\end{matrix}\right)\right)$$

03.04.26.0087.01

$$K_\nu\left(2\sqrt{z}\right){}_0F_1(;\nu-1;z) = \frac{\Gamma(\nu-1)}{4\sqrt{2}}\left(2\sqrt{2}(\nu-1)z^{-\frac{\nu}{2}} + 2^\nu\pi^{3/2}\csc\left(\pi\left(\nu-\frac{5}{4}\right)\right)G_{4,6}^{2,2}\left(4z\,\middle|\,\begin{matrix}1-\frac{\nu}{2}, \frac{3}{2}-\frac{\nu}{2}, \frac{5}{4}-\frac{\nu}{2}, \frac{\nu}{2}-\frac{1}{4}\\ 2-\frac{\nu}{2}, \frac{\nu}{2}, 2-\frac{3\nu}{2}, \frac{5}{4}-\frac{\nu}{2}, -\frac{\nu}{2}, \frac{\nu}{2}-\frac{1}{4}\end{matrix}\right)\right)$$

03.04.26.0088.01

$$K_\nu\left(2\sqrt{z}\right){}_0F_1(;-\nu;z) = \frac{1}{2}\Gamma(-\nu)\left(z^{\nu/2} + 2^{-\nu-\frac{1}{2}}\pi^{3/2}\csc\left(\pi\left(\nu-\frac{1}{4}\right)\right)G_{4,6}^{2,2}\left(4z\,\middle|\,\begin{matrix}\frac{\nu+1}{2}, \frac{\nu+2}{2}, \frac{1}{4}(1-2\nu), \frac{1}{4}(2\nu+3)\\ -\frac{\nu}{2}, \frac{\nu+2}{2}, \frac{1}{4}(1-2\nu), \frac{\nu}{2}, \frac{3\nu}{2}+1, \frac{1}{4}(2\nu+3)\end{matrix}\right)\right)$$

03.04.26.0089.01

$$K_\nu\left(2\sqrt{z}\right){}_0F_1(;-\nu-1;z) =$$

$$-2^{-\nu-\frac{5}{2}}\Gamma(-\nu-1)\left(2^{\nu+\frac{3}{2}}(\nu+1)z^{\nu/2} + \pi^{3/2}\csc\left(\pi\left(\nu+\frac{5}{4}\right)\right)G_{4,6}^{2,2}\left(4z\,\middle|\,\begin{matrix}\frac{\nu+2}{2}, \frac{\nu+3}{2}, \frac{1}{4}(-2\nu-1), \frac{1}{4}(2\nu+5)\\ -\frac{\nu}{2}, \frac{\nu+4}{2}, \frac{1}{4}(-2\nu-1), \frac{\nu}{2}, \frac{3\nu}{2}+2, \frac{1}{4}(2\nu+5)\end{matrix}\right)\right)$$

03.04.26.0090.01

$$K_\nu\left(2\sqrt{z}\right){}_0F_1(;\nu+1;-z) = \frac{2^{\frac{1}{2}(-\nu-4)}\Gamma(\nu+1)}{\sqrt{\pi}}G_{0,4}^{3,0}\left(\frac{z^2}{4}\,\middle|\,\frac{2-\nu}{4}, -\frac{\nu}{4}, \frac{\nu}{4}, -\frac{1}{4}(3\nu)\right) /; -\frac{\pi}{2} < \arg(z) \leq \frac{\pi}{2}$$

03.04.26.0091.01

$$K_\nu\left(2\sqrt{z}\right){}_0F_1(;1-\nu;-z) = \frac{2^{\frac{\nu-4}{2}}\Gamma(1-\nu)}{\sqrt{\pi}}G_{0,4}^{3,0}\left(\frac{z^2}{4}\,\middle|\,-\frac{\nu}{4}, \frac{\nu}{4}, \frac{\nu+2}{4}, \frac{3\nu}{4}\right) /; -\frac{\pi}{2} < \arg(z) \leq \frac{\pi}{2}$$

Classical cases involving ${}_0\tilde{F}_1$

03.04.26.0034.01

$$K_\nu(z)\,{}_0\tilde{F}_1\left(;b;\frac{z^2}{4}\right) = \frac{2^{b-2}}{\sqrt{\pi}}G_{2,4}^{2,2}\left(z^2\,\middle|\,\begin{matrix}\frac{1-b}{2}, 1-\frac{b}{2}\\ -\frac{\nu}{2}, \frac{\nu}{2}, 1-b-\frac{\nu}{2}, 1-b+\frac{\nu}{2}\end{matrix}\right) /; -b-\nu \notin \mathbb{N} \bigwedge \nu-b \notin \mathbb{N} \bigwedge -\frac{\pi}{2} < \arg(z) \leq \frac{\pi}{2}$$

03.04.26.0092.01

$$K_\nu(z)\,{}_0\tilde{F}_1\left(;-n-\nu;\frac{z^2}{4}\right) =$$

$$2^{-n-\nu-\frac{3}{2}}\pi\left(\sqrt{\pi}\csc\left(\frac{1}{4}\pi(4\nu-(-1)^n)\right)G_{4,6}^{2,2}\left(z^2\,\middle|\,\begin{matrix}\frac{1}{2}(n+\nu+1), \frac{1}{2}(n+\nu+2), \frac{1}{4}(-2n-2\nu+1), \frac{1}{4}(2n+2\nu+3)\\ n+\frac{\nu}{2}+1, -\frac{\nu}{2}, \frac{1}{4}(-2n-2\nu+1), \frac{1}{4}(2n+2\nu+3), \frac{\nu}{2}, n+\frac{3\nu}{2}+1\end{matrix}\right) -\right.$$

$$\left.\sqrt{\frac{2}{\pi}}\csc(\nu\pi)\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\frac{(-1)^{\lfloor\frac{n+1}{2}\rfloor}z^{2k+\nu}\Gamma\left(k-n+\lfloor\frac{n}{2}\rfloor+\frac{1}{2}\right)\left(1-k+\lfloor\frac{n}{2}\rfloor\right)_{n-\lfloor\frac{n}{2}\rfloor}}{k!\,\Gamma(k-n-\nu)\,\Gamma(k+\nu+1)}\right) /; n \in \mathbb{N} \bigwedge -\frac{\pi}{2} < \arg(z) \leq \frac{\pi}{2}$$

03.04.26.0093.01

$$K_\nu(z)\,{}_0\tilde{F}_1\left(;\nu-n;\frac{z^2}{4}\right)=$$
$$2^{-n+\nu-\frac{3}{2}}\,\pi\left(\sqrt{\frac{2}{\pi}}\,\csc(\nu\pi)\sum_{k=0}^{\left\lfloor\frac{n}{2}\right\rfloor}\frac{(-1)^{\left\lfloor\frac{n+1}{2}\right\rfloor}z^{2k-\nu}\,\Gamma\left(k-n+\left\lfloor\frac{n}{2}\right\rfloor+\frac{1}{2}\right)\left(1-k+\left\lfloor\frac{n}{2}\right\rfloor\right)_{n-\left\lfloor\frac{n}{2}\right\rfloor}}{k!\,\Gamma(k-n+\nu)\,\Gamma(k-\nu+1)}-\sqrt{\pi}\,\csc\left(\frac{1}{4}\pi\left(4\nu+(-1)^n\right)\right)\right.$$
$$\left.G_{4,6}^{2,2}\left(z^2\,\middle|\,\begin{array}{l}\frac{1}{2}(n-\nu+1),\,\frac{1}{2}(n-\nu+2),\,\frac{1}{4}(-2n+2\nu+1),\,\frac{1}{4}(2n-2\nu+3)\\ n-\frac{\nu}{2}+1,\,\frac{\nu}{2},\,\frac{1}{4}(-2n+2\nu+1),\,\frac{1}{4}(2n-2\nu+3),\,-\frac{\nu}{2},\,n-\frac{3\nu}{2}+1\end{array}\right)\right)\,/;\,n\in\mathbb{N}\bigwedge-\frac{\pi}{2}<\arg(z)\leq\frac{\pi}{2}$$

03.04.26.0094.01

$$K_\nu(z)\,{}_0\tilde{F}_1\left(;\nu;\frac{z^2}{4}\right)=2^{\nu-1}\left(z^{-\nu}-\frac{\pi^{3/2}\csc\left(\frac{1}{4}(4\nu+1)\pi\right)}{\sqrt{2}}G_{4,6}^{2,2}\left(z^2\,\middle|\,\begin{array}{c}\frac{1-\nu}{2},\,1-\frac{\nu}{2},\,\frac{1}{4}(3-2\nu),\,\frac{1}{4}(2\nu+1)\\ 1-\frac{\nu}{2},\,\frac{\nu}{2},\,\frac{1}{4}(3-2\nu),\,1-\frac{3\nu}{2},\,-\frac{\nu}{2},\,\frac{1}{4}(2\nu+1)\end{array}\right)\right)\,/;$$
$$-\frac{\pi}{2}<\arg(z)\leq\frac{\pi}{2}$$

03.04.26.0095.01

$$K_\nu(z)\,{}_0\tilde{F}_1\left(;\nu-1;\frac{z^2}{4}\right)=2^{\nu-2}\left(2(\nu-1)z^{-\nu}+\frac{\pi^{3/2}\csc\left(\pi\left(\nu-\frac{5}{4}\right)\right)}{\sqrt{2}}G_{4,6}^{2,2}\left(z^2\,\middle|\,\begin{array}{c}\frac{3-\nu}{2},\,1-\frac{\nu}{2},\,\frac{1}{4}(5-2\nu),\,\frac{1}{4}(2\nu-1)\\ 2-\frac{\nu}{2},\,\frac{\nu}{2},\,\frac{1}{4}(5-2\nu),\,2-\frac{3\nu}{2},\,-\frac{\nu}{2},\,\frac{1}{4}(2\nu-1)\end{array}\right)\right)\,/;$$
$$-\frac{\pi}{2}<\arg(z)\leq\frac{\pi}{2}$$

03.04.26.0096.01

$$K_\nu(z)\,{}_0\tilde{F}_1\left(;-\nu;\frac{z^2}{4}\right)=2^{-\nu-1}\left(z^{\nu}-\frac{\pi^{3/2}\csc\left(\frac{1}{4}\pi(1-4\nu)\right)}{\sqrt{2}}G_{4,6}^{2,2}\left(z^2\,\middle|\,\begin{array}{c}\frac{\nu+1}{2},\,\frac{\nu+2}{2},\,\frac{1}{4}(1-2\nu),\,\frac{1}{4}(2\nu+3)\\ -\frac{\nu}{2},\,\frac{\nu+2}{2},\,\frac{1}{4}(1-2\nu),\,\frac{\nu}{2},\,\frac{3\nu}{2}+1,\,\frac{1}{4}(2\nu+3)\end{array}\right)\right)\,/;$$
$$-\frac{\pi}{2}<\arg(z)\leq\frac{\pi}{2}$$

03.04.26.0097.01

$$K_\nu(z)\,{}_0\tilde{F}_1\left(;-\nu-1;\frac{z^2}{4}\right)=2^{-\nu-2}\left(\frac{\pi^{3/2}\csc\left(\pi\left(\nu+\frac{1}{4}\right)\right)}{\sqrt{2}}G_{4,6}^{2,2}\left(z^2\,\middle|\,\begin{array}{c}\frac{\nu+2}{2},\,\frac{\nu+3}{2},\,\frac{1}{4}(-2\nu-1),\,\frac{1}{4}(2\nu+5)\\ -\frac{\nu}{2},\,\frac{\nu+4}{2},\,\frac{1}{4}(-2\nu-1),\,\frac{\nu}{2},\,\frac{3\nu}{2}+2,\,\frac{1}{4}(2\nu+5)\end{array}\right)-2z^{\nu}(\nu+1)\right)\,/;$$
$$-\frac{\pi}{2}<\arg(z)\leq\frac{\pi}{2}$$

03.04.26.0035.01

$$K_\nu(z)\,{}_0\tilde{F}_1\left(;\nu+1;-\frac{z^2}{4}\right)=\frac{2^{-\frac{\nu}{2}-2}}{\sqrt{\pi}}G_{0,4}^{3,0}\left(\frac{z^4}{64}\,\middle|\,\frac{2-\nu}{4},\,-\frac{\nu}{4},\,\frac{\nu}{4},\,-\frac{3\nu}{4}\right)\,/;\,-\frac{\pi}{2}<\arg(z)\leq\frac{\pi}{2}$$

03.04.26.0036.01

$$K_\nu(z)\,{}_0\tilde{F}_1\left(;1-\nu;-\frac{z^2}{4}\right)=\frac{2^{\frac{\nu}{2}-2}}{\sqrt{\pi}}G_{0,4}^{3,0}\left(\frac{z^4}{64}\,\middle|\,\frac{\nu+2}{4},\,-\frac{\nu}{4},\,\frac{\nu}{4},\,\frac{3\nu}{4}\right)\,/;\,-\frac{\pi}{2}<\arg(z)\leq\frac{\pi}{2}$$

03.04.26.0098.01

$$K_\nu\left(2\sqrt{z}\right){}_0\tilde{F}_1(;b;z)=\frac{2^{b-2}}{\sqrt{\pi}}\,G_{2,4}^{2,2}\left(4z\,\middle|\,\begin{array}{c}\frac{1-b}{2},1-\frac{b}{2}\\ -\frac{\nu}{2},\frac{\nu}{2},-b-\frac{\nu}{2}+1,-b+\frac{\nu}{2}+1\end{array}\right)/;-b-\nu\notin\mathbb{N}\bigwedge -b+\nu\notin\mathbb{N}$$

03.04.26.0099.01

$$K_\nu\left(2\sqrt{z}\right){}_0\tilde{F}_1(;\nu-n;z)=2^{-n-\frac{3}{2}}\sqrt{\pi}\,z^{-\frac{\nu}{2}}\left(\sqrt{2}\csc(\nu\pi)\sum_{k=0}^{\left\lfloor\frac{n}{2}\right\rfloor}\frac{(-1)^{\left\lfloor\frac{n+1}{2}\right\rfloor}4^k z^k\,\Gamma\left(k-n+\left\lfloor\frac{n}{2}\right\rfloor+\frac{1}{2}\right)\left(1-k+\left\lfloor\frac{n}{2}\right\rfloor\right)_{n-\left\lfloor\frac{n}{2}\right\rfloor}}{k!\,\Gamma(k-\nu+1)\,\Gamma(k-n+\nu)}-\right.$$

$$\left.2^\nu\pi z^{\nu/2}\csc\left(\frac{1}{4}\pi\left(4\nu+(-1)^n\right)\right)G_{4,6}^{2,2}\left(4z\,\middle|\,\begin{array}{c}\frac{1}{2}(n-\nu+1),\frac{1}{2}(n-\nu+2),\frac{1}{4}(-2n+2\nu+1),\frac{1}{4}(2n-2\nu+3)\\ n-\frac{\nu}{2}+1,\frac{\nu}{2},\frac{1}{4}(-2n+2\nu+1),\frac{1}{4}(2n-2\nu+3),-\frac{\nu}{2},n-\frac{3\nu}{2}+1\end{array}\right)\right)/;n\in\mathbb{N}$$

03.04.26.0100.01

$$K_\nu\left(2\sqrt{z}\right){}_0\tilde{F}_1(;-n-\nu;z)=2^{-n-\frac{3}{2}}\sqrt{\pi}$$

$$z^{\nu/2}\left(2^{-\nu}\pi z^{-\frac{\nu}{2}}\csc\left(\frac{1}{4}\pi\left(4\nu-(-1)^n\right)\right)G_{4,6}^{2,2}\left(4z\,\middle|\,\begin{array}{c}\frac{1}{2}(n+\nu+1),\frac{1}{2}(n+\nu+2),\frac{1}{4}(-2n-2\nu+1),\frac{1}{4}(2n+2\nu+3)\\ n+\frac{\nu}{2}+1,-\frac{\nu}{2},\frac{1}{4}(-2n-2\nu+1),\frac{1}{4}(2n+2\nu+3),\frac{\nu}{2},n+\frac{3\nu}{2}+1\end{array}\right)-\right.$$

$$\left.\sqrt{2}\csc(\nu\pi)\sum_{k=0}^{\left\lfloor\frac{n}{2}\right\rfloor}\frac{(-1)^{\left\lfloor\frac{n+1}{2}\right\rfloor}4^k z^k\,\Gamma\left(k-n+\left\lfloor\frac{n}{2}\right\rfloor+\frac{1}{2}\right)\left(1-k+\left\lfloor\frac{n}{2}\right\rfloor\right)_{n-\left\lfloor\frac{n}{2}\right\rfloor}}{k!\,\Gamma(k+\nu+1)\,\Gamma(k-n-\nu)}\right)/;n\in\mathbb{N}$$

03.04.26.0101.01

$$K_\nu\left(2\sqrt{z}\right){}_0\tilde{F}_1(;\nu;z)=\frac{z^{-\frac{\nu}{2}}}{2}-2^{\nu-\frac{3}{2}}\pi^{3/2}\csc\left(\frac{1}{4}(4\nu+1)\pi\right)G_{4,6}^{2,2}\left(4z\,\middle|\,\begin{array}{c}\frac{1-\nu}{2},1-\frac{\nu}{2},\frac{1}{4}(3-2\nu),\frac{1}{4}(2\nu+1)\\ 1-\frac{\nu}{2},\frac{\nu}{2},\frac{1}{4}(3-2\nu),1-\frac{3\nu}{2},-\frac{\nu}{2},\frac{1}{4}(2\nu+1)\end{array}\right)$$

03.04.26.0102.01

$$K_\nu\left(2\sqrt{z}\right){}_0\tilde{F}_1(;\nu-1;z)=\frac{1}{2}(\nu-1)\,z^{-\frac{\nu}{2}}+2^{\nu-\frac{5}{2}}\pi^{3/2}\csc\left(\pi\left(\nu-\frac{5}{4}\right)\right)G_{4,6}^{2,2}\left(4z\,\middle|\,\begin{array}{c}\frac{3-\nu}{2},1-\frac{\nu}{2},\frac{1}{4}(5-2\nu),\frac{1}{4}(2\nu-1)\\ 2-\frac{\nu}{2},\frac{\nu}{2},\frac{1}{4}(5-2\nu),2-\frac{3\nu}{2},-\frac{\nu}{2},\frac{1}{4}(2\nu-1)\end{array}\right)$$

03.04.26.0103.01

$$K_\nu\left(2\sqrt{z}\right){}_0\tilde{F}_1(;-\nu;z)=\frac{z^{\nu/2}}{2}+2^{-\nu-\frac{3}{2}}\pi^{3/2}\csc\left(\pi\left(\nu-\frac{1}{4}\right)\right)G_{4,6}^{2,2}\left(4z\,\middle|\,\begin{array}{c}\frac{\nu+1}{2},\frac{\nu+2}{2},\frac{1}{4}(1-2\nu),\frac{1}{4}(2\nu+3)\\ -\frac{\nu}{2},\frac{\nu+2}{2},\frac{1}{4}(1-2\nu),\frac{\nu}{2},\frac{3\nu}{2}+1,\frac{1}{4}(2\nu+3)\end{array}\right)$$

03.04.26.0104.01

$$K_\nu\left(2\sqrt{z}\right){}_0\tilde{F}_1(;-\nu-1;-z)=$$

$$-\frac{1}{2}(\nu+1)\,z^{\nu/2}-2^{-\nu-\frac{5}{2}}\pi^{3/2}\csc\left(\pi\left(\nu+\frac{5}{4}\right)\right)G_{4,6}^{2,2}\left(4z\,\middle|\,\begin{array}{c}\frac{\nu+2}{2},\frac{\nu+3}{2},\frac{1}{4}(-2\nu-1),\frac{1}{4}(2\nu+5)\\ -\frac{\nu}{2},\frac{\nu+4}{2},\frac{1}{4}(-2\nu-1),\frac{\nu}{2},\frac{3\nu}{2}+2,\frac{1}{4}(2\nu+5)\end{array}\right)$$

03.04.26.0105.01

$$K_\nu\left(2\sqrt{z}\right){}_0\tilde{F}_1(;\nu+1;-z)=\frac{2^{\frac{1}{2}(-\nu-4)}}{\sqrt{\pi}}\,G_{0,4}^{3,0}\left(\frac{z^2}{4}\,\middle|\,\frac{2-\nu}{4},-\frac{\nu}{4},\frac{\nu}{4},-\frac{1}{4}(3\nu)\right)/;-\frac{\pi}{2}<\arg(z)\leq\frac{\pi}{2}$$

03.04.26.0106.01

$$K_\nu\left(2\sqrt{z}\right)\,{}_0\tilde{F}_1(;1-\nu;-z)=\frac{2^{\frac{\nu-4}{2}}}{\sqrt{\pi}}\,G_{0,4}^{3,0}\left(\frac{z^2}{4}\,\middle|\,-\frac{\nu}{4},\frac{\nu}{4},\frac{\nu+2}{4},\frac{3\nu}{4}\right)\;/;\;-\frac{\pi}{2}<\arg(z)\leq\frac{\pi}{2}$$

Generalized cases for the direct function itself

03.04.26.0037.01

$$K_\nu(z)=\frac{1}{2}\,G_{0,2}^{2,0}\left(\frac{z}{2},\frac{1}{2}\,\middle|\,\frac{\nu}{2},-\frac{\nu}{2}\right)$$

Generalized cases involving cosh

03.04.26.0038.01

$$\cosh(z)\,K_\nu(z)=\frac{1}{2\sqrt{2}}\,G_{2,4}^{2,2}\left(z,\frac{1}{2}\,\middle|\,\begin{matrix}\frac{1}{4},\frac{3}{4}\\-\frac{\nu}{2},\frac{\nu}{2},\frac{1-\nu}{2},\frac{\nu+1}{2}\end{matrix}\right)$$

Generalized cases involving sinh

03.04.26.0039.01

$$\sinh(z)\,K_\nu(z)=\frac{1}{2\sqrt{2}}\,G_{2,4}^{2,2}\left(z,\frac{1}{2}\,\middle|\,\begin{matrix}\frac{1}{4},\frac{3}{4}\\\frac{1-\nu}{2},\frac{\nu+1}{2},-\frac{\nu}{2},\frac{\nu}{2}\end{matrix}\right)$$

Generalized cases involving Ai

03.04.26.0040.01

$$\mathrm{Ai}\left(\left(\frac{3}{2}\right)^{2/3}z^{2/3}\right)K_\nu(z)=\frac{1}{2\sqrt[3]{2}\,\sqrt[6]{3}\,\sqrt{\pi}}\,G_{2,4}^{4,0}\left(z^{2/3},\frac{1}{3}\,\middle|\,\begin{matrix}\frac{1}{6},\frac{2}{3}\\-\frac{\nu}{2},\frac{1}{6}(2-3\nu),\frac{\nu}{2},\frac{1}{6}(3\nu+2)\end{matrix}\right)$$

03.04.26.0107.01

$$\mathrm{Ai}(z)\,K_\nu\left(\frac{2z^{3/2}}{3}\right)=\frac{1}{2\sqrt[3]{2}\,\sqrt[6]{3}\,\sqrt{\pi}}\,G_{2,4}^{4,0}\left(\left(\frac{2}{3}\right)^{2/3}z,\frac{1}{3}\,\middle|\,\begin{matrix}\frac{1}{6},\frac{2}{3}\\-\frac{\nu}{2},\frac{1}{6}(2-3\nu),\frac{\nu}{2},\frac{1}{6}(3\nu+2)\end{matrix}\right)\;/;\;-\frac{1}{3}(2\pi)<\arg(z)\leq\frac{2\pi}{3}$$

Generalized cases involving Ai′

03.04.26.0067.01

$$\mathrm{Ai}'\left(\left(\frac{3}{2}\right)^{2/3}z^{2/3}\right)K_\nu(z)=-\frac{\sqrt[6]{3}}{2\,2^{2/3}\sqrt{\pi}}\,G_{2,4}^{4,0}\left(z^{2/3},\frac{1}{3}\,\middle|\,\begin{matrix}\frac{1}{3},\frac{5}{6}\\\frac{\nu}{2},-\frac{\nu}{2},\frac{1}{6}(3\nu+4),\frac{1}{6}(4-3\nu)\end{matrix}\right)$$

03.04.26.0041.01

$$\mathrm{Ai}'\left(\left(\frac{3}{2}\right)^{2/3}z^{2/3}\right)K_\nu(z)=$$

$$-\frac{\sqrt[6]{3}\,\csc(\pi\nu)}{4\,2^{2/3}\sqrt{\pi}}\left(G_{2,4}^{2,2}\left(z,\frac{1}{2}\,\middle|\,\begin{matrix}\frac{1}{3},\frac{5}{6}\\-\frac{\nu}{2},\frac{1}{6}(4-3\nu),\frac{\nu}{2},\frac{1}{6}(3\nu+4)\end{matrix}\right)-G_{2,4}^{2,2}\left(z,\frac{1}{2}\,\middle|\,\begin{matrix}\frac{1}{3},\frac{5}{6}\\\frac{\nu}{2},\frac{1}{6}(3\nu+4),-\frac{\nu}{2},\frac{1}{6}(4-3\nu)\end{matrix}\right)\right)$$

03.04.26.0108.01

$$\mathrm{Ai}'(z)\,K_\nu\left(\frac{2z^{3/2}}{3}\right)=-\frac{\sqrt[6]{3}}{2\,2^{2/3}\sqrt{\pi}}\,G_{2,4}^{4,0}\left(\left(\frac{2}{3}\right)^{2/3}z,\frac{1}{3}\,\middle|\,\begin{matrix}\frac{1}{3},\frac{5}{6}\\\frac{\nu}{2},-\frac{\nu}{2},\frac{1}{6}(3\nu+4),\frac{1}{6}(4-3\nu)\end{matrix}\right)\;/;\;-\frac{1}{3}(2\pi)<\arg(z)\leq\frac{2\pi}{3}$$

Generalized cases involving Bi

03.04.26.0042.01

$$\mathrm{Bi}\left(\left(\frac{3}{2}\right)^{2/3} z^{2/3}\right) K_\nu(z) = \frac{\pi^{3/2}\csc(\pi\nu)}{\sqrt[3]{2}\,\sqrt[6]{3}}\left(G_{4,6}^{2,2}\left(z, \frac{1}{2} \middle| \begin{matrix} \frac{1}{6}, \frac{2}{3}, \frac{1}{6}(1-3\nu), \frac{1}{6}(4-3\nu) \\ -\frac{\nu}{2}, \frac{1}{6}(2-3\nu), \frac{1}{6}(1-3\nu), \frac{1}{6}(4-3\nu), \frac{\nu}{2}, \frac{1}{6}(3\nu+2) \end{matrix}\right) - G_{4,6}^{2,2}\left(z, \frac{1}{2} \middle| \begin{matrix} \frac{1}{6}, \frac{2}{3}, \frac{1}{6}(3\nu+1), \frac{1}{6}(3\nu+4) \\ \frac{\nu}{2}, \frac{1}{6}(3\nu+2), -\frac{\nu}{2}, \frac{1}{6}(2-3\nu), \frac{1}{6}(3\nu+1), \frac{1}{6}(3\nu+4) \end{matrix}\right)\right)$$

03.04.26.0109.01

$$\mathrm{Bi}(z) K_\nu\left(\frac{2z^{3/2}}{3}\right) = \frac{\pi^{3/2}\csc(\pi\nu)}{\sqrt[3]{2}\,\sqrt[6]{3}}\left(G_{4,6}^{2,2}\left(\left(\frac{2}{3}\right)^{2/3} z, \frac{1}{3} \middle| \begin{matrix} \frac{1}{6}, \frac{2}{3}, \frac{1}{6}(1-3\nu), \frac{1}{6}(4-3\nu) \\ -\frac{\nu}{2}, \frac{1}{6}(2-3\nu), \frac{\nu}{2}, \frac{1}{6}(3\nu+2), \frac{1}{6}(1-3\nu), \frac{1}{6}(4-3\nu) \end{matrix}\right) - G_{4,6}^{2,2}\left(\left(\frac{2}{3}\right)^{2/3} z, \frac{1}{3} \middle| \begin{matrix} \frac{1}{6}, \frac{2}{3}, \frac{1}{6}(3\nu+1), \frac{1}{6}(3\nu+4) \\ \frac{\nu}{2}, \frac{1}{6}(3\nu+2), -\frac{\nu}{2}, \frac{1}{6}(2-3\nu), \frac{1}{6}(3\nu+1), \frac{1}{6}(3\nu+4) \end{matrix}\right)\right) /; -\frac{2\pi}{3} < \arg(z) \le \frac{2\pi}{3}$$

Generalized cases involving Bi′

03.04.26.0043.01

$$\mathrm{Bi}'\left(\left(\frac{3}{2}\right)^{2/3} z^{2/3}\right) K_\nu(z) = \frac{\sqrt[6]{3}\,\pi^{3/2}\csc(\pi\nu)}{2^{2/3}}\left(G_{4,6}^{2,2}\left(z^{2/3}, \frac{1}{3} \middle| \begin{matrix} \frac{1}{3}, \frac{5}{6}, \frac{1}{3}-\frac{\nu}{2}, \frac{5}{6}-\frac{\nu}{2} \\ -\frac{\nu}{2}, \frac{2}{3}-\frac{\nu}{2}, \frac{1}{3}-\frac{\nu}{2}, \frac{5}{6}-\frac{\nu}{2}, \frac{\nu}{2}, \frac{\nu}{2}+\frac{2}{3} \end{matrix}\right) - G_{4,6}^{2,2}\left(z^{2/3}, \frac{1}{3} \middle| \begin{matrix} \frac{1}{3}, \frac{5}{6}, \frac{\nu}{2}+\frac{1}{3}, \frac{\nu}{2}+\frac{5}{6} \\ \frac{\nu}{2}, \frac{\nu}{2}+\frac{2}{3}, -\frac{\nu}{2}, \frac{2}{3}-\frac{\nu}{2}, \frac{\nu}{2}+\frac{1}{3}, \frac{\nu}{2}+\frac{5}{6} \end{matrix}\right)\right)$$

03.04.26.0110.01

$$\mathrm{Bi}'(z) K_\nu\left(\frac{2z^{3/2}}{3}\right) = \frac{\sqrt[6]{3}\,\pi^{3/2}\csc(\pi\nu)}{2^{2/3}}\left(G_{4,6}^{2,2}\left(\left(\frac{2}{3}\right)^{2/3} z, \frac{1}{3} \middle| \begin{matrix} \frac{1}{3}, \frac{5}{6}, \frac{1}{3}-\frac{\nu}{2}, \frac{5}{6}-\frac{\nu}{2} \\ -\frac{\nu}{2}, \frac{2}{3}-\frac{\nu}{2}, \frac{1}{3}-\frac{\nu}{2}, \frac{5}{6}-\frac{\nu}{2}, \frac{\nu}{2}, \frac{\nu}{2}+\frac{2}{3} \end{matrix}\right) - G_{4,6}^{2,2}\left(\left(\frac{2}{3}\right)^{2/3} z, \frac{1}{3} \middle| \begin{matrix} \frac{1}{3}, \frac{5}{6}, \frac{\nu}{2}+\frac{1}{3}, \frac{\nu}{2}+\frac{5}{6} \\ \frac{\nu}{2}, \frac{\nu}{2}+\frac{2}{3}, -\frac{\nu}{2}, \frac{2}{3}-\frac{\nu}{2}, \frac{\nu}{2}+\frac{1}{3}, \frac{\nu}{2}+\frac{5}{6} \end{matrix}\right)\right) /; -\frac{2\pi}{3} < \arg(z) \le \frac{2\pi}{3}$$

Generalized cases for powers of Bessel K

03.04.26.0044.01

$$K_\nu(z)^2 = \frac{1}{2}\sqrt{\pi}\, G_{1,3}^{3,0}\left(z, \frac{1}{2} \middle| \begin{matrix} \frac{1}{2} \\ 0, -\nu, \nu \end{matrix}\right)$$

Generalized cases for products of Bessel K

03.04.26.0045.01

$$K_{\nu-1}(z) K_\nu(z) = \frac{1}{2}\sqrt{\pi}\, G_{1,3}^{3,0}\left(z, \frac{1}{2} \middle| \begin{matrix} 0 \\ -\frac{1}{2}, \nu-\frac{1}{2}, \frac{1}{2}-\nu \end{matrix}\right)$$

03.04.26.0046.01

$$K_\mu(z) K_\nu(z) = \frac{1}{2}\sqrt{\pi}\, G_{2,4}^{4,0}\left(z, \frac{1}{2} \middle| \begin{matrix} 0, \frac{1}{2} \\ \frac{\mu+\nu}{2}, \frac{\mu-\nu}{2}, \frac{\nu-\mu}{2}, -\frac{1}{2}(\mu+\nu) \end{matrix}\right)$$

03.04.26.0047.01

$$K_\nu(-z)\,K_\nu(z) = \frac{\cos(\nu\pi)}{2\sqrt{\pi}}\,G_{1,3}^{3,1}\!\left(\sqrt{-z^2}\,,\,\frac{1}{2}\,\middle|\,\begin{matrix}\frac{1}{2}\\ 0,\,-\nu,\,\nu\end{matrix}\right)$$

03.04.26.0048.01

$$K_\nu(z)\,K_\nu(-i\,z) = \frac{1}{8\sqrt{\pi}}\,G_{0,4}^{4,0}\!\left(\frac{e^{-\frac{\pi i}{4}}\,z}{2\sqrt{2}},\,\frac{1}{4}\,\middle|\,0,\,\frac{1}{2},\,-\frac{\nu}{2},\,\frac{\nu}{2}\right)/;\,-\frac{\pi}{2}<\arg(z)\leq\pi$$

Generalized cases involving Bessel *J*

03.04.26.0049.01

$$J_\nu(z)\,K_\nu(z) = \frac{1}{4\sqrt{\pi}}\,G_{0,4}^{3,0}\!\left(\frac{z}{2\sqrt{2}},\,\frac{1}{4}\,\middle|\,0,\,\frac{1}{2},\,\frac{\nu}{2},\,-\frac{\nu}{2}\right)$$

03.04.26.0111.01

$$J_{-\nu}(z)\,K_\nu(z) = \frac{1}{4\sqrt{\pi}}\,G_{0,4}^{3,0}\!\left(\frac{z}{2\sqrt{2}},\,\frac{1}{4}\,\middle|\,0,\,\frac{1}{2},\,-\frac{\nu}{2},\,\frac{\nu}{2}\right)$$

03.04.26.0050.01

$$(J_{-\nu}(z)+J_\nu(z))\,K_\nu(z) = \frac{\cos\left(\frac{\pi\nu}{2}\right)}{2\sqrt{\pi}}\,G_{0,4}^{3,0}\!\left(\frac{z}{2\sqrt{2}},\,\frac{1}{4}\,\middle|\,\frac{1}{2},\,-\frac{\nu}{2},\,\frac{\nu}{2},\,0\right)$$

03.04.26.0051.01

$$(J_{-\nu}(z)-J_\nu(z))\,K_\nu(z) = \frac{\sin\left(\frac{\pi\nu}{2}\right)}{2\sqrt{\pi}}\,G_{0,4}^{3,0}\!\left(\frac{z}{2\sqrt{2}},\,\frac{1}{4}\,\middle|\,0,\,-\frac{\nu}{2},\,\frac{\nu}{2},\,\frac{1}{2}\right)$$

Generalized cases involving Bessel *I*

03.04.26.0052.01

$$I_\nu(z)\,K_\nu(z) = \frac{1}{2\sqrt{\pi}}\,G_{1,3}^{2,1}\!\left(z,\,\frac{1}{2}\,\middle|\,\begin{matrix}\frac{1}{2}\\ 0,\,\nu,\,-\nu\end{matrix}\right)$$

03.04.26.0112.01

$$I_{-\nu}(z)\,K_\nu(z) = \frac{1}{2\sqrt{\pi}}\,G_{1,3}^{2,1}\!\left(z,\,\frac{1}{2}\,\middle|\,\begin{matrix}\frac{1}{2}\\ 0,\,-\nu,\,\nu\end{matrix}\right)$$

03.04.26.0053.01

$$I_\mu(z)\,K_\nu(z) = \frac{1}{2\sqrt{\pi}}\,G_{2,4}^{2,2}\!\left(z,\,\frac{1}{2}\,\middle|\,\begin{matrix}0,\,\frac{1}{2}\\ \frac{\mu-\nu}{2},\,\frac{\mu+\nu}{2},\,\frac{1}{2}(-\mu-\nu),\,\frac{\nu-\mu}{2}\end{matrix}\right)/;\,-\mu-\nu-1\notin\mathbb{N}\bigwedge\nu-\mu-1\notin\mathbb{N}$$

03.04.26.0113.01

$$I_\nu(z)\,K_{n+\nu+1}(z) = \frac{1}{\sqrt{2}}\left((-1)^n\,\pi^{3/2}\csc\!\left(\frac{1}{4}\,\pi\,(4\,\nu+(-1)^n)\right)\right)G_{4,6}^{2,2}\!\left(z,\,\frac{1}{2}\,\middle|\,\begin{matrix}0,\,\frac{1}{2},\,\nu+\frac{3}{4},\,\frac{1}{4}\\ \frac{n+1}{2},\,\frac{n+1}{2}+\nu,\,\nu+\frac{3}{4},\,\frac{1}{4},\,-\frac{1}{2}(n+1),\,-\frac{1}{2}(n+1)-\nu\end{matrix}\right)-$$

$$(-1)^n\sqrt{\pi}\,\csc(\nu\pi)\sum_{k=0}^{\left\lfloor\frac{n}{2}\right\rfloor}\frac{(-1)^{\left\lfloor\frac{n+1}{2}\right\rfloor}\,z^{2k-n-1}\,\Gamma\!\left(k-n+\left\lfloor\frac{n}{2}\right\rfloor+\frac{1}{2}\right)\left(1-k+\left\lfloor\frac{n}{2}\right\rfloor\right)_{n-\left\lfloor\frac{n}{2}\right\rfloor}}{k!\,\Gamma(k-n-\nu)\,\Gamma(k+\nu+1)}\;/;\,n\in\mathbb{N}$$

03.04.26.0114.01

$$I_\nu(z)\,K_{-n-\nu-1}(z)=\frac{1}{\sqrt{2}}\left((-1)^n\,\pi^{3/2}\csc\left(\frac{1}{4}\,\pi\,(4\,\nu+(-1)^n)\right)\right)G_{4,6}^{2,2}\left(z,\,\frac{1}{2}\,\middle|\,\begin{matrix}0,\,\frac{1}{2},\,\nu+\frac{3}{4},\,\frac{1}{4}\\ \frac{n+1}{2},\,\frac{n+1}{2}+\nu,\,\nu+\frac{3}{4},\,\frac{1}{4},\,-\frac{1}{2}\,(n+1),\,-\frac{1}{2}\,(n+1)-\nu\end{matrix}\right)-(-1)^n\,\sqrt{\pi}\,\csc(\nu\,\pi)\sum_{k=0}^{\left\lfloor\frac{n}{2}\right\rfloor}\frac{(-1)^{\left\lfloor\frac{n+1}{2}\right\rfloor}\,z^{2\,k-n-1}\,\Gamma\left(k-n+\left\lfloor\frac{n}{2}\right\rfloor+\frac{1}{2}\right)\left(1-k+\left\lfloor\frac{n}{2}\right\rfloor\right)_{n-\left\lfloor\frac{n}{2}\right\rfloor}}{k!\,\Gamma(k-n-\nu)\,\Gamma(k+\nu+1)}\;/;\;n\in\mathbb{N}$$

03.04.26.0115.01

$$I_\nu(z)\,K_{\nu+1}(z)=\frac{\pi^{3/2}}{\cos(\pi\,\nu)+\sin(\pi\,\nu)}\,G_{4,6}^{2,2}\left(z,\,\frac{1}{2}\,\middle|\,\begin{matrix}0,\,\frac{1}{2},\,\nu+\frac{3}{4},\,\frac{1}{4}\\ \frac{1}{2},\,\nu+\frac{1}{2},\,\nu+\frac{3}{4},\,\frac{1}{4},\,-\frac{1}{2},\,-\nu-\frac{1}{2}\end{matrix}\right)+\frac{1}{z}$$

03.04.26.0116.01

$$I_\nu(z)\,K_{\nu+2}(z)=\frac{2\,(\nu+1)}{z^2}+\frac{\pi^{3/2}\csc(\pi\,\nu)}{\cot(\pi\,\nu)-1}\,G_{4,6}^{2,2}\left(z,\,\frac{1}{2}\,\middle|\,\begin{matrix}0,\,\frac{1}{2},\,\nu+\frac{3}{4},\,\frac{1}{4}\\ 1,\,\nu+1,\,\nu+\frac{3}{4},\,\frac{1}{4},\,-1,\,-\nu-1\end{matrix}\right)$$

03.04.26.0054.01

$$(I_{-\nu}\,(z)+I_\nu\,(z))\,K_\nu\,(z)=\frac{\cos\,(\pi\,\nu)}{\sqrt{\pi}}\,G_{1,3}^{2,1}\left(z,\,\frac{1}{2}\,\middle|\,\begin{matrix}\frac{1}{2}\\ -\nu,\,\nu,\,0\end{matrix}\right)$$

03.04.26.0117.01

$$(I_{-\nu}(z)-I_\nu(z))\,K_\nu(z)=\frac{\sin(\pi\,\nu)}{\sqrt{\pi}}\,G_{1,3}^{3,0}\left(z,\,\frac{1}{2}\,\middle|\,\begin{matrix}\frac{1}{2}\\ 0,\,\nu,\,-\nu\end{matrix}\right)$$

03.04.26.0055.01

$$I_\nu\,(z)\,K_\mu\,(z)+I_\mu\,(z)\,K_\nu\,(z)=\frac{\cos\left(\frac{1}{2}\,\pi\,(\mu-\nu)\right)}{\sqrt{\pi}}\,G_{2,4}^{3,1}\left(z,\,\frac{1}{2}\,\middle|\,\begin{matrix}\frac{1}{2},\,0\\ \frac{\mu+\nu}{2},\,\frac{\mu-\nu}{2},\,\frac{\nu-\mu}{2},\,-\frac{\mu+\nu}{2}\end{matrix}\right)$$

03.04.26.0056.01

$$I_\nu\,(z)\,K_\mu\,(z)-I_\mu\,(z)\,K_\nu\,(z)=\frac{\sin\left(\frac{1}{2}\,\pi\,(\mu-\nu)\right)}{\sqrt{\pi}}\,G_{2,4}^{3,1}\left(z,\,\frac{1}{2}\,\middle|\,\begin{matrix}0,\,\frac{1}{2}\\ \frac{\mu+\nu}{2},\,\frac{\mu-\nu}{2},\,\frac{\nu-\mu}{2},\,-\frac{\mu+\nu}{2}\end{matrix}\right)$$

03.04.26.0118.01

$$I_\nu(z)^2-\frac{1}{\pi^2}\,K_\nu(z)^2=-\sec(\pi\,\nu)\,\sqrt{\pi}\;G_{3,5}^{3,0}\left(z,\,\frac{1}{2}\,\middle|\,\begin{matrix}\frac{1}{2},\,\frac{1}{4}\,(1-2\,\nu),\,\frac{1}{4}\,(3-2\,\nu)\\ 0,\,-\nu,\,\nu,\,\frac{1}{4}\,(1-2\,\nu),\,\frac{1}{4}\,(3-2\,\nu)\end{matrix}\right)$$

Generalized cases involving Bessel Y

03.04.26.0057.01

$$K_\nu\,(z)\,Y_\nu\,(z)=-\frac{1}{4\,\sqrt{\pi}}\,G_{1,5}^{4,0}\left(\frac{z}{2\,\sqrt{2}},\,\frac{1}{4}\,\middle|\,\begin{matrix}\frac{1-\nu}{2}\\ 0,\,\frac{1}{2},\,\frac{\nu}{2},\,-\frac{\nu}{2},\,\frac{1-\nu}{2}\end{matrix}\right)$$

03.04.26.0119.01

$$K_\nu(z)\,Y_{-\nu}(z)=-\frac{1}{4\,\sqrt{\pi}}\,G_{1,5}^{4,0}\left(\frac{z}{2\,\sqrt{2}},\,\frac{1}{4}\,\middle|\,\begin{matrix}\frac{\nu+1}{2}\\ 0,\,\frac{1}{2},\,\frac{\nu}{2},\,-\frac{\nu}{2},\,\frac{\nu+1}{2}\end{matrix}\right)$$

03.04.26.0058.01

$$K_0(z) - \frac{\pi}{2} Y_0(z) = \frac{\pi}{2} G_{0,4}^{2,0}\left(\frac{z}{4}, \frac{1}{4} \,\middle|\, 0, 0, \tfrac{1}{2}, \tfrac{1}{2}\right)$$

03.04.26.0059.01

$$K_n(z) + \frac{\pi}{2} Y_n(z) = \frac{\pi}{2} G_{1,5}^{3,0}\left(\frac{z}{4}, \frac{1}{4} \,\middle|\, \begin{matrix} -\frac{n}{4} \\ \frac{2-n}{4}, \frac{n}{4}, \frac{n+2}{4}, -\frac{n}{4}, -\frac{n}{4} \end{matrix}\right) /; n \in \mathbb{N}$$

03.04.26.0060.01

$$K_n(z) - \frac{\pi}{2} Y_n(z) = \frac{\pi}{2} G_{1,5}^{3,0}\left(\frac{z}{4}, \frac{1}{4} \,\middle|\, \begin{matrix} \frac{2-n}{4} \\ -\frac{n}{4}, \frac{n}{4}, \frac{n+2}{4}, \frac{2-n}{4}, \frac{2-n}{4} \end{matrix}\right) /; n \in \mathbb{N}$$

Generalized cases involving $_0F_1$

03.04.26.0062.01

$$K_\nu(z)\, _0F_1\left(; \nu+1; -\frac{z^2}{4}\right) = \frac{2^{-\frac{\nu}{2}-2}\,\Gamma(\nu+1)}{\sqrt{\pi}} G_{0,4}^{3,0}\left(\frac{z}{2\sqrt{2}}, \frac{1}{4} \,\middle|\, \frac{2-\nu}{4}, -\frac{\nu}{4}, \frac{\nu}{4}, -\frac{3\nu}{4}\right)$$

03.04.26.0063.01

$$K_\nu(z)\, _0F_1\left(; 1-\nu; -\frac{z^2}{4}\right) = \frac{2^{\frac{\nu}{2}-2}\,\Gamma(1-\nu)}{\sqrt{\pi}} G_{0,4}^{3,0}\left(\frac{z}{2\sqrt{2}}, \frac{1}{4} \,\middle|\, \frac{\nu+2}{4}, -\frac{\nu}{4}, \frac{\nu}{4}, \frac{3\nu}{4}\right)$$

03.04.26.0061.01

$$K_\nu(z)\, _0F_1\left(; b; \frac{z^2}{4}\right) = \frac{2^{b-2}\,\Gamma(b)}{\sqrt{\pi}} G_{2,4}^{2,2}\left(z, \frac{1}{2} \,\middle|\, \begin{matrix} \frac{1-b}{2}, 1-\frac{b}{2} \\ -\frac{\nu}{2}, \frac{\nu}{2}, 1-b-\frac{\nu}{2}, 1-b+\frac{\nu}{2} \end{matrix}\right) /; -b-\nu \notin \mathbb{N} \bigwedge \nu-b \notin \mathbb{N}$$

03.04.26.0120.01

$$K_\nu(z)\, _0F_1\left(; \nu-n; \frac{z^2}{4}\right) = 2^{-n+\nu-\frac{3}{2}}\, \pi\, \Gamma(\nu-n) \left(\sqrt{\frac{2}{\pi}}\, \csc(\nu\pi) \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{(-1)^{\left\lfloor \frac{n+1}{2} \right\rfloor} z^{2k-\nu}\, \Gamma\left(k-n+\left\lfloor \frac{n}{2} \right\rfloor + \frac{1}{2}\right) \left(1-k+\left\lfloor \frac{n}{2} \right\rfloor\right)_{n-\left\lfloor \frac{n}{2} \right\rfloor}}{k!\, \Gamma(k-\nu+1)\, \Gamma(k-n+\nu)} - \right.$$

$$\left. \sqrt{\pi}\, \csc\left(\frac{1}{4}\pi(4\nu+(-1)^n)\right) G_{4,6}^{2,2}\left(z, \frac{1}{2} \,\middle|\, \begin{matrix} \frac{1}{2}(n-\nu+1), \frac{1}{2}(n-\nu+2), \frac{1}{4}(-2n+2\nu+1), \frac{1}{4}(2n-2\nu+3) \\ n-\frac{\nu}{2}+1, \frac{\nu}{2}, \frac{1}{4}(-2n+2\nu+1), \frac{1}{4}(2n-2\nu+3), -\frac{\nu}{2}, n-\frac{3\nu}{2}+1 \end{matrix}\right)\right) /; n \in \mathbb{N}$$

03.04.26.0121.01

$$K_\nu(z)\, _0F_1\left(; -n-\nu; \frac{z^2}{4}\right) = 2^{-n-\nu-\frac{3}{2}}\, \pi\, \Gamma(-n-\nu)$$

$$\left(\sqrt{\pi}\, \csc\left(\frac{1}{4}\pi(4\nu-(-1)^n)\right) G_{4,6}^{2,2}\left(z, \frac{1}{2} \,\middle|\, \begin{matrix} \frac{1}{2}(n+\nu+1), \frac{1}{2}(n+\nu+2), \frac{1}{4}(-2n-2\nu+1), \frac{1}{4}(2n+2\nu+3) \\ n+\frac{\nu}{2}+1, -\frac{\nu}{2}, \frac{1}{4}(-2n-2\nu+1), \frac{1}{4}(2n+2\nu+3), \frac{\nu}{2}, n+\frac{3\nu}{2}+1 \end{matrix}\right) - \right.$$

$$\left. \sqrt{\frac{2}{\pi}}\, \csc(\nu\pi) \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{(-1)^{\left\lfloor \frac{n+1}{2} \right\rfloor} z^{2k+\nu}\, \Gamma\left(k-n+\left\lfloor \frac{n}{2} \right\rfloor + \frac{1}{2}\right) \left(1-k+\left\lfloor \frac{n}{2} \right\rfloor\right)_{n-\left\lfloor \frac{n}{2} \right\rfloor}}{k!\, \Gamma(k+\nu+1)\, \Gamma(k-n-\nu)}\right) /; n \in \mathbb{N}$$

03.04.26.0122.01

$$K_\nu(z)\,{}_0F_1\left(;\nu;\frac{z^2}{4}\right) = 2^{\nu-1}\,\Gamma(\nu)\left(z^{-\nu} - \frac{\pi^{3/2}\csc\left(\frac{1}{4}(4\nu+1)\pi\right)}{\sqrt{2}}\,G_{4,6}^{2,2}\left(z,\frac{1}{2}\,\middle|\,\begin{matrix}\frac{1-\nu}{2},1-\frac{\nu}{2},\frac{1}{4}(3-2\nu),\frac{1}{4}(2\nu+1)\\ 1-\frac{\nu}{2},\frac{\nu}{2},\frac{1}{4}(3-2\nu),1-\frac{3\nu}{2},-\frac{\nu}{2},\frac{1}{4}(2\nu+1)\end{matrix}\right)\right)$$

03.04.26.0123.01

$$K_\nu(z)\,{}_0F_1\left(;\nu-1;\frac{z^2}{4}\right) =$$

$$2^{\nu-2}\,\Gamma(\nu-1)\left(2(\nu-1)z^{-\nu} + \frac{\pi^{3/2}\csc\left(\pi\left(\nu-\frac{5}{4}\right)\right)}{\sqrt{2}}\,G_{4,6}^{2,2}\left(z,\frac{1}{2}\,\middle|\,\begin{matrix}\frac{3-\nu}{2},1-\frac{\nu}{2},\frac{1}{4}(5-2\nu),\frac{1}{4}(2\nu-1)\\ 2-\frac{\nu}{2},\frac{\nu}{2},\frac{1}{4}(5-2\nu),2-\frac{3\nu}{2},-\frac{\nu}{2},\frac{1}{4}(2\nu-1)\end{matrix}\right)\right)$$

03.04.26.0124.01

$$K_\nu(z)\,{}_0F_1\left(;-\nu;\frac{z^2}{4}\right) = 2^{-\nu-1}\,\Gamma(-\nu)\left(z^{\nu} - \frac{\pi^{3/2}\csc\left(\frac{1}{4}\pi(1-4\nu)\right)}{\sqrt{2}}\,G_{4,6}^{2,2}\left(z,\frac{1}{2}\,\middle|\,\begin{matrix}\frac{\nu+1}{2},\frac{\nu+2}{2},\frac{1}{4}(1-2\nu),\frac{1}{4}(2\nu+3)\\ -\frac{\nu}{2},\frac{\nu+2}{2},\frac{1}{4}(1-2\nu),\frac{\nu}{2},\frac{3\nu}{2}+1,\frac{1}{4}(2\nu+3)\end{matrix}\right)\right)$$

03.04.26.0125.01

$$K_\nu(z)\,{}_0F_1\left(;-\nu-1;\frac{z^2}{4}\right) =$$

$$2^{-\nu-2}\,\Gamma(-\nu-1)\left(\frac{\pi^{3/2}\csc\left(\pi\left(\nu+\frac{1}{4}\right)\right)}{\sqrt{2}}\,G_{4,6}^{2,2}\left(z,\frac{1}{2}\,\middle|\,\begin{matrix}\frac{\nu+2}{2},\frac{\nu+3}{2},\frac{1}{4}(-2\nu-1),\frac{1}{4}(2\nu+5)\\ -\frac{\nu}{2},\frac{\nu+4}{2},\frac{1}{4}(-2\nu-1),\frac{\nu}{2},\frac{3\nu}{2}+2,\frac{1}{4}(2\nu+5)\end{matrix}\right) - 2z^{\nu}(\nu+1)\right)$$

Generalized cases involving ${}_0\tilde{F}_1$

03.04.26.0065.01

$$K_\nu(z)\,{}_0\tilde{F}_1\left(;\nu+1;-\frac{z^2}{4}\right) = \frac{2^{-\frac{\nu}{2}-2}}{\sqrt{\pi}}\,G_{0,4}^{3,0}\left(\frac{z}{2\sqrt{2}},\frac{1}{4}\,\middle|\,\frac{2-\nu}{4},-\frac{\nu}{4},\frac{\nu}{4},-\frac{3\nu}{4}\right)$$

03.04.26.0066.01

$$K_\nu(z)\,{}_0\tilde{F}_1\left(;1-\nu;-\frac{z^2}{4}\right) = \frac{2^{\frac{\nu}{2}-2}}{\sqrt{\pi}}\,G_{0,4}^{3,0}\left(\frac{z}{2\sqrt{2}},\frac{1}{4}\,\middle|\,\frac{\nu+2}{4},-\frac{\nu}{4},\frac{\nu}{4},\frac{3\nu}{4}\right)$$

03.04.26.0064.01

$$K_\nu(z)\,{}_0\tilde{F}_1\left(;b;\frac{z^2}{4}\right) = \frac{2^{b-2}}{\sqrt{\pi}}\,G_{2,4}^{2,2}\left(z,\frac{1}{2}\,\middle|\,\begin{matrix}\frac{1-b}{2},1-\frac{b}{2}\\ -\frac{\nu}{2},\frac{\nu}{2},1-b-\frac{\nu}{2},1-b+\frac{\nu}{2}\end{matrix}\right)\,/;\,-b-\nu\notin\mathbb{N}\land\nu-b\notin\mathbb{N}$$

03.04.26.0126.01

$$K_\nu(z)\,{}_0\tilde{F}_1\left(;\nu-n;\frac{z^2}{4}\right) = 2^{-n+\nu-\frac{3}{2}}\,\pi\left(\sqrt{\frac{2}{\pi}}\csc(\nu\pi)\sum_{k=0}^{\left\lfloor\frac{n}{2}\right\rfloor}\frac{(-1)^{\left\lfloor\frac{n+1}{2}\right\rfloor}z^{2k-\nu}\,\Gamma\left(k-n+\left\lfloor\frac{n}{2}\right\rfloor+\frac{1}{2}\right)\left(1-k+\left\lfloor\frac{n}{2}\right\rfloor\right)_{n-\left\lfloor\frac{n}{2}\right\rfloor}}{k!\,\Gamma(k-\nu+1)\,\Gamma(k-n+\nu)} -\right.$$

$$\left.\sqrt{\pi}\,\csc\left(\frac{1}{4}\pi(4\nu+(-1)^n)\right)G_{4,6}^{2,2}\left(z,\frac{1}{2}\,\middle|\,\begin{matrix}\frac{1}{2}(n-\nu+1),\frac{1}{2}(n-\nu+2),\frac{1}{4}(-2n+2\nu+1),\frac{1}{4}(2n-2\nu+3)\\ n-\frac{\nu}{2}+1,\frac{\nu}{2},\frac{1}{4}(-2n+2\nu+1),\frac{1}{4}(2n-2\nu+3),-\frac{\nu}{2},n-\frac{3\nu}{2}+1\end{matrix}\right)\right)\,/;\,n\in\mathbb{N}$$

03.04.26.0127.01

$$K_\nu(z)\,{}_0\tilde{F}_1\left(;-n-\nu;\frac{z^2}{4}\right)=2^{-n-\nu-\frac{3}{2}}\pi\left(\sqrt{\pi}\csc\left(\frac{1}{4}\pi\left(4\nu-(-1)^n\right)\right)G_{4,6}^{2,2}\left(z,\frac{1}{2}\,\middle|\,\begin{matrix}\frac{1}{2}(n+\nu+1),\frac{1}{2}(n+\nu+2),\frac{1}{4}(-2n-2\nu+1),\frac{1}{4}(2n+2\nu+3)\\ n+\frac{\nu}{2}+1,-\frac{\nu}{2},\frac{1}{4}(-2n-2\nu+1),\frac{1}{4}(2n+2\nu+3),\frac{\nu}{2},n+\frac{3\nu}{2}+1\end{matrix}\right)-\sqrt{\frac{2}{\pi}}\csc(\nu\pi)\sum_{k=0}^{\left\lfloor\frac{n}{2}\right\rfloor}\frac{(-1)^{\left\lfloor\frac{n+1}{2}\right\rfloor}z^{2k+\nu}\,\Gamma\left(k-n+\left\lfloor\frac{n}{2}\right\rfloor+\frac{1}{2}\right)\left(1-k+\left\lfloor\frac{n}{2}\right\rfloor\right)_{n-\left\lfloor\frac{n}{2}\right\rfloor}}{k!\,\Gamma(k+\nu+1)\,\Gamma(k-n-\nu)}\right)\,/;\,n\in\mathbb{N}$$

03.04.26.0128.01

$$K_\nu(z)\,{}_0\tilde{F}_1\left(;\nu;\frac{z^2}{4}\right)=2^{\nu-1}\left(z^{-\nu}-\frac{\pi^{3/2}\csc\left(\frac{1}{4}(4\nu+1)\pi\right)}{\sqrt{2}}G_{4,6}^{2,2}\left(z,\frac{1}{2}\,\middle|\,\begin{matrix}\frac{1-\nu}{2},1-\frac{\nu}{2},\frac{1}{4}(3-2\nu),\frac{1}{4}(2\nu+1)\\ 1-\frac{\nu}{2},\frac{\nu}{2},\frac{1}{4}(3-2\nu),1-\frac{3\nu}{2},-\frac{\nu}{2},\frac{1}{4}(2\nu+1)\end{matrix}\right)\right)$$

03.04.26.0129.01

$$K_\nu(z)\,{}_0\tilde{F}_1\left(;\nu-1;\frac{z^2}{4}\right)=2^{\nu-2}\left(2(\nu-1)z^{-\nu}+\frac{\pi^{3/2}\csc\left(\pi\left(\nu-\frac{5}{4}\right)\right)}{\sqrt{2}}G_{4,6}^{2,2}\left(z,\frac{1}{2}\,\middle|\,\begin{matrix}\frac{3-\nu}{2},1-\frac{\nu}{2},\frac{1}{4}(5-2\nu),\frac{1}{4}(2\nu-1)\\ 2-\frac{\nu}{2},\frac{\nu}{2},\frac{1}{4}(5-2\nu),2-\frac{3\nu}{2},-\frac{\nu}{2},\frac{1}{4}(2\nu-1)\end{matrix}\right)\right)$$

03.04.26.0130.01

$$K_\nu(z)\,{}_0\tilde{F}_1\left(;-\nu;\frac{z^2}{4}\right)=2^{-\nu-1}\left(z^{\nu}-\frac{\pi^{3/2}\csc\left(\frac{1}{4}\pi(1-4\nu)\right)}{\sqrt{2}}G_{4,6}^{2,2}\left(z,\frac{1}{2}\,\middle|\,\begin{matrix}\frac{\nu+1}{2},\frac{\nu+2}{2},\frac{1}{4}(1-2\nu),\frac{1}{4}(2\nu+3)\\ -\frac{\nu}{2},\frac{\nu+2}{2},\frac{1}{4}(1-2\nu),\frac{\nu}{2},\frac{3\nu}{2}+1,\frac{1}{4}(2\nu+3)\end{matrix}\right)\right)$$

03.04.26.0131.01

$$K_\nu(z)\,{}_0\tilde{F}_1\left(;-\nu-1;\frac{z^2}{4}\right)=2^{-\nu-2}\left(\frac{\pi^{3/2}\csc\left(\pi\left(\nu+\frac{1}{4}\right)\right)}{\sqrt{2}}G_{4,6}^{2,2}\left(z,\frac{1}{2}\,\middle|\,\begin{matrix}\frac{\nu+2}{2},\frac{\nu+3}{2},\frac{1}{4}(-2\nu-1),\frac{1}{4}(2\nu+5)\\ -\frac{\nu}{2},\frac{\nu+4}{2},\frac{1}{4}(-2\nu-1),\frac{\nu}{2},\frac{3\nu}{2}+2,\frac{1}{4}(2\nu+5)\end{matrix}\right)-2z^{\nu}(\nu+1)\right)$$

Representations through equivalent functions

With related functions

03.04.27.0001.01

$$K_\nu(z)=\frac{\pi}{2}\csc(\pi\nu)\left(I_{-\nu}(z)-I_\nu(z)\right)\,/;\,\nu\notin\mathbb{Z}$$

03.04.27.0002.01

$$I_\nu(z)\,K_{\nu+1}(z)+I_{\nu+1}(z)\,K_\nu(z)=\frac{1}{z}$$

03.04.27.0003.01

$$K_\nu(z)=\frac{\pi}{2}\left(\frac{(i z)^\nu\cos(\pi\nu)}{z^\nu}-\frac{z^\nu}{(i z)^\nu}\right)\csc(\pi\nu)\,J_\nu(i z)-\frac{\pi(i z)^\nu}{2z^\nu}Y_\nu(i z)\,/;\,\nu\notin\mathbb{Z}$$

03.04.27.0006.01

$$K_\nu(z)=i^\nu\left((\log(i z)-\log(z))\,J_\nu(i z)-\frac{1}{2}\pi\,Y_\nu(i z)\right)/;\,\nu\in\mathbb{Z}$$

03.04.27.0007.01

$$K_\nu(z) = \frac{\pi}{2}\left(\frac{(i\,z)^{2\nu}\cos(\pi\,\nu)}{z^{2\nu}} - 1\right)\csc(\pi\,\nu)\,I_\nu(z) - \frac{\pi\,(i\,z)^\nu}{2\,z^\nu}\,Y_\nu(i\,z)\;/;\;\nu \notin \mathbb{Z}$$

03.04.27.0008.01

$$K_\nu(z) = -\frac{1}{2}\,\pi\,i^\nu\,Y_\nu(i\,z) + (-1)^\nu\,(\log(i\,z) - \log(z))\,I_\nu(z)\;/;\;\nu \in \mathbb{Z}$$

03.04.27.0004.01

$$K_\nu(z) = 2^{\nu-1}\,\pi\,z^{-\nu}\csc(\pi\,\nu)\,{}_0\tilde{F}_1\left(;1-\nu;\frac{z^2}{4}\right) - 2^{-\nu-1}\,\pi\,z^\nu\csc(\pi\,\nu)\,{}_0\tilde{F}_1\left(;\nu+1;\frac{z^2}{4}\right)\;/;\;\nu \notin \mathbb{Z}$$

03.04.27.0005.01

$$K_\nu(z) = 2^{\nu-1}\,\Gamma(\nu)\,z^{-\nu}\,{}_0F_1\left(;1-\nu;\frac{z^2}{4}\right) + 2^{-\nu-1}\,\Gamma(-\nu)\,z^\nu\,{}_0F_1\left(;\nu+1;\frac{z^2}{4}\right)\;/;\;\nu \notin \mathbb{Z}$$

03.04.27.0009.01

$$K_\nu(z) = \frac{1}{2}\,\pi\csc(\pi\,\nu)\left(e^{\frac{3\,i\,\pi\,\nu}{4}}\,z^{-\nu}\left(-(-1)^{3/4}\,z\right)^\nu\left(i\,\mathrm{bei}_{-\nu}\left(-(-1)^{3/4}\,z\right) + \mathrm{ber}_{-\nu}\left(-(-1)^{3/4}\,z\right)\right) - e^{\frac{1}{4}(-3)\,i\,\pi\,\nu}\,z^\nu\left(-(-1)^{3/4}\,z\right)^{-\nu}\left(i\,\mathrm{bei}_\nu\left(-(-1)^{3/4}\,z\right) + \mathrm{ber}_\nu\left(-(-1)^{3/4}\,z\right)\right)\right)\;/;\;\nu \notin \mathbb{Z}$$

03.04.27.0010.01

$$K_\nu\left(\sqrt[4]{-1}\,z\right) = \frac{1}{2}\,\pi\csc(\pi\,\nu)\left(e^{\frac{3\,i\,\pi\,\nu}{4}}\,z^\nu\left(\sqrt[4]{-1}\,z\right)^{-\nu}(i\,\mathrm{bei}_{-\nu}(z) + \mathrm{ber}_{-\nu}(z)) - e^{\frac{1}{4}(-3)\,i\,\pi\,\nu}\,z^{-\nu}\left(\sqrt[4]{-1}\,z\right)^\nu(i\,\mathrm{bei}_\nu(z) + \mathrm{ber}_\nu(z))\right)\;/;\;\nu \notin \mathbb{Z}$$

03.04.27.0011.01

$$K_\nu(z) = i^\nu\left(\mathrm{ker}_\nu\left(-(-1)^{3/4}\,z\right) + i\,\mathrm{kei}_\nu\left(-(-1)^{3/4}\,z\right)\right) - \frac{1}{4}\,i^\nu\left(\mathrm{bei}_\nu\left(-(-1)^{3/4}\,z\right) - i\,\mathrm{ber}_\nu\left(-(-1)^{3/4}\,z\right)\right)\left(4\,i\log(z) - 4\,i\log\left(-(-1)^{3/4}\,z\right) + \pi\right)\;/;\;\nu \in \mathbb{Z}$$

03.04.27.0012.01

$$K_\nu\left(\sqrt[4]{-1}\,z\right) = i^\nu\,(i\,\mathrm{kei}_\nu(z) + \mathrm{ker}_\nu(z)) - \frac{1}{4}\,i^\nu\,(\mathrm{bei}_\nu(z) - i\,\mathrm{ber}_\nu(z))\left(-4\,i\log(z) + 4\,i\log\left(\sqrt[4]{-1}\,z\right) + \pi\right)\;/;\;\nu \in \mathbb{Z}$$

Zeros

The function $K_\nu(z)$ has no zeros in the region $|\mathrm{Arg}(z)| \leq \frac{\pi}{2}$ for any real ν.

Theorems

Kontorovich-Lebedev transformation

$$\hat{f}(y) = \int_0^\infty f(x)\,K_{i\,y}(x)\,dx \Leftrightarrow f(x) = \frac{2}{\pi^2\,x}\int_0^\infty \hat{f}(y)\,y\sinh(\pi\,y)\,K_{i\,y}(x)\,dy.$$

Meijer transformation

For $\mathrm{Re}(\nu) \geq -\frac{1}{2}$ the following identity holds:

$$\hat{f}_\nu(y) = \int_0^\infty f(x)\sqrt{x\,y}\,K_\nu(x\,y)\,dx \Leftrightarrow f(x) = \frac{1}{\pi\,i}\int_{\gamma-i\infty}^{\gamma+i\infty}\hat{f}_\nu(y)\sqrt{x\,y}\,I_\nu(x\,y)\,dy$$

The Green's function of the time independent Schrödinger equation

The Green's function $G(\boldsymbol{x}', \boldsymbol{x}; \varepsilon)$ of the time–independent Schrödinger equation for a free particle in d dimensions $-\Delta\,G(\boldsymbol{x}', \boldsymbol{x}; \varepsilon) - \varepsilon\,G(\boldsymbol{x}', \boldsymbol{x}; \varepsilon) = -i\,\delta(\boldsymbol{x}' - \boldsymbol{x})$ is given by

$$G(\boldsymbol{x}', \boldsymbol{x}; \varepsilon) = -\frac{i\left(\sqrt{-\varepsilon}\right)^{d-2}}{(2\,\pi)^{d/2}}\,\frac{K_{d/2-1}\left(\sqrt{-\varepsilon}\,\left|\boldsymbol{x}' - \boldsymbol{x}\right|\right)}{\left(\sqrt{-\varepsilon}\,\left|\boldsymbol{x}' - \boldsymbol{x}\right|\right)^{d/2-1}}.$$

The Newton-Wigner wave function

The Newton-Wigner wave function $\psi_{\text{NW}}(t, \boldsymbol{x})$ is related to the covariant wave function $\psi(t, \boldsymbol{x})$ by $\psi_{\text{NW}}(t, \boldsymbol{x}) = \int\sqrt{\frac{\pi}{2}}\left(\frac{2\,\lambda_C}{|\boldsymbol{x}-\boldsymbol{y}|}\right)^{\frac{5}{4}} K_{\frac{5}{4}}\left(\frac{|\boldsymbol{x}-\boldsymbol{y}|}{\lambda_C}\right)\psi(t, \boldsymbol{y})\,d\boldsymbol{y}$, where λ_C is the Compton wavelength.

Green's function of the Helmholtz operator

Green's function of the Helmholtz operator in the xy–plane $\left(\partial_{x\,x}+\partial_{y\,y}+k^2\right)G(x, y) = \delta(x)\;\;\delta(y)$: $G(x, y) = -\frac{1}{2\pi}\,K_1\left(-i\,k\sqrt{x^2+y^2}\right)$.

History

– H. M. MacDonald (1899)

Copyright