BesselJ

Notations

Traditional name

Bessel function of the first kind

Traditional notation

$J_\nu(z)$

Mathematica StandardForm notation

`BesselJ[ν, z]`

Primary definition

03.01.02.0001.01

$$J_\nu(z) == \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(k+\nu+1)\,k!} \left(\frac{z}{2}\right)^{2k+\nu}$$

Specific values

Specialized values

For fixed ν

03.01.03.0001.01

$J_\nu(0) == 0 \;/;\; \mathrm{Re}(\nu) > 0 \lor \nu \in \mathbb{Z}$

03.01.03.0002.01

$J_\nu(0) == \tilde{\infty} \;/;\; \mathrm{Re}(\nu) < 0 \bigwedge \nu \notin \mathbb{Z}$

03.01.03.0003.01

$J_\nu(0) == ¿ \;/;\; \mathrm{Re}(\nu) == 0 \land \nu \neq 0$

For fixed z

Explicit rational ν

03.01.03.0014.01

$$J_{-\frac{14}{3}}(z) = \frac{1}{81\, 2^{2/3}\, 3^{5/6}\, z^{14/3}} \left(-288 \sqrt{3} \left(9 z^2 - 110\right) \mathrm{Ai}\left(-\left(\frac{3}{2}\right)^{2/3} z^{2/3}\right) z^{4/3} + 288 \left(9 z^2 - 110\right) \mathrm{Bi}\left(-\left(\frac{3}{2}\right)^{2/3} z^{2/3}\right) z^{4/3} - 3 \sqrt[3]{2} \sqrt[6]{3} \left(81 z^4 - 4320 z^2 + 14\,080\right) \mathrm{Ai}'\left(-\left(\frac{3}{2}\right)^{2/3} z^{2/3}\right) + \sqrt[3]{2}\, 3^{2/3} \left(81 z^4 - 4320 z^2 + 14\,080\right) \mathrm{Bi}'\left(-\left(\frac{3}{2}\right)^{2/3} z^{2/3}\right)\right)$$

03.01.03.0015.01

$$J_{-\frac{9}{2}}(z) = \sqrt{\frac{2}{\pi}} \frac{\left(z^4 - 45 z^2 + 105\right) \cos(z) + 5 z \left(21 - 2 z^2\right) \sin(z)}{z^{9/2}}$$

03.01.03.0016.01

$$J_{-\frac{13}{3}}(z) = \frac{1}{54 \sqrt[3]{2}\, 3^{5/6}\, z^{13/3}} \left(-168 \sqrt[6]{3} \left(9 z^2 - 80\right) \mathrm{Ai}'\left(-\left(\frac{3}{2}\right)^{2/3} z^{2/3}\right) z^{2/3} + 56\, 3^{2/3} \left(80 - 9 z^2\right) \mathrm{Bi}'\left(-\left(\frac{3}{2}\right)^{2/3} z^{2/3}\right) z^{2/3} + 2^{2/3} \sqrt{3} \left(81 z^4 - 3024 z^2 + 4480\right) \mathrm{Ai}\left(-\left(\frac{3}{2}\right)^{2/3} z^{2/3}\right) + 2^{2/3} \left(81 z^4 - 3024 z^2 + 4480\right) \mathrm{Bi}\left(-\left(\frac{3}{2}\right)^{2/3} z^{2/3}\right)\right)$$

03.01.03.0017.01

$$J_{-\frac{11}{3}}(z) = \frac{1}{27\, 2^{2/3}\, 3^{5/6}\, z^{11/3}} \left(9 \sqrt{3} \left(9 z^2 - 160\right) \mathrm{Ai}\left(-\left(\frac{3}{2}\right)^{2/3} z^{2/3}\right) z^{4/3} + 9 \left(160 - 9 z^2\right) \mathrm{Bi}\left(-\left(\frac{3}{2}\right)^{2/3} z^{2/3}\right) z^{4/3} - 60 \sqrt[3]{2} \sqrt[6]{3} \left(9 z^2 - 32\right) \mathrm{Ai}'\left(-\left(\frac{3}{2}\right)^{2/3} z^{2/3}\right) + 20 \sqrt[3]{2}\, 3^{2/3} \left(9 z^2 - 32\right) \mathrm{Bi}'\left(-\left(\frac{3}{2}\right)^{2/3} z^{2/3}\right)\right)$$

03.01.03.0018.01

$$J_{-\frac{7}{2}}(z) = \sqrt{\frac{2}{\pi}} \frac{3 \left(2 z^2 - 5\right) \cos(z) + z \left(z^2 - 15\right) \sin(z)}{z^{7/2}}$$

03.01.03.0019.01

$$J_{-\frac{10}{3}}(z) = \frac{1}{9 \sqrt[3]{2}\, 3^{5/6}\, z^{10/3}} \left(3 \sqrt[6]{3} \left(9 z^2 - 112\right) \mathrm{Ai}'\left(-\left(\frac{3}{2}\right)^{2/3} z^{2/3}\right) z^{2/3} + 3^{2/3} \left(9 z^2 - 112\right) \mathrm{Bi}'\left(-\left(\frac{3}{2}\right)^{2/3} z^{2/3}\right) z^{2/3} + 8\, 2^{2/3} \sqrt{3} \left(9 z^2 - 14\right) \mathrm{Ai}\left(-\left(\frac{3}{2}\right)^{2/3} z^{2/3}\right) + 8\, 2^{2/3} \left(9 z^2 - 14\right) \mathrm{Bi}\left(-\left(\frac{3}{2}\right)^{2/3} z^{2/3}\right)\right)$$

03.01.03.0020.01

$$J_{-\frac{8}{3}}(z) = -\frac{1}{9\, 2^{2/3}\, 3^{5/6}\, z^{8/3}} \left(-90 \sqrt{3}\, \mathrm{Ai}\left(-\left(\frac{3}{2}\right)^{2/3} z^{2/3}\right) z^{4/3} + 90\, \mathrm{Bi}\left(-\left(\frac{3}{2}\right)^{2/3} z^{2/3}\right) z^{4/3} - 3 \sqrt[3]{2} \sqrt[6]{3} \left(9 z^2 - 40\right) \mathrm{Ai}'\left(-\left(\frac{3}{2}\right)^{2/3} z^{2/3}\right) + \sqrt[3]{2}\, 3^{2/3} \left(9 z^2 - 40\right) \mathrm{Bi}'\left(-\left(\frac{3}{2}\right)^{2/3} z^{2/3}\right)\right)$$

03.01.03.0021.01

$$J_{-\frac{5}{2}}(z) = \sqrt{\frac{2}{\pi}} \frac{3 z \sin(z) - \left(z^2 - 3\right) \cos(z)}{z^{5/2}}$$

03.01.03.0022.01

$$J_{-\frac{7}{3}}(z) = -\frac{1}{6\sqrt[3]{2}\,3^{5/6}\,z^{7/3}}\left(-48\sqrt[6]{3}\,\mathrm{Ai}'\left(-\left(\frac{3}{2}\right)^{2/3} z^{2/3}\right) z^{2/3} - 16\,3^{2/3}\,\mathrm{Bi}'\left(-\left(\frac{3}{2}\right)^{2/3} z^{2/3}\right) z^{2/3} + 2^{2/3}\sqrt{3}\,(9z^2-16)\,\mathrm{Ai}\left(-\left(\frac{3}{2}\right)^{2/3} z^{2/3}\right) + 2^{2/3}(9z^2-16)\,\mathrm{Bi}\left(-\left(\frac{3}{2}\right)^{2/3} z^{2/3}\right)\right)$$

03.01.03.0023.01

$$J_{-\frac{5}{3}}(z) = \frac{1}{3\,2^{2/3}\,3^{5/6}\,z^{5/3}}\left(-9\sqrt{3}\,\mathrm{Ai}\left(-\left(\frac{3}{2}\right)^{2/3} z^{2/3}\right) z^{4/3} + 9\,\mathrm{Bi}\left(-\left(\frac{3}{2}\right)^{2/3} z^{2/3}\right) z^{4/3} + 12\sqrt[3]{2}\,\sqrt[6]{3}\,\mathrm{Ai}'\left(-\left(\frac{3}{2}\right)^{2/3} z^{2/3}\right) - 4\sqrt[3]{2}\,3^{2/3}\,\mathrm{Bi}'\left(-\left(\frac{3}{2}\right)^{2/3} z^{2/3}\right)\right)$$

03.01.03.0024.01

$$J_{-\frac{3}{2}}(z) = -\sqrt{\frac{2}{\pi}}\,\frac{\cos(z) + z\sin(z)}{z^{3/2}}$$

03.01.03.0025.01

$$J_{-\frac{4}{3}}(z) = -\frac{1}{\sqrt[3]{2}\,3^{5/6}\,z^{4/3}}\left(3\sqrt[6]{3}\,\mathrm{Ai}'\left(-\left(\frac{3}{2}\right)^{2/3} z^{2/3}\right) z^{2/3} + 3^{2/3}\,\mathrm{Bi}'\left(-\left(\frac{3}{2}\right)^{2/3} z^{2/3}\right) z^{2/3} + 2^{2/3}\sqrt{3}\,\mathrm{Ai}\left(-\left(\frac{3}{2}\right)^{2/3} z^{2/3}\right) + 2^{2/3}\,\mathrm{Bi}\left(-\left(\frac{3}{2}\right)^{2/3} z^{2/3}\right)\right)$$

03.01.03.0010.01

$$J_{-\frac{2}{3}}(z) = \frac{1}{\sqrt[3]{2}\,\sqrt[6]{3}\,z^{2/3}}\left(\mathrm{Bi}'\left(-\left(\left(\frac{3}{2}\right)^{2/3} z^{2/3}\right)\right) - \sqrt{3}\,\mathrm{Ai}'\left(-\left(\left(\frac{3}{2}\right)^{2/3} z^{2/3}\right)\right)\right)$$

03.01.03.0005.01

$$J_{-\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi}}\,\frac{\cos(z)}{\sqrt{z}}$$

03.01.03.0008.01

$$J_{-\frac{1}{3}}(z) = \frac{1}{2^{2/3}\sqrt[3]{3}\,\sqrt[3]{z}}\left(3\,\mathrm{Ai}\left(-\left(\left(\frac{3}{2}\right)^{2/3} z^{2/3}\right)\right) + \sqrt{3}\,\mathrm{Bi}\left(-\left(\left(\frac{3}{2}\right)^{2/3} z^{2/3}\right)\right)\right)$$

03.01.03.0007.01

$$J_{\frac{1}{3}}(z) = \frac{1}{2^{2/3}\sqrt[3]{3}\,\sqrt[3]{z}}\left(3\,\mathrm{Ai}\left(-\left(\left(\frac{3}{2}\right)^{2/3} z^{2/3}\right)\right) - \sqrt{3}\,\mathrm{Bi}\left(-\left(\left(\frac{3}{2}\right)^{2/3} z^{2/3}\right)\right)\right)$$

03.01.03.0004.01

$$J_{\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi}}\,\frac{\sin(z)}{\sqrt{z}}$$

03.01.03.0009.01

$$J_{\frac{2}{3}}(z) = \frac{1}{\sqrt[3]{2}\,\sqrt[6]{3}\,z^{2/3}}\left(\sqrt{3}\,\mathrm{Ai}'\left(-\left(\left(\frac{3}{2}\right)^{2/3} z^{2/3}\right)\right) + \mathrm{Bi}'\left(-\left(\left(\frac{3}{2}\right)^{2/3} z^{2/3}\right)\right)\right)$$

03.01.03.0026.01

$$J_{\frac{4}{3}}(z) = -\frac{1}{\sqrt[3]{2}\, 3^{5/6}\, z^{4/3}} \left(-3\sqrt[6]{3}\,\text{Ai}'\left(-\left(\frac{3}{2}\right)^{2/3} z^{2/3}\right) z^{2/3} + 3^{2/3}\,\text{Bi}'\left(-\left(\frac{3}{2}\right)^{2/3} z^{2/3}\right) z^{2/3} - 2^{2/3}\sqrt{3}\,\text{Ai}\left(-\left(\frac{3}{2}\right)^{2/3} z^{2/3}\right) + 2^{2/3}\,\text{Bi}\left(-\left(\frac{3}{2}\right)^{2/3} z^{2/3}\right)\right)$$

03.01.03.0027.01

$$J_{\frac{3}{2}}(z) = \sqrt{\frac{2}{\pi}}\;\frac{\sin(z) - z\cos(z)}{z^{3/2}}$$

03.01.03.0028.01

$$J_{\frac{5}{3}}(z) = -\frac{1}{3\, 2^{2/3}\, 3^{5/6}\, z^{5/3}} \left(9\sqrt{3}\,\text{Ai}\left(-\left(\frac{3}{2}\right)^{2/3} z^{2/3}\right) z^{4/3} + 9\,\text{Bi}\left(-\left(\frac{3}{2}\right)^{2/3} z^{2/3}\right) z^{4/3} - 12\sqrt[3]{2}\,\sqrt[6]{3}\,\text{Ai}'\left(-\left(\frac{3}{2}\right)^{2/3} z^{2/3}\right) - 4\sqrt[3]{2}\, 3^{2/3}\,\text{Bi}'\left(-\left(\frac{3}{2}\right)^{2/3} z^{2/3}\right)\right)$$

03.01.03.0029.01

$$J_{\frac{7}{3}}(z) = -\frac{1}{6\sqrt[3]{2}\, 3^{5/6}\, z^{7/3}} \left(-48\sqrt[6]{3}\,\text{Ai}'\left(-\left(\frac{3}{2}\right)^{2/3} z^{2/3}\right) z^{2/3} + 16\, 3^{2/3}\,\text{Bi}'\left(-\left(\frac{3}{2}\right)^{2/3} z^{2/3}\right) z^{2/3} + 2^{2/3}\sqrt{3}\,(9z^2 - 16)\,\text{Ai}\left(-\left(\frac{3}{2}\right)^{2/3} z^{2/3}\right) + 2^{2/3}\,(16 - 9z^2)\,\text{Bi}\left(-\left(\frac{3}{2}\right)^{2/3} z^{2/3}\right)\right)$$

03.01.03.0030.01

$$J_{\frac{5}{2}}(z) = -\sqrt{\frac{2}{\pi}}\;\frac{3z\cos(z) + (z^2 - 3)\sin(z)}{z^{5/2}}$$

03.01.03.0031.01

$$J_{\frac{8}{3}}(z) = -\frac{1}{9\, 2^{2/3}\, 3^{5/6}\, z^{8/3}} \left(90\sqrt{3}\,\text{Ai}\left(-\left(\frac{3}{2}\right)^{2/3} z^{2/3}\right) z^{4/3} + 90\,\text{Bi}\left(-\left(\frac{3}{2}\right)^{2/3} z^{2/3}\right) z^{4/3} + 3\sqrt[3]{2}\,\sqrt[6]{3}\,(9z^2 - 40)\,\text{Ai}'\left(-\left(\frac{3}{2}\right)^{2/3} z^{2/3}\right) + \sqrt[3]{2}\, 3^{2/3}\,(9z^2 - 40)\,\text{Bi}'\left(-\left(\frac{3}{2}\right)^{2/3} z^{2/3}\right)\right)$$

03.01.03.0032.01

$$J_{\frac{10}{3}}(z) = -\frac{1}{9\sqrt[3]{2}\, 3^{5/6}\, z^{10/3}} \left(3\sqrt[6]{3}\,(9z^2 - 112)\,\text{Ai}'\left(-\left(\frac{3}{2}\right)^{2/3} z^{2/3}\right) z^{2/3} + 3^{2/3}\,(112 - 9z^2)\,\text{Bi}'\left(-\left(\frac{3}{2}\right)^{2/3} z^{2/3}\right) z^{2/3} + 8\, 2^{2/3}\sqrt{3}\,(9z^2 - 14)\,\text{Ai}\left(-\left(\frac{3}{2}\right)^{2/3} z^{2/3}\right) + 8\, 2^{2/3}\,(14 - 9z^2)\,\text{Bi}\left(-\left(\frac{3}{2}\right)^{2/3} z^{2/3}\right)\right)$$

03.01.03.0033.01

$$J_{\frac{7}{2}}(z) = \sqrt{\frac{2}{\pi}}\;\frac{z(z^2 - 15)\cos(z) + 3(5 - 2z^2)\sin(z)}{z^{7/2}}$$

03.01.03.0034.01

$$J_{\frac{11}{3}}(z) = \frac{1}{27\, 2^{2/3}\, 3^{5/6}\, z^{11/3}} \left(9\sqrt{3}\,(9z^2 - 160)\,\text{Ai}\left(-\left(\frac{3}{2}\right)^{2/3} z^{2/3}\right) z^{4/3} + 9\,(9z^2 - 160)\,\text{Bi}\left(-\left(\frac{3}{2}\right)^{2/3} z^{2/3}\right) z^{4/3} - 60\sqrt[3]{2}\,\sqrt[6]{3}\,(9z^2 - 32)\,\text{Ai}'\left(-\left(\frac{3}{2}\right)^{2/3} z^{2/3}\right) + 20\sqrt[3]{2}\, 3^{2/3}\,(32 - 9z^2)\,\text{Bi}'\left(-\left(\frac{3}{2}\right)^{2/3} z^{2/3}\right)\right)$$

03.01.03.0035.01

$$J_{\frac{13}{3}}(z) = -\frac{1}{54\sqrt[3]{2}\,3^{5/6}\,z^{13/3}}\left(168\sqrt[6]{3}\,(9z^2-80)\,\mathrm{Ai}'\left(-\left(\frac{3}{2}\right)^{2/3}z^{2/3}\right)z^{2/3}+56\,3^{2/3}\,(80-9z^2)\,\mathrm{Bi}'\left(-\left(\frac{3}{2}\right)^{2/3}z^{2/3}\right)z^{2/3}+2^{2/3}\sqrt{3}\,(-81z^4+3024z^2-4480)\,\mathrm{Ai}\left(-\left(\frac{3}{2}\right)^{2/3}z^{2/3}\right)+2^{2/3}\,(81z^4-3024z^2+4480)\,\mathrm{Bi}\left(-\left(\frac{3}{2}\right)^{2/3}z^{2/3}\right)\right)$$

03.01.03.0036.01

$$J_{\frac{9}{2}}(z) = \sqrt{\frac{2}{\pi}}\,\frac{5z\,(2z^2-21)\cos(z)+(z^4-45z^2+105)\sin(z)}{z^{9/2}}$$

03.01.03.0037.01

$$J_{\frac{14}{3}}(z) = \frac{1}{81\,2^{2/3}\,3^{5/6}\,z^{14/3}}\left(288\sqrt{3}\,(9z^2-110)\,\mathrm{Ai}\left(-\left(\frac{3}{2}\right)^{2/3}z^{2/3}\right)z^{4/3}+288\,(9z^2-110)\,\mathrm{Bi}\left(-\left(\frac{3}{2}\right)^{2/3}z^{2/3}\right)z^{4/3}+3\sqrt[3]{2}\,\sqrt[6]{3}\,(81z^4-4320z^2+14080)\,\mathrm{Ai}'\left(-\left(\frac{3}{2}\right)^{2/3}z^{2/3}\right)+\sqrt[3]{2}\,3^{2/3}\,(81z^4-4320z^2+14080)\,\mathrm{Bi}'\left(-\left(\frac{3}{2}\right)^{2/3}z^{2/3}\right)\right)$$

Symbolic rational ν

03.01.03.0006.01

$$J_\nu(z) = \sqrt{\frac{2}{\pi}}\,\frac{1}{\sqrt{z}}\left(\cos\left(\frac{\pi}{2}\left(\nu-\frac{1}{2}\right)-z\right)\sum_{j=0}^{\left\lfloor\frac{2|\nu|-3}{4}\right\rfloor}\frac{(-1)^j\left(2j+|\nu|+\frac{1}{2}\right)!\,(2z)^{-2j-1}}{(2j+1)!\left(-2j+|\nu|-\frac{3}{2}\right)!}-\sin\left(\frac{\pi}{2}\left(\nu-\frac{1}{2}\right)-z\right)\sum_{j=0}^{\left\lfloor\frac{2|\nu|-1}{4}\right\rfloor}\frac{(-1)^j\left(2j+|\nu|-\frac{1}{2}\right)!}{(2j)!\left(-2j+|\nu|-\frac{1}{2}\right)!\,(2z)^{2j}}\right)/;\ \nu-\frac{1}{2}\in\mathbb{Z}$$

03.01.03.0011.01

$$J_\nu(z) = \frac{\mathrm{sgn}(\nu)\,(-1)^{\frac{1}{2}\left(|\nu|-\frac{1}{3}\right)(\mathrm{sgn}(\nu)+1)}\,2^{|\nu|-\frac{5}{3}}\,z^{-|\nu|}\,\Gamma\left(-\frac{1}{3}\right)}{3^{5/6}\,\Gamma(1-|\nu|)}\left(\sqrt[6]{3}\,z^{2/3}\left(\sqrt{3}\,\mathrm{Bi}'\left(-\left(\frac{3}{2}\right)^{2/3}z^{2/3}\right)-3\,\mathrm{sgn}(\nu)\,\mathrm{Ai}'\left(-\left(\frac{3}{2}\right)^{2/3}z^{2/3}\right)\right)\sum_{k=0}^{|\nu|-\frac{4}{3}}\frac{\left(|\nu|-k-\frac{4}{3}\right)!}{k!\left(|\nu|-2k-\frac{4}{3}\right)!\left(\frac{4}{3}\right)_k(1-|\nu|)_k}\left(\frac{z^2}{4}\right)^k+2^{2/3}\left(\mathrm{Bi}\left(-\left(\frac{3}{2}\right)^{2/3}z^{2/3}\right)-\mathrm{sgn}(\nu)\sqrt{3}\,\mathrm{Ai}\left(-\left(\frac{3}{2}\right)^{2/3}z^{2/3}\right)\right)\sum_{k=0}^{|\nu|-\frac{1}{3}}\frac{\left(|\nu|-k-\frac{1}{3}\right)!}{k!\left(|\nu|-2k-\frac{1}{3}\right)!\left(\frac{1}{3}\right)_k(1-|\nu|)_k}\left(\frac{z^2}{4}\right)^k\right)/;\ |\nu|-\frac{1}{3}\in\mathbb{Z}$$

03.01.03.0012.01

$$J_\nu(z) = \frac{\text{sgn}(\nu)\,(-1)^{\frac{1}{2}\left(|\nu|-\frac{2}{3}\right)(\text{sgn}(\nu)+1)}\,2^{|\nu|-\frac{7}{3}}\,z^{-|\nu|}\,\Gamma\left(-\frac{2}{3}\right)}{3\,3^{5/6}\,\Gamma(1-|\nu|)} \left(9\,z^{4/3}\left(\sqrt{3}\,\text{Ai}\left(-\left(\frac{3}{2}\right)^{2/3} z^{2/3}\right) + \text{sgn}(\nu)\,\text{Bi}\left(-\left(\frac{3}{2}\right)^{2/3} z^{2/3}\right)\right)\sum_{k=0}^{|\nu|-\frac{5}{3}} \frac{\left(|\nu|-k-\frac{5}{3}\right)!}{k!\left(|\nu|-2k-\frac{5}{3}\right)!\left(\frac{5}{3}\right)_k (1-|\nu|)_k}\left(\frac{z^2}{4}\right)^k - 4\sqrt[3]{2}\,\sqrt[6]{3}\left(3\,\text{Ai}'\left(-\left(\frac{3}{2}\right)^{2/3} z^{2/3}\right) + \text{sgn}(\nu)\sqrt{3}\,\text{Bi}'\left(-\left(\frac{3}{2}\right)^{2/3} z^{2/3}\right)\right)\sum_{k=0}^{|\nu|-\frac{2}{3}} \frac{\left(|\nu|-k-\frac{2}{3}\right)!}{k!\left(|\nu|-2k-\frac{2}{3}\right)!\left(\frac{2}{3}\right)_k (1-|\nu|)_k}\left(\frac{z^2}{4}\right)^k\right) /;\ |\nu|-\frac{2}{3}\in\mathbb{Z}$$

Values at fixed points

03.01.03.0013.01

$$J_0(0) = 1$$

Values at infinities

03.01.03.0038.01

$$\lim_{x\to\infty} J_\nu(x) = 0$$

03.01.03.0039.01

$$\lim_{x\to-\infty} J_\nu(x) = 0$$

03.01.03.0040.01

$$J_\nu\left(e^{i\lambda}\,\infty\right) = \begin{cases} 0 & \lambda = 0 \lor \lambda = \pi \\ \tilde{\infty} & \text{True} \end{cases} /;\ \text{Im}(\lambda) = 0$$

03.01.03.0041.01

$$J_\nu(\infty) = 0$$

03.01.03.0042.01

$$J_\nu(-\infty) = 0$$

General characteristics

Domain and analyticity

$J_\nu(z)$ is an analytical function of ν and z which is defined over $\mathbb{C}^2$.

03.01.04.0001.01

$$(\nu * z) \longrightarrow J_\nu(z) :: (\mathbb{C}\otimes\mathbb{C}) \longrightarrow \mathbb{C}$$

Symmetries and periodicities

Parity

03.01.04.0002.01

$$J_\nu(-z) = (-z)^\nu\, z^{-\nu}\, J_\nu(z)$$

03.01.04.0003.01

$J_{-n}(z) = (-1)^n J_n(z) /; n \in \mathbb{Z}$

Mirror symmetry

03.01.04.0004.01

$J_{\bar{\nu}}(\bar{z}) = \overline{J_\nu(z)} /; z \notin (-\infty, 0)$

Periodicity

No periodicity

Poles and essential singularities

With respect to *z*

For fixed ν, the function $J_\nu(z)$ has an essential singularity at $z = \tilde{\infty}$. At the same time, the point $z = \tilde{\infty}$ is a branch point for generic ν.

03.01.04.0005.01

$\mathcal{Sing}_z(J_\nu(z)) = \{\{\tilde{\infty}, \infty\}\}$

With respect to *ν*

For fixed z, the function $J_\nu(z)$ has only one singular point at $\nu = \tilde{\infty}$. It is an essential singular point.

03.01.04.0006.01

$\mathcal{Sing}_\nu(J_\nu(z)) = \{\{\tilde{\infty}, \infty\}\}$

Branch points

With respect to *z*

For fixed noninteger ν, the function $J_\nu(z)$ has two branch points: $z = 0$, $z = \tilde{\infty}$. At the same time, the point $z = \tilde{\infty}$ is an essential singularity.

03.01.04.0007.01

$\mathcal{BP}_z(J_\nu(z)) = \{0, \tilde{\infty}\} /; \nu \notin \mathbb{Z}$

03.01.04.0008.01

$\mathcal{BP}_z(J_\nu(z)) = \{\} /; \nu \in \mathbb{Z}$

03.01.04.0009.01

$\mathcal{R}_z(J_\nu(z), 0) = \log /; \nu \notin \mathbb{Q}$

03.01.04.0010.01

$\mathcal{R}_z\left(J_{\frac{p}{q}}(z), 0\right) = q /; p \in \mathbb{Z} \bigwedge q - 1 \in \mathbb{N}^+ \bigwedge \gcd(p, q) = 1$

03.01.04.0011.01

$\mathcal{R}_z(J_\nu(z), \tilde{\infty}) = \log /; \nu \notin \mathbb{Q}$

03.01.04.0012.01

$\mathcal{R}_z\left(J_{\frac{p}{q}}(z), \tilde{\infty}\right) = q /; p \in \mathbb{Z} \bigwedge q - 1 \in \mathbb{N}^+ \bigwedge \gcd(p, q) = 1$

With respect to *ν*

For fixed z, the function $J_\nu(z)$ does not have branch points.

03.01.04.0013.01

$$\mathcal{BP}_\nu(J_\nu(z)) = \{\}$$

Branch cuts

With respect to z

When ν is an integer, $J_\nu(z)$ is an entire function of z. For fixed noninteger ν, it has one infinitely long branch cut. For fixed noninteger ν, the function $J_\nu(z)$ is a single-valued function on the z-plane cut along the interval $(-\infty, 0)$, where it is continuous from above.

03.01.04.0014.01

$$\mathcal{BC}_z(J_\nu(z)) = \{\{(-\infty, 0), -i\}\} /; \nu \notin \mathbb{Z}$$

03.01.04.0015.01

$$\mathcal{BC}_z(J_\nu(z)) = \{\} /; \nu \in \mathbb{Z}$$

03.01.04.0016.01

$$\lim_{\epsilon \to +0} J_\nu(x + i\,\epsilon) = J_\nu(x) /; x \in \mathbb{R} \land x < 0$$

03.01.04.0017.01

$$\lim_{\epsilon \to +0} J_\nu(x - i\,\epsilon) = e^{-2 i \pi \nu} J_\nu(x) /; x \in \mathbb{R} \land x < 0$$

With respect to ν

For fixed z, the function $J_\nu(z)$ is an entire function of ν and does not have branch cuts.

03.01.04.0018.01

$$\mathcal{BC}_\nu(J_\nu(z)) = \{\}$$

Series representations

Generalized power series

Expansions at $\nu = \pm n$

03.01.06.0019.01

$$J_\nu(z) \propto J_n(z) + \left(\frac{\pi}{2} Y_n(z) + \frac{n!}{2}\left(\frac{z}{2}\right)^{-n} \sum_{k=0}^{n-1} \frac{1}{(n-k)\,k!} J_k(z)\left(\frac{z}{2}\right)^k\right)(\nu - n) + \ldots /; (\nu \to n) \land n \in \mathbb{N}$$

03.01.06.0020.01

$$J_\nu(z) \propto (-1)^n J_n(z) + \left(\frac{n!}{2}\left(-\frac{z}{2}\right)^{-n} \sum_{k=0}^{n-1} \frac{1}{(n-k)\,k!} J_k(z)\left(\frac{z}{2}\right)^k + \frac{1}{n!}\left(-\frac{z}{2}\right)^n \sum_{j=1}^{n} \frac{1}{j}\, {}_1F_2\left(j; j+1, n+1; -\frac{z^2}{4}\right) + \frac{\pi (-1)^n}{2} Y_n(z) + (-1)^{n-1} \sum_{k=0}^{n-1} \frac{(n-k-1)!}{k!}\left(\frac{z}{2}\right)^{2k-n}\right)(\nu + n) + \ldots /; (\nu \to -n) \bigwedge n \in \mathbb{N}^+$$

Expansions at generic point $z = z_0$

For the function itself

03.01.06.0021.01

$$J_\nu(z) \propto \left(\frac{1}{z_0}\right)^{\nu \left\lfloor \frac{\arg(z-z_0)}{2\pi} \right\rfloor} z_0^{\nu \left\lfloor \frac{\arg(z-z_0)}{2\pi} \right\rfloor} \left(J_\nu(z_0) + \left(\frac{\nu}{z_0} J_\nu(z_0) - J_{\nu+1}(z_0)\right)(z - z_0) + \frac{J_{\nu+1}(z_0)\, z_0 + J_\nu(z_0)\left((\nu - 1)\,\nu - z_0^2\right)}{2\, z_0^2} (z - z_0)^2 + \ldots\right) /;$$

$$(z \to z_0)$$

03.01.06.0022.01

$$J_\nu(z) \propto \left(\frac{1}{z_0}\right)^{\nu \left\lfloor \frac{\arg(z-z_0)}{2\pi} \right\rfloor} z_0^{\nu \left\lfloor \frac{\arg(z-z_0)}{2\pi} \right\rfloor} \left(J_\nu(z_0) + \left(\frac{\nu}{z_0} J_\nu(z_0) - J_{\nu+1}(z_0)\right)(z - z_0) + \frac{J_{\nu+1}(z_0)\, z_0 + J_\nu(z_0)\left((\nu - 1)\,\nu - z_0^2\right)}{2\, z_0^2} (z - z_0)^2 + O\left((z - z_0)^3\right)\right)$$

03.01.06.0023.01

$$J_\nu(z) = \left(\frac{1}{z_0}\right)^{\nu \left\lfloor \frac{\arg(z-z_0)}{2\pi} \right\rfloor} z_0^{\nu \left\lfloor \frac{\arg(z-z_0)}{2\pi} \right\rfloor} \sum_{k=0}^{\infty} \frac{J_\nu^{(0,k)}(z_0)}{k!} (z - z_0)^k$$

03.01.06.0024.01

$$J_\nu(z) = \sqrt{\pi}\; \Gamma(\nu + 1) \left(\frac{z_0}{4}\right)^{\nu} \left(\frac{1}{z_0}\right)^{\nu \left\lfloor \frac{\arg(z-z_0)}{2\pi} \right\rfloor} z_0^{\nu \left\lfloor \frac{\arg(z-z_0)}{2\pi} \right\rfloor}$$

$$\sum_{k=0}^{\infty} \frac{z_0^{-k}\, 2^k}{k!}\; {}_2\tilde{F}_3\left(\frac{\nu + 1}{2}, \frac{\nu}{2} + 1; \frac{1}{2}(\nu - k + 1), \frac{1}{2}(\nu - k + 2), \nu + 1; -\frac{z_0^2}{4}\right) (z - z_0)^k$$

03.01.06.0025.01

$$J_\nu(z) = \left(\frac{1}{z_0}\right)^{\nu \left\lfloor \frac{\arg(z-z_0)}{2\pi} \right\rfloor} z_0^{\nu \left\lfloor \frac{\arg(z-z_0)}{2\pi} \right\rfloor} \sum_{k=0}^{\infty} \frac{2^{-k}}{k!} \sum_{j=0}^{k} (-1)^j \binom{k}{j} J_{2j-k+\nu}(z_0)\, (z - z_0)^k$$

03.01.06.0026.01

$$J_\nu(z) = \left(\frac{1}{z_0}\right)^{\nu \left\lfloor \frac{\arg(z-z_0)}{2\pi} \right\rfloor} z_0^{\nu \left\lfloor \frac{\arg(z-z_0)}{2\pi} \right\rfloor}$$

$$\sum_{k=0}^{\infty} \frac{z_0^{-k}}{k!} \sum_{m=0}^{k} (-1)^{k+m} \binom{k}{m} (-\nu)_{k-m} \sum_{i=0}^{m} \frac{(-1)^{i-1}\, 2^{2i-m}\, (-m)_{2(m-i)}\, (\nu)_i}{(m - i)!} \left(\frac{z_0}{2} \sum_{j=0}^{i-1} \frac{(i - j - 1)!}{j!\,(i - 2j - 1)!\,(-i - \nu + 1)_j\, (\nu)_{j+1}} \left(\frac{z_0^2}{4}\right)^j J_{\nu-1}(z_0) - \right.$$

$$\left. \sum_{j=0}^{i} \frac{(i - j)!}{j!\,(i - 2j)!\,(-i - \nu + 1)_j\, (\nu)_j} \left(\frac{z_0^2}{4}\right)^j J_\nu(z_0)\right) (z - z_0)^k$$

03.01.06.0027.01

$$J_\nu(z) \propto \left(\frac{1}{z_0}\right)^{\nu \left\lfloor \frac{\arg(z-z_0)}{2\pi} \right\rfloor} z_0^{\nu \left\lfloor \frac{\arg(z-z_0)}{2\pi} \right\rfloor} J_\nu(z_0)\, (1 + O(z - z_0))$$

Expansions on branch cuts

For the function itself

03.01.06.0028.01

$$J_\nu(z) \propto e^{2\nu\pi i \left\lfloor \frac{\arg(z-x)}{2\pi} \right\rfloor} \left(J_\nu(x) + \left(\frac{\nu}{x} J_\nu(x) - J_{\nu+1}(x) \right)(z-x) + \frac{J_{\nu+1}(x)\, x + J_\nu(x)\left((\nu-1)\,\nu - x^2\right)}{2\,x^2} (z-x)^2 + \ldots \right) /; (z \to x) \land x \in \mathbb{R} \land x < 0$$

03.01.06.0029.01

$$J_\nu(z) \propto e^{2\nu\pi i \left\lfloor \frac{\arg(z-x)}{2\pi} \right\rfloor} \left(J_\nu(x) + \left(\frac{\nu}{x} J_\nu(x) - J_{\nu+1}(x) \right)(z-x) + \frac{J_{\nu+1}(x)\, x + J_\nu(x)\left((\nu-1)\,\nu - x^2\right)}{2\,x^2} (z-x)^2 + O\left((z-x)^3\right) \right) /; x \in \mathbb{R} \land x < 0$$

03.01.06.0030.01

$$J_\nu(z) = 2^{-2\nu} \sqrt{\pi}\, \Gamma(\nu+1)\, e^{2\nu\pi i \left\lfloor \frac{\arg(z-x)}{2\pi} \right\rfloor} x^\nu \sum_{k=0}^{\infty} {}_2\tilde{F}_3\left(\frac{\nu+1}{2}, \frac{\nu}{2}+1; \frac{1}{2}(\nu+1-k), \frac{1}{2}(\nu+2-k), \nu+1; -\frac{x^2}{4} \right) \frac{2^k (z-x)^k}{x^k\, k!} /;$$

$$x \in \mathbb{R} \land x < 0$$

03.01.06.0031.01

$$J_\nu(z) = e^{2\nu\pi i \left\lfloor \frac{\arg(z-x)}{2\pi} \right\rfloor} \sum_{k=0}^{\infty} \frac{2^{-k}}{k!} \sum_{j=0}^{k} (-1)^j \binom{k}{j} J_{2j-k+\nu}(x)\, (z-x)^k /; x \in \mathbb{R} \land x < 0$$

03.01.06.0032.01

$$J_\nu(z) = e^{2\nu\pi i \left\lfloor \frac{\arg(z-x)}{2\pi} \right\rfloor} \sum_{k=0}^{\infty} \frac{x^{-k}}{k!}$$

$$\sum_{m=0}^{k} (-1)^{k+m} \binom{k}{m} (-\nu)_{k-m} \sum_{i=0}^{m} \frac{(-1)^{i-1}\, 2^{2i-m}\, (-m)_{2(m-i)}\, (\nu)_i}{(m-i)!} \left(\frac{x}{2} \sum_{j=0}^{i-1} \frac{(i-j-1)!}{j!\,(i-2j-1)!\,(-i-\nu+1)_j\, (\nu)_{j+1}} \left(\frac{x^2}{4}\right)^j J_{\nu-1}(x) - \right.$$

$$\left. \sum_{j=0}^{i} \frac{(i-j)!}{j!\,(i-2j)!\,(-i-\nu+1)_j\, (\nu)_j} \left(\frac{x^2}{4}\right)^j J_\nu(x) \right) (z-x)^k /; x \in \mathbb{R} \land x < 0$$

03.01.06.0033.01

$$J_\nu(z) \propto e^{2\nu\pi i \left\lfloor \frac{\arg(z-x)}{2\pi} \right\rfloor} J_\nu(x)\, (1 + O(z-x)) /; x \in \mathbb{R} \land x < 0$$

Expansions at $z == 0$

For the function itself

General case

03.01.06.0001.02

$$J_\nu(z) \propto \frac{1}{\Gamma(\nu+1)} \left(\frac{z}{2}\right)^\nu \left(1 - \frac{z^2}{4\,(\nu+1)} + \frac{z^4}{32\,(\nu+1)\,(\nu+2)} - \ldots \right) /; (z \to 0)$$

03.01.06.0034.01

$$J_\nu(z) \propto \frac{1}{\Gamma(\nu+1)} \left(\frac{z}{2}\right)^\nu \left(1 - \frac{z^2}{4\,(\nu+1)} + \frac{z^4}{32\,(\nu+1)\,(\nu+2)} + O\left(z^6\right) \right)$$

03.01.06.0002.01

$$J_\nu(z) == \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(k+\nu+1)\, k!} \left(\frac{z}{2}\right)^{2k+\nu}$$

03.01.06.0035.01

$$J_\nu(z) = \frac{1}{\Gamma(\nu+1)} \left(\frac{z}{2}\right)^\nu \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k}}{4^k (\nu+1)_k k!}$$

03.01.06.0036.01

$$J_\nu(z) = \frac{1}{\Gamma(\nu+1)} \left(\frac{z}{2}\right)^\nu {}_0F_1\left(; \nu+1; -\frac{z^2}{4}\right)$$

03.01.06.0003.01

$$J_\nu(z) = \left(\frac{z}{2}\right)^\nu {}_0\tilde{F}_1\left(; \nu+1; -\frac{z^2}{4}\right)$$

03.01.06.0004.02

$$J_\nu(z) \propto \frac{1}{\Gamma(\nu+1)} \left(\frac{z}{2}\right)^\nu + O\left(z^{\nu+2}\right) /; -\nu \notin \mathbb{N}^+$$

03.01.06.0037.01

$$J_\nu(z) = F_\infty(z, \nu) /; \left(\left(F_n(z, \nu) = \sum_{k=0}^{n} \frac{(-1)^k \left(\frac{z}{2}\right)^{2k+\nu}}{\Gamma(k+\nu+1) k!} = J_\nu(z) + \frac{(-1)^n 2^{-2n-\nu-2} z^{2n+\nu+2}}{\Gamma(n+\nu+2)(n+1)!} {}_1F_2\left(1; n+2, n+\nu+2; -\frac{z^2}{4}\right)\right) \bigwedge n \in \mathbb{N}\right)$$

Summed form of the truncated series expansion.

Special cases

03.01.06.0038.01

$$J_\nu(z) \propto \frac{1}{\Gamma(1-\nu)} \left(-\frac{z}{2}\right)^{-\nu} \left(1 - \frac{z^2}{4(1-\nu)} + \frac{z^4}{32(1-\nu)(2-\nu)} + \ldots\right) /; (z \to 0) \bigwedge -\nu \in \mathbb{N}^+$$

03.01.06.0039.01

$$J_\nu(z) \propto \frac{1}{\Gamma(1-\nu)} \left(-\frac{z}{2}\right)^{-\nu} \left(1 - \frac{z^2}{4(1-\nu)} + \frac{z^4}{32(1-\nu)(2-\nu)} + O\left(z^6\right)\right) /; (z \to 0) \bigwedge -\nu \in \mathbb{N}^+$$

03.01.06.0040.01

$$J_\nu(z) = (-1)^\nu \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(k-\nu+1) k!} \left(\frac{z}{2}\right)^{2k-\nu} /; -\nu \in \mathbb{N}^+$$

03.01.06.0041.01

$$J_\nu(z) = (-1)^{\frac{1}{2}(|\nu|-\nu)} \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(k+|\nu|+1) k!} \left(\frac{z}{2}\right)^{2k+|\nu|} /; \nu \in \mathbb{Z}$$

03.01.06.0042.01

$$J_\nu(z) = \frac{1}{\Gamma(1-\nu)} \left(-\frac{z}{2}\right)^{-\nu} \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k}}{4^k (1-\nu)_k k!} /; -\nu \in \mathbb{N}^+$$

03.01.06.0043.01

$$J_\nu(z) = \frac{1}{\Gamma(1-\nu)} \left(-\frac{z}{2}\right)^{-\nu} {}_0F_1\left(; 1-\nu; -\frac{z^2}{4}\right) /; -\nu \in \mathbb{N}^+$$

03.01.06.0044.01

$$J_\nu(z) = \left(-\frac{z}{2}\right)^{-\nu} {}_0\tilde{F}_1\left(;1-\nu;-\frac{z^2}{4}\right) /; -\nu \in \mathbb{N}^+$$

03.01.06.0005.02

$$J_\nu(z) \propto \frac{1}{\Gamma(1-\nu)} \left(-\frac{z}{2}\right)^{-\nu} + O\left(z^{2-\nu}\right) /; -\nu \in \mathbb{N}^+$$

Generic formulas for main term

03.01.06.0045.01

$$J_\nu(z) \propto \begin{cases} \frac{1}{\Gamma(1-\nu)} \left(\frac{z}{2}\right)^{-\nu} & -\nu \in \mathbb{N}^+ \\ \frac{1}{\Gamma(\nu+1)} \left(\frac{z}{2}\right)^{\nu} & \text{True} \end{cases} /; (z \to 0)$$

For small integer powers of the function

For the second power

03.01.06.0046.01

$$J_\nu(z)^2 \propto \frac{2^{-2\nu} z^{2\nu}}{\Gamma(\nu+1)^2} \left(1 - \frac{z^2}{2+2\nu} + \frac{(3+2\nu) z^4}{16(1+\nu)^2 (2+\nu)} + \ldots\right) /; (z \to 0)$$

03.01.06.0047.01

$$J_\nu(z)^2 \propto \frac{2^{-2\nu} z^{2\nu}}{\Gamma(\nu+1)^2} \left(1 - \frac{z^2}{2+2\nu} + \frac{(3+2\nu) z^4}{16(1+\nu)^2 (2+\nu)} + O\left(z^6\right)\right)$$

03.01.06.0048.01

$$J_\nu(z)^2 = \frac{z^{2\nu}}{2^{2\nu}\,\Gamma(\nu+1)^2} \sum_{k=0}^{\infty} \frac{(-1)^k \left(\nu+\frac{1}{2}\right)_k z^{2k}}{(\nu+1)_k (2\nu+1)_k\, k!}$$

03.01.06.0049.01

$$J_\nu(z)^2 = \frac{z^{2\nu}}{2^{2\nu}\,\Gamma(\nu+1)^2}\, {}_1F_2\left(\nu+\frac{1}{2}; \nu+1, 2\nu+1; -z^2\right)$$

03.01.06.0050.01

$$J_\nu(z)^2 = \frac{\sec(\pi\nu)\sqrt{\pi}\, z^{2\nu}}{\Gamma\left(\frac{1}{2}-\nu\right)}\, {}_1\tilde{F}_2\left(\nu+\frac{1}{2}; \nu+1, 2\nu+1; -z^2\right)$$

03.01.06.0051.01

$$J_\nu(z)^2 \propto \frac{2^{-2\nu} z^{2\nu}}{\Gamma(\nu+1)^2} \left(1 + O\left(z^2\right)\right)$$

03.01.06.0052.01

$$J_\nu(z)^2 = F_\infty(z.\nu) /; \left(\left(F_n(z,\nu) = \frac{z^{2\nu}}{2^{2\nu}\,\Gamma(\nu+1)^2}\sum_{k=0}^{n}\frac{(-1)^k\left(\nu+\frac{1}{2}\right)_k z^{2k}}{(\nu+1)_k\,(2\nu+1)_k\,k!} = J_\nu(z)^2 + \frac{(-1)^n z^{2n+2\nu+2}\,\Gamma\left(n+\nu+\frac{3}{2}\right)}{\sqrt{\pi}\;\Gamma(n+\nu+2)\,\Gamma(n+2\nu+2)\,(n+1)!}\,{}_2F_3\left(1, n+\nu+\frac{3}{2}; n+2, n+\nu+2, n+2\nu+2; -z^2\right)\right)\bigwedge n\in\mathbb{N}\right)$$

Summed form of the truncated series expansion.

Asymptotic series expansions

Expansions inside Stokes sectors

Expansions containing $z \to \infty$

In exponential form ||| In exponential form

03.01.06.0053.01

$$J_\nu(z) \propto \frac{1}{\sqrt{2\pi z}}\left(e^{-i\left(z-\frac{1}{4}(2\nu+1)\pi\right)}\left(1+\frac{i\left(1-4\nu^2\right)}{8z}-\frac{9-40\nu^2+16\nu^4}{128z^2}+\ldots\right)+ e^{i\left(z-\frac{1}{4}(2\nu+1)\pi\right)}\left(1-\frac{i\left(1-4\nu^2\right)}{8z}-\frac{9-40\nu^2+16\nu^4}{128z^2}+\ldots\right)\right) /; |\arg(z)|<\pi \bigwedge (|z|\to\infty)$$

03.01.06.0054.01

$$J_\nu(z) \propto \frac{1}{\sqrt{2\pi z}}\left(e^{i\left(\frac{1}{4}(2\nu+1)\pi-z\right)}\left(\sum_{k=0}^{n}\frac{\left(\nu+\frac{1}{2}\right)_k\left(\frac{1}{2}-\nu\right)_k}{k!}\left(\frac{i}{2z}\right)^k+O\left(\frac{1}{z^{n+1}}\right)\right)+ e^{-i\left(\frac{1}{4}(2\nu+1)\pi-z\right)}\left(\sum_{k=0}^{n}\frac{\left(\nu+\frac{1}{2}\right)_k\left(\frac{1}{2}-\nu\right)_k}{k!}\left(-\frac{i}{2z}\right)^k+O\left(\frac{1}{z^{n+1}}\right)\right)\right) /; |\arg(z)|<\pi \bigwedge (|z|\to\infty)$$

03.01.06.0055.01

$$J_\nu(z) \propto \frac{1}{\sqrt{2\pi z}}\left(\exp\left(i\left(\frac{(2\nu+1)\pi}{4}-z\right)\right){}_2F_0\left(\nu+\frac{1}{2}, \frac{1}{2}-\nu;;\frac{i}{2z}\right)+\exp\left(-i\left(\frac{(2\nu+1)\pi}{4}-z\right)\right){}_2F_0\left(\nu+\frac{1}{2}, \frac{1}{2}-\nu;;-\frac{i}{2z}\right)\right) /; |\arg(z)|<\pi \bigwedge (|z|\to\infty)$$

03.01.06.0056.01

$$J_\nu(z) \propto \frac{1}{\sqrt{2\pi z}}\left(\exp\left(-i\left(\frac{(2\nu+1)\pi}{4}-z\right)\right)\left(1+O\left(\frac{1}{z}\right)\right)+\exp\left(i\left(\frac{(2\nu+1)\pi}{4}-z\right)\right)\left(1+O\left(\frac{1}{z}\right)\right)\right) /; |\arg(z)|<\pi \bigwedge (|z|\to\infty)$$

In trigonometric form ||| In trigonometric form

03.01.06.0057.01

$$J_\nu(z) \propto \frac{\sqrt{2}}{\sqrt{\pi z}} \left(\cos\left(z - \frac{\pi(2\nu+1)}{4}\right) \left(1 - \frac{9 - 40\nu^2 + 16\nu^4}{128 z^2} + \frac{11025 - 51664\nu^2 + 31584\nu^4 - 5376\nu^6 + 256\nu^8}{98304 z^4} + \ldots\right) + \frac{1-4\nu^2}{8z} \sin\left(z - \frac{\pi(2\nu+1)}{4}\right) \left(1 - \frac{225 - 136\nu^2 + 16\nu^4}{384 z^2} + \frac{893025 - 656784\nu^2 + 137824\nu^4 - 10496\nu^6 + 256\nu^8}{491520 z^4} + \ldots\right)\right) /; |\arg(z)| < \pi \land (|z| \to \infty)$$

03.01.06.0058.01

$$J_\nu(z) \propto \frac{\sqrt{2}}{\sqrt{\pi z}} \left(\cos\left(z - \frac{\pi(2\nu+1)}{4}\right) \left(\sum_{k=0}^{n} \frac{\left(\frac{1}{4}(1-2\nu)\right)_k \left(\frac{1}{4}(3-2\nu)\right)_k \left(\frac{1}{4}(2\nu+1)\right)_k \left(\frac{1}{4}(2\nu+3)\right)_k}{\left(\frac{1}{2}\right)_k k!} \left(-\frac{1}{z^2}\right)^k + O\left(\frac{1}{z^{2n+2}}\right)\right) + \frac{1-4\nu^2}{8z} \sin\left(z - \frac{\pi(2\nu+1)}{4}\right) \left(\sum_{k=0}^{n} \frac{\left(\frac{1}{4}(3-2\nu)\right)_k \left(\frac{1}{4}(5-2\nu)\right)_k \left(\frac{1}{4}(2\nu+3)\right)_k \left(\frac{1}{4}(2\nu+5)\right)_k}{\left(\frac{3}{2}\right)_k k!} \left(-\frac{1}{z^2}\right)^k + O\left(\frac{1}{z^{2n+2}}\right)\right)\right) /; |\arg(z)| < \pi \land (|z| \to \infty)$$

03.01.06.0059.01

$$J_\nu(z) \propto \frac{2 e^{i\pi\nu}}{\sqrt{2\pi}\sqrt{-z}} \left(\cos\left(\frac{1}{4}\pi(2\nu+1) + z\right) \left(\sum_{k=0}^{n} \frac{\left(\frac{1}{4}(1-2\nu)\right)_k \left(\frac{1}{4}(3-2\nu)\right)_k \left(\frac{1}{4}(2\nu+1)\right)_k \left(\frac{1}{4}(2\nu+3)\right)_k}{\left(\frac{1}{2}\right)_k k!} \left(-\frac{1}{z^2}\right)^k + O\left(\frac{1}{z^{2n+2}}\right)\right) + \frac{1-4\nu^2}{8z} \sin\left(\frac{1}{4}\pi(2\nu+1) + z\right) \left(\sum_{k=0}^{n} \frac{\left(\frac{1}{4}(3-2\nu)\right)_k \left(\frac{1}{4}(5-2\nu)\right)_k \left(\frac{1}{4}(2\nu+3)\right)_k \left(\frac{1}{4}(2\nu+5)\right)_k}{\left(\frac{3}{2}\right)_k k!} \left(-\frac{1}{z^2}\right)^k + O\left(\frac{1}{z^{2n+2}}\right)\right)\right) /; 0 < \arg(z) \leq \pi \land (|z| \to \infty)$$

03.01.06.0060.01

$$J_\nu(z) \propto \frac{\sqrt{2}}{\sqrt{\pi z}} \left(\cos\left(z - \frac{\pi(2\nu+1)}{4}\right) {}_4F_1\left(\frac{1-2\nu}{4}, \frac{3-2\nu}{4}, \frac{2\nu+1}{4}, \frac{2\nu+3}{4}; \frac{1}{2}; -\frac{1}{z^2}\right) + \frac{1-4\nu^2}{8z} \sin\left(z - \frac{\pi(2\nu+1)}{4}\right) {}_4F_1\left(\frac{3-2\nu}{4}, \frac{5-2\nu}{4}, \frac{2\nu+3}{4}, \frac{2\nu+5}{4}; \frac{3}{2}; -\frac{1}{z^2}\right)\right) /; |\arg(z)| < \pi \land (|z| \to \infty)$$

03.01.06.0061.01

$$J_\nu(z) \propto \frac{\sqrt{2}}{\sqrt{\pi z}} \left(\cos\left(z - \frac{\pi(2\nu+1)}{4}\right) \left(1 + O\left(\frac{1}{z^2}\right)\right) + \frac{1-4\nu^2}{8z} \sin\left(z - \frac{\pi(2\nu+1)}{4}\right) \left(1 + O\left(\frac{1}{z^2}\right)\right)\right) /; |\arg(z)| < \pi \land (|z| \to \infty)$$

Expansions containing $z \to -\infty$

In exponential form ||| In exponential form

03.01.06.0062.01

$$J_\nu(z) \propto \frac{(-1)^\nu}{\sqrt{2\pi z}} \left(e^{i\left(z+\frac{1}{4}(2\nu+1)\pi\right)} \left(1 - \frac{i\,(1-4\nu^2)}{8z} - \frac{9-40\nu^2+16\nu^4}{128 z^2} + \ldots\right) + e^{-i\left(z+\frac{1}{4}(2\nu+1)\pi\right)} \left(1 + \frac{i\,(1-4\nu^2)}{8z} - \frac{9-40\nu^2+16\nu^4}{128 z^2} + \ldots\right)\right) /;\ 0 < \arg(z) \le \pi \bigwedge (|z| \to \infty)$$

03.01.06.0063.01

$$J_\nu(z) \propto \frac{(-1)^\nu}{\sqrt{-2\pi z}} \left(e^{i\left(\frac{1}{4}(2\nu+1)\pi+z\right)} \left(\sum_{k=0}^{n} \frac{\left(\nu+\frac{1}{2}\right)_k \left(\frac{1}{2}-\nu\right)_k}{k!} \left(-\frac{i}{2z}\right)^k + O\left(\frac{1}{z^{n+1}}\right)\right) + e^{-i\left(\frac{1}{4}(2\nu+1)\pi+z\right)} \left(\sum_{k=0}^{n} \frac{\left(\nu+\frac{1}{2}\right)_k \left(\frac{1}{2}-\nu\right)_k}{k!} \left(\frac{i}{2z}\right)^k + O\left(\frac{1}{z^{n+1}}\right)\right)\right) /;\ 0 < \arg(z) \le \pi \bigwedge (|z| \to \infty)$$

03.01.06.0064.01

$$J_\nu(z) \propto \frac{(-1)^\nu}{\sqrt{-2\pi z}} \left(\exp\left(i\left(\frac{(2\nu+1)\pi}{4}+z\right)\right) {}_2F_0\left(\nu+\frac{1}{2}, \frac{1}{2}-\nu;;\frac{i}{2z}\right) + \exp\left(-i\left(\frac{(2\nu+1)\pi}{4}+z\right)\right) {}_2F_0\left(\nu+\frac{1}{2}, \frac{1}{2}-\nu;;-\frac{i}{2z}\right)\right) /;\ 0 < \arg(z) \le \pi \bigwedge (|z| \to \infty)$$

03.01.06.0065.01

$$J_\nu(z) \propto \frac{(-1)^\nu}{\sqrt{-2\pi z}} \left(\exp\left(-i\left(\frac{(2\nu+1)\pi}{4}+z\right)\right)\left(1+O\left(\frac{1}{z}\right)\right) + \exp\left(i\left(\frac{(2\nu+1)\pi}{4}+z\right)\right)\left(1+O\left(\frac{1}{z}\right)\right)\right) /;\ 0 < \arg(z) \le \pi \bigwedge (|z| \to \infty)$$

In trigonometric form ||| In trigonometric form

03.01.06.0066.01

$$J_\nu(z) \propto \frac{\sqrt{2}\,(-1)^\nu}{\sqrt{-\pi z}} \left(\cos\left(z+\frac{\pi(2\nu+1)}{4}\right)\left(1 - \frac{9-40\nu^2+16\nu^4}{128z^2} + \frac{11025-51664\nu^2+31584\nu^4-5376\nu^6+256\nu^8}{98304 z^4} + \ldots\right) + \frac{1-4\nu^2}{8z}\sin\left(z+\frac{\pi(2\nu+1)}{4}\right)\left(1 - \frac{225-136\nu^2+16\nu^4}{384z^2} + \frac{893025-656784\nu^2+137824\nu^4-10496\nu^6+256\nu^8}{491520 z^4} + \ldots\right)\right) /;\ 0 < \arg(z) \le \pi \bigwedge (|z| \to \infty)$$

03.01.06.0067.01

$$J_\nu(z) \propto \frac{\sqrt{2}\,(-1)^\nu}{\sqrt{-\pi z}} \left(\cos\left(z+\frac{\pi(2\nu+1)}{4}\right) \left(\sum_{k=0}^{n} \frac{\left(\frac{1}{4}(1-2\nu)\right)_k \left(\frac{1}{4}(3-2\nu)\right)_k \left(\frac{1}{4}(2\nu+1)\right)_k \left(\frac{1}{4}(2\nu+3)\right)_k}{\left(\frac{1}{2}\right)_k k!} \left(-\frac{1}{z^2}\right)^k + O\left(\frac{1}{z^{2n+2}}\right)\right) + \frac{1-4\nu^2}{8z}\sin\left(z+\frac{\pi(2\nu+1)}{4}\right) \left(\sum_{k=0}^{n} \frac{\left(\frac{1}{4}(3-2\nu)\right)_k \left(\frac{1}{4}(5-2\nu)\right)_k \left(\frac{1}{4}(2\nu+3)\right)_k \left(\frac{1}{4}(2\nu+5)\right)_k}{\left(\frac{3}{2}\right)_k k!} \left(-\frac{1}{z^2}\right)^k + O\left(\frac{1}{z^{2n+2}}\right)\right)\right) /;\ 0 < \arg(z) \le \pi \bigwedge (|z| \to \infty)$$

03.01.06.0068.01

$$J_\nu(z) \propto \frac{\sqrt{2}\,(-1)^\nu}{\sqrt{-\pi z}} \left(\cos\left(z + \frac{\pi(2\nu+1)}{4}\right) {}_4F_1\left(\frac{1-2\nu}{4}, \frac{3-2\nu}{4}, \frac{2\nu+1}{4}, \frac{2\nu+3}{4}; \frac{1}{2}; -\frac{1}{z^2}\right) + \frac{1-4\nu^2}{8z} \sin\left(z + \frac{\pi(2\nu+1)}{4}\right) {}_4F_1\left(\frac{3-2\nu}{4}, \frac{5-2\nu}{4}, \frac{2\nu+3}{4}, \frac{2\nu+5}{4}; \frac{3}{2}; -\frac{1}{z^2}\right)\right) /; 0 < \arg(z) \leq \pi \bigwedge (|z| \to \infty)$$

03.01.06.0069.01

$$J_\nu(z) \propto \frac{\sqrt{2}\,(-1)^\nu}{\sqrt{-\pi z}} \left(\cos\left(z + \frac{\pi(2\nu+1)}{4}\right)\left(1 + O\left(\frac{1}{z^2}\right)\right) + \frac{1-4\nu^2}{8z} \sin\left(z + \frac{\pi(2\nu+1)}{4}\right)\left(1 + O\left(\frac{1}{z^2}\right)\right)\right) /; 0 < \arg(z) \leq \pi \bigwedge (|z| \to \infty)$$

Expansions for any *z* in exponential form

Using exponential function with branch cut-containing arguments

03.01.06.0006.01

$$J_\nu(z) \propto \frac{1}{\sqrt{2\pi}} z^\nu \left(z^2\right)^{-\frac{2\nu+1}{4}} \left(e^{-i\left(\sqrt{z^2} - \frac{1}{4}(2\nu+1)\pi\right)} \left(1 + \frac{i\left(1-4\nu^2\right)}{8\sqrt{z^2}} - \frac{9 - 40\nu^2 + 16\nu^4}{128 z^2} + \ldots\right) + e^{i\left(\sqrt{z^2} - \frac{1}{4}(2\nu+1)\pi\right)} \left(1 - \frac{i\left(1-4\nu^2\right)}{8\sqrt{z^2}} - \frac{9 - 40\nu^2 + 16\nu^4}{128 z^2} + \ldots\right)\right) /; (|z| \to \infty)$$

03.01.06.0070.01

$$J_\nu(z) \propto \frac{1}{\sqrt{2\pi}} z^\nu \left(z^2\right)^{-\frac{1}{4}(2\nu+1)} \left(e^{i\left(\frac{1}{4}(2\nu+1)\pi - \sqrt{z^2}\right)} \left(\sum_{k=0}^{n} \frac{\left(\nu+\frac{1}{2}\right)_k \left(\frac{1}{2}-\nu\right)_k}{k!} \left(\frac{i}{2\sqrt{z^2}}\right)^k + O\left(\frac{1}{z^{n+1}}\right)\right) + e^{-i\left(\frac{1}{4}(2\nu+1)\pi - \sqrt{z^2}\right)} \left(\sum_{k=0}^{n} \frac{\left(\nu+\frac{1}{2}\right)_k \left(\frac{1}{2}-\nu\right)_k}{k!} \left(-\frac{i}{2\sqrt{z^2}}\right)^k + O\left(\frac{1}{z^{n+1}}\right)\right)\right) /; (|z| \to \infty)$$

03.01.06.0007.01

$$J_\nu(z) \propto \frac{1}{\sqrt{2\pi}} z^\nu \left(z^2\right)^{-\frac{2\nu+1}{4}} \left(\exp\left(i\left(\frac{(2\nu+1)\pi}{4} - \sqrt{z^2}\right)\right) {}_2F_0\left(\nu + \frac{1}{2}, \frac{1}{2} - \nu;; \frac{i}{2\sqrt{z^2}}\right) + \exp\left(-i\left(\frac{(2\nu+1)\pi}{4} - \sqrt{z^2}\right)\right) {}_2F_0\left(\nu + \frac{1}{2}, \frac{1}{2} - \nu;; -\frac{i}{2\sqrt{z^2}}\right)\right) /; (|z| \to \infty)$$

03.01.06.0008.01

$$J_\nu(z) \propto \frac{1}{\sqrt{2\pi}} z^\nu \left(z^2\right)^{-\frac{2\nu+1}{4}} \left(\exp\left(-i\left(\frac{(2\nu+1)\pi}{4} - \sqrt{z^2}\right)\right)\left(1 + O\left(\frac{1}{z}\right)\right) + \exp\left(i\left(\frac{(2\nu+1)\pi}{4} - \sqrt{z^2}\right)\right)\left(1 + O\left(\frac{1}{z}\right)\right)\right) /; (|z| \to \infty)$$

Using exponential function with branch cut-free arguments

03.01.06.0071.01

$$J_\nu(z) \propto \frac{\sqrt[4]{-1}\, i^\nu\, (-z)^{-\nu-\frac{1}{2}}\, z^\nu}{\sqrt{2\pi}} \left(e^{iz} \left(1 + \frac{i(-1+4\nu^2)}{8z} - \frac{9-40\nu^2+16\nu^4}{128z^2} + \ldots\right) + e^{-iz} \left(\frac{\sqrt{-z^2}}{z}\cos(\pi\nu) - \sin(\pi\nu)\right) \left(1 - \frac{i(-1+4\nu^2)}{8z} - \frac{9-40\nu^2+16\nu^4}{128z^2} + \ldots\right)\right) /; (|z| \to \infty)$$

03.01.06.0072.01

$$J_\nu(z) \propto \frac{\sqrt[4]{-1}\, i^\nu\, (-z)^{-\nu-\frac{1}{2}}\, z^\nu}{\sqrt{2\pi}} \left(e^{iz} \left(\sum_{k=0}^{n} \frac{\left(\nu+\frac{1}{2}\right)_k \left(\frac{1}{2}-\nu\right)_k}{k!} \left(-\frac{i}{2z}\right)^k + O\left(\frac{1}{z^{n+1}}\right)\right) + e^{-iz} \left(\frac{\sqrt{-z^2}}{z}\cos(\pi\nu) - \sin(\pi\nu)\right) \left(\sum_{k=0}^{n} \frac{\left(\nu+\frac{1}{2}\right)_k \left(\frac{1}{2}-\nu\right)_k}{k!} \left(\frac{i}{2z}\right)^k + O\left(\frac{1}{z^{n+1}}\right)\right)\right) /; (|z| \to \infty)$$

03.01.06.0073.01

$$J_\nu(z) \propto \frac{\sqrt[4]{-1}\, i^\nu\, (-z)^{-\nu-\frac{1}{2}}\, z^\nu}{\sqrt{2\pi}} \left(e^{iz}\, {}_2F_0\left(\nu+\frac{1}{2}, \frac{1}{2}-\nu;\, ; -\frac{i}{2z}\right) + e^{-iz} \left(\frac{\sqrt{-z^2}\, \cos(\pi\nu)}{z} - \sin(\pi\nu)\right) {}_2F_0\left(\nu+\frac{1}{2}, \frac{1}{2}-\nu;\, ; \frac{i}{2z}\right)\right) /; (|z| \to \infty) /; (|z| \to \infty)$$

03.01.06.0074.01

$$J_\nu(z) \propto \frac{\sqrt[4]{-1}\, i^\nu\, (-z)^{-\nu-\frac{1}{2}}\, z^\nu}{\sqrt{2\pi}} \left(e^{iz} \left(1 + O\left(\frac{1}{z}\right)\right) + e^{-iz} \left(\frac{\sqrt{-z^2}}{z}\cos(\pi\nu) - \sin(\pi\nu)\right) \left(1 + O\left(\frac{1}{z}\right)\right)\right) /; (|z| \to \infty)$$

03.01.06.0075.01

$$J_\nu(z) \propto \begin{cases} \frac{\sqrt[4]{-1}\, i^\nu\, e^{-iz}}{\sqrt{2\pi}\,\sqrt{z}} - \frac{(-1)^{3/4}\, i^\nu\, e^{iz-i\pi\nu}}{\sqrt{2\pi}\,\sqrt{z}} & \arg(z) \le 0 \\ \frac{\sqrt[4]{-1}\, i^\nu\, e^{-iz}}{\sqrt{2\pi}\,\sqrt{z}} + \frac{(-1)^{3/4}\, i^\nu\, e^{iz+i\pi\nu}}{\sqrt{2\pi}\,\sqrt{z}} & \text{True} \end{cases} /; (|z| \to \infty)$$

Expansions for any *z* in trigonometric form

Using trigonometric functions with branch cut-containing arguments

03.01.06.0009.01

$$J_\nu(z) \propto \frac{\sqrt{2}}{\sqrt{\pi}} z^\nu \left(z^2\right)^{-\frac{2\nu+1}{4}} \left(\cos\left(\sqrt{z^2} - \frac{\pi(2\nu+1)}{4}\right)\left(1 - \frac{9 - 40\nu^2 + 16\nu^4}{128 z^2} + \frac{11025 - 51664\nu^2 + 31584\nu^4 - 5376\nu^6 + 256\nu^8}{98304 z^4} + \ldots\right) + \frac{1 - 4\nu^2}{8\sqrt{z^2}} \sin\left(\sqrt{z^2} - \frac{\pi(2\nu+1)}{4}\right) \left(1 - \frac{225 - 136\nu^2 + 16\nu^4}{384 z^2} + \frac{893025 - 656784\nu^2 + 137824\nu^4 - 10496\nu^6 + 256\nu^8}{491520 z^4} + \ldots\right)\right) /; (|z| \to \infty)$$

03.01.06.0076.01

$$J_\nu(z) \propto \frac{\sqrt{2}\, z^\nu \left(z^2\right)^{-\frac{1}{4}(2\nu+1)}}{\sqrt{\pi}} \left(\cos\left(\sqrt{z^2} - \frac{\pi(2\nu+1)}{4}\right)\left(\sum_{k=0}^{n} \frac{\left(\frac{1}{4}(1-2\nu)\right)_k \left(\frac{1}{4}(3-2\nu)\right)_k \left(\frac{1}{4}(2\nu+1)\right)_k \left(\frac{1}{4}(2\nu+3)\right)_k}{\left(\frac{1}{2}\right)_k k!} \left(-\frac{1}{z^2}\right)^k + O\left(\frac{1}{z^{2n+2}}\right)\right) + \frac{1 - 4\nu^2}{8\sqrt{z^2}} \sin\left(\sqrt{z^2} - \frac{\pi(2\nu+1)}{4}\right)\left(\sum_{k=0}^{n} \frac{\left(\frac{1}{4}(3-2\nu)\right)_k \left(\frac{1}{4}(5-2\nu)\right)_k \left(\frac{1}{4}(2\nu+3)\right)_k \left(\frac{1}{4}(2\nu+5)\right)_k}{\left(\frac{3}{2}\right)_k k!} \left(-\frac{1}{z^2}\right)^k + O\left(\frac{1}{z^{2n+2}}\right)\right)\right) /; (|z| \to \infty)$$

03.01.06.0010.01

$$J_\nu(z) \propto \frac{\sqrt{2}\, z^\nu \left(z^2\right)^{-\frac{2\nu+1}{4}}}{\sqrt{\pi}} \left(\cos\left(\sqrt{z^2} - \frac{\pi(2\nu+1)}{4}\right) {}_4F_1\left(\frac{1-2\nu}{4}, \frac{3-2\nu}{4}, \frac{2\nu+1}{4}, \frac{2\nu+3}{4}; \frac{1}{2}; -\frac{1}{z^2}\right) + \frac{1-4\nu^2}{8\sqrt{z^2}} \sin\left(\sqrt{z^2} - \frac{\pi(2\nu+1)}{4}\right) {}_4F_1\left(\frac{3-2\nu}{4}, \frac{5-2\nu}{4}, \frac{2\nu+3}{4}, \frac{2\nu+5}{4}; \frac{3}{2}; -\frac{1}{z^2}\right)\right) /; (|z| \to \infty)$$

03.01.06.0011.01

$$J_\nu(z) \propto \frac{\sqrt{2}}{\sqrt{\pi}} z^\nu \left(z^2\right)^{-\frac{2\nu+1}{4}} \left(\cos\left(\sqrt{z^2} - \frac{\pi(2\nu+1)}{4}\right)\left(1 + O\left(\frac{1}{z^2}\right)\right) + \frac{1-4\nu^2}{8\sqrt{z^2}} \sin\left(\sqrt{z^2} - \frac{\pi(2\nu+1)}{4}\right)\left(1 + O\left(\frac{1}{z^2}\right)\right)\right) /; (|z| \to \infty)$$

Using trigonometric functions with branch cut-free arguments

03.01.06.0077.01

$$J_\nu(z) \propto \frac{1}{\sqrt{2\pi}} \left(\frac{1}{\sqrt{z}} \left(1 - \frac{i\sqrt{-z^2}}{z}\right) \cos\left(z - \frac{\pi(2\nu+1)}{4}\right) + \frac{e^{i\pi\nu}}{\sqrt{-z}} \left(1 + \frac{i\sqrt{-z^2}}{z}\right) \cos\left(z + \frac{\pi(2\nu+1)}{4}\right)\right)$$
$$\left(1 - \frac{16\nu^4 - 40\nu^2 + 9}{128 z^2} + \frac{256\nu^8 - 5376\nu^6 + 31584\nu^4 - 51664\nu^2 + 11025}{98304 z^4} + \ldots\right) +$$
$$\frac{1 - 4\nu^2}{8z\sqrt{2\pi}} \left(\frac{1}{\sqrt{z}} \left(1 - \frac{i\sqrt{-z^2}}{z}\right) \sin\left(z - \frac{\pi(2\nu+1)}{4}\right) + \frac{e^{i\pi\nu}}{\sqrt{-z}} \left(1 + \frac{i\sqrt{-z^2}}{z}\right) \sin\left(z + \frac{\pi(2\nu+1)}{4}\right)\right)$$
$$\left(1 - \frac{16\nu^4 - 136\nu^2 + 225}{384 z^2} + \frac{256\nu^8 - 10496\nu^6 + 137824\nu^4 - 656784\nu^2 + 893025}{491520 z^4} + \ldots\right) /; (|z| \to \infty)$$

03.01.06.0078.01

$$J_\nu(z) \propto \frac{1}{\sqrt{2\pi}} \left(\frac{1}{\sqrt{-z}} \left(1 + \frac{i\sqrt{-z^2}}{z}\right) e^{i\pi\nu} \cos\left(z + \frac{\pi(2\nu+1)}{4}\right) + \frac{1}{\sqrt{z}} \left(1 - \frac{i\sqrt{-z^2}}{z}\right) \cos\left(z - \frac{\pi(2\nu+1)}{4}\right)\right)$$
$$\left(\sum_{k=0}^{n} \frac{\left(\frac{1}{4}(1-2\nu)\right)_k \left(\frac{1}{4}(3-2\nu)\right)_k \left(\frac{1}{4}(2\nu+1)\right)_k \left(\frac{1}{4}(2\nu+3)\right)_k}{\left(\frac{1}{2}\right)_k k!} \left(-\frac{1}{z^2}\right)^k + O\left(\frac{1}{z^{2n+2}}\right)\right) +$$
$$\frac{1 - 4\nu^2}{8z\sqrt{2\pi}} \left(\frac{1}{\sqrt{-z}} \left(1 + \frac{i\sqrt{-z^2}}{z}\right) e^{i\pi\nu} \sin\left(z + \frac{\pi(2\nu+1)}{4}\right) + \frac{1}{\sqrt{z}} \left(1 - \frac{i\sqrt{-z^2}}{z}\right) \sin\left(z - \frac{\pi(2\nu+1)}{4}\right)\right)$$
$$\left(\sum_{k=0}^{n} \frac{\left(\frac{1}{4}(3-2\nu)\right)_k \left(\frac{1}{4}(5-2\nu)\right)_k \left(\frac{1}{4}(2\nu+3)\right)_k \left(\frac{1}{4}(2\nu+5)\right)_k}{\left(\frac{3}{2}\right)_k k!} \left(-\frac{1}{z^2}\right)^k + O\left(\frac{1}{z^{2n+2}}\right)\right) /; (|z| \to \infty)$$

03.01.06.0012.01

$$J_\nu(z) \propto \frac{1}{\sqrt{2\pi}} \left(\frac{1}{\sqrt{-z}} \left(1 + \frac{i\sqrt{-z^2}}{z}\right) e^{i\pi\nu} \cos\left(z + \frac{\pi(2\nu+1)}{4}\right) + \frac{1}{\sqrt{z}} \left(1 - \frac{i\sqrt{-z^2}}{z}\right) \cos\left(z - \frac{\pi(2\nu+1)}{4}\right)\right)$$
$${}_4F_1\left(\frac{1-2\nu}{4}, \frac{3-2\nu}{4}, \frac{2\nu+1}{4}, \frac{2\nu+3}{4}; \frac{1}{2}; -\frac{1}{z^2}\right) +$$
$$\frac{1 - 4\nu^2}{8z\sqrt{2\pi}} \left(\frac{1}{\sqrt{-z}} \left(1 + \frac{i\sqrt{-z^2}}{z}\right) e^{i\pi\nu} \sin\left(z + \frac{\pi(2\nu+1)}{4}\right) + \frac{1}{\sqrt{z}} \left(1 - \frac{i\sqrt{-z^2}}{z}\right) \sin\left(z - \frac{\pi(2\nu+1)}{4}\right)\right)$$
$${}_4F_1\left(\frac{3-2\nu}{4}, \frac{5-2\nu}{4}, \frac{2\nu+3}{4}, \frac{2\nu+5}{4}; \frac{3}{2}; -\frac{1}{z^2}\right) /; (|z| \to \infty)$$

03.01.06.0013.01

$$J_\nu(z) \propto \frac{1}{\sqrt{2\pi}}\left(\frac{1}{\sqrt{-z}}\left(1+\frac{i\sqrt{-z^2}}{z}\right)e^{i\pi\nu}\cos\left(z+\frac{\pi(2\nu+1)}{4}\right)+\frac{1}{\sqrt{z}}\left(1-\frac{i\sqrt{-z^2}}{z}\right)\cos\left(z-\frac{\pi(2\nu+1)}{4}\right)\right)\left(1+O\left(\frac{1}{z^2}\right)\right)+$$
$$\frac{1-4\nu^2}{8z\sqrt{2\pi}}\left(\frac{1}{\sqrt{-z}}\left(1+\frac{i\sqrt{-z^2}}{z}\right)e^{i\pi\nu}\sin\left(z+\frac{\pi(2\nu+1)}{4}\right)+\frac{1}{\sqrt{z}}\left(1-\frac{i\sqrt{-z^2}}{z}\right)\sin\left(z-\frac{\pi(2\nu+1)}{4}\right)\right)$$
$$\left(1+O\left(\frac{1}{z^2}\right)\right) /; (|z|\to\infty)$$

03.01.06.0079.01

$$J_\nu(z) \propto \begin{cases} \frac{\sqrt{2}}{\sqrt{\pi z}}\cos\left(z-\frac{\pi(2\nu+1)}{4}\right) & \arg(z)\le 0 \\ \frac{i e^{i\pi\nu}\sqrt{2}}{\sqrt{\pi z}}\cos\left(z+\frac{\pi(2\nu+1)}{4}\right) & \text{True} \end{cases} /; (|z|\to\infty)$$

Residue representations

03.01.06.0014.01

$$J_\nu(z) = \pi\, 2^{-\nu} z^\nu \sum_{j=0}^{\infty} \operatorname{res}_s\left(\frac{\left(-\frac{z^2}{4}\right)^{-s}}{\Gamma\left(s+\frac{1}{2}\right)\Gamma(1+\nu-s)\Gamma\left(\frac{1}{2}-s\right)}\Gamma(s)\right)(-j)$$

03.01.06.0015.01

$$J_\nu(z) = z^\nu \left(z^2\right)^{-\frac{\nu}{2}} \sum_{j=0}^{\infty} \operatorname{res}_s\left(\frac{\left(\frac{z^2}{4}\right)^{-s}}{\Gamma\left(1+\frac{\nu}{2}-s\right)}\Gamma\left(s+\frac{\nu}{2}\right)\right)\left(-j-\frac{\nu}{2}\right)$$

03.01.06.0016.01

$$J_\nu(z) = \pi z^\nu \left(-z^2\right)^{-\frac{\nu}{2}} \sum_{j=0}^{\infty} \operatorname{res}_s\left(\frac{\left(-\frac{z^2}{4}\right)^{-s}}{\Gamma\left(\frac{\nu+1}{2}+s\right)\Gamma\left(1+\frac{\nu}{2}-s\right)\Gamma\left(\frac{1-\nu}{2}-s\right)}\Gamma\left(s+\frac{\nu}{2}\right)\right)\left(-j-\frac{\nu}{2}\right)$$

03.01.06.0017.01

$$J_\nu(z) = \sum_{j=0}^{\infty} \operatorname{res}_s\left(\frac{\left(\frac{z}{2}\right)^{-2s}}{\Gamma\left(1+\frac{\nu}{2}-s\right)}\Gamma\left(s+\frac{\nu}{2}\right)\right)\left(-j-\frac{\nu}{2}\right)$$

03.01.06.0018.01

$$J_\nu(z) = \pi z^\nu (i z)^{-\nu} \sum_{j=0}^{\infty} \operatorname{res}_s\left(\frac{\left(\frac{i z}{2}\right)^{-2s}}{\Gamma\left(\frac{\nu+1}{2}+s\right)\Gamma\left(1+\frac{\nu}{2}-s\right)\Gamma\left(\frac{1-\nu}{2}-s\right)}\Gamma\left(s+\frac{\nu}{2}\right)\right)\left(-j-\frac{\nu}{2}\right)$$

Integral representations

On the real axis

Of the direct function

03.01.07.0001.01

$$J_\nu(z) = \frac{2^{1-\nu}}{\sqrt{\pi}\,\Gamma\left(\nu+\frac{1}{2}\right)} z^\nu \int_0^1 \left(1-t^2\right)^{\nu-\frac{1}{2}} \cos(z\,t)\,dt \;/;\; \operatorname{Re}(\nu) > -\frac{1}{2}$$

03.01.07.0002.01

$$J_\nu(z) = \frac{1}{\Gamma\left(\nu+\frac{1}{2}\right)\sqrt{\pi}} \left(\frac{z}{2}\right)^\nu \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(z\sin(t)) \cos^{2\nu}(t)\,dt \;/;\; \operatorname{Re}(\nu) > -\frac{1}{2}$$

03.01.07.0003.01

$$J_\nu(z) = \frac{1}{\Gamma\left(\nu+\frac{1}{2}\right)\sqrt{\pi}} \left(\frac{z}{2}\right)^\nu \int_0^\pi \cos(z\cos(t)) \sin^{2\nu}(t)\,dt \;/;\; \operatorname{Re}(\nu) > -\frac{1}{2}$$

03.01.07.0004.01

$$J_\nu(z) = \frac{1}{\pi} \int_0^\pi \cos(z\sin(t) - t\,\nu)\,dt - \frac{\sin(\pi\,\nu)}{\pi} \int_0^\infty e^{-t\,\nu - z\sinh(t)}\,dt \;/;\; \arg(z) < \frac{\pi}{2}$$

03.01.07.0005.01

$$J_n(z) = \frac{i^{-n}}{\pi} \int_0^\pi e^{i\,z\cos(t)} \cos(n\,t)\,dt \;/;\; n \in \mathbb{N}^+$$

03.01.07.0006.02

$$J_n(z) = \frac{1}{\pi} \int_0^\pi \cos(t\,n - z\sin(t))\,dt \;/;\; n \in \mathbb{Z}$$

Contour integral representations

03.01.07.0007.01

$$J_\nu(z) = \frac{1}{2\pi i} \left(\frac{z}{2}\right)^\nu \int_{\gamma-i\infty}^{\gamma+i\infty} e^{t-\frac{z^2}{4t}} t^{-\nu-1}\,dt \;/;\; \gamma > 0 \bigwedge \operatorname{Re}(\nu) > 0$$

03.01.07.0008.01

$$J_\nu(x) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\Gamma(s)}{\Gamma(1+\nu-s)} \left(\frac{x}{2}\right)^{\nu-2s} ds \;/;\; x > 0 \bigwedge 0 < \gamma < \frac{3}{4} + \frac{\operatorname{Re}(\nu)}{2}$$

03.01.07.0009.01

$$J_\nu(z) = \frac{z^\nu \left(-z^2\right)^{-\frac{\nu}{2}}}{2\pi i} \int_{\mathcal{L}} \frac{\Gamma\left(s+\frac{\nu}{2}\right)}{\Gamma\left(1+\frac{\nu}{2}-s\right)} \left(\frac{z^2}{4}\right)^{-s} ds$$

03.01.07.0010.01

$$J_\nu(z) = \frac{z^\nu \left(-z^2\right)^{-\frac{\nu}{2}}}{2\,i} \int_{\mathcal{L}} \frac{\Gamma\left(s+\frac{\nu}{2}\right)}{\Gamma\left(s+\frac{\nu+1}{2}\right)\Gamma\left(1+\frac{\nu}{2}-s\right)\Gamma\left(\frac{1-\nu}{2}-s\right)} \left(-\frac{z^2}{4}\right)^{-s} ds$$

03.01.07.0011.01

$$J_\nu(z) = \frac{1}{2\pi i} \int_{\mathcal{L}} \frac{\Gamma\left(s+\frac{\nu}{2}\right)}{\Gamma\left(1+\frac{\nu}{2}-s\right)} \left(\frac{z}{2}\right)^{-2s} ds$$

03.01.07.0012.01

$$J_\nu(z) = \frac{\pi z^\nu (i\,z)^{-\nu}}{2\pi i} \int_{\mathcal{L}} \frac{\Gamma\left(s+\frac{\nu}{2}\right)}{\Gamma\left(s+\frac{\nu+1}{2}\right)\Gamma\left(1+\frac{\nu}{2}-s\right)\Gamma\left(\frac{1-\nu}{2}-s\right)} \left(\frac{i\,z}{2}\right)^{-2s} ds$$

Integral representations of negative integer order

For spherical Bessel functions

03.01.07.0013.01

$$j_n(z) = \sum_{k=0}^{\left\lfloor \frac{l}{2} \right\rfloor} c_{k,n} \frac{\partial^{n-2k} j_0(z)}{\partial x^{n-2k}} /; n \in \mathbb{N}^+ \bigwedge j_n(z) = \sqrt{\frac{\pi}{2z}} J_{n+\frac{1}{2}}(z) \bigwedge c_{0,0} = 0 \bigwedge c_{1,2} = \frac{1}{2} \bigwedge$$
$$c_{0,n} = \frac{(-1)^n (2n-1)!!}{n!} \bigwedge \left(c_{k,n} = \frac{(n-1)\, c_{k-1,n-2}}{n} - \frac{(2n-1)\, c_{k,n-1}}{n} /; n \geq 1 \land k \leq n\right) \bigwedge (c_{k,n} = 0 /; k > n)$$

Limit representations

03.01.09.0001.01

$$J_\nu(z) = \lim_{\lambda \to \infty} \lambda^\nu P_\lambda^{-\nu}\left(\cos\left(\frac{z}{\lambda}\right)\right)$$

03.01.09.0002.01

$$J_\nu(z) = \lim_{n \to \infty} \frac{1}{n^\nu} \left(\frac{2}{z}\right)^{-\nu} P_n^{(\nu,b)}\left(\cos\left(\frac{z}{n}\right)\right)$$

03.01.09.0003.01

$$J_\nu(z) = \left(\frac{z}{2}\right)^\nu \lim_{n \to \infty} \frac{1}{n^\nu} L_n^\nu\left(\frac{z^2}{4n}\right)$$

03.01.09.0004.01

$$J_\nu(z) = \frac{z^\nu}{2^\nu \Gamma(\nu+1)} \lim_{a \to \infty} {}_1F_1\left(a; \nu+1; -\frac{z^2}{4a}\right)$$

Generating functions

03.01.11.0001.01

$$\sum_{k=-\infty}^{\infty} t^k J_k(z) = e^{\frac{1}{2} z \left(t - \frac{1}{t}\right)}$$

03.01.11.0002.01

$$\sum_{k=-\infty}^{\infty} J_k(z)\, e^{i k q} = e^{i z \sin(q)}$$

P. Abbott

Differential equations

Ordinary linear differential equations and wronskians

For the direct function itself

03.01.13.0001.01

$$w''(z)\, z^2 + w'(z)\, z + \left(z^2 - \nu^2\right) w(z) = 0 /; w(z) = c_1 J_\nu(z) + c_2 Y_\nu(z)$$

03.01.13.0002.01

$$W_z(J_\nu(z),\ Y_\nu(z)) = \frac{2}{\pi z}$$

03.01.13.0003.01

$$w''(z)\, z^2 + w'(z)\, z + \left(z^2 - \nu^2\right) w(z) = 0 \,/;\, w(z) = c_1\, J_\nu(z) + c_2\, J_{-\nu}(z) \bigwedge \nu \notin \mathbb{Z}$$

03.01.13.0004.01

$$W_z(J_\nu(z),\ J_{-\nu}(z)) = -\frac{2\sin(\pi\, \nu)}{\pi\, z}$$

03.01.13.0005.01

$$w''(z) + a\, z^n\, w(z) = 0 \,/;\, w(z) = \sqrt{z} \left(c_1\, J_{\frac{1}{n+2}}\left(\frac{2\sqrt{a}}{n+2}\, z^{\frac{n+2}{2}}\right) + c_2\, Y_{\frac{1}{n+2}}\left(\frac{2\sqrt{a}}{n+2}\, z^{\frac{n+2}{2}}\right)\right)$$

03.01.13.0006.01

$$W_z\left(\sqrt{z}\ J_{\frac{1}{n+2}}\left(\frac{2\sqrt{a}}{n+2}\, z^{\frac{n+2}{2}}\right),\ \sqrt{z}\ Y_{\frac{1}{n+2}}\left(\frac{2\sqrt{a}}{n+2}\, z^{\frac{n+2}{2}}\right)\right) = \frac{n+2}{\pi}$$

03.01.13.0007.01

$$w''(z) + a\, z^n\, w(z) = 0 \,/;\, w(z) = \sqrt{z} \left(c_1\, J_{\frac{1}{n+2}}\left(\frac{2\sqrt{a}}{n+2}\, z^{\frac{n+2}{2}}\right) + c_2\, J_{-\frac{1}{n+2}}\left(\frac{2\sqrt{a}}{n+2}\, z^{\frac{n+2}{2}}\right)\right) \bigwedge \frac{1}{n+2} \notin \mathbb{Z}$$

03.01.13.0008.01

$$W_z\left(\sqrt{z}\ J_{\frac{1}{n+2}}\left(\frac{2\sqrt{a}\ z^{\frac{n+2}{2}}}{n+2}\right),\ \sqrt{z}\ J_{-\frac{1}{n+2}}\left(\frac{2\sqrt{a}\ z^{\frac{n+2}{2}}}{n+2}\right)\right) = -\frac{n+2}{\pi} \sin\left(\frac{\pi}{n+2}\right)$$

03.01.13.0009.01

$$w''(z) + \left(m^2 - \frac{1}{z^2}\left(\nu^2 - \frac{1}{4}\right)\right) w(z) = 0 \,/;\, w(z) = c_1 \sqrt{z}\ J_\nu\left(\sqrt{m^2}\ z\right) + c_2 \sqrt{z}\ Y_\nu\left(\sqrt{m^2}\ z\right)$$

03.01.13.0010.01

$$W_z\left(\sqrt{z}\ J_\nu\left(\sqrt{m^2}\ z\right),\ \sqrt{z}\ Y_\nu\left(\sqrt{m^2}\ z\right)\right) = \frac{2}{\pi}$$

03.01.13.0011.01

$$w''(z) + \left(m^2 - \frac{1}{z^2}\left(\nu^2 - \frac{1}{4}\right)\right) w(z) = 0 \,/;\, w(z) = c_1 \sqrt{z}\ J_\nu\left(\sqrt{m^2}\ z\right) + c_2 \sqrt{z}\ J_{-\nu}\left(\sqrt{m^2}\ z\right) \bigwedge \nu \notin \mathbb{Z}$$

03.01.13.0012.01

$$W_z\left(\sqrt{z}\ J_\nu\left(\sqrt{m^2}\ z\right),\ \sqrt{z}\ J_{-\nu}\left(\sqrt{m^2}\ z\right)\right) = -\frac{2\sin(\pi\, \nu)}{\pi}$$

03.01.13.0013.01

$$w''(z) + \left(\frac{m^2}{4\, z} - \frac{\nu^2 - 1}{4\, z^2}\right) w(z) = 0 \,/;\, w(z) = c_1 \sqrt{z}\ J_\nu\left(\sqrt{m^2}\ \sqrt{z}\right) + c_2 \sqrt{z}\ Y_\nu\left(\sqrt{m^2}\ \sqrt{z}\right)$$

03.01.13.0014.01

$$W_z\left(\sqrt{z}\ J_\nu\left(\sqrt{m^2}\ \sqrt{z}\right),\ \sqrt{z}\ Y_\nu\left(\sqrt{m^2}\ \sqrt{z}\right)\right) = \frac{1}{\pi}$$

03.01.13.0015.01

$$w''(z)+\left(\frac{m^2}{4z}-\frac{\nu^2-1}{4z^2}\right)w(z)=0\;/;\;w(z)=c_1\sqrt{z}\;J_\nu\left(\sqrt{m^2}\;\sqrt{z}\right)+c_2\sqrt{z}\;J_{-\nu}\left(\sqrt{m^2}\;\sqrt{z}\right)\bigwedge\nu\notin\mathbb{Z}$$

03.01.13.0016.01

$$W_z\left(\sqrt{z}\;J_\nu\left(\sqrt{m^2}\;\sqrt{z}\right),\sqrt{z}\;J_{-\nu}\left(\sqrt{m^2}\;\sqrt{z}\right)\right)=-\frac{\sin(\pi\nu)}{\pi}$$

03.01.13.0017.01

$$w''(z)-\frac{2\nu-1}{z}w'(z)+m^2w(z)=0\;/;\;w(z)=c_1z^\nu J_\nu(mz)+c_2z^\nu Y_\nu(mz)$$

03.01.13.0018.01

$$W_z(z^\nu J_\nu(mz),z^\nu Y_\nu(mz))=\frac{2}{\pi}z^{2\nu-1}$$

03.01.13.0019.01

$$w''(z)-\frac{2\nu-1}{z}w'(z)+m^2w(z)=0\;/;\;w(z)=c_1z^\nu J_\nu(mz)+c_2z^\nu J_{-\nu}(mz)\bigwedge\nu\notin\mathbb{Z}$$

03.01.13.0020.01

$$W_z(z^\nu J_\nu(mz),z^\nu J_{-\nu}(mz))=-\frac{2}{\pi}z^{2\nu-1}\sin(\pi\nu)$$

03.01.13.0023.01

$$z^2w''(z)+(1-2p)z\,w'(z)+\left(m^2q^2z^{2q}+p^2-\nu^2q^2\right)w(z)=0\;/;\;w(z)=c_1z^pJ_\nu(mz^q)+c_2z^pJ_{-\nu}(mz^q)\bigwedge\nu\notin\mathbb{Z}$$

03.01.13.0024.01

$$W_z(J_\nu(me^z),Y_\nu(me^z))=\frac{2}{\pi}$$

03.01.13.0025.01

$$w''(z)+\left(m^2e^{2z}-\nu^2\right)w(z)=0\;/;\;w(z)=c_1J_\nu(me^z)+c_2Y_\nu(me^z)$$

03.01.13.0026.01

$$W_z(J_\nu(me^z),J_{-\nu}(me^z))=-\frac{2}{\pi}\sin(\pi\nu)$$

03.01.13.0027.01

$$\left(z^2-\nu^2\right)z^2w''(z)+\left(z^2-3\nu^2\right)z\,w'(z)+\left(\left(z^2-\nu^2\right)^2-z^2-\nu^2\right)w(z)=0\;/;\;w(z)=c_1\frac{\partial J_\nu(z)}{\partial z}+c_2\frac{\partial Y_\nu(z)}{\partial z}$$

03.01.13.0028.01

$$W_z\left(\frac{\partial J_\nu(z)}{\partial z},\frac{\partial Y_\nu(z)}{\partial z}\right)=\frac{1}{2\pi z}(4-\pi\nu J_{\nu+1}(z)Y_{\nu-1}(z)+\pi\nu J_{\nu-1}(z)Y_{\nu+1}(z))$$

03.01.13.0029.01

$$w^{(4)}(z)-\frac{m^4}{z^2}w(z)=0\;/;\;w(z)=c_1z\left(I_2\left(2m\sqrt{z}\right)-J_2\left(2m\sqrt{z}\right)\right)+ c_2z\left(I_2\left(2m\sqrt{z}\right)+J_2\left(2m\sqrt{z}\right)\right)+c_3G_{0,4}^{2,0}\left(\frac{m^4z^2}{16}\,\middle|\,0,1,\tfrac{1}{2},\tfrac{3}{2}\right)+c_4G_{0,4}^{2,0}\left(\frac{m^4z^2}{16}\,\middle|\,\tfrac{1}{2},\tfrac{3}{2},0,1\right)$$

03.01.13.0034.01

$$w''(z)-\left(\frac{g''(z)}{g'(z)}-\frac{g'(z)}{g(z)}\right)w'(z)-\left(\frac{\nu^2}{g(z)^2}-1\right)g'(z)^2w(z)=0\;/;\;w(z)=c_1J_\nu(g(z))+c_2Y_\nu(g(z))$$

03.01.13.0035.01

$$W_z(J_\nu(g(z)),\, Y_\nu(g(z))) = \frac{2\,g'(z)}{\pi\, g(z)}$$

03.01.13.0036.01

$$w''(z) - \left(-\frac{g'(z)}{g(z)} + \frac{2\,h'(z)}{h(z)} + \frac{g''(z)}{g'(z)}\right) w'(z) - \left(\left(\frac{\nu^2}{g(z)^2} - 1\right) g'(z)^2 + \frac{h'(z)\,g'(z)}{g(z)\,h(z)} + \frac{h(z)\,h''(z) - 2\,h'(z)^2}{h(z)^2} - \frac{h'(z)\,g''(z)}{h(z)\,g'(z)}\right) w(z) = 0 \;/;\; w(z) = c_1\, h(z)\, J_\nu(g(z)) + c_2\, h(z)\, Y_\nu(g(z))$$

03.01.13.0037.01

$$W_z(h(z)\,J_\nu(g(z)),\, h(z)\,Y_\nu(g(z))) = \frac{2\,h(z)^2\,g'(z)}{\pi\, g(z)}$$

03.01.13.0021.01

$$z^2\, w''(z) + (1 - 2\,s)\,z\,w'(z) + \left(a^2\,r^2\,z^{2\,r} - \nu^2\,r^2 + s^2\right) w(z) = 0 \;/;\; w(z) = c_1\, z^s\, J_\nu(a\,z^r) + c_2\, z^s\, Y_\nu(a\,z^r)$$

03.01.13.0022.01

$$W_z(z^s\,J_\nu(a\,z^r),\, z^s\,Y_\nu(a\,z^r)) = \frac{2\,r\,z^{2\,s-1}}{\pi}$$

03.01.13.0038.01

$$w''(z) - 2\log(s)\,w'(z) + \left(\left(a^2\,r^{2\,z} - \nu^2\right)\log^2(r) + \log^2(s)\right) w(z) = 0 \;/;\; w(z) = c_1\, s^z\, J_\nu(a\,r^z) + c_2\, s^z\, Y_\nu(a\,r^z)$$

03.01.13.0039.01

$$W_z(s^z\,J_\nu(a\,r^z),\, s^z\,Y_\nu(a\,r^z)) = \frac{2\,s^{2\,z}\log(r)}{\pi}$$

Involving related functions

03.01.13.0030.01

$$\left(\prod_{k=1}^{4}\left(z\,\frac{d}{d\,z}\right)\right) w(z) - 2\left(\mu^2 + \nu^2\right)\left(\prod_{k=1}^{2}\left(z\,\frac{d}{d\,z}\right)\right) w(z) + \left(\nu^2 - \mu^2\right)^2 w(z) + 4\,z^2\left(\left(\prod_{k=1}^{2}\left(z\,\frac{d}{d\,z}\right)\right) w(z) + 2\,w(z) + 3\,z\,w'(z)\right) = 0 \;/;\; w(z) = c_1\, J_\mu(z)\, J_\nu(z) + c_2\, J_\nu(z)\, Y_\mu(z) + c_3\, J_\mu(z)\, Y_\nu(z) + c_4\, Y_\mu(z)\, Y_\nu(z)$$

03.01.13.0031.01

$$\left(\prod_{k=1}^{4}\left(z\,\frac{d}{d\,z}\right)\right) w(z) - 2\left(\mu^2 + \nu^2\right)\left(\prod_{k=1}^{2}\left(z\,\frac{d}{d\,z}\right)\right) w(z) + \left(\nu^2 - \mu^2\right)^2 w(z) + 4\,z^2\left(\left(\prod_{k=1}^{2}\left(z\,\frac{d}{d\,z}\right)\right) w(z) + 2\,w(z) + 3\,z\,w'(z)\right) = 0 \;/;\; w(z) = c_1\, J_\mu(z)\, J_\nu(z) + c_2\, J_{-\mu}(z)\, J_\nu(z) + c_3\, J_\mu(z)\, J_{-\nu}(z) + c_4\, J_{-\mu}(z)\, J_{-\nu}(z)$$

03.01.13.0032.01

$$\left(\prod_{k=1}^{3}\left(z\,\frac{d}{d\,z}\right)\right) w(z) + 4\left(z^2 - \nu^2\right) z\; w'(z) + 4\,z^2\,w(z) = 0 \;/;\; w(z) = c_1\, J_\nu(z)^2 + c_2\, Y_\nu(z)\, J_\nu(z) + c_3\, Y_\nu(z)^2$$

03.01.13.0033.01

$$z^3\, w^{(3)}(z) + \left(4\,z^2 - 4\,\nu^2 + 1\right) z\, w'(z) + \left(4\,\nu^2 - 1\right) w(z) = 0 \;/;\; w(z) = c_1\, z\, J_\nu(z)^2 + c_2\, z\, Y_\nu(z)\, J_\nu(z) + c_3\, z\, Y_\nu(z)^2$$

Transformations

Transformations and argument simplifications

Argument involving basic arithmetic operations

03.01.16.0001.01

$$J_\nu(-z) = (-z)^\nu z^{-\nu} J_\nu(z)$$

03.01.16.0002.01

$$J_\nu(i z) = \frac{(i z)^\nu}{z^\nu} I_\nu(z)$$

03.01.16.0003.01

$$J_\nu(-i z) = \frac{(-i z)^\nu}{z^\nu} I_\nu(z)$$

03.01.16.0004.01

$$J_\nu\left(\sqrt{z^2}\right) = z^{-\nu} \left(z^2\right)^{\nu/2} J_\nu(z)$$

03.01.16.0005.01

$$J_\nu(c (d z^n)^m) = \frac{(c (d z^n)^m)^\nu}{(c d^m z^{m n})^\nu} J_\nu(c d^m z^{m n}) /; 2 m \in \mathbb{Z}$$

Addition formulas

03.01.16.0006.01

$$J_\nu(z_1 - z_2) = \sum_{k=-\infty}^{\infty} J_{k+\nu}(z_1) J_k(z_2) /; \left|\frac{z_2}{z_1}\right| < 1 \bigvee \nu \in \mathbb{Z}$$

03.01.16.0007.01

$$J_\nu(z_1 + z_2) = \sum_{k=-\infty}^{\infty} J_{\nu-k}(z_1) J_k(z_2) /; \left|\frac{z_2}{z_1}\right| < 1 \bigvee \nu \in \mathbb{Z}$$

Multiple arguments

03.01.16.0008.01

$$J_\nu(z_1 z_2) = z_1^\nu \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \left(z_1^2 - 1\right)^k J_{k+\nu}(z_2) \left(\frac{z_2}{2}\right)^k$$

03.01.16.0009.01

$$J_\nu(z_1 z_2) = z_1^{-\nu} \sum_{k=0}^{\infty} \frac{1}{k!} \left(z_1^2 - 1\right)^k J_{\nu-k}(z_2) \left(\frac{z_2}{2}\right)^k$$

Identities

Recurrence identities

Consecutive neighbors

03.01.17.0001.01

$$J_\nu(z) = \frac{2(\nu+1)}{z} J_{\nu+1}(z) - J_{\nu+2}(z)$$

03.01.17.0002.01

$$J_\nu(z) = \frac{2(\nu-1)}{z} J_{\nu-1}(z) - J_{\nu-2}(z)$$

Distant neighbors

Increasing

03.01.17.0003.01

$$J_\nu(z) = 2^{n-1} z^{-n} (\nu+1)_{n-1} \left(2(n+\nu) \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(n-k)!}{k!\,(n-2k)!\,(-n-\nu)_k\,(\nu+1)_k} \left(\frac{z^2}{4}\right)^k J_{n+\nu}(z) - z \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \frac{(n-k-1)!}{k!\,(n-2k-1)!\,(1-\nu-n)_k\,(\nu+1)_k} \left(\frac{z^2}{4}\right)^k J_{n+\nu+1}(z)\right) /;\ n \in \mathbb{N}$$

03.01.17.0015.01

$$J_\nu(z) = 2^{n-1} z^{-n} (\nu+1)_{n-1} \left(2(n+\nu)\, {}_3F_4\!\left(1, \frac{1-n}{2}, -\frac{n}{2}; 1, -n, -n-\nu, \nu+1; -z^2\right) J_{n+\nu}(z) - z\, {}_3F_4\!\left(1, \frac{1-n}{2}, 1-\frac{n}{2}; 1, 1-n, -n-\nu+1, \nu+1; -z^2\right) J_{n+\nu+1}(z)\right) /;\ n \in \mathbb{N}$$

03.01.17.0007.01

$$J_\nu(z) = \frac{\left(4(\nu+1)(\nu+2) - z^2\right) J_{\nu+2}(z) - 2z(\nu+1) J_{\nu+3}(z)}{z^2}$$

03.01.17.0008.01

$$J_\nu(z) = \frac{4(\nu+2)\left(2(\nu+1)(\nu+3) - z^2\right) J_{\nu+3}(z) + z\left(z^2 - 4(\nu+1)(\nu+2)\right) J_{\nu+4}(z)}{z^3}$$

03.01.17.0009.01

$$J_\nu(z) = \frac{1}{z^4}\left(\left(z^4 - 12(\nu+2)(\nu+3)z^2 + 16(\nu+1)(\nu+2)(\nu+3)(\nu+4)\right) J_{\nu+4}(z) + 4z(\nu+2)\left(z^2 - 2(\nu+1)(\nu+3)\right) J_{\nu+5}(z)\right)$$

03.01.17.0010.01

$$J_\nu(z) = \frac{1}{z^5}\left(2(\nu+3)\left(3z^4 - 16(\nu+2)(\nu+4)z^2 + 16(\nu+1)(\nu+2)(\nu+4)(\nu+5)\right) J_{\nu+5}(z) - z\left(z^4 - 12(\nu+2)(\nu+3)z^2 + 16(\nu+1)(\nu+2)(\nu+3)(\nu+4)\right) J_{\nu+6}(z)\right)$$

03.01.17.0016.01

$$J_\nu(z) = C_n(\nu, z)\, J_{\nu+n}(z) - C_{n-1}(\nu, z)\, J_{\nu+n+1}(z) /;\ C_0(\nu, z) = 1 \bigwedge C_1(\nu, z) = \frac{2(\nu+1)}{z} \bigwedge C_n(\nu, z) = \frac{2(n+\nu)}{z} C_{n-1}(\nu, z) - C_{n-2}(\nu, z) \bigwedge n \in \mathbb{N}^+$$

03.01.17.0017.01

$$J_\nu(z) = C_n(\nu, z)\, J_{\nu+n}(z) - C_{n-1}(\nu, z)\, J_{\nu+n+1}(z) /;\ C_n(\nu, z) = 2^n z^{-n} (\nu+1)_n\, {}_2F_3\!\left(\frac{1-n}{2}, -\frac{n}{2}; \nu+1, -n, -n-\nu; -z^2\right) \bigwedge n \in \mathbb{N}^+$$

Decreasing

03.01.17.0004.01

$$J_\nu(z) = 2^{n-1}(-z)^{-n}(1-\nu)_{n-1}\left(z\sum_{k=0}^{\left\lfloor\frac{n-1}{2}\right\rfloor}\frac{(n-k-1)!}{k!\,(n-2k-1)!\,(1-\nu)_k\,(\nu-n+1)_k}\left(\frac{z^2}{4}\right)^k J_{\nu-n-1}(z) + 2(n-\nu)\sum_{k=0}^{\left\lfloor\frac{n}{2}\right\rfloor}\frac{(n-k)!}{k!\,(n-2k)!\,(1-\nu)_k\,(\nu-n)_k}\left(\frac{z^2}{4}\right)^k J_{\nu-n}(z)\right) /;\ n\in\mathbb{N}$$

03.01.17.0018.01

$$J_\nu(z) = 2^{n-1}(-z)^{-n}(1-\nu)_{n-1}\left(2(n-\nu)\ {}_3F_4\left(1, \frac{1-n}{2}, -\frac{n}{2}; 1, -n, 1-\nu, \nu-n; -z^2\right)J_{\nu-n}(z) + z\ {}_3F_4\left(1, \frac{1-n}{2}, 1-\frac{n}{2}; 1, 1-n, 1-\nu, -n+\nu+1; -z^2\right)J_{\nu-n-1}(z)\right) /;\ n\in\mathbb{N}$$

03.01.17.0011.01

$$J_\nu(z) = -\frac{2z(\nu-1)J_{\nu-3}(z) + \left(z^2-4(\nu-2)(\nu-1)\right)J_{\nu-2}(z)}{z^2}$$

03.01.17.0012.01

$$J_\nu(z) = \frac{z\left(z^2-4(\nu-2)(\nu-1)\right)J_{\nu-4}(z) - 4\left(z^2-2(\nu-3)(\nu-1)\right)(\nu-2)J_{\nu-3}(z)}{z^3}$$

03.01.17.0013.01

$$J_\nu(z) = \frac{1}{z^4}\left(4z\left(z^2-2(\nu-3)(\nu-1)\right)(\nu-2)J_{\nu-5}(z) + \left(z^4-12(\nu-3)(\nu-2)z^2+16(\nu-4)(\nu-3)(\nu-2)(\nu-1)\right)J_{\nu-4}(z)\right)$$

03.01.17.0014.01

$$J_\nu(z) = -\frac{1}{z^5}\left(z\left(z^4-12(\nu-3)(\nu-2)z^2+16(\nu-4)(\nu-3)(\nu-2)(\nu-1)\right)J_{\nu-6}(z) - 2\left(3z^4-16(\nu-4)(\nu-2)z^2+16(\nu-5)(\nu-4)(\nu-2)(\nu-1)\right)(\nu-3)J_{\nu-5}(z)\right)$$

03.01.17.0019.01

$$J_\nu(z) = C_n(\nu, z)J_{\nu-n}(z) - C_{n-1}(\nu, z)J_{\nu-n-1}(z)\ /;\ C_0(\nu, z) = 1 \bigwedge C_1(\nu, z) = \frac{2(\nu-1)}{z} \bigwedge C_n(\nu, z) = \frac{2(-n+\nu)}{z}C_{n-1}(\nu, z) - C_{n-2}(\nu, z) \bigwedge n\in\mathbb{N}^+$$

03.01.17.0020.01

$$J_\nu(z) = C_n(\nu, z)J_{\nu-n}(z) - C_{n-1}(\nu, z)J_{-n+\nu-1}(z)\ /;\ C_n(\nu, z) = (-2)^n z^{-n}(1-\nu)_n\ {}_2F_3\left(\frac{1-n}{2}, -\frac{n}{2}; 1-\nu, -n, \nu-n; -z^2\right) \bigwedge n\in\mathbb{N}^+$$

Functional identities

Relations between contiguous functions

03.01.17.0005.01

$$J_\nu(z) = \frac{z\left(J_{\nu-1}(z)+J_{\nu+1}(z)\right)}{2\nu}$$

Relations of special kind

03.01.17.0006.01

$$J_{\nu-1}(z)J_{-\nu}(z) + J_{1-\nu}(z)J_\nu(z) = \frac{2\sin(\nu\pi)}{\pi z}$$

Differentiation

Low-order differentiation

With respect to ν

03.01.20.0001.01

$$J_\nu^{(1,0)}(z) = J_\nu(z)\log\left(\frac{z}{2}\right) - \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(k+\nu+1)}\,\psi(k+\nu+1)\left(\frac{z}{2}\right)^{2k+\nu}$$

03.01.20.0002.01

$$J_\nu^{(1,0)}(z) = \frac{1}{(\nu+1)\,\Gamma(\nu+2)}\left(\frac{z}{2}\right)^{\nu+2} F_{2\times0\times1}^{0\times1\times2}\left(\begin{matrix} ;\,1;\,1,\,1+\nu; \\ 2,\,2+\nu;;\,2+\nu; \end{matrix}\; -\frac{z^2}{4},\, -\frac{z^2}{4}\right) + J_\nu(z)\,(\log(z)-\log(2)-\psi(\nu+1))$$

03.01.20.0003.02

$$J_n^{(1,0)}(z) = \frac{n!}{2}\left(\frac{z}{2}\right)^{-n}\sum_{k=0}^{n-1}\frac{1}{(n-k)\,k!}\,J_k(z)\left(\frac{z}{2}\right)^k + \frac{\pi}{2}\,Y_n(z)\;/;\; n\in\mathbb{N}$$

03.01.20.0018.01

$$J_{-n}^{(1,0)}(z) = \frac{n!}{2}\left(-\frac{z}{2}\right)^{-n}\sum_{k=0}^{n-1}\frac{1}{(n-k)\,k!}\,J_k(z)\left(\frac{z}{2}\right)^k + \frac{1}{n!}\left(-\frac{z}{2}\right)^n \sum_{j=1}^{n}\frac{1}{j}\,{}_1F_2\left(j;\,j+1,\,n+1;\,-\frac{z^2}{4}\right) + \frac{(-1)^n\,\pi}{2}\,Y_n(z) + (-1)^{n-1}\sum_{k=0}^{n-1}\frac{(n-k-1)!}{k!}\left(\frac{z}{2}\right)^{2k-n}\;/;\; n\in\mathbb{N}$$

03.01.20.0019.01

$$J_{n+\frac{1}{2}}^{(1,0)}(z) = \frac{2\,(2z)^{\frac{1}{2}-n}}{n!\,\sqrt{\pi}}\sum_{k=0}^{\left\lfloor\frac{n-1}{2}\right\rfloor}(-1)^k\,2^{2k}\binom{n}{2k+1}(2n-2k-1)!\left(\cos(z)\left(\psi\left(k+\frac{3}{2}\right)-\psi\left(k-n+\frac{1}{2}\right)\right)-\cos(z)\,\mathrm{Ci}(2z)-\sin(z)\,\mathrm{Si}(2z)\right)z^{2k} - \frac{2\,(2z)^{-n-\frac{1}{2}}}{n!\,\sqrt{\pi}}\sum_{k=0}^{\left\lfloor\frac{n}{2}\right\rfloor}(-1)^k\,2^{2k}\binom{n}{2k}(2n-2k)!\left(\left(\psi\left(k+\frac{1}{2}\right)-\psi\left(k-n+\frac{1}{2}\right)\right)\sin(z)+\cos(z)\,\mathrm{Si}(2z)-\mathrm{Ci}(2z)\sin(z)\right)z^{2k}\;/;\; n\in\mathbb{N}$$

Brychkov Yu.A. (2005)

03.01.20.0020.01

$$J_{-n-\frac{1}{2}}^{(1,0)}(z) = \frac{(-1)^n\,2\,(2z)^{-n-\frac{1}{2}}}{n!\,\sqrt{\pi}}\sum_{k=0}^{\left\lfloor\frac{n}{2}\right\rfloor}(-1)^k\,2^{2k}\binom{n}{2k}(2n-2k)!\left(\cos(z)\left(\psi\left(k+\frac{1}{2}\right)-\psi\left(k-n+\frac{1}{2}\right)\right)+\cos(z)\,\mathrm{Ci}(2z)+\sin(z)\,\mathrm{Si}(2z)\right)z^{2k} + \frac{(-1)^n\,2\,(2z)^{\frac{1}{2}-n}}{n!\,\sqrt{\pi}}\sum_{k=0}^{\left\lfloor\frac{n-1}{2}\right\rfloor}(-1)^k\,2^{2k}\binom{n}{2k+1}(2n-2k-1)!\left(\left(\psi\left(k+\frac{3}{2}\right)-\psi\left(k-n+\frac{1}{2}\right)\right)\sin(z)+\mathrm{Ci}(2z)\sin(z)-\cos(z)\,\mathrm{Si}(2z)\right)z^{2k}\;/;\; n\in\mathbb{N}$$

Brychkov Yu.A. (2005)

With respect to *z*

03.01.20.0004.01

$$\frac{\partial J_\nu(z)}{\partial z} = J_{\nu-1}(z) - \frac{\nu}{z} J_\nu(z)$$

03.01.20.0005.01

$$\frac{\partial J_\nu(z)}{\partial z} = \frac{\nu}{z} J_\nu(z) - J_{\nu+1}(z)$$

03.01.20.0006.01

$$\frac{\partial J_\nu(z)}{\partial z} = \frac{1}{2} (J_{\nu-1}(z) - J_{\nu+1}(z))$$

03.01.20.0007.01

$$\frac{\partial J_0(z)}{\partial z} = -J_1(z)$$

03.01.20.0008.01

$$\frac{\partial (z^\nu J_\nu(z))}{\partial z} = z^\nu J_{\nu-1}(z)$$

03.01.20.0009.01

$$\frac{\partial (z^{-\nu} J_\nu(z))}{\partial z} = -z^{-\nu} J_{\nu+1}(z)$$

03.01.20.0010.01

$$\frac{\partial^2 J_\nu(z)}{\partial z^2} = \frac{1}{4} (J_{\nu-2}(z) - 2 J_\nu(z) + J_{\nu+2}(z))$$

Symbolic differentiation

With respect to ν

03.01.20.0011.02

$$J_\nu^{(m,0)}(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \left(\frac{z}{2}\right)^{2k} \frac{\partial^m}{\partial \nu^m} \left(\frac{1}{\Gamma(k+\nu+1)} \left(\frac{z}{2}\right)^\nu \right) /; m \in \mathbb{N}$$

With respect to *z*

03.01.20.0012.01

$$J_\nu^{(0,n)}(0) = 0 /; n \in \mathbb{N}^+ \bigwedge \left(\nu \in \mathbb{Z} \land |\nu| > n \bigvee \frac{n-\nu-1}{2} \in \mathbb{N} \right)$$

03.01.20.0013.01

$$J_\nu^{(0,n)}(0) = \frac{(-1)^{\frac{n-\nu}{2}} 2^{-n} n!}{\Gamma\left(\frac{1}{2}(n-\nu+2)\right) \Gamma\left(\frac{1}{2}(n+\nu+2)\right)} /; n \in \mathbb{N}^+ \bigwedge \frac{n-\nu}{2} \in \mathbb{Z} \bigwedge |\nu| \le n$$

03.01.20.0021.01

$$\frac{\partial^n J_\nu(z)}{\partial z^n} = z^{-n} \sum_{m=0}^{n} (-1)^{m+n} \binom{n}{m} (-\nu)_{n-m} \sum_{k=0}^{m} \frac{(-1)^{k-1} 2^{2k-m} (-m)_{2(m-k)} (\nu)_k}{(m-k)!} \left(\frac{z}{2} \sum_{j=0}^{k-1} \frac{(k-j-1)!}{j!\,(k-2j-1)!\,(1-k-\nu)_j\,(\nu)_{j+1}} \left(\frac{z^2}{4}\right)^j J_{\nu-1}(z) - \sum_{j=0}^{k} \frac{(k-j)!}{j!\,(k-2j)!\,(1-k-\nu)_j\,(\nu)_j} \left(\frac{z^2}{4}\right)^j J_\nu(z) \right) /; n \in \mathbb{N}$$

03.01.20.0014.02

$$\frac{\partial^n J_\nu(z)}{\partial z^n} = 2^{n-2\nu} \sqrt{\pi}\, z^{\nu-n}\, \Gamma(\nu+1)\, {}_2\tilde{F}_3\left(\frac{\nu+1}{2}, \frac{\nu+2}{2}; \frac{1}{2}(-n+\nu+1), \frac{1}{2}(-n+\nu+2), \nu+1; -\frac{z^2}{4}\right) /; n \in \mathbb{N}$$

03.01.20.0015.02

$$\frac{\partial^n J_\nu(z)}{\partial z^n} = 2^{-n} \sum_{k=0}^{n} (-1)^k \binom{n}{k} J_{2k-n+\nu}(z) /; n \in \mathbb{N}$$

Fractional integro-differentiation

With respect to z

03.01.20.0016.01

$$\frac{\partial^\alpha J_\nu(z)}{\partial z^\alpha} = 2^{\alpha-2\nu} \sqrt{\pi}\, z^{\nu-\alpha}\, \Gamma(\nu+1)\, {}_2\tilde{F}_3\left(\frac{\nu+1}{2}, \frac{\nu+2}{2}; \frac{1}{2}(-\alpha+\nu+1), \frac{1}{2}(-\alpha+\nu+2), \nu+1; -\frac{z^2}{4}\right) /; -\nu \notin \mathbb{N}^+$$

03.01.20.0017.01

$$\frac{\partial^\alpha J_{-n}(z)}{\partial z^\alpha} = (-1)^n\, 2^{\alpha-2n} \sqrt{\pi}\, z^{n-\alpha}\, \Gamma(n+1)\, {}_2\tilde{F}_3\left(\frac{n+1}{2}, \frac{n+2}{2}; \frac{1}{2}(n-\alpha+1), \frac{1}{2}(n-\alpha+2), n+1; -\frac{z^2}{4}\right) /; n \in \mathbb{N}^+$$

Integration

Indefinite integration

Involving only one direct function

03.01.21.0001.01

$$\int J_\nu(a z)\, dz = 2^{-\nu-1} z\, (a z)^\nu\, \Gamma\left(\frac{\nu}{2}+\frac{1}{2}\right) {}_1\tilde{F}_2\left(\frac{\nu}{2}+\frac{1}{2}; \nu+1, \frac{\nu}{2}+\frac{3}{2}; -\frac{1}{4} a^2 z^2\right)$$

03.01.21.0002.01

$$\int J_\nu(z)\, dz = 2^{-\nu-1} z^{\nu+1}\, \Gamma\left(\frac{\nu}{2}+\frac{1}{2}\right) {}_1\tilde{F}_2\left(\frac{\nu}{2}+\frac{1}{2}; \nu+1, \frac{\nu}{2}+\frac{3}{2}; -\frac{z^2}{4}\right)$$

03.01.21.0003.01

$$\int J_0(z)\, dz = \frac{1}{2} z\, (\pi J_1(z)\, \boldsymbol{H}_0(z) + J_0(z)\, (2 - \pi\, \boldsymbol{H}_1(z)))$$

03.01.21.0004.01

$$\int J_1(z)\, dz = -J_0(z)$$

Involving one direct function and elementary functions

Involving power function

Involving power

Linear arguments

03.01.21.0005.01

$$\int z^{\alpha-1} J_\nu(a z) \, dz = 2^{-\nu-1} z^\alpha (a z)^\nu \, \Gamma\left(\frac{\alpha+\nu}{2}\right) {}_1\tilde{F}_2\left(\frac{\alpha+\nu}{2}; \nu+1, \frac{1}{2}(\alpha+\nu+2); -\frac{1}{4} a^2 z^2\right)$$

03.01.21.0006.01

$$\int z^{\alpha-1} J_\nu(z) \, dz = 2^{-\nu-1} z^{\alpha+\nu} \, \Gamma\left(\frac{\alpha}{2}+\frac{\nu}{2}\right) {}_1\tilde{F}_2\left(\frac{\alpha}{2}+\frac{\nu}{2}; \nu+1, \frac{\alpha}{2}+\frac{\nu}{2}+1; -\frac{z^2}{4}\right)$$

03.01.21.0007.01

$$\int z^{\alpha-1} J_0(z) \, dz = \frac{z^\alpha}{\alpha} \, {}_1F_2\left(\frac{\alpha}{2}; 1, \frac{\alpha}{2}+1; -\frac{z^2}{4}\right)$$

03.01.21.0008.01

$$\int z^{1-\nu} J_\nu(z) \, dz = -z^{1-\nu} J_{\nu-1}(z)$$

03.01.21.0009.01

$$\int z^{-\nu} J_\nu(z) \, dz = \frac{2^{-\nu} z}{\Gamma(\nu+1)} \, {}_1F_2\left(\frac{1}{2}; \frac{3}{2}, \nu+1; -\frac{z^2}{4}\right)$$

03.01.21.0010.01

$$\int z^{\nu+3} J_\nu(z) \, dz = \frac{z^{\nu+2} \, \Gamma(\nu+2)}{(\nu+1)\,\Gamma(\nu+1)} \left(2(\nu+1) J_{\nu+2}(z) - z J_{\nu+3}(z)\right)$$

03.01.21.0011.01

$$\int z^{\nu+1} J_\nu(z) \, dz = z^{\nu+1} J_{\nu+1}(z)$$

03.01.21.0012.01

$$\int z^\nu J_\nu(z) \, dz = 2^{-\nu-1} z^{2\nu+1} \, \Gamma\left(\nu+\frac{1}{2}\right) {}_1\tilde{F}_2\left(\nu+\frac{1}{2}; \nu+1, \nu+\frac{3}{2}; -\frac{z^2}{4}\right)$$

03.01.21.0013.01

$$\int z J_0(z) \, dz = z J_1(z)$$

03.01.21.0014.01

$$\int \frac{J_0(z)}{z} \, dz = -\frac{1}{2} G_{1,3}^{2,0}\left(\frac{z^2}{4} \,\middle|\, \begin{matrix} 1 \\ 0, 0, 0 \end{matrix}\right)$$

Power arguments

03.01.21.0015.01

$$\int z^{\alpha-1} J_\nu(a z^r) \, dz = \frac{2^{-\nu-1} z^\alpha (a z^r)^\nu}{r} \, \Gamma\left(\frac{\alpha+r\nu}{2r}\right) {}_1\tilde{F}_2\left(\frac{\alpha+r\nu}{2r}; \nu+1, \frac{\alpha+r(\nu+2)}{2r}; -\frac{1}{4} a^2 z^{2r}\right)$$

Involving exponential function

Involving exp

Linear arguments

03.01.21.0016.01

$$\int e^{-i a z} J_\nu(a z) \, dz = \frac{2^{-\nu} z (a z)^\nu}{(\nu+1)\Gamma(\nu+1)} \, {}_2F_2\left(\nu+\frac{1}{2}, \nu+1; \nu+2, 2\nu+1; -2 i a z\right)$$

03.01.21.0017.01

$$\int e^{i a z} J_\nu(a z) \, dz = \frac{2^{-\nu} z (a z)^\nu}{(\nu+1)\Gamma(\nu+1)} \, {}_2F_2\left(\nu+\frac{1}{2}, \nu+1; \nu+2, 2\nu+1; 2 i a z\right)$$

Power arguments

03.01.21.0018.01

$$\int e^{-i a z^r} J_\nu(a z^r) \, dz = \frac{2^{-\nu} z (a z^r)^\nu}{(r \nu+1)\Gamma(\nu+1)} \, {}_2F_2\left(\nu+\frac{1}{2}, \nu+\frac{1}{r}; \nu+\frac{1}{r}+1, 2\nu+1; -2 i a z^r\right)$$

03.01.21.0019.01

$$\int e^{i a z^r} J_\nu(a z^r) \, dz = \frac{2^{-\nu} z (a z^r)^\nu}{(r \nu+1)\Gamma(\nu+1)} \, {}_2F_2\left(\nu+\frac{1}{2}, \nu+\frac{1}{r}; \nu+\frac{1}{r}+1, 2\nu+1; 2 i a z^r\right)$$

Involving exponential function and a power function

Involving exp and power

Linear arguments

03.01.21.0020.01

$$\int z^{\alpha-1} e^{-i a z} J_\nu(a z) \, dz = \frac{2^{-\nu} z^\alpha (a z)^\nu}{(\alpha+\nu)\Gamma(\nu+1)} \, {}_2F_2\left(\nu+\frac{1}{2}, \alpha+\nu; \alpha+\nu+1, 2\nu+1; -2 i a z\right)$$

03.01.21.0021.01

$$\int z^{-\nu} e^{-i a z} J_\nu(a z) \, dz = \frac{i\, 2^{-\nu} e^{-i a z} z^{-\nu}}{a (2\nu-1)\Gamma(\nu)} \left(-2 e^{i a z} (a z)^\nu + 2^\nu a z J_{\nu-1}(a z)\Gamma(\nu) + 2^\nu a i z J_\nu(a z)\Gamma(\nu)\right)$$

03.01.21.0022.01

$$\int z^\nu e^{-i a z} J_\nu(a z) \, dz = \frac{e^{-i a z} z^{\nu+1}}{2\nu+1} \left(J_\nu(a z) + i J_{\nu+1}(a z)\right)$$

03.01.21.0023.01

$$\int z^{\alpha-1} e^{i a z} J_\nu(a z) \, dz = \frac{2^{-\nu} z^\alpha (a z)^\nu}{(\alpha+\nu)\Gamma(\nu+1)} \, {}_2F_2\left(\nu+\frac{1}{2}, \alpha+\nu; \alpha+\nu+1, 2\nu+1; 2 i a z\right)$$

03.01.21.0024.01

$$\int z^{-\nu} e^{i a z} J_\nu(a z) \, dz = \frac{2^{-\nu} z^{-\nu}}{a (2\nu-1)\Gamma(\nu)} \left(2 i (a z)^\nu - i\, 2^\nu a e^{i a z} z J_{\nu-1}(a z)\Gamma(\nu) - 2^\nu a e^{i a z} z J_\nu(a z)\Gamma(\nu)\right)$$

03.01.21.0025.01

$$\int z^\nu e^{i a z} J_\nu(a z)\, dz = \frac{e^{i a z} z^{\nu+1}}{2\nu+1} \left(J_\nu(a z) - i J_{\nu+1}(a z)\right)$$

Power arguments

03.01.21.0026.01

$$\int z^{\alpha-1} e^{-i a z^r} J_\nu(a z^r)\, dz = \frac{2^{-\nu} z^\alpha (a z^r)^\nu}{(\alpha + r\nu)\,\Gamma(\nu+1)}\, {}_2F_2\left(\nu + \frac{1}{2}, \frac{\alpha}{r} + \nu; \frac{\alpha}{r} + \nu + 1, 2\nu + 1; -2 i a z^r\right)$$

03.01.21.0027.01

$$\int z^{\alpha-1} e^{i a z^r} J_\nu(a z^r)\, dz = \frac{2^{-\nu} z^\alpha (a z^r)^\nu}{(\alpha + r\nu)\,\Gamma(\nu+1)}\, {}_2F_2\left(\nu + \frac{1}{2}, \frac{\alpha}{r} + \nu; \frac{\alpha}{r} + \nu + 1, 2\nu + 1; 2 i a z^r\right)$$

Involving trigonometric functions

Involving sin

Linear arguments

03.01.21.0028.01

$$\int \sin(a z) J_\nu(a z)\, dz = \frac{2^{-\nu} z (a z)^{\nu+1}}{(\nu+2)\,\Gamma(\nu+1)}\, {}_3F_4\left(\frac{\nu}{2} + \frac{3}{4}, \frac{\nu}{2} + 1, \frac{\nu}{2} + \frac{5}{4}; \frac{3}{2}, \frac{\nu}{2} + 2, \nu + 1, \nu + \frac{3}{2}; -a^2 z^2\right)$$

03.01.21.0029.01

$$\int \sin(b + a z) J_\nu(a z)\, dz = \frac{1}{(\nu+2)\,\Gamma(\nu+1)\,\Gamma(\nu+2)} \left(2^{-\nu} z (a z)^\nu \left(a z \cos(b)\, \Gamma(\nu+2)\, {}_3F_4\left(\frac{\nu}{2} + \frac{3}{4}, \frac{\nu}{2} + 1, \frac{\nu}{2} + \frac{5}{4}; \frac{3}{2}, \frac{\nu}{2} + 2, \nu + 1, \nu + \frac{3}{2}; -a^2 z^2\right) + (\nu+2)\,\Gamma(\nu+1)\, {}_3F_4\left(\frac{\nu}{2} + \frac{1}{4}, \frac{\nu}{2} + \frac{1}{2}, \frac{\nu}{2} + \frac{3}{4}; \frac{1}{2}, \frac{\nu}{2} + \frac{3}{2}, \nu + \frac{1}{2}, \nu + 1; -a^2 z^2\right) \sin(b)\right)\right)$$

Power arguments

03.01.21.0030.01

$$\int \sin(a z^r) J_\nu(a z^r)\, dz = \frac{2^{-\nu} z (a z^r)^{\nu+1}}{(\nu r + r + 1)\,\Gamma(\nu+1)}\, {}_3F_4\left(\frac{\nu}{2} + \frac{3}{4}, \frac{\nu}{2} + \frac{5}{4}, \frac{\nu}{2} + \frac{1}{2} + \frac{1}{2r}; \frac{3}{2}, \frac{\nu}{2} + \frac{3}{2} + \frac{1}{2r}, \nu + 1, \nu + \frac{3}{2}; -a^2 z^{2r}\right)$$

03.01.21.0031.01

$$\int \sin(a z^r + b) J_\nu(a z^r)\, dz = \frac{1}{(r\nu+1)(\nu r + r + 1)\,\Gamma(\nu+1)} \left(2^{-\nu} z (a z^r)^\nu \left(a (r\nu+1) \cos(b)\, {}_3F_4\left(\frac{\nu}{2} + \frac{3}{4}, \frac{\nu}{2} + \frac{5}{4}, \frac{\nu}{2} + \frac{1}{2} + \frac{1}{2r}; \frac{3}{2}, \frac{\nu}{2} + \frac{3}{2} + \frac{1}{2r}, \nu + 1, \nu + \frac{3}{2}; -a^2 z^{2r}\right) z^r + (\nu r + r + 1)\, {}_3F_4\left(\frac{\nu}{2} + \frac{1}{4}, \frac{\nu}{2} + \frac{3}{4}, \frac{\nu}{2} + \frac{1}{2r}; \frac{1}{2}, \frac{\nu}{2} + \frac{1}{2r} + 1, \nu + \frac{1}{2}, \nu + 1; -a^2 z^{2r}\right) \sin(b)\right)\right)$$

Involving cos

Linear arguments

03.01.21.0032.01

$$\int \cos(a z) J_\nu(a z) \, dz = \frac{2^{-\nu} z (a z)^\nu}{(\nu+1) \Gamma(\nu+1)} \, {}_3F_4\left(\frac{\nu}{2}+\frac{1}{4}, \frac{\nu}{2}+\frac{1}{2}, \frac{\nu}{2}+\frac{3}{4}; \frac{1}{2}, \frac{\nu}{2}+\frac{3}{2}, \nu+\frac{1}{2}, \nu+1; -a^2 z^2\right)$$

03.01.21.0033.01

$$\int \cos(b + a z) J_\nu(a z) \, dz = -\frac{1}{(\nu+1)(\nu+2)\Gamma(\nu+1)} \left(2^{-\nu} z (a z)^\nu \left(a z (\nu+1) \, {}_3F_4\left(\frac{\nu}{2}+\frac{3}{4}, \frac{\nu}{2}+1, \frac{\nu}{2}+\frac{5}{4}; \frac{3}{2}, \frac{\nu}{2}+2, \nu+1, \nu+\frac{3}{2}; -a^2 z^2\right) \sin(b) - (\nu+2)\cos(b) \, {}_3F_4\left(\frac{\nu}{2}+\frac{1}{4}, \frac{\nu}{2}+\frac{1}{2}, \frac{\nu}{2}+\frac{3}{4}; \frac{1}{2}, \frac{\nu}{2}+\frac{3}{2}, \nu+\frac{1}{2}, \nu+1; -a^2 z^2\right)\right)\right)$$

Power arguments

03.01.21.0034.01

$$\int \cos(a z^r) J_\nu(a z^r) \, dz = \frac{2^{-\nu} z (a z^r)^\nu}{r \nu \Gamma(\nu+1) + \Gamma(\nu+1)} \, {}_3F_4\left(\frac{\nu}{2}+\frac{1}{4}, \frac{\nu}{2}+\frac{3}{4}, \frac{\nu}{2}+\frac{1}{2r}; \frac{1}{2}, \frac{\nu}{2}+\frac{1}{2r}+1, \nu+\frac{1}{2}, \nu+1; -a^2 z^{2r}\right)$$

03.01.21.0035.01

$$\int \cos(a z^r + b) J_\nu(a z^r) \, dz = \frac{1}{(r\nu+1)(\nu r + r + 1)\Gamma(\nu+1)} \left(2^{-\nu} z (a z^r)^\nu \left((\nu r + r + 1)\cos(b) \, {}_3F_4\left(\frac{\nu}{2}+\frac{1}{4}, \frac{\nu}{2}+\frac{3}{4}, \frac{\nu}{2}+\frac{1}{2r}; \frac{1}{2}, \frac{\nu}{2}+\frac{1}{2r}+1, \nu+\frac{1}{2}, \nu+1; -a^2 z^{2r}\right) - a z^r (r\nu+1) \, {}_3F_4\left(\frac{\nu}{2}+\frac{3}{4}, \frac{\nu}{2}+\frac{5}{4}, \frac{\nu}{2}+\frac{1}{2}+\frac{1}{2r}; \frac{3}{2}, \frac{\nu}{2}+\frac{3}{2}+\frac{1}{2r}, \nu+1, \nu+\frac{3}{2}; -a^2 z^{2r}\right)\sin(b)\right)\right)$$

Involving trigonometric functions and a power function

Involving sin and power

Linear arguments

03.01.21.0036.01

$$\int z^{\alpha-1} \sin(a z) J_\nu(a z) \, dz = \frac{2^{-\nu} z^\alpha (a z)^{\nu+1}}{(\alpha+\nu+1)\Gamma(\nu+1)} \, {}_3F_4\left(\frac{\nu}{2}+\frac{3}{4}, \frac{\nu}{2}+\frac{5}{4}, \frac{\alpha}{2}+\frac{\nu}{2}+\frac{1}{2}; \frac{3}{2}, \frac{\alpha}{2}+\frac{\nu}{2}+\frac{3}{2}, \nu+1, \nu+\frac{3}{2}; -a^2 z^2\right)$$

03.01.21.0037.01

$$\int z^{\alpha-1} \sin(b + a z) J_\nu(a z) \, dz = \frac{1}{(\alpha+\nu)(\alpha+\nu+1)\Gamma(\nu+1)} \left(2^{-\nu} z^\alpha (a z)^\nu \left(a z (\alpha+\nu)\cos(b) \, {}_3F_4\left(\frac{\nu}{2}+\frac{3}{4}, \frac{\nu}{2}+\frac{5}{4}, \frac{\alpha}{2}+\frac{\nu}{2}+\frac{1}{2}; \frac{3}{2}, \frac{\alpha}{2}+\frac{\nu}{2}+\frac{3}{2}, \nu+1, \nu+\frac{3}{2}; -a^2 z^2\right) + (\alpha+\nu+1) \, {}_3F_4\left(\frac{\nu}{2}+\frac{1}{4}, \frac{\nu}{2}+\frac{3}{4}, \frac{\alpha}{2}+\frac{\nu}{2}; \frac{1}{2}, \frac{\alpha}{2}+\frac{\nu}{2}+1, \nu+\frac{1}{2}, \nu+1; -a^2 z^2\right)\sin(b)\right)\right)$$

Power arguments

03.01.21.0038.01

$$\int z^{\alpha-1}\sin(a z^r)\,J_\nu(a z^r)\,dz = \frac{2^{-\nu} z^\alpha (a z^r)^{\nu+1}}{(\nu r + r + \alpha)\,\Gamma(\nu+1)}\,{}_3F_4\!\left(\frac{\nu}{2}+\frac{3}{4}, \frac{\nu}{2}+\frac{5}{4}, \frac{\alpha}{2r}+\frac{\nu}{2}+\frac{1}{2}; \frac{3}{2}, \frac{\alpha}{2r}+\frac{\nu}{2}+\frac{3}{2}, \nu+1, \nu+\frac{3}{2}; -a^2 z^{2r}\right)$$

03.01.21.0039.01

$$\int z^{\alpha-1}\sin(a z^r + b)\,J_\nu(a z^r)\,dz = \frac{1}{(\alpha + r\nu)(\nu r + r + \alpha)\,\Gamma(\nu+1)} \left(2^{-\nu} z^\alpha (a z^r)^\nu \left(a(\alpha + r\nu)\cos(b)\,{}_3F_4\!\left(\frac{\nu}{2}+\frac{3}{4}, \frac{\nu}{2}+\frac{5}{4}, \frac{\alpha}{2r}+\frac{\nu}{2}+\frac{1}{2}; \frac{3}{2}, \frac{\alpha}{2r}+\frac{\nu}{2}+\frac{3}{2}, \nu+1, \nu+\frac{3}{2}; -a^2 z^{2r}\right) z^r + (\nu r + r + \alpha)\,{}_3F_4\!\left(\frac{\nu}{2}+\frac{1}{4}, \frac{\nu}{2}+\frac{3}{4}, \frac{\alpha}{2r}+\frac{\nu}{2}; \frac{1}{2}, \frac{\alpha}{2r}+\frac{\nu}{2}+1, \nu+\frac{1}{2}, \nu+1; -a^2 z^{2r}\right)\sin(b)\right)\right)$$

Involving cos and power

Linear arguments

03.01.21.0040.01

$$\int z^{\alpha-1}\cos(a z)\,J_\nu(a z)\,dz = \frac{2^{-\nu} z^\alpha (a z)^\nu}{(\alpha+\nu)\,\Gamma(\nu+1)}\,{}_3F_4\!\left(\frac{\nu}{2}+\frac{1}{4}, \frac{\nu}{2}+\frac{3}{4}, \frac{\alpha}{2}+\frac{\nu}{2}; \frac{1}{2}, \frac{\alpha}{2}+\frac{\nu}{2}+1, \nu+\frac{1}{2}, \nu+1; -a^2 z^2\right)$$

03.01.21.0041.01

$$\int z^{\alpha-1}\cos(b + a z)\,J_\nu(a z)\,dz = \frac{1}{(\alpha+\nu)(\alpha+\nu+1)\,\Gamma(\nu+1)} \left(2^{-\nu} z^\alpha (a z)^\nu \left((\alpha+\nu+1)\cos(b)\,{}_3F_4\!\left(\frac{\nu}{2}+\frac{1}{4}, \frac{\nu}{2}+\frac{3}{4}, \frac{\alpha}{2}+\frac{\nu}{2}; \frac{1}{2}, \frac{\alpha}{2}+\frac{\nu}{2}+1, \nu+\frac{1}{2}, \nu+1; -a^2 z^2\right) - a z (\alpha+\nu)\,{}_3F_4\!\left(\frac{\nu}{2}+\frac{3}{4}, \frac{\nu}{2}+\frac{5}{4}, \frac{\alpha}{2}+\frac{\nu}{2}+\frac{1}{2}; \frac{3}{2}, \frac{\alpha}{2}+\frac{\nu}{2}+\frac{3}{2}, \nu+1, \nu+\frac{3}{2}; -a^2 z^2\right)\sin(b)\right)\right)$$

Power arguments

03.01.21.0042.01

$$\int z^{\alpha-1}\cos(a z^r)\,J_\nu(a z^r)\,dz = \frac{2^{-\nu} z^\alpha (a z^r)^\nu}{(\alpha + r\nu)\,\Gamma(\nu+1)}\,{}_3F_4\!\left(\frac{\nu}{2}+\frac{1}{4}, \frac{\nu}{2}+\frac{3}{4}, \frac{\alpha}{2r}+\frac{\nu}{2}; \frac{1}{2}, \frac{\alpha}{2r}+\frac{\nu}{2}+1, \nu+\frac{1}{2}, \nu+1; -a^2 z^{2r}\right)$$

03.01.21.0043.01

$$\int z^{\alpha-1}\cos(a z^r + b)\,J_\nu(a z^r)\,dz = \frac{1}{(\alpha + r\nu)(\nu r + r + \alpha)\,\Gamma(\nu+1)} \left(2^{-\nu} z^\alpha (a z^r)^\nu \left((\nu r + r + \alpha)\cos(b)\,{}_3F_4\!\left(\frac{\nu}{2}+\frac{1}{4}, \frac{\nu}{2}+\frac{3}{4}, \frac{\alpha}{2r}+\frac{\nu}{2}; \frac{1}{2}, \frac{\alpha}{2r}+\frac{\nu}{2}+1, \nu+\frac{1}{2}, \nu+1; -a^2 z^{2r}\right) - a z^r (\alpha + r\nu)\,{}_3F_4\!\left(\frac{\nu}{2}+\frac{3}{4}, \frac{\nu}{2}+\frac{5}{4}, \frac{\alpha}{2r}+\frac{\nu}{2}+\frac{1}{2}; \frac{3}{2}, \frac{\alpha}{2r}+\frac{\nu}{2}+\frac{3}{2}, \nu+1, \nu+\frac{3}{2}; -a^2 z^{2r}\right)\sin(b)\right)\right)$$

Involving functions of the direct function

Involving elementary functions of the direct function

Involving powers of the direct function

Linear arguments

03.01.21.0044.01

$$\int J_\nu(a z)^2 \, dz = \frac{4^{-\nu} z (a z)^{2\nu}}{(2\nu+1)\,\Gamma(\nu+1)^2} \, {}_2F_3\!\left(\nu+\frac{1}{2}, \nu+\frac{1}{2}; \nu+1, \nu+\frac{3}{2}, 2\nu+1; -a^2 z^2\right)$$

03.01.21.0045.01

$$\int J_\nu(z)^2 \, dz = \frac{2^{-2\nu-1} z^{2\nu+1}}{\left(\nu+\frac{1}{2}\right)\Gamma(\nu+1)^2} \, {}_2F_3\!\left(\nu+\frac{1}{2}, \nu+\frac{1}{2}; \nu+1, \nu+\frac{3}{2}, 2\nu+1; -z^2\right)$$

03.01.21.0046.01

$$\int \frac{1}{z \, J_{-\nu}(z) \, J_\nu(z)} \, dz = -\frac{1}{2} \pi \csc(\pi \nu) \log\!\left(\frac{J_{-\nu}(z)}{J_\nu(z)}\right)$$

Power arguments

03.01.21.0047.01

$$\int J_\nu(a z^r)^2 \, dz = \frac{4^{-\nu} z (a z^r)^{2\nu}}{(2 r \nu+1)\,\Gamma(\nu+1)^2} \, {}_2F_3\!\left(\nu+\frac{1}{2}, \nu+\frac{1}{2r}; \nu+1, \nu+\frac{1}{2r}+1, 2\nu+1; -a^2 z^{2r}\right)$$

Involving products of the direct function

Linear arguments

03.01.21.0048.01

$$\int J_\mu(a z) \, J_\nu(a z) \, dz = \frac{2^{-\mu-\nu} z (a z)^{\mu+\nu}}{(\mu+\nu+1)\,\Gamma(\mu+1)\,\Gamma(\nu+1)} \, {}_3F_4\!\left(\frac{\mu}{2}+\frac{\nu}{2}+\frac{1}{2}, \frac{\mu}{2}+\frac{\nu}{2}+\frac{1}{2}, \frac{\mu}{2}+\frac{\nu}{2}+1; \mu+1, \frac{\mu}{2}+\frac{\nu}{2}+\frac{3}{2}, \nu+1, \mu+\nu+1; -a^2 z^2\right)$$

03.01.21.0049.01

$$\int J_\nu(a z) \, J_{\nu+1}(a z) \, dz = 4^{-\nu-1} z (a z)^{2\nu+1} \, \Gamma(2\nu+2) \, {}_2\tilde{F}_3\!\left(\nu+1, \nu+\frac{3}{2}; \nu+2, \nu+2, 2\nu+2; -a^2 z^2\right)$$

03.01.21.0050.01

$$\int J_0(a z) \, J_1(a z) \, dz = -\frac{J_0(a z)^2}{2a}$$

Power arguments

03.01.21.0051.01

$$\int J_\mu(a z^r) \, J_\nu(a z^r) \, dz = \frac{2^{-\mu-\nu} z (a z^r)^{\mu+\nu}}{(r(\mu+\nu)+1)\,\Gamma(\mu+1)\,\Gamma(\nu+1)} \, {}_3F_4\!\left(\frac{\mu}{2}+\frac{\nu}{2}+\frac{1}{2}, \frac{\mu}{2}+\frac{\nu}{2}+1, \frac{\mu}{2}+\frac{\nu}{2}+\frac{1}{2r}; \mu+1, \frac{\mu}{2}+\frac{\nu}{2}+\frac{1}{2r}+1, \nu+1, \mu+\nu+1; -a^2 z^{2r}\right)$$

Involving functions of the direct function and elementary functions

Involving elementary functions of the direct function and elementary functions

Involving powers of the direct function and a power function

Linear arguments

03.01.21.0052.01

$$\int z^{\alpha-1} J_\nu(a z)^2 \, dz = \frac{4^{-\nu} z^\alpha (a z)^{2\nu}}{(\alpha + 2\nu)\,\Gamma(\nu+1)^2} \, {}_2F_3\!\left(\nu + \frac{1}{2}, \frac{\alpha}{2} + \nu; \nu + 1, \frac{\alpha}{2} + \nu + 1, 2\nu + 1; -a^2 z^2\right)$$

03.01.21.0053.01

$$\int z^{1-2\nu} J_\nu(a z)^2 \, dz = -\frac{2^{-2\nu-1} z^{-2\nu}}{a^2 (2\nu - 1)\,\Gamma(\nu)^2} \left(-4 (a z)^{2\nu} + 4^\nu a^2 z^2 J_{\nu-1}(a z)^2 \Gamma(\nu)^2 + 4^\nu a^2 z^2 J_\nu(a z)^2 \Gamma(\nu)^2\right)$$

03.01.21.0054.01

$$\int z^{2\nu+1} J_\nu(a z)^2 \, dz = \frac{z^{2(\nu+1)}}{4\nu + 2} \left(J_\nu(a z)^2 + J_{\nu+1}(a z)^2\right)$$

03.01.21.0055.01

$$\int z\, J_\nu(a z)^2 \, dz = \frac{1}{2} z^2 \left(J_\nu(a z)^2 - J_{\nu-1}(a z)\, J_{\nu+1}(a z)\right)$$

03.01.21.0056.01

$$\int z\, J_0(a z)^2 \, dz = \frac{1}{2} z^2 \left(J_0(a z)^2 + J_1(a z)^2\right)$$

03.01.21.0057.01

$$\int \frac{1}{z\, J_\nu(z)^2} \, dz = \frac{\pi\, Y_\nu(z)}{2\, J_\nu(z)}$$

03.01.21.0058.01

$$\int \frac{J_\nu(a z)^2}{z} \, dz = \frac{2^{-2\nu-1} (a z)^{2\nu}}{\nu^3\, \Gamma(\nu)^2} \, {}_2F_3\!\left(\nu, \nu + \frac{1}{2}; \nu + 1, \nu + 1, 2\nu + 1; -a^2 z^2\right)$$

03.01.21.0059.01

$$\int \frac{J_\nu(a z)^2}{z^2} \, dz = \frac{1}{z\left(4\nu^2 - 1\right)} \left(2 a^2 z^2 J_{\nu-1}(a z)^2 - 2 a z\, J_\nu(a z)\, J_{\nu-1}(a z) + J_\nu(a z) \left((1 - 2\nu)\, J_\nu(a z) - 2 a^2 z^2 J_{\nu-2}(a z)\right)\right)$$

Power arguments

03.01.21.0060.01

$$\int z^{\alpha-1} J_\nu(a z^r)^2 \, dz = \frac{4^{-\nu} z^\alpha (a z^r)^{2\nu}}{(\alpha + 2 r \nu)\,\Gamma(\nu+1)^2} \, {}_2F_3\!\left(\nu + \frac{1}{2}, \frac{\alpha}{2 r} + \nu; \nu + 1, \frac{\alpha}{2 r} + \nu + 1, 2\nu + 1; -a^2 z^{2 r}\right)$$

Involving products of the direct function and a power function

Linear arguments

03.01.21.0061.01

$$\int z^{\alpha-1} J_\mu(a z) J_\nu(a z)\, dz = 2^{-\mu-\nu-1} z^\alpha (a z)^{\mu+\nu} \Gamma(\mu+\nu+1) \Gamma\left(\frac{1}{2}(\alpha+\mu+\nu)\right) {}_3\tilde{F}_4\left(\frac{1}{2}(\mu+\nu+1), \frac{1}{2}(\mu+\nu+2), \frac{1}{2}(\alpha+\mu+\nu); \mu+1, \frac{1}{2}(\alpha+\mu+\nu+2), \nu+1, \mu+\nu+1; -a^2 z^2\right)$$

03.01.21.0062.01

$$\int z^{1-\mu-\nu} J_\mu(a z) J_\nu(a z)\, dz = \frac{2^{-\mu-\nu+1} z^{-\mu-\nu} (a z)^{\mu+\nu}}{a^2} \left(\frac{1}{(\mu+\nu-1)\Gamma(\mu)\Gamma(\nu)} - \Gamma(\mu+\nu-1)\, {}_2\tilde{F}_3\left(\frac{1}{2}(\mu+\nu-1), \frac{\mu+\nu}{2}; \mu, \nu, \mu+\nu; -a^2 z^2\right)\right)$$

03.01.21.0063.01

$$\int z^{\mu+\nu+1} J_\mu(a z) J_\nu(a z)\, dz = 2^{-\mu-\nu-1} z^{\mu+\nu+2} (a z)^{\mu+\nu} \Gamma(\mu+\nu+1)\, {}_2\tilde{F}_3\left(\frac{1}{2}(\mu+\nu+1), \frac{1}{2}(\mu+\nu+2); \mu+1, \nu+1, \mu+\nu+2; -a^2 z^2\right)$$

03.01.21.0064.01

$$\int z J_\nu(a z) J_\nu(b z)\, dz = \frac{z}{a^2-b^2} (b J_{\nu-1}(b z) J_\nu(a z) - a J_{\nu-1}(a z) J_\nu(b z))$$

03.01.21.0065.01

$$\int z J_{-\nu}(a z) J_\nu(b z)\, dz = \frac{z}{a^2-b^2} (-a J_{-\nu-1}(a z) J_\nu(b z) - b J_{\nu+1}(b z) J_{-\nu}(a z))$$

03.01.21.0066.01

$$\int \frac{\left(\left(a^2-b^2\right) z^2 + \mu^2 - \nu^2\right) J_\nu(a z) J_\mu(b z)}{z}\, dz = b z J_{\mu-1}(b z) J_\nu(a z) - J_\mu(b z) (a z J_{\nu-1}(a z) + (\mu-\nu) J_\nu(a z))$$

03.01.21.0067.01

$$\int \frac{J_\mu(a z) J_\nu(a z)}{z}\, dz = \frac{1}{\mu^2-\nu^2} \left(a z J_{\mu-1}(a z) J_\nu(a z) - J_\mu(a z) (a z J_{\nu-1}(a z) + (\mu-\nu) J_\nu(a z))\right)$$

03.01.21.0068.01

$$\int \frac{J_\mu(a z) J_\nu(a z)}{z^2}\, dz = \frac{2^{-\mu-\nu} a (a z)^{\mu+\nu-1}}{(\mu+\nu-1)\Gamma(\mu+1)\Gamma(\nu+1)}\, {}_2F_3\left(\frac{\mu}{2}+\frac{\nu}{2}-\frac{1}{2}, \frac{\mu}{2}+\frac{\nu}{2}+1; \mu+1, \nu+1, \mu+\nu+1; -a^2 z^2\right)$$

03.01.21.0069.01

$$\int \frac{J_{\nu-1}(a z) J_\nu(a z)}{z^2}\, dz = \frac{4^{-\nu} a (a z)^{2(\nu-1)} \Gamma(2\nu)}{\nu-1}\, {}_2\tilde{F}_3\left(\nu-1, \nu+\frac{1}{2}; \nu, 2\nu, \nu+1; -a^2 z^2\right)$$

Power arguments

03.01.21.0070.01

$$\int z^{\alpha-1} J_\mu(a z^r) J_\nu(a z^r)\, dz = \frac{1}{r}\left(2^{-\mu-\nu-1} z^\alpha (a z^r)^{\mu+\nu} \Gamma(\mu+\nu+1) \Gamma\left(\frac{\alpha+r(\mu+\nu)}{2r}\right) {}_3\tilde{F}_4\left(\frac{1}{2}(\mu+\nu+1), \frac{1}{2}(\mu+\nu+2), \frac{\alpha+r(\mu+\nu)}{2r}; \mu+1, \frac{\alpha+r(\mu+\nu+2)}{2r}, \nu+1, \mu+\nu+1; -a^2 z^{2r}\right)\right)$$

03.01.21.0071.01

$$\int z^{\alpha-1} J_{\nu-1}(a z^r) J_\nu(a z^r)\, dz = \frac{2^{1-2\nu} z^\alpha (a z^r)^{2\nu-1}}{(\alpha+r(2\nu-1))\Gamma(\nu)\Gamma(\nu+1)}\, {}_2F_3\left(\nu+\frac{1}{2}, \frac{\alpha}{2r}+\nu-\frac{1}{2}; 2\nu, \nu+1, \frac{\alpha}{2r}+\nu+\frac{1}{2}; -a^2 z^{2r}\right)$$

03.01.21.0072.01

$$\int z^{\alpha-1} J_{-\nu}(a z^r) J_\nu(a z^r) \, dz = \frac{z^\alpha \sin(\pi \nu)}{\pi \alpha \nu} \, {}_2F_3\left(\frac{1}{2}, \frac{\alpha}{2r}; \frac{\alpha}{2r}+1, 1-\nu, \nu+1; -a^2 z^{2r}\right)$$

03.01.21.0073.01

$$\int J_\nu(a\sqrt{z}) J_\nu(b\sqrt{z}) \, dz = \frac{2\sqrt{z}}{a^2-b^2} \left(b J_{\nu-1}(b\sqrt{z}) J_\nu(a\sqrt{z}) - a J_{\nu-1}(a\sqrt{z}) J_\nu(b\sqrt{z})\right)$$

03.01.21.0074.01

$$\int J_{-\nu}(a\sqrt{z}) J_\nu(b\sqrt{z}) \, dz = \frac{2\sqrt{z}}{a^2-b^2} \left(-a J_{-\nu-1}(a\sqrt{z}) J_\nu(b\sqrt{z}) - b J_{\nu+1}(b\sqrt{z}) J_{-\nu}(a\sqrt{z})\right)$$

Definite integration

For the direct function itself

03.01.21.0075.01

$$\int_0^\infty J_\nu(t) \, dt = 1 \;/;\; \mathrm{Re}(\nu) > 0$$

03.01.21.0076.01

$$\int_0^\infty t^{\alpha-1} J_\nu(t) \, dt = \frac{2^{\alpha-1}}{\Gamma\left(\frac{1}{2}(2-\alpha+\nu)\right)} \Gamma\left(\frac{\alpha+\nu}{2}\right) \;/;\; \mathrm{Re}(\alpha+\nu) > 0 \bigwedge \mathrm{Re}(\alpha) < \frac{3}{2}$$

Involving the direct function

03.01.21.0081.01

$$\int_0^\infty t^{\alpha-1} J_\mu(a t) J_\nu(b t) \, dt = \begin{cases} \frac{2^{\alpha-1} b^{-\alpha-\mu} a^\mu \Gamma\left(\frac{1}{2}(\alpha+\mu+\nu)\right)}{\Gamma\left(\frac{1}{2}(-\alpha-\mu+\nu)+1\right)\Gamma(\mu+1)} \, {}_2F_1\left(\frac{1}{2}(\alpha+\mu+\nu), \frac{1}{2}(\alpha+\mu-\nu); \mu+1; \frac{a^2}{b^2}\right) & b > a \land \mathrm{Re}(\alpha) < 2 \\ \frac{2^{\alpha-1} a^{-\alpha-\nu} b^\nu \Gamma\left(\frac{1}{2}(\alpha+\mu+\nu)\right)}{\Gamma\left(\frac{1}{2}(-\alpha+\mu-\nu)+1\right)\Gamma(\nu+1)} \, {}_2F_1\left(\frac{1}{2}(\alpha+\mu+\nu), \frac{1}{2}(\alpha-\mu+\nu); \nu+1; \frac{b^2}{a^2}\right) & a > b \land \mathrm{Re}(\alpha) < 2 \\ \frac{2^{\alpha-1} a^{-\alpha} \Gamma\left(\frac{1}{2}(\alpha+\mu+\nu)\right)\Gamma(1-\alpha)}{\Gamma\left(\frac{1}{2}(-\alpha+\mu-\nu)+1\right)\Gamma\left(\frac{1}{2}(-\alpha-\mu+\nu)+1\right)\Gamma\left(\frac{1}{2}(-\alpha+\mu+\nu)+1\right)} & b = a \land \mathrm{Re}(\alpha) < 1 \\ \int_0^\infty t^{\alpha-1} J_\mu(a t) J_\nu(b t) \, dt & \text{True} \end{cases} \;/;$$

$$a > 0 \bigwedge b > 0 \bigwedge \mathrm{Re}(\alpha+\mu+\nu) > 0$$

03.01.21.0082.01

$$\int_0^\infty J_\mu(a t) J_\nu(b t) \, dt = \begin{cases} \frac{a^\mu b^{-\mu-1} \Gamma\left(\frac{1}{2}(\mu+\nu+1)\right)}{\Gamma(\mu+1)\Gamma\left(\frac{1}{2}(-\mu+\nu-1)+1\right)} \, {}_2F_1\left(\frac{1}{2}(\mu+\nu+1), \frac{1}{2}(\mu-\nu+1); \mu+1; \frac{a^2}{b^2}\right) & b > a \\ \frac{a^{-\nu-1} b^\nu \Gamma\left(\frac{1}{2}(\mu+\nu+1)\right)}{\Gamma\left(\frac{1}{2}(\mu-\nu-1)+1\right)\Gamma(\nu+1)} \, {}_2F_1\left(\frac{1}{2}(\mu+\nu+1), \frac{1}{2}(-\mu+\nu+1); \nu+1; \frac{b^2}{a^2}\right) & a > b \\ \int_0^\infty J_\mu(a t) J_\nu(b t) \, dt & \text{True} \end{cases} \;/;$$

$$a > 0 \bigwedge b > 0 \bigwedge \mathrm{Re}(\mu+\nu) > -1$$

03.01.21.0077.01

$$\int_0^\infty \frac{1}{t} J_\nu(a t) J_\nu(b t) \, dt = \frac{1}{2\nu} \left(\frac{b}{a}\right)^\nu \;/;\; \mathrm{Re}(\nu) > 0 \bigwedge 0 < b < a$$

03.01.21.0078.01

$$\int_0^\infty \frac{J_\nu(t)^2}{t}\,dt = \frac{1}{2\nu}\; /;\; \mathrm{Re}(\nu) > 0$$

03.01.21.0079.01

$$\int_0^\infty t^{\alpha-1} J_\nu(t)^2\,dt = -\frac{\Gamma\left(\frac{1}{2}-\frac{\alpha}{2}\right)\Gamma\left(\frac{\alpha}{2}+\nu\right)}{\sqrt{\pi}\;\alpha\,\Gamma\left(-\frac{\alpha}{2}\right)\Gamma\left(-\frac{\alpha}{2}+\nu+1\right)}\; /;\; \mathrm{Re}(\alpha+2\nu) > 0 \land \mathrm{Re}(\alpha) < 1$$

03.01.21.0080.01

$$\int_0^\infty J_\nu(a\,t)\,J_\nu(b\,t)\,t\,dt = \frac{\delta(a-b)}{a}\; /;\; a \in \mathbb{R} \bigwedge b \in \mathbb{R} \bigwedge \nu \in \mathbb{R}$$

03.01.21.0083.01

$$\int_0^\infty t^{\alpha-1} J_\lambda(a\,t)\,J_\mu(b\,t)\,J_\nu(c\,t)\,dt = \frac{2^{\alpha-1} a^\lambda b^\mu c^{-\alpha-\lambda-\mu}\,\Gamma\left(\frac{1}{2}(\alpha+\lambda+\mu+\nu)\right)}{\Gamma(\lambda+1)\,\Gamma(\mu+1)\,\Gamma\left(1-\frac{1}{2}(\alpha+\lambda+\mu-\nu)\right)} F_{0\times1\times1}^{2\times0\times0}\left(\begin{matrix}\frac{\alpha+\lambda+\mu+\nu}{2}, \frac{\alpha+\lambda+\mu-\nu}{2};;; \\ ;\lambda+1;\,\mu+1;\end{matrix}\; \frac{a^2}{c^2}, \frac{b^2}{c^2}\right) /;$$

$$a \in \mathbb{R} \bigwedge b \in \mathbb{R} \bigwedge c \in \mathbb{R} \bigwedge a > 0 \bigwedge b > 0 \bigwedge c > a+b \bigwedge \mathrm{Re}(\alpha+\lambda+\mu+\nu) > 0 \bigwedge \mathrm{Re}(\alpha) < \frac{5}{2}$$

Integral transforms

Fourier exp transforms

03.01.22.0006.01

$$\mathcal{F}_t[j_n(t)](z) = \sqrt{\frac{\pi}{2}}\, i^n\, \theta\left(1-z^2\right) P_n(z),\; /;\; n \in \mathbb{N} \bigwedge j_n(z) = \sqrt{\frac{\pi}{2z}}\, J_{n+\frac{1}{2}}(z) \bigwedge z \in \mathbb{R}$$

Fourier cos transforms

03.01.22.0001.02

$$\mathcal{F}c_t[J_\nu(t)](z) = \frac{\theta(1-z)}{\sqrt{1-z^2}}\sqrt{\frac{2}{\pi}}\cos\left(\nu\sin^{-1}(z)\right) - \frac{\theta(z-1)\,2^{\frac{1}{2}-\nu} z^{-\nu-1}}{\sqrt{\pi}}\sin\left(\frac{\pi\nu}{2}\right){}_2F_1\left(\frac{\nu+1}{2}, \frac{\nu+2}{2}; \nu+1; \frac{1}{z^2}\right) /;$$

$$z > 0 \land z \neq 1 \land \mathrm{Re}(\nu) > -1$$

Fourier sin transforms

03.01.22.0002.02

$$\mathcal{F}s_t[J_\nu(t)](z) = \frac{2^{\frac{1}{2}-\nu} z^{-\nu-1}\,\theta(z-1)}{\sqrt{\pi}}\cos\left(\frac{\pi\nu}{2}\right){}_2F_1\left(\frac{\nu+1}{2}, \frac{\nu+2}{2}; \nu+1; \frac{1}{z^2}\right) + \frac{\theta(1-z)}{\sqrt{1-z^2}}\sqrt{\frac{2}{\pi}}\sin\left(\nu\sin^{-1}(z)\right) /;$$

$$z > 0 \land z \neq 1 \land \mathrm{Re}(\nu) > -2$$

Laplace transforms

03.01.22.0003.01

$$\mathcal{L}_t[J_\nu(t)](z) = 2^{-\nu} z^{-\nu-1}\,{}_2F_1\left(\frac{\nu+1}{2}, \frac{\nu+2}{2}; \nu+1; -\frac{1}{z^2}\right) /;\; \mathrm{Re}(\nu) > -1$$

For spherical Bessel functions

03.01.22.0007.01

$$\mathcal{L}_t[t^\mu\, j_n(t)]\,(z) = 2^{-n-1}\sqrt{\pi}\, z^{-n-\mu-1}\,\frac{\Gamma(n+\mu+1)}{\Gamma\left(n+\frac{3}{2}\right)}\,{}_2F_1\left(\frac{1}{2}(n+\mu+1),\,\frac{1}{2}(n+\mu+2);\,n+\frac{3}{2};\,-\frac{1}{z^2}\right)\,/;\; n\in\mathbb{N}\bigwedge j_n(z) = \sqrt{\frac{\pi}{2\,z}}\,J_{n+\frac{1}{2}}(z)\bigwedge \mathrm{Re}(\mu)>0$$

03.01.22.0008.01

$$\mathcal{L}_t[j_n(t)]\,(z) = i^{n+1}\,Q_l(i\,z)\,/;\; n\in\mathbb{N}\bigwedge j_n(z) = \sqrt{\frac{\pi}{2\,z}}\,J_{n+\frac{1}{2}}(z)$$

03.01.22.0009.01

$$\mathcal{L}_t[j_n(t)]\,(z) = 2^{-n-1}\sqrt{\pi}\, z^{-n-1}\,\frac{\Gamma(n+1)}{\Gamma\left(n+\frac{3}{2}\right)}\,{}_2F_1\left(\frac{n+1}{2},\,\frac{n+2}{2};\,n+\frac{3}{2};\,-\frac{1}{z^2}\right)\,/;\; n\in\mathbb{N}\bigwedge j_n(z) = \sqrt{\frac{\pi}{2\,z}}\,J_{n+\frac{1}{2}}(z)$$

03.01.22.0010.01

$$\mathcal{L}_t[j_n(t)]\,(z) = z^n\left(\tan^{-1}\left(\frac{1}{z}\right)\sum_{k=0}^{\left\lfloor\frac{n}{2}\right\rfloor} c_{k,l}\,z^{-2k} - \sum_{k=0}^{\left\lfloor\frac{n}{2}\right\rfloor} c_{k,l}\sum_{m=0}^{\left\lfloor\frac{n-1}{2}\right\rfloor-k}\frac{(-1)^m}{2\,m+1}\,z^{-2k-2m-1}\right)/;$$
$$n\in\mathbb{N}^+\bigwedge j_n(z) = \sqrt{\frac{\pi}{2\,z}}\,J_{n+\frac{1}{2}}(z)\bigwedge c_{0,0} = 0\bigwedge c_{1,2} = \frac{1}{2}\bigwedge c_{0,n} = \frac{(-1)^n\,(2\,n-1)!!}{n!}\bigwedge$$
$$\left(c_{k,n} = \frac{(n-1)\,c_{k-1,n-2}}{n} - \frac{(2\,n-1)\,c_{k,n-1}}{n}\,/;\; n\geq 1\land k\leq n\right)\bigwedge (c_{k,n} = 0\,/;\; k>n)$$

Mellin transforms

03.01.22.0004.01

$$\mathcal{M}_t[J_\nu(t)]\,(z) = \frac{1}{\Gamma\left(\frac{1}{2}(-z+\nu+2)\right)}\,2^{z-1}\,\Gamma\left(\frac{z+\nu}{2}\right)\,/;\; \mathrm{Re}(z+\nu)>0\bigwedge \mathrm{Re}(z)<\frac{3}{2}$$

Hankel transforms

03.01.22.0005.02

$$\mathcal{H}_{t;\mu}[J_\nu(t)]\,(z) = \frac{\sqrt{2}\,z^{-\nu-1}\,\Gamma\left(\frac{1}{4}(2\,\mu+2\,\nu+3)\right)\theta(z-1)}{\Gamma(\nu+1)\,\Gamma\left(\frac{1}{4}(2\,\mu-2\,\nu+1)\right)}\,{}_2F_1\left(\frac{1}{4}(-2\,\mu+2\,\nu+3),\,\frac{1}{4}(2\,\mu+2\,\nu+3);\,\nu+1;\,\frac{1}{z^2}\right)+$$
$$\frac{\sqrt{2}\,z^{\mu+\frac{1}{2}}\,\Gamma\left(\frac{1}{4}(2\,\mu+2\,\nu+3)\right)\theta(1-z)}{\Gamma(\mu+1)\,\Gamma\left(\frac{1}{4}(-2\,\mu+2\,\nu+1)\right)}\,{}_2F_1\left(\frac{1}{4}(2\,\mu-2\,\nu+3),\,\frac{1}{4}(2\,\mu+2\,\nu+3);\,\mu+1;\,z^2\right)/;\; z>$$
$$0\bigwedge z\neq 1\bigwedge \mathrm{Re}(\mu+\nu)>-\frac{3}{2}$$

Summation

Infinite summation

03.01.23.0001.01

$$\sum_{k=0}^{\infty} \frac{1}{k!} J_{k+\nu}(x)\, x^k = I_\nu(x)$$

03.01.23.0002.01

$$\sum_{k=0}^{\infty} \frac{(2k+\nu)\,\Gamma(k+\nu)}{k!} J_{2k+\nu}(x) = \left(\frac{x}{2}\right)^\nu$$

03.01.23.0003.01

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k} J_{2k}(x) = \frac{1}{2}\left(\log\left(\frac{x}{2}\right)+\gamma\right) J_0(x) - \frac{\pi}{4} Y_0(x)$$

03.01.23.0004.01

$$\sum_{k=1}^{\infty} \frac{(-1)^k (2k+\nu)}{k(k+\nu)} J_{2k+\nu}(x) = -\frac{\nu!}{2}\left(\sum_{k=0}^{\nu-1} \frac{1}{(\nu-k)\,k!}\left(\frac{x}{2}\right)^k J_k(x)\right)\left(\frac{x}{2}\right)^{-\nu} - \frac{\pi}{2} Y_\nu(x) + J_\nu(x)\left(\log\left(\frac{x}{2}\right)-\psi(\nu+1)\right) /;\ \nu \in \mathbb{N}$$

03.01.23.0005.01

$$\sum_{k=-\infty}^{\infty} J_k(x)\, t^k = e^{\frac{1}{2} x \left(t-\frac{1}{t}\right)}$$

03.01.23.0006.01

$$\sum_{k=1}^{\infty} \cos(2kt)\, J_{2k}(x) = \frac{1}{2}\left(\cos(x\sin(t)) - J_0(x)\right)$$

03.01.23.0007.01

$$\sum_{k=1}^{\infty} J_k(x)^2 = \frac{1}{2}\left(1 - J_0(x)^2\right)$$

03.01.23.0008.01

$$\sum_{k=0}^{\infty} \sin((2k+1)t)\, J_{2k+1}(x) = \frac{1}{2}\sin(x\sin(t))$$

03.01.23.0009.01

$$\sum_{k=1}^{\infty} (-1)^k \cos(2kt)\, J_{2k}(x) = \frac{1}{2}\left(\cos(x\cos(t)) - J_0(x)\right)$$

03.01.23.0010.01

$$\sum_{k=0}^{\infty} (-1)^k \cos((2k+1)t)\, J_{2k+1}(x) = \frac{1}{2}\sin(x\cos(t))$$

03.01.23.0011.01

$$\sum_{k=1}^{\infty} J_{2k}(x) = \frac{1}{2}\left(1 - J_0(x)\right)$$

03.01.23.0012.01

$$\sum_{k=1}^{\infty} (-1)^k J_{2k}(x) = \frac{1}{2}\left(\cos(x) - J_0(x)\right)$$

03.01.23.0013.01

$$\sum_{k=0}^{\infty} (-1)^k J_{2k+1}(x) = \frac{\sin(x)}{2}$$

03.01.23.0014.01

$$\sum_{k=0}^{\infty} i^{n k} J_{n k}(z) = \frac{J_0(z)}{2} + \frac{1}{2 n} \sum_{k=0}^{n-1} e^{i z \cos\left(\frac{2 \pi k}{n}\right)} /; n \in \mathbb{N}$$

03.01.23.0015.01

$$\sum_{k=0}^{\infty} (-i)^{n k} J_{n k}(z) = \frac{J_0(z)}{2} + \frac{1}{2 n} \sum_{k=0}^{n-1} e^{-i z \cos\left(\frac{2 \pi k}{n}\right)} /; n \in \mathbb{N}$$

03.01.23.0016.01

$$\sum_{k=0}^{\infty} (4 k + 2 \nu + 2) J_{2 k+\nu+1}(z) J_{2 k+\nu+1}(w) = \frac{z w}{z^2 - w^2} (z J_{\nu+1}(z) J_\nu(w) - w J_\nu(z) J_{\nu+1}(w)) /; \mathrm{Re}(\nu) > -1$$

Representations through more general functions

Through hypergeometric functions

Involving ${}_0\tilde{F}_1$

03.01.26.0001.01

$$J_\nu(z) = \left(\frac{z}{2}\right)^\nu {}_0\tilde{F}_1\left(; \nu + 1; -\frac{z^2}{4}\right)$$

Involving ${}_0F_1$

03.01.26.0002.01

$$J_\nu(z) = \frac{1}{\Gamma(\nu + 1)} \left(\frac{z}{2}\right)^\nu {}_0F_1\left(; \nu + 1; -\frac{z^2}{4}\right) /; -\nu \notin \mathbb{N}^+$$

Involving ${}_1F_1$

03.01.26.0003.01

$$J_\nu(z) = \frac{z^\nu}{2^\nu e^{i z} \Gamma(\nu + 1)} {}_1F_1\left(\nu + \frac{1}{2}; 2 \nu + 1; 2 i z\right)$$

03.01.26.0004.01

$$J_\nu(z) = \frac{z^\nu}{2^\nu \Gamma(\nu + 1)} \lim_{a \to \infty} {}_1F_1\left(a; \nu + 1; -\frac{z^2}{4 a}\right)$$

Through Meijer G

Classical cases for the direct function itself

03.01.26.0005.01

$$J_\nu(z) = z^\nu \left(z^2\right)^{-\frac{\nu}{2}} G_{0,2}^{1,0}\left(\frac{z^2}{4} \middle| \frac{\nu}{2}, -\frac{\nu}{2}\right)$$

03.01.26.0006.01

$$J_\nu(z) = \pi z^\nu \left(-z^2\right)^{-\frac{\nu}{2}} G_{1,3}^{1,0}\left(-\frac{z^2}{4} \middle| \begin{array}{c} \frac{\nu+1}{2} \\ \frac{\nu}{2}, -\frac{\nu}{2}, \frac{\nu+1}{2} \end{array}\right)$$

03.01.26.0107.01

$$J_\nu(z) = G_{0,2}^{1,0}\left(\frac{z^2}{4} \,\middle|\, \frac{\nu}{2}, -\frac{\nu}{2}\right) /; -\frac{\pi}{2} < \arg(z) \leq \frac{\pi}{2}$$

03.01.26.0007.01

$$J_\nu\left(\sqrt{z}\right) = G_{0,2}^{1,0}\left(\frac{z}{4} \,\middle|\, \frac{\nu}{2}, -\frac{\nu}{2}\right)$$

03.01.26.0008.01

$$J_{-\nu}\left(\sqrt{z}\right) + J_\nu\left(\sqrt{z}\right) = 2\cos\left(\frac{\nu\pi}{2}\right) G_{1,3}^{2,0}\left(\frac{z}{4} \,\middle|\, \begin{matrix} 0 \\ -\frac{\nu}{2}, \frac{\nu}{2}, 0 \end{matrix}\right)$$

03.01.26.0009.01

$$J_{-\nu}\left(\sqrt{z}\right) - J_\nu\left(\sqrt{z}\right) = 2\sin\left(\frac{\nu\pi}{2}\right) G_{1,3}^{2,0}\left(\frac{z}{4} \,\middle|\, \begin{matrix} \frac{1}{2} \\ -\frac{\nu}{2}, \frac{\nu}{2}, \frac{1}{2} \end{matrix}\right)$$

Classical cases involving cos

03.01.26.0010.01

$$\cos\left(\sqrt{z}\right) J_\nu\left(\sqrt{z}\right) = \frac{1}{\sqrt{2}} G_{2,4}^{1,2}\left(z \,\middle|\, \begin{matrix} \frac{1}{4}, \frac{3}{4} \\ \frac{\nu}{2}, -\frac{\nu}{2}, \frac{\nu+1}{2}, \frac{1-\nu}{2} \end{matrix}\right)$$

03.01.26.0011.01

$$\cos\left(a + \sqrt{z}\right) J_\nu\left(\sqrt{z}\right) = \frac{1}{\sqrt{2}} G_{3,5}^{2,2}\left(z \,\middle|\, \begin{matrix} \frac{1}{4}, \frac{3}{4}, \frac{a}{\pi} + \frac{\nu+1}{2} \\ \frac{\nu}{2}, \frac{\nu+1}{2}, -\frac{\nu}{2}, \frac{1-\nu}{2}, \frac{a}{\pi} + \frac{\nu+1}{2} \end{matrix}\right)$$

Classical cases involving sin

03.01.26.0012.01

$$\sin\left(\sqrt{z}\right) J_\nu\left(\sqrt{z}\right) = \frac{1}{\sqrt{2}} G_{2,4}^{1,2}\left(z \,\middle|\, \begin{matrix} \frac{1}{4}, \frac{3}{4} \\ \frac{\nu+1}{2}, -\frac{\nu}{2}, \frac{1-\nu}{2}, \frac{\nu}{2} \end{matrix}\right)$$

03.01.26.0013.01

$$\sin\left(a + \sqrt{z}\right) J_\nu\left(\sqrt{z}\right) = \frac{1}{\sqrt{2}} G_{3,5}^{2,2}\left(z \,\middle|\, \begin{matrix} \frac{1}{4}, \frac{3}{4}, \frac{a}{\pi} + \frac{\nu}{2} \\ \frac{\nu}{2}, \frac{\nu+1}{2}, -\frac{\nu}{2}, \frac{1-\nu}{2}, \frac{a}{\pi} + \frac{\nu}{2} \end{matrix}\right)$$

Classical cases involving cos, sin

03.01.26.0014.01

$$\cos\left(\sqrt{z}\right) J_{-\nu}\left(\sqrt{z}\right) + \sin\left(\sqrt{z}\right) J_\nu\left(\sqrt{z}\right) = -\sqrt{2}\sin\left(\frac{1}{4}(2\nu-1)\pi\right) G_{2,4}^{2,1}\left(z \,\middle|\, \begin{matrix} \frac{3}{4}, \frac{1}{4} \\ -\frac{\nu}{2}, \frac{\nu+1}{2}, \frac{\nu}{2}, \frac{1-\nu}{2} \end{matrix}\right)$$

03.01.26.0015.01

$$\cos\left(\sqrt{z}\right) J_{-\nu}\left(\sqrt{z}\right) - \sin\left(\sqrt{z}\right) J_\nu\left(\sqrt{z}\right) = \sqrt{2}\sin\left(\frac{1}{4}\pi(2\nu+1)\right) G_{2,4}^{2,1}\left(z \,\middle|\, \begin{matrix} \frac{1}{4}, \frac{3}{4} \\ -\frac{\nu}{2}, \frac{\nu+1}{2}, \frac{1-\nu}{2}, \frac{\nu}{2} \end{matrix}\right)$$

03.01.26.0108.01

$$J_\nu\left(\sqrt{z}\right)\cos\left(\sqrt{z}\right) + J_{-\nu}\left(\sqrt{z}\right)\sin\left(\sqrt{z}\right) = \sqrt{2}\sin\left(\frac{1}{4}\pi(2\nu+1)\right) G_{2,4}^{2,1}\left(z \,\middle|\, \begin{matrix} \frac{3}{4}, \frac{1}{4} \\ \frac{\nu}{2}, \frac{1-\nu}{2}, \frac{\nu+1}{2}, -\frac{\nu}{2} \end{matrix}\right)$$

03.01.26.0109.01

$$\sin\left(\sqrt{z}\right) J_{-\nu}\left(\sqrt{z}\right) - \cos\left(\sqrt{z}\right) J_{\nu}\left(\sqrt{z}\right) = \sqrt{2}\,\sin\left(\frac{1}{4}\pi(2\nu-1)\right) G_{2,4}^{2,1}\left(z \,\middle|\, \begin{matrix} \frac{1}{4}, \frac{3}{4} \\ \frac{\nu}{2}, \frac{1-\nu}{2}, \frac{\nu+1}{2}, -\frac{\nu}{2} \end{matrix}\right)$$

Classical cases for powers of Bessel J

03.01.26.0016.01

$$J_{\nu}\left(\sqrt{z}\right)^2 = \frac{1}{\sqrt{\pi}} G_{1,3}^{1,1}\left(z \,\middle|\, \begin{matrix} \frac{1}{2} \\ \nu, 0, -\nu \end{matrix}\right)$$

03.01.26.0017.01

$$J_{-\nu}\left(\sqrt{z}\right)^2 + J_{\nu}\left(\sqrt{z}\right)^2 = \frac{2\cos(\pi\nu)}{\sqrt{\pi}} G_{2,4}^{2,1}\left(z \,\middle|\, \begin{matrix} \frac{1}{2}, 0 \\ -\nu, \nu, 0, 0 \end{matrix}\right)$$

03.01.26.0018.01

$$J_{-\nu}\left(\sqrt{z}\right)^2 - J_{\nu}\left(\sqrt{z}\right)^2 = \frac{2\sin(\pi\nu)}{\sqrt{\pi}} G_{1,3}^{2,0}\left(z \,\middle|\, \begin{matrix} \frac{1}{2} \\ -\nu, \nu, 0 \end{matrix}\right)$$

Classical cases for products of Bessel J

03.01.26.0019.01

$$J_{-\nu}\left(\sqrt{z}\right) J_{\nu}\left(\sqrt{z}\right) = \frac{1}{\sqrt{\pi}} G_{1,3}^{1,1}\left(z \,\middle|\, \begin{matrix} \frac{1}{2} \\ 0, \nu, -\nu \end{matrix}\right)$$

03.01.26.0020.01

$$J_{\nu-1}\left(\sqrt{z}\right) J_{\nu}\left(\sqrt{z}\right) = \frac{1}{\sqrt{\pi}} G_{1,3}^{1,1}\left(z \,\middle|\, \begin{matrix} 0 \\ \nu-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}-\nu \end{matrix}\right)$$

03.01.26.0021.01

$$J_{\mu}\left(\sqrt{z}\right) J_{\nu}\left(\sqrt{z}\right) = \frac{1}{\sqrt{\pi}} G_{2,4}^{1,2}\left(z \,\middle|\, \begin{matrix} 0, \frac{1}{2} \\ \frac{\mu+\nu}{2}, -\frac{\mu+\nu}{2}, \frac{\mu-\nu}{2}, \frac{\nu-\mu}{2} \end{matrix}\right) /; -\mu-\nu-1 \notin \mathbb{N}$$

03.01.26.0110.01

$$J_{-n-\nu-1}\left(\sqrt{z}\right) J_{\nu}\left(\sqrt{z}\right) = \frac{2}{\sqrt{\pi}} \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{(-1)^{k+\left\lfloor \frac{n+1}{2} \right\rfloor} z^{\frac{1}{2}(2k-n-1)}\, \Gamma\left(k-n+\left\lfloor \frac{n}{2} \right\rfloor+\frac{1}{2}\right) \left(1-k+\left\lfloor \frac{n}{2} \right\rfloor\right)_{n-\left\lfloor \frac{n}{2} \right\rfloor}}{k!\,\Gamma(k-n-\nu)\,\Gamma(k+\nu+1)} -$$
$$\frac{(-1)^n}{\sqrt{\pi}} G_{2,4}^{1,2}\left(z \,\middle|\, \begin{matrix} 0, \frac{1}{2} \\ \frac{n+1}{2}, \frac{n+1}{2}+\nu, -\frac{1}{2}(n+1), -\frac{1}{2}(n+1)-\nu \end{matrix}\right) /; n \in \mathbb{N}$$

03.01.26.0111.01

$$J_{-\nu-1}\left(\sqrt{z}\right) J_{\nu}\left(\sqrt{z}\right) = -\frac{1}{\sqrt{\pi}} G_{2,4}^{1,2}\left(z \,\middle|\, \begin{matrix} 0, \frac{1}{2} \\ \frac{1}{2}, \nu+\frac{1}{2}, -\frac{1}{2}, -\nu-\frac{1}{2} \end{matrix}\right) - \frac{2\sin(\pi\nu)}{\pi\sqrt{z}}$$

03.01.26.0112.01

$$J_{-\nu-2}\left(\sqrt{z}\right) J_{\nu}\left(\sqrt{z}\right) = \frac{1}{\sqrt{\pi}} G_{2,4}^{1,2}\left(z \,\middle|\, \begin{matrix} 0, \frac{1}{2} \\ 1, \nu+1, -1, -\nu-1 \end{matrix}\right) + \frac{4(\nu+1)\sin(\pi\nu)}{\pi z}$$

03.01.26.0022.01

$$J_{-\mu}\left(\sqrt{z}\right) J_{-\nu}\left(\sqrt{z}\right) + J_{\mu}\left(\sqrt{z}\right) J_{\nu}\left(\sqrt{z}\right) = \frac{2}{\sqrt{\pi}} \cos\left(\frac{1}{2}(\mu+\nu)\pi\right) G_{2,4}^{2,1}\left(z \,\middle|\, \begin{matrix} \frac{1}{2}, 0 \\ \frac{\mu+\nu}{2}, -\frac{\mu+\nu}{2}, \frac{\mu-\nu}{2}, \frac{\nu-\mu}{2} \end{matrix}\right) /; \frac{\mu+\nu-1}{2} \notin \mathbb{Z}$$

03.01.26.0023.01

$$J_{\mu}\left(\sqrt{z}\right) J_{\nu}\left(\sqrt{z}\right) - J_{-\mu}\left(\sqrt{z}\right) J_{-\nu}\left(\sqrt{z}\right) = -\frac{2}{\sqrt{\pi}} \sin\left(\frac{1}{2}(\mu+\nu)\pi\right) G_{2,4}^{2,1}\left(z \,\middle|\, \begin{matrix} 0, \frac{1}{2} \\ \frac{\mu+\nu}{2}, -\frac{\mu+\nu}{2}, \frac{\mu-\nu}{2}, \frac{\nu-\mu}{2} \end{matrix}\right) /; \frac{\mu+\nu}{2} \notin \mathbb{Z}$$

Classical cases involving Bessel *I*

03.01.26.0024.01

$$J_{\nu}\left(\sqrt[4]{z}\right) I_{\nu}\left(\sqrt[4]{z}\right) = \sqrt{\pi}\, G_{0,4}^{1,0}\left(\frac{z}{64} \,\middle|\, \frac{\nu}{2}, 0, \frac{1}{2}, -\frac{\nu}{2}\right)$$

03.01.26.0025.01

$$J_{-\nu}\left(\sqrt[4]{z}\right) I_{\nu}\left(\sqrt[4]{z}\right) = \sqrt{\pi}\, G_{1,5}^{2,0}\left(\frac{z}{64} \,\middle|\, \begin{matrix} \frac{\nu+1}{2} \\ 0, \frac{1}{2}, -\frac{\nu}{2}, \frac{\nu}{2}, \frac{\nu+1}{2} \end{matrix}\right)$$

Classical cases involving Bessel *K*

03.01.26.0026.01

$$J_{\nu}\left(\sqrt[4]{z}\right) K_{\nu}\left(\sqrt[4]{z}\right) = \frac{1}{4\sqrt{\pi}} G_{0,4}^{3,0}\left(\frac{z}{64} \,\middle|\, 0, \frac{1}{2}, \frac{\nu}{2}, -\frac{\nu}{2}\right)$$

03.01.26.0027.01

$$J_{-\nu}\left(\sqrt[4]{z}\right) K_{\nu}\left(\sqrt[4]{z}\right) = \frac{1}{4\sqrt{\pi}} G_{0,4}^{3,0}\left(\frac{z}{64} \,\middle|\, 0, \frac{1}{2}, -\frac{\nu}{2}, \frac{\nu}{2}\right)$$

03.01.26.0028.01

$$\left(J_{-\nu}\left(\sqrt[4]{z}\right) + J_{\nu}\left(\sqrt[4]{z}\right)\right) K_{\nu}\left(\sqrt[4]{z}\right) = \frac{1}{2\sqrt{\pi}} \cos\left(\frac{\pi\nu}{2}\right) G_{0,4}^{3,0}\left(\frac{z}{64} \,\middle|\, \frac{1}{2}, -\frac{\nu}{2}, \frac{\nu}{2}, 0\right)$$

03.01.26.0029.01

$$\left(J_{-\nu}\left(\sqrt[4]{z}\right) - J_{\nu}\left(\sqrt[4]{z}\right)\right) K_{\nu}\left(\sqrt[4]{z}\right) = \frac{1}{2\sqrt{\pi}} \sin\left(\frac{\pi\nu}{2}\right) G_{0,4}^{3,0}\left(\frac{z}{64} \,\middle|\, 0, -\frac{\nu}{2}, \frac{\nu}{2}, \frac{1}{2}\right)$$

Classical cases involving Bessel *Y*

03.01.26.0030.01

$$\cos(a\pi) J_{\nu}\left(\sqrt{z}\right) + \sin(a\pi) Y_{\nu}\left(\sqrt{z}\right) = G_{1,3}^{2,0}\left(\frac{z}{4} \,\middle|\, \begin{matrix} -a-\frac{\nu}{2} \\ -\frac{\nu}{2}, \frac{\nu}{2}, -a-\frac{\nu}{2} \end{matrix}\right)$$

03.01.26.0113.01

$$\cos(a\pi) J_{\nu}\left(\sqrt{z}\right) - \sin(a\pi) Y_{\nu}\left(\sqrt{z}\right) = G_{1,3}^{2,0}\left(\frac{z}{4} \,\middle|\, \begin{matrix} a-\frac{\nu}{2} \\ -\frac{\nu}{2}, \frac{\nu}{2}, a-\frac{\nu}{2} \end{matrix}\right)$$

03.01.26.0031.01

$$J_{\nu}\left(\sqrt{z}\right) Y_{\nu}\left(\sqrt{z}\right) = -\frac{1}{\sqrt{\pi}} G_{1,3}^{2,0}\left(z \,\middle|\, \begin{matrix} \frac{1}{2} \\ 0, \nu, -\nu \end{matrix}\right)$$

03.01.26.0032.01

$$J_{-\nu}\left(\sqrt{z}\right)Y_{\nu}\left(\sqrt{z}\right)=\frac{1}{\sqrt{\pi}}G_{2,4}^{2,1}\left(z\left|\begin{array}{c}\frac{1}{2},-\nu-\frac{1}{2}\\0,-\nu,-\nu-\frac{1}{2},\nu\end{array}\right.\right)$$

03.01.26.0033.01

$$J_{\nu+1}\left(\sqrt{z}\right)Y_{\nu}\left(\sqrt{z}\right)=\frac{1}{\sqrt{\pi}}G_{2,4}^{2,1}\left(z\left|\begin{array}{c}\frac{1}{2},0\\\frac{1}{2},\nu+\frac{1}{2},-\frac{1}{2},-\frac{1}{2}-\nu\end{array}\right.\right)$$

03.01.26.0034.01

$$J_{\nu+2}\left(\sqrt{z}\right)Y_{\nu}\left(\sqrt{z}\right)=\frac{1}{\sqrt{\pi}}G_{2,4}^{2,1}\left(z\left|\begin{array}{c}0,\frac{1}{2}\\1,\nu+1,-1,-\nu-1\end{array}\right.\right)$$

03.01.26.0035.01

$$J_{\mu}\left(\sqrt{z}\right)Y_{\nu}\left(\sqrt{z}\right)=-\frac{1}{\sqrt{\pi}}G_{3,5}^{2,2}\left(z\left|\begin{array}{c}0,\frac{1}{2},\frac{\mu-\nu+1}{2}\\\frac{\mu+\nu}{2},\frac{\mu-\nu}{2},\frac{\mu-\nu+1}{2},-\frac{\mu+\nu}{2},\frac{\nu-\mu}{2}\end{array}\right.\right)/;-\mu-\nu-1\notin\mathbb{N}$$

03.01.26.0114.01

$$J_{\nu}\left(\sqrt{z}\right)Y_{-n-\nu-1}\left(\sqrt{z}\right)=\frac{(-1)^n}{\sqrt{\pi}}G_{3,5}^{2,2}\left(z\left|\begin{array}{c}0,\frac{1}{2},\frac{n}{2}+\nu+1\\\frac{n+1}{2},\frac{n+1}{2}+\nu,-\frac{1}{2}(n+1)-\nu,-\frac{1}{2}(n+1),\frac{n}{2}+\nu+1\end{array}\right.\right)-$$
$$\frac{2\cot(\nu\pi)}{\sqrt{\pi}}\sum_{k=0}^{\left\lfloor\frac{n}{2}\right\rfloor}\frac{(-1)^{k+\left\lfloor\frac{n+1}{2}\right\rfloor}z^{\frac{2k-n-1}{2}}\,\Gamma\left(k-n+\left\lfloor\frac{n}{2}\right\rfloor+\frac{1}{2}\right)\left(1-k+\left\lfloor\frac{n}{2}\right\rfloor\right)_{n-\left\lfloor\frac{n}{2}\right\rfloor}}{k!\,\Gamma(k-n-\nu)\,\Gamma(k+\nu+1)}/;n\in\mathbb{N}$$

03.01.26.0115.01

$$J_{\nu}\left(\sqrt{z}\right)Y_{-\nu-1}\left(\sqrt{z}\right)=\frac{2\cos(\pi\nu)}{\pi\sqrt{z}}+\frac{1}{\sqrt{\pi}}G_{3,5}^{2,2}\left(z\left|\begin{array}{c}0,\frac{1}{2},\nu+1\\\frac{1}{2},\nu+\frac{1}{2},-\frac{1}{2},-\nu-\frac{1}{2},\nu+1\end{array}\right.\right)$$

03.01.26.0116.01

$$J_{\nu}\left(\sqrt{z}\right)Y_{-\nu-2}\left(\sqrt{z}\right)=-\frac{4(\nu+1)\cos(\pi\nu)}{\pi z}-\frac{1}{\sqrt{\pi}}G_{3,5}^{2,2}\left(z\left|\begin{array}{c}0,\frac{1}{2},\nu+\frac{3}{2}\\1,\nu+1,-1,-\nu-1,\nu+\frac{3}{2}\end{array}\right.\right)$$

03.01.26.0036.01

$$J_{\nu}\left(\sqrt{z}\right)Y_{-\nu}\left(\sqrt{z}\right)+J_{-\nu}\left(\sqrt{z}\right)Y_{\nu}\left(\sqrt{z}\right)=-\frac{2}{\sqrt{\pi}}G_{1,3}^{2,0}\left(z\left|\begin{array}{c}\frac{1}{2}\\\nu,-\nu,0\end{array}\right.\right)$$

03.01.26.0037.01

$$J_{\nu}\left(\sqrt{z}\right)Y_{-\nu}\left(\sqrt{z}\right)-J_{-\nu}\left(\sqrt{z}\right)Y_{\nu}\left(\sqrt{z}\right)=\frac{\sin(2\pi\nu)}{\pi^{5/2}}G_{1,3}^{3,1}\left(z\left|\begin{array}{c}\frac{1}{2}\\0,\nu,-\nu\end{array}\right.\right)$$

03.01.26.0038.01

$$J_{\nu}\left(\sqrt{z}\right)Y_{\mu}\left(\sqrt{z}\right)+J_{\mu}\left(\sqrt{z}\right)Y_{\nu}\left(\sqrt{z}\right)=-\frac{2}{\sqrt{\pi}}G_{2,4}^{3,0}\left(z\left|\begin{array}{c}0,\frac{1}{2}\\\frac{\mu+\nu}{2},\frac{\mu-\nu}{2},\frac{\nu-\mu}{2},-\frac{\mu+\nu}{2}\end{array}\right.\right)$$

03.01.26.0039.01

$$J_{\nu}\left(\sqrt{z}\right)Y_{\mu}\left(\sqrt{z}\right)-J_{\mu}\left(\sqrt{z}\right)Y_{\nu}\left(\sqrt{z}\right)=\frac{\sin(\pi(\nu-\mu))}{\pi^{5/2}}G_{2,4}^{3,2}\left(z\left|\begin{array}{c}0,\frac{1}{2}\\\frac{\mu+\nu}{2},\frac{\mu-\nu}{2},\frac{\nu-\mu}{2},-\frac{\mu+\nu}{2}\end{array}\right.\right)/;-\mu-\nu-1\notin\mathbb{N}$$

03.01.26.0040.01

$$J_\nu\left(\sqrt{z}\right)^2 + Y_\nu\left(\sqrt{z}\right)^2 = \frac{2\cos(\pi\nu)}{\pi^{5/2}} G_{1,3}^{3,1}\left(z \,\middle|\, \begin{array}{c} \frac{1}{2} \\ 0, -\nu, \nu \end{array}\right)$$

03.01.26.0041.01

$$J_\nu\left(\sqrt{z}\right)^2 - Y_\nu\left(\sqrt{z}\right)^2 = -\frac{2}{\sqrt{\pi}} G_{2,4}^{3,0}\left(z \,\middle|\, \begin{array}{c} \frac{1}{2}, \frac{1}{2}-\nu \\ 0, -\nu, \nu, \frac{1}{2}-\nu \end{array}\right)$$

03.01.26.0042.01

$$J_\nu\left(\sqrt{z}\right) J_{-\nu}\left(\sqrt{z}\right) - Y_\nu\left(\sqrt{z}\right) Y_{-\nu}\left(\sqrt{z}\right) = -\frac{2}{\sqrt{\pi}} G_{2,4}^{3,0}\left(z \,\middle|\, \begin{array}{c} \frac{1}{2}, \frac{1}{2} \\ 0, -\nu, \nu, \frac{1}{2} \end{array}\right)$$

03.01.26.0043.01

$$J_\mu\left(\sqrt{z}\right) J_\nu\left(\sqrt{z}\right) - Y_\mu\left(\sqrt{z}\right) Y_\nu\left(\sqrt{z}\right) = -\frac{2}{\sqrt{\pi}} G_{3,5}^{4,0}\left(z \,\middle|\, \begin{array}{c} 0, \frac{1}{2}, \frac{1-\mu-\nu}{2} \\ \frac{\mu+\nu}{2}, \frac{\mu-\nu}{2}, \frac{\nu-\mu}{2}, -\frac{\mu+\nu}{2}, \frac{1-\mu-\nu}{2} \end{array}\right)$$

03.01.26.0044.01

$$J_\mu\left(\sqrt{z}\right) J_\nu\left(\sqrt{z}\right) + Y_\mu\left(\sqrt{z}\right) Y_\nu\left(\sqrt{z}\right) = \frac{\cos(\pi\mu)}{\pi^{5/2}} G_{2,4}^{3,2}\left(z \,\middle|\, \begin{array}{c} 0, \frac{1}{2} \\ \frac{\mu+\nu}{2}, \frac{\mu-\nu}{2}, -\frac{\mu+\nu}{2}, \frac{\nu-\mu}{2} \end{array}\right) + \frac{\cos(\pi\nu)}{\pi^{5/2}} G_{2,4}^{3,2}\left(z \,\middle|\, \begin{array}{c} 0, \frac{1}{2} \\ \frac{\mu+\nu}{2}, -\frac{\mu+\nu}{2}, \frac{\nu-\mu}{2}, \frac{\mu-\nu}{2} \end{array}\right) /; \mu+\nu \notin \mathbb{Z}$$

Classical cases involving cos, sin, *Y*

03.01.26.0045.01

$$\sin\left(\sqrt{z}\right) J_\nu\left(\sqrt{z}\right) + \cos\left(\sqrt{z}\right) Y_\nu\left(\sqrt{z}\right) = -\sqrt{2}\, G_{2,4}^{3,0}\left(z \,\middle|\, \begin{array}{c} \frac{1}{4}, \frac{3}{4} \\ -\frac{\nu}{2}, \frac{\nu}{2}, \frac{\nu+1}{2}, \frac{1-\nu}{2} \end{array}\right)$$

03.01.26.0046.01

$$\cos\left(\sqrt{z}\right) J_\nu\left(\sqrt{z}\right) - \sin\left(\sqrt{z}\right) Y_\nu\left(\sqrt{z}\right) = \sqrt{2}\, G_{2,4}^{3,0}\left(z \,\middle|\, \begin{array}{c} \frac{1}{4}, \frac{3}{4} \\ \frac{1-\nu}{2}, \frac{\nu}{2}, \frac{\nu+1}{2}, -\frac{\nu}{2} \end{array}\right)$$

03.01.26.0047.01

$$\sin\left(\sqrt{z}\right) J_\nu\left(\sqrt{z}\right) - \cos\left(\sqrt{z}\right) Y_\nu\left(\sqrt{z}\right) = -\frac{\cos(\nu\pi)}{\pi^2\sqrt{2}} G_{2,4}^{3,2}\left(z \,\middle|\, \begin{array}{c} \frac{1}{4}, \frac{3}{4} \\ -\frac{\nu}{2}, \frac{\nu}{2}, \frac{\nu+1}{2}, \frac{1-\nu}{2} \end{array}\right)$$

03.01.26.0048.01

$$\cos\left(\sqrt{z}\right) J_\nu\left(\sqrt{z}\right) + \sin\left(\sqrt{z}\right) Y_\nu\left(\sqrt{z}\right) = \frac{\cos(\nu\pi)}{\pi^2\sqrt{2}} G_{2,4}^{3,2}\left(z \,\middle|\, \begin{array}{c} \frac{1}{4}, \frac{3}{4} \\ \frac{1-\nu}{2}, \frac{\nu}{2}, \frac{\nu+1}{2}, -\frac{\nu}{2} \end{array}\right)$$

03.01.26.0049.01

$$\sin\left(a+\sqrt{z}\right) J_\nu\left(\sqrt{z}\right) - \cos\left(a+\sqrt{z}\right) Y_\nu\left(\sqrt{z}\right) = \frac{\cos(\pi\nu)}{\pi^2\sqrt{2}} G_{3,5}^{4,2}\left(z \,\middle|\, \begin{array}{c} \frac{1}{4}, \frac{3}{4}, \frac{1-\nu}{2}-\frac{a}{\pi} \\ -\frac{\nu}{2}, \frac{1-\nu}{2}, \frac{\nu}{2}, \frac{\nu+1}{2}, \frac{1-\nu}{2}-\frac{a}{\pi} \end{array}\right)$$

03.01.26.0117.01

$$\cos\left(a+\sqrt{z}\right) J_\nu\left(\sqrt{z}\right) + \sin\left(a+\sqrt{z}\right) Y_\nu\left(\sqrt{z}\right) = \frac{\cos(\pi\nu)}{\sqrt{2}\,\pi^2} G_{3,5}^{4,2}\left(z \,\middle|\, \begin{array}{c} \frac{1}{4}, \frac{3}{4}, -\frac{a}{\pi}-\frac{\nu}{2} \\ \frac{1-\nu}{2}, -\frac{\nu}{2}, \frac{\nu}{2}, \frac{\nu+1}{2}, -\frac{a}{\pi}-\frac{\nu}{2} \end{array}\right)$$

Classical cases involving ${}_0F_1$

03.01.26.0050.01

$$J_\nu(z)\,{}_0F_1\left(;b;-\frac{z^2}{4}\right)=\frac{\Gamma(b)\,2^{b-1}}{\sqrt{\pi}}\,G_{2,4}^{1,2}\left(z^2\,\middle|\,\begin{array}{c}\frac{1-b}{2},\,1-\frac{b}{2}\\ \frac{\nu}{2},\,-\frac{\nu}{2},\,1-b+\frac{\nu}{2},\,1-b-\frac{\nu}{2}\end{array}\right)/;\,-b-\nu\notin\mathbb{N}\bigwedge-\frac{\pi}{2}<\arg(z)\leq\frac{\pi}{2}$$

03.01.26.0118.01

$$J_\nu(z)\,{}_0F_1\left(;-n-\nu;-\frac{z^2}{4}\right)=\frac{2^{-n-\nu-1}\,\Gamma(-n-\nu)}{\sqrt{\pi}}\left((-1)^{n-1}\,G_{2,4}^{1,2}\left(z^2\,\middle|\,\begin{array}{c}\frac{1}{2}(n+\nu+1),\,\frac{n+\nu}{2}+1\\ n+\frac{\nu}{2}+1,\,-\frac{\nu}{2},\,\frac{\nu}{2},\,n+\frac{3\nu}{2}+1\end{array}\right)+2\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\frac{(-1)^{k+\lfloor\frac{n+1}{2}\rfloor}\,z^{2k+\nu}\,\Gamma\left(k-n+\lfloor\frac{n}{2}\rfloor+\frac{1}{2}\right)\left(1-k+\lfloor\frac{n}{2}\rfloor\right)_{n-\lfloor\frac{n}{2}\rfloor}}{k!\,\Gamma(k-n-\nu)\,\Gamma(k+\nu+1)}\right)/;\,n\in\mathbb{N}\bigwedge-\frac{\pi}{2}<\arg(z)\leq\frac{\pi}{2}$$

03.01.26.0119.01

$$J_\nu(z)\,{}_0F_1\left(;-\nu;-\frac{z^2}{4}\right)=-\frac{2^{-\nu-1}\,\Gamma(-\nu)}{\pi}\left(2\sin(\pi\nu)\,z^\nu+\sqrt{\pi}\,G_{2,4}^{1,2}\left(z^2\,\middle|\,\begin{array}{c}\frac{\nu+1}{2},\,\frac{\nu+2}{2}\\ \frac{\nu+2}{2},\,-\frac{\nu}{2},\,\frac{\nu}{2},\,\frac{3\nu}{2}+1\end{array}\right)\right)/;\,-\frac{\pi}{2}<\arg(z)\leq\frac{\pi}{2}$$

03.01.26.0120.01

$$J_\nu(z)\,{}_0F_1\left(;-\nu-1;-\frac{z^2}{4}\right)=2^{-\nu-2}\left(\frac{4\,z^\nu}{\Gamma(\nu+1)}+\frac{\Gamma(-\nu-1)}{\sqrt{\pi}}\,G_{2,4}^{1,2}\left(z^2\,\middle|\,\begin{array}{c}\frac{\nu}{2}+1,\,\frac{\nu+3}{2}\\ \frac{\nu}{2}+2,\,-\frac{\nu}{2},\,\frac{\nu}{2},\,\frac{3\nu}{2}+2\end{array}\right)\right)/;\,-\frac{\pi}{2}<\arg(z)\leq\frac{\pi}{2}$$

03.01.26.0121.01

$$J_\nu(z)\,{}_0F_1\left(;\nu;-\frac{z^2}{4}\right)=\frac{2^{\nu-1}\,\Gamma(\nu)}{\sqrt{\pi}}\,G_{1,3}^{1,1}\left(z^2\,\middle|\,\begin{array}{c}\frac{1-\nu}{2}\\ \frac{\nu}{2},\,-\frac{\nu}{2},\,1-\frac{3\nu}{2}\end{array}\right)/;\,-\frac{\pi}{2}<\arg(z)\leq\frac{\pi}{2}$$

03.01.26.0122.01

$$J_\nu(z)\,{}_0F_1\left(;\nu+1;-\frac{z^2}{4}\right)=\frac{2^\nu\,\Gamma(\nu+1)}{\sqrt{\pi}}\,G_{1,3}^{1,1}\left(z^2\,\middle|\,\begin{array}{c}\frac{1-\nu}{2}\\ \frac{\nu}{2},\,-\frac{\nu}{2},\,-\frac{1}{2}(3\nu)\end{array}\right)/;\,-\frac{\pi}{2}<\arg(z)\leq\frac{\pi}{2}$$

03.01.26.0123.01

$$J_\nu(z)\,{}_0F_1\left(;1-\nu;-\frac{z^2}{4}\right)=\frac{2^{-\nu}\,\Gamma(1-\nu)}{\sqrt{\pi}}\,G_{1,3}^{1,1}\left(z^2\,\middle|\,\begin{array}{c}\frac{1+\nu}{2}\\ \frac{\nu}{2},\,-\frac{\nu}{2},\,\frac{3\nu}{2}\end{array}\right)/;\,-\frac{\pi}{2}<\arg(z)\leq\frac{\pi}{2}$$

03.01.26.0051.01

$$J_\nu(z)\,{}_0F_1\left(;\nu+1;\frac{z^2}{4}\right)=\sqrt{\pi}\,2^{-\frac{\nu}{2}}\,\Gamma(\nu+1)\,z^\nu\left(z^4\right)^{-\frac{\nu}{4}}\,G_{0,4}^{1,0}\left(\frac{z^4}{64}\,\middle|\,\frac{\nu}{4},\,-\frac{\nu}{4},\,\frac{2-\nu}{4},\,-\frac{3\nu}{4}\right)$$

03.01.26.0052.01

$$J_\nu(z)\,{}_0F_1\left(;1-\nu;\frac{z^2}{4}\right)=2^{\nu/2}\,\sqrt{\pi}\,\Gamma(1-\nu)\,G_{1,5}^{2,0}\left(\frac{z^4}{64}\,\middle|\,\begin{array}{c}\frac{2-\nu}{4}\\ \frac{\nu}{4},\,\frac{\nu+2}{4},\,\frac{3\nu}{4},\,-\frac{\nu}{4},\,\frac{2-\nu}{4}\end{array}\right)/;\,-\frac{\pi}{2}<\arg(z)\leq\frac{\pi}{2}$$

03.01.26.0124.01

$$J_\nu\left(2\sqrt{z}\right)\,{}_0F_1(;b;-z)=\frac{\Gamma(b)\,2^{b-1}}{\sqrt{\pi}}\,G_{2,4}^{1,2}\left(4z\,\middle|\,\begin{array}{c}\frac{1-b}{2},\,1-\frac{b}{2}\\ \frac{\nu}{2},\,-\frac{\nu}{2},\,1-b+\frac{\nu}{2},\,1-b-\frac{\nu}{2}\end{array}\right)/;\,-b-\nu\notin\mathbb{N}$$

03.01.26.0125.01

$$J_\nu\left(2\sqrt{z}\right)\,{}_0F_1(;-n-\nu;-z)=\frac{\Gamma(-n-\nu)}{\sqrt{\pi}}\left(2^{-n}\,z^{\nu/2}\sum_{k=0}^{\left\lfloor\frac{n}{2}\right\rfloor}\frac{(-1)^{k+\left\lfloor\frac{n+1}{2}\right\rfloor}\,4^k\,z^k\,\Gamma\left(k-n+\left\lfloor\frac{n}{2}\right\rfloor+\frac{1}{2}\right)\left(1-k+\left\lfloor\frac{n}{2}\right\rfloor\right)_{n-\left\lfloor\frac{n}{2}\right\rfloor}}{k!\,\Gamma(k-n-\nu)\,\Gamma(k+\nu+1)}-\right.$$

$$\left.(-1)^n\,2^{-n-\nu-1}\,G_{2,4}^{1,2}\left(4\,z\,\middle|\,\begin{array}{c}\frac{1}{2}(n+\nu+1),\ \frac{n+\nu}{2}+1\\ n+\frac{\nu}{2}+1,\ -\frac{\nu}{2},\ \frac{\nu}{2},\ n+\frac{3\nu}{2}+1\end{array}\right)\right)/;\ n\in\mathbb{N}$$

03.01.26.0126.01

$$J_\nu\left(2\sqrt{z}\right)\,{}_0F_1(;-\nu;-z)=\frac{\Gamma(-\nu)}{2\,\pi}\left(-2\sin(\pi\,\nu)\,z^{\nu/2}-2^{-\nu}\sqrt{\pi}\;G_{3,5}^{2,2}\left(4\,z\,\middle|\,\begin{array}{c}\frac{\nu+1}{2},\ \frac{\nu}{2}+1,\ -\frac{\nu}{2}\\ \frac{\nu}{2}+1,\ -\frac{\nu}{2},\ -\frac{\nu}{2},\ \frac{\nu}{2},\ \frac{3\nu}{2}+1\end{array}\right)\right)$$

03.01.26.0127.01

$$J_\nu\left(2\sqrt{z}\right)\,{}_0F_1(;-\nu-1;-z)=\frac{z^{\nu/2}}{\Gamma(\nu+1)}+\frac{2^{-\nu-2}\,\Gamma(-\nu-1)}{\sqrt{\pi}}\,G_{2,4}^{1,2}\left(4\,z\,\middle|\,\begin{array}{c}\frac{\nu}{2}+1,\ \frac{\nu+3}{2}\\ \frac{\nu}{2}+2,\ -\frac{\nu}{2},\ \frac{\nu}{2},\ \frac{3\nu}{2}+2\end{array}\right)$$

03.01.26.0128.01

$$J_\nu\left(2\sqrt{z}\right)\,{}_0F_1(;\nu;-z)=\frac{2^{\nu-1}\,\Gamma(\nu)}{\sqrt{\pi}}\,G_{1,3}^{1,1}\left(4\,z\,\middle|\,\begin{array}{c}\frac{1-\nu}{2}\\ \frac{\nu}{2},\ -\frac{\nu}{2},\ 1-\frac{3\nu}{2}\end{array}\right)$$

03.01.26.0129.01

$$J_\nu\left(2\sqrt{z}\right)\,{}_0F_1(;\nu+1;z)=2^{-\frac{\nu}{2}}\sqrt{\pi}\;z^{\nu/2}\left(z^2\right)^{-\frac{\nu}{4}}\,\Gamma(\nu+1)\,G_{0,4}^{1,0}\left(\frac{z^2}{4}\,\middle|\,\frac{\nu}{4},\ -\frac{\nu}{4},\ \frac{2-\nu}{4},\ -\frac{3\nu}{4}\right)$$

03.01.26.0130.01

$$J_\nu\left(2\sqrt{z}\right)\,{}_0F_1(;1-\nu;z)=2^{\nu/2}\sqrt{\pi}\;\Gamma(1-\nu)\,G_{1,5}^{2,0}\left(\frac{z^2}{4}\,\middle|\,\begin{array}{c}\frac{2-\nu}{4}\\ \frac{\nu}{4},\ \frac{\nu+2}{4},\ \frac{3\nu}{4},\ -\frac{\nu}{4},\ \frac{2-\nu}{4}\end{array}\right)/;\ -\frac{\pi}{2}<\arg(z)\leq\frac{\pi}{2}$$

Classical cases involving ${}_0\tilde{F}_1$

03.01.26.0053.01

$$J_\nu(z)\,{}_0\tilde{F}_1\left(;b;-\frac{z^2}{4}\right)=\frac{2^{b-1}}{\sqrt{\pi}}\,G_{2,4}^{1,2}\left(z^2\,\middle|\,\begin{array}{c}\frac{1-b}{2},\ 1-\frac{b}{2}\\ \frac{\nu}{2},\ -\frac{\nu}{2},\ 1-b+\frac{\nu}{2},\ 1-b-\frac{\nu}{2}\end{array}\right)/;\ -b-\nu\notin\mathbb{N}\bigwedge-\frac{\pi}{2}<\arg(z)\leq\frac{\pi}{2}$$

03.01.26.0131.01

$$J_\nu(z)\,{}_0\tilde{F}_1\left(;-n-\nu;-\frac{z^2}{4}\right)=$$

$$\frac{2^{-n-\nu-1}}{\sqrt{\pi}}\left((-1)^{n-1}\,G_{2,4}^{1,2}\left(z^2\,\middle|\,\begin{array}{c}\frac{1}{2}(n+\nu+1),\ \frac{n+\nu}{2}+1\\ n+\frac{\nu}{2}+1,\ -\frac{\nu}{2},\ \frac{\nu}{2},\ n+\frac{3\nu}{2}+1\end{array}\right)+2\sum_{k=0}^{\left\lfloor\frac{n}{2}\right\rfloor}\frac{(-1)^{k+\left\lfloor\frac{n+1}{2}\right\rfloor}\,z^{2k+\nu}\,\Gamma\left(k-n+\left\lfloor\frac{n}{2}\right\rfloor+\frac{1}{2}\right)\left(1-k+\left\lfloor\frac{n}{2}\right\rfloor\right)_{n-\left\lfloor\frac{n}{2}\right\rfloor}}{k!\,\Gamma(k-n-\nu)\,\Gamma(k+\nu+1)}\right)/;$$

$$n\in\mathbb{N}\bigwedge-\frac{\pi}{2}<\arg(z)\leq\frac{\pi}{2}$$

03.01.26.0132.01

$$J_\nu(z)\,{}_0\tilde{F}_1\left(;-\nu;-\frac{z^2}{4}\right)=-\frac{2^{-\nu-1}}{\pi}\left(2\sin(\pi\nu)\,z^\nu+\sqrt{\pi}\;G_{2,4}^{1,2}\left(z^2\,\middle|\,\begin{array}{c}\frac{\nu+1}{2},\frac{\nu+2}{2}\\ \frac{\nu+2}{2},-\frac{\nu}{2},\frac{\nu}{2},\frac{3\nu}{2}+1\end{array}\right)\right)\,/;\,-\frac{\pi}{2}<\arg(z)\leq\frac{\pi}{2}$$

03.01.26.0133.01

$$J_\nu(z)\,{}_0\tilde{F}_1\left(;-\nu-1;-\frac{z^2}{4}\right)=2^{-\nu-2}\left(\frac{4(\nu+1)\sin(\pi\nu)\,z^\nu}{\pi}+\frac{1}{\sqrt{\pi}}\,G_{2,4}^{1,2}\left(z^2\,\middle|\,\begin{array}{c}\frac{\nu}{2}+1,\frac{\nu+3}{2}\\ \frac{\nu}{2}+2,-\frac{\nu}{2},\frac{\nu}{2},\frac{3\nu}{2}+2\end{array}\right)\right)\,/;\,-\frac{\pi}{2}<\arg(z)\leq\frac{\pi}{2}$$

03.01.26.0134.01

$$J_\nu(z)\,{}_0\tilde{F}_1\left(;\nu;-\frac{z^2}{4}\right)=\frac{2^{\nu-1}}{\sqrt{\pi}}\,G_{1,3}^{1,1}\left(z^2\,\middle|\,\begin{array}{c}\frac{1-\nu}{2}\\ \frac{\nu}{2},-\frac{\nu}{2},1-\frac{3\nu}{2}\end{array}\right)\,/;\,-\frac{\pi}{2}<\arg(z)\leq\frac{\pi}{2}$$

03.01.26.0135.01

$$J_\nu(z)\,{}_0\tilde{F}_1\left(;\nu+1;-\frac{z^2}{4}\right)=\frac{2^{\nu}}{\sqrt{\pi}}\,G_{1,3}^{1,1}\left(z^2\,\middle|\,\begin{array}{c}\frac{1-\nu}{2}\\ \frac{\nu}{2},-\frac{\nu}{2},-\frac{1}{2}(3\nu)\end{array}\right)\,/;\,-\frac{\pi}{2}<\arg(z)\leq\frac{\pi}{2}$$

03.01.26.0136.01

$$J_\nu(z)\,{}_0\tilde{F}_1\left(;1-\nu;-\frac{z^2}{4}\right)=\frac{2^{-\nu}}{\sqrt{\pi}}\,G_{1,3}^{1,1}\left(z^2\,\middle|\,\begin{array}{c}\frac{1+\nu}{2}\\ \frac{\nu}{2},-\frac{\nu}{2},\frac{3\nu}{2}\end{array}\right)\,/;\,-\frac{\pi}{2}<\arg(z)\leq\frac{\pi}{2}$$

03.01.26.0054.01

$$J_\nu(z)\,{}_0\tilde{F}_1\left(;\nu+1;\frac{z^2}{4}\right)=\sqrt{\pi}\;2^{-\frac{\nu}{2}}\,z^\nu\left(z^4\right)^{-\frac{\nu}{4}}\,G_{0,4}^{1,0}\left(\frac{z^4}{64}\,\middle|\,\frac{\nu}{4},-\frac{\nu}{4},\frac{2-\nu}{4},-\frac{3\nu}{4}\right)$$

03.01.26.0055.01

$$J_\nu(z)\,{}_0\tilde{F}_1\left(;1-\nu;\frac{z^2}{4}\right)=2^{\nu/2}\sqrt{\pi}\;G_{1,5}^{2,0}\left(\frac{z^4}{64}\,\middle|\,\begin{array}{c}\frac{2-\nu}{4}\\ \frac{\nu}{4},\frac{\nu+2}{4},\frac{3\nu}{4},-\frac{\nu}{4},\frac{2-\nu}{4}\end{array}\right)\,/;\,-\frac{\pi}{2}<\arg(z)\leq\frac{\pi}{2}$$

03.01.26.0137.01

$$J_\nu\left(2\sqrt{z}\right)\,{}_0\tilde{F}_1(;b;-z)=\frac{2^{b-1}}{\sqrt{\pi}}\;G_{2,4}^{1,2}\left(4z\,\middle|\,\begin{array}{c}\frac{1-b}{2},1-\frac{b}{2}\\ \frac{\nu}{2},-\frac{\nu}{2},1-b+\frac{\nu}{2},1-b-\frac{\nu}{2}\end{array}\right)\,/;\,1-b-\nu\notin\mathbb{N}$$

03.01.26.0138.01

$$J_\nu\left(2\sqrt{z}\right)\,{}_0\tilde{F}_1(;-n-\nu;-z)=\frac{1}{\sqrt{\pi}}\left(2^{-n}\,z^{\nu/2}\sum_{k=0}^{\left\lfloor\frac{n}{2}\right\rfloor}\frac{(-1)^{k+\left\lfloor\frac{n+1}{2}\right\rfloor}\,4^k\,z^k\,\Gamma\left(k-n+\left\lfloor\frac{n}{2}\right\rfloor+\frac{1}{2}\right)\left(1-k+\left\lfloor\frac{n}{2}\right\rfloor\right)_{n-\left\lfloor\frac{n}{2}\right\rfloor}}{k!\,\Gamma(k-n-\nu)\,\Gamma(k+\nu+1)}-\right.$$

$$\left.(-1)^n\,2^{-n-\nu-1}\,G_{2,4}^{1,2}\left(4z\,\middle|\,\begin{array}{c}\frac{1}{2}(n+\nu+1),\frac{n+\nu}{2}+1\\ n+\frac{\nu}{2}+1,-\frac{\nu}{2},\frac{\nu}{2},n+\frac{3\nu}{2}+1\end{array}\right)\right)\,/;\,n\in\mathbb{N}$$

03.01.26.0139.01

$$J_\nu\left(2\sqrt{z}\right)\,{}_0\tilde{F}_1(;-\nu;-z)=\frac{1}{2\pi}\left(-2\sin(\pi\nu)\,z^{\nu/2}-2^{-\nu}\sqrt{\pi}\;G_{3,5}^{2,2}\left(4z\,\middle|\,\begin{array}{c}\frac{\nu+1}{2},\frac{\nu}{2}+1,-\frac{\nu}{2}\\ \frac{\nu}{2}+1,-\frac{\nu}{2},-\frac{\nu}{2},\frac{\nu}{2},\frac{3\nu}{2}+1\end{array}\right)\right)$$

03.01.26.0140.01

$$J_\nu\left(2\sqrt{z}\right)\,{}_0\tilde{F}_1(;-\nu-1;-z)=\frac{(\nu+1)\sin(\pi\nu)\,z^{\nu/2}}{\pi}+\frac{2^{-\nu-2}}{\sqrt{\pi}}\,G_{2,4}^{1,2}\left(4z\,\middle|\,\begin{array}{c}\frac{\nu}{2}+1,\frac{\nu+3}{2}\\ \frac{\nu}{2}+2,-\frac{\nu}{2},\frac{\nu}{2},\frac{3\nu}{2}+2\end{array}\right)$$

03.01.26.0141.01

$$J_\nu\left(2\sqrt{z}\right)\,{}_0\tilde{F}_1(;\nu;-z)=\frac{2^{\nu-1}}{\sqrt{\pi}}\,G_{1,3}^{1,1}\left(4z\,\middle|\,\begin{array}{c}\frac{1-\nu}{2}\\ \frac{\nu}{2},-\frac{\nu}{2},1-\frac{3\nu}{2}\end{array}\right)$$

03.01.26.0142.01

$$J_\nu\left(2\sqrt{z}\right)\,{}_0\tilde{F}_1(;\nu+1;-z)=2^{-\frac{\nu}{2}}\sqrt{\pi}\,z^{\nu/2}\left(z^2\right)^{-\frac{\nu}{4}}\,G_{0,4}^{1,0}\left(\frac{z^2}{4}\,\middle|\,\frac{\nu}{4},-\frac{\nu}{4},\frac{2-\nu}{4},-\frac{3\nu}{4}\right)$$

03.01.26.0143.01

$$J_\nu\left(2\sqrt{z}\right)\,{}_0\tilde{F}_1(;1-\nu;-z)=2^{\nu/2}\sqrt{\pi}\;G_{1,5}^{2,0}\left(\frac{z^2}{4}\,\middle|\,\begin{array}{c}\frac{2-\nu}{4}\\ \frac{\nu}{4},\frac{\nu+2}{4},\frac{3\nu}{4},-\frac{\nu}{4},\frac{2-\nu}{4}\end{array}\right)\,/;\,-\frac{\pi}{2}<\arg(z)\leq\frac{\pi}{2}$$

Generalized cases for the direct function itself

03.01.26.0056.01

$$J_\nu(z)=G_{0,2}^{1,0}\left(\frac{z}{2},\frac{1}{2}\,\middle|\,\frac{\nu}{2},-\frac{\nu}{2}\right)$$

03.01.26.0057.01

$$J_\nu(z)=\pi\,z^\nu\,(i\,z)^{-\nu}\,G_{1,3}^{1,0}\left(\frac{i\,z}{2},\frac{1}{2}\,\middle|\,\begin{array}{c}\frac{\nu+1}{2}\\ \frac{\nu}{2},-\frac{\nu}{2},\frac{\nu+1}{2}\end{array}\right)$$

03.01.26.0144.01

$$J_{-\nu}(z)+J_\nu(z)=2\cos\left(\frac{\pi\nu}{2}\right)G_{1,3}^{2,0}\left(\frac{z}{2},\frac{1}{2}\,\middle|\,\begin{array}{c}0\\ -\frac{\nu}{2},\frac{\nu}{2},0\end{array}\right)$$

03.01.26.0145.01

$$J_{-\nu}(z)-J_\nu(z)=2\sin\left(\frac{\pi\nu}{2}\right)G_{1,3}^{2,0}\left(\frac{z}{2},\frac{1}{2}\,\middle|\,\begin{array}{c}\frac{1}{2}\\ -\frac{\nu}{2},\frac{\nu}{2},\frac{1}{2}\end{array}\right)$$

Generalized cases involving cos

03.01.26.0058.01

$$\cos(z)\,J_\nu(z)=\frac{1}{\sqrt{2}}\,G_{2,4}^{1,2}\left(z,\frac{1}{2}\,\middle|\,\begin{array}{c}\frac{1}{4},\frac{3}{4}\\ \frac{\nu}{2},-\frac{\nu}{2},\frac{\nu+1}{2},\frac{1-\nu}{2}\end{array}\right)$$

03.01.26.0059.01

$$\cos(a+z)\,J_\nu(z)=\frac{1}{\sqrt{2}}\,G_{3,5}^{2,2}\left(z,\frac{1}{2}\,\middle|\,\begin{array}{c}\frac{1}{4},\frac{3}{4},\frac{a}{\pi}+\frac{\nu+1}{2}\\ \frac{\nu}{2},\frac{\nu+1}{2},-\frac{\nu}{2},\frac{1-\nu}{2},\frac{a}{\pi}+\frac{\nu+1}{2}\end{array}\right)$$

Generalized cases involving sin

03.01.26.0060.01

$$\sin(z)\,J_\nu(z)=\frac{1}{\sqrt{2}}\,G_{2,4}^{1,2}\left(z,\frac{1}{2}\,\middle|\,\begin{array}{c}\frac{1}{4},\frac{3}{4}\\ \frac{\nu+1}{2},-\frac{\nu}{2},\frac{1-\nu}{2},\frac{\nu}{2}\end{array}\right)$$

03.01.26.0061.01

$$\sin(a+z)\,J_\nu(z) = \frac{1}{\sqrt{2}}\,G_{3,5}^{2,2}\!\left(z,\ \frac{1}{2}\ \middle|\ \begin{matrix}\frac{1}{4},\ \frac{3}{4},\ \frac{a}{\pi}+\frac{\nu}{2}\\ \frac{\nu}{2},\ \frac{\nu+1}{2},\ -\frac{\nu}{2},\ \frac{1-\nu}{2},\ \frac{a}{\pi}+\frac{\nu}{2}\end{matrix}\right)$$

Generalized cases involving cos, sin

03.01.26.0062.01

$$\cos(z)\,J_{-\nu}(z) + \sin(z)\,J_\nu(z) = -\sqrt{2}\,\sin\left(\frac{1}{4}\pi(2\nu-1)\right)G_{2,4}^{2,1}\!\left(z,\ \frac{1}{2}\ \middle|\ \begin{matrix}\frac{3}{4},\ \frac{1}{4}\\ -\frac{\nu}{2},\ \frac{\nu+1}{2},\ \frac{1-\nu}{2},\ \frac{\nu}{2}\end{matrix}\right)$$

03.01.26.0063.01

$$\cos(z)\,J_{-\nu}(z) - \sin(z)\,J_\nu(z) = \sqrt{2}\,\sin\left(\frac{1}{4}\pi(2\nu+1)\right)G_{2,4}^{2,1}\!\left(z,\ \frac{1}{2}\ \middle|\ \begin{matrix}\frac{1}{4},\ \frac{3}{4}\\ -\frac{\nu}{2},\ \frac{\nu+1}{2},\ \frac{1-\nu}{2},\ \frac{\nu}{2}\end{matrix}\right)$$

03.01.26.0146.01

$$\sin(z)\,J_{-\nu}(z) + \cos(z)\,J_\nu(z) = \sqrt{2}\,\sin\left(\frac{1}{4}\pi(2\nu+1)\right)G_{2,4}^{2,1}\!\left(z,\ \frac{1}{2}\ \middle|\ \begin{matrix}\frac{3}{4},\ \frac{1}{4}\\ \frac{\nu}{2},\ \frac{1-\nu}{2},\ \frac{\nu+1}{2},\ -\frac{\nu}{2}\end{matrix}\right)$$

03.01.26.0147.01

$$\sin(z)\,J_{-\nu}(z) - \cos(z)\,J_\nu(z) = \sqrt{2}\,\sin\left(\frac{1}{4}\pi(2\nu-1)\right)G_{2,4}^{2,1}\!\left(z,\ \frac{1}{2}\ \middle|\ \begin{matrix}\frac{1}{4},\ \frac{3}{4}\\ \frac{\nu}{2},\ \frac{1-\nu}{2},\ \frac{\nu+1}{2},\ -\frac{\nu}{2}\end{matrix}\right)$$

Generalized cases for powers of Bessel *J*

03.01.26.0064.01

$$J_\nu(z)^2 = \frac{1}{\sqrt{\pi}}\,G_{1,3}^{1,1}\!\left(z,\ \frac{1}{2}\ \middle|\ \begin{matrix}\frac{1}{2}\\ \nu,\ 0,\ -\nu\end{matrix}\right)$$

03.01.26.0065.01

$$J_{-\nu}(z)^2 + J_\nu(z)^2 = \frac{2\cos(\pi\nu)}{\sqrt{\pi}}\,G_{2,4}^{2,1}\!\left(z,\ \frac{1}{2}\ \middle|\ \begin{matrix}\frac{1}{2},\ 0\\ -\nu,\ \nu,\ 0,\ 0\end{matrix}\right)$$

03.01.26.0066.01

$$J_{-\nu}(z)^2 - J_\nu(z)^2 = \frac{2\sin(\pi\nu)}{\sqrt{\pi}}\,G_{1,3}^{2,0}\!\left(z,\ \frac{1}{2}\ \middle|\ \begin{matrix}\frac{1}{2}\\ -\nu,\ \nu,\ 0\end{matrix}\right)$$

Generalized cases for products of Bessel *J*

03.01.26.0067.01

$$J_{-\nu}(z)\,J_\nu(z) = \frac{1}{\sqrt{\pi}}\,G_{1,3}^{1,1}\!\left(z,\ \frac{1}{2}\ \middle|\ \begin{matrix}\frac{1}{2}\\ 0,\ \nu,\ -\nu\end{matrix}\right)$$

03.01.26.0068.01

$$J_{\nu-1}(z)\,J_\nu(z) = \frac{1}{\sqrt{\pi}}\,G_{1,3}^{1,1}\!\left(z,\ \frac{1}{2}\ \middle|\ \begin{matrix}0\\ \nu-\frac{1}{2},\ -\frac{1}{2},\ \frac{1}{2}-\nu\end{matrix}\right)$$

03.01.26.0069.01

$$J_\mu(z)\,J_\nu(z) = \frac{1}{\sqrt{\pi}}\,G_{2,4}^{1,2}\!\left(z,\ \frac{1}{2}\ \middle|\ \begin{matrix}0,\ \frac{1}{2}\\ \frac{\mu+\nu}{2},\ -\frac{\mu+\nu}{2},\ \frac{\mu-\nu}{2},\ \frac{\nu-\mu}{2}\end{matrix}\right)\ /;\ -\mu-\nu-1\notin\mathbb{N}$$

03.01.26.0148.01

$$J_{-n-\nu-1}(z)\,J_\nu(z) = \frac{2}{\sqrt{\pi}} \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{(-1)^{k+\left\lfloor \frac{n+1}{2} \right\rfloor} z^{2k-n-1}\,\Gamma\!\left(k-n+\left\lfloor \frac{n}{2} \right\rfloor+\frac{1}{2}\right)\left(1-k+\left\lfloor \frac{n}{2} \right\rfloor\right)_{n-\left\lfloor \frac{n}{2} \right\rfloor}}{k!\,\Gamma(k-n-\nu)\,\Gamma(k+\nu+1)} - \frac{(-1)^n}{\sqrt{\pi}}\, G_{2,4}^{1,2}\!\left(z, \frac{1}{2} \,\middle|\, \begin{array}{c} 0, \frac{1}{2} \\ \frac{n+1}{2}, \frac{n+1}{2}+\nu, -\frac{1}{2}(n+1), -\frac{1}{2}(n+1)-\nu \end{array}\right) /;\, n \in \mathbb{N}$$

03.01.26.0149.01

$$J_{-\nu-1}(z)\,J_\nu(z) = -\frac{1}{\sqrt{\pi}}\, G_{2,4}^{1,2}\!\left(z, \frac{1}{2} \,\middle|\, \begin{array}{c} 0, \frac{1}{2} \\ \frac{1}{2}, \nu+\frac{1}{2}, -\frac{1}{2}, -\nu-\frac{1}{2} \end{array}\right) - \frac{2\sin(\pi\,\nu)}{\pi\,z}$$

03.01.26.0150.01

$$J_{-\nu-2}(z)\,J_\nu(z) = \frac{1}{\sqrt{\pi}}\, G_{2,4}^{1,2}\!\left(z, \frac{1}{2} \,\middle|\, \begin{array}{c} 0, \frac{1}{2} \\ 1, \nu+1, -1, -\nu-1 \end{array}\right) + \frac{4(\nu+1)\sin(\pi\,\nu)}{\pi\,z^2}$$

03.01.26.0070.01

$$J_{-\mu}(z)\,J_{-\nu}(z) + J_\mu(z)\,J_\nu(z) = \frac{2\cos\left(\frac{1}{2}(\mu+\nu)\pi\right)}{\sqrt{\pi}}\, G_{2,4}^{2,1}\!\left(z, \frac{1}{2} \,\middle|\, \begin{array}{c} \frac{1}{2}, 0 \\ \frac{\mu+\nu}{2}, -\frac{\mu+\nu}{2}, \frac{\mu-\nu}{2}, \frac{\nu-\mu}{2} \end{array}\right) /;\, \frac{\mu+\nu-1}{2} \notin \mathbb{Z}$$

03.01.26.0071.01

$$J_\mu(z)\,J_\nu(z) - J_{-\mu}(z)\,J_{-\nu}(z) = -\frac{2\sin\left(\frac{1}{2}(\mu+\nu)\pi\right)}{\sqrt{\pi}}\, G_{2,4}^{2,1}\!\left(z, \frac{1}{2} \,\middle|\, \begin{array}{c} 0, \frac{1}{2} \\ \frac{\mu+\nu}{2}, -\frac{\mu+\nu}{2}, \frac{\mu-\nu}{2}, \frac{\nu-\mu}{2} \end{array}\right) /;\, \frac{\mu+\nu}{2} \notin \mathbb{Z}$$

Generalized cases involving Bessel *I*

03.01.26.0072.01

$$J_\nu(z)\,I_\nu(z) = \sqrt{\pi}\, G_{0,4}^{1,0}\!\left(\frac{z}{2\sqrt{2}}, \frac{1}{4} \,\middle|\, \frac{\nu}{2}, 0, \frac{1}{2}, -\frac{\nu}{2}\right)$$

03.01.26.0073.01

$$J_{-\nu}(z)\,I_\nu(z) = \sqrt{\pi}\, G_{1,5}^{2,0}\!\left(\frac{z}{2\sqrt{2}}, \frac{1}{4} \,\middle|\, \begin{array}{c} \frac{\nu+1}{2} \\ 0, \frac{1}{2}, -\frac{\nu}{2}, \frac{\nu}{2}, \frac{\nu+1}{2} \end{array}\right)$$

03.01.26.0104.01

$$(J_{-\nu}(z) + J_\nu(z))\,I_\nu(z) = 2\cos\left(\frac{\pi\,\nu}{2}\right)\sqrt{\pi}\, G_{2,6}^{3,0}\!\left(\frac{z}{2\sqrt{2}}, \frac{1}{4} \,\middle|\, \begin{array}{c} \frac{\nu}{4}, \frac{\nu+2}{4} \\ 0, \frac{1}{2}, \frac{\nu}{2}, -\frac{\nu}{2}, \frac{\nu}{4}, \frac{\nu+2}{4} \end{array}\right)$$

03.01.26.0105.01

$$(J_{-\nu}(z) - J_\nu(z))\,I_\nu(z) = 2\sin\left(\frac{\pi\,\nu}{2}\right)\sqrt{\pi}\, G_{2,6}^{3,0}\!\left(\frac{z}{2\sqrt{2}}, \frac{1}{4} \,\middle|\, \begin{array}{c} \frac{\nu+1}{4}, \frac{\nu+3}{4} \\ 0, \frac{1}{2}, \frac{\nu}{2}, -\frac{\nu}{2}, \frac{\nu+3}{4}, \frac{\nu+1}{4} \end{array}\right)$$

Generalized cases involving Bessel *K*

03.01.26.0074.01

$$J_\nu(z)\,K_\nu(z) = \frac{1}{4\sqrt{\pi}}\, G_{0,4}^{3,0}\!\left(\frac{z}{2\sqrt{2}}, \frac{1}{4} \,\middle|\, 0, \frac{1}{2}, \frac{\nu}{2}, -\frac{\nu}{2}\right)$$

03.01.26.0151.01

$$J_{-\nu}(z)\,K_\nu(z)=\frac{1}{4\sqrt{\pi}}\,G_{0,4}^{3,0}\!\left(\frac{z}{2\sqrt{2}},\frac{1}{4}\,\middle|\,0,\frac{1}{2},-\frac{\nu}{2},\frac{\nu}{2}\right)$$

03.01.26.0075.01

$$(J_{-\nu}(z)+J_\nu(z))\,K_\nu(z)=\frac{\cos\left(\frac{\pi\nu}{2}\right)}{2\sqrt{\pi}}\,G_{0,4}^{3,0}\!\left(\frac{z}{2\sqrt{2}},\frac{1}{4}\,\middle|\,\frac{1}{2},-\frac{\nu}{2},\frac{\nu}{2},0\right)$$

03.01.26.0076.01

$$(J_{-\nu}(z)-J_\nu(z))\,K_\nu(z)=\frac{\sin\left(\frac{\pi\nu}{2}\right)}{2\sqrt{\pi}}\,G_{0,4}^{3,0}\!\left(\frac{z}{2\sqrt{2}},\frac{1}{4}\,\middle|\,0,-\frac{\nu}{2},\frac{\nu}{2},\frac{1}{2}\right)$$

Generalized cases involving Bessel *Y*

03.01.26.0077.01

$$J_\nu(z)\cos(a\pi)+Y_\nu(z)\sin(a\pi)=G_{1,3}^{2,0}\!\left(\frac{z}{2},\frac{1}{2}\,\middle|\,\begin{matrix}-a-\frac{\nu}{2}\\ -\frac{\nu}{2},\frac{\nu}{2},-a-\frac{\nu}{2}\end{matrix}\right)$$

03.01.26.0152.01

$$\cos(a\pi)\,J_\nu(z)-\sin(a\pi)\,Y_\nu(z)=G_{1,3}^{2,0}\!\left(\frac{z}{2},\frac{1}{2}\,\middle|\,\begin{matrix}a-\frac{\nu}{2}\\ -\frac{\nu}{2},\frac{\nu}{2},a-\frac{\nu}{2}\end{matrix}\right)$$

03.01.26.0078.01

$$J_\nu(z)\,Y_\nu(z)=-\frac{1}{\sqrt{\pi}}\,G_{1,3}^{2,0}\!\left(z,\frac{1}{2}\,\middle|\,\begin{matrix}\frac{1}{2}\\ 0,\nu,-\nu\end{matrix}\right)$$

03.01.26.0079.01

$$J_{-\nu}(z)\,Y_\nu(z)=\frac{1}{\sqrt{\pi}}\,G_{2,4}^{2,1}\!\left(z,\frac{1}{2}\,\middle|\,\begin{matrix}\frac{1}{2},-\nu-\frac{1}{2}\\ 0,-\nu,-\nu-\frac{1}{2},\nu\end{matrix}\right)$$

03.01.26.0080.01

$$J_{\nu+1}(z)\,Y_\nu(z)=\frac{1}{\sqrt{\pi}}\,G_{2,4}^{2,1}\!\left(z,\frac{1}{2}\,\middle|\,\begin{matrix}\frac{1}{2},0\\ \frac{1}{2},\nu+\frac{1}{2},-\frac{1}{2},-\nu-\frac{1}{2}\end{matrix}\right)$$

03.01.26.0081.01

$$J_{\nu+2}(z)\,Y_\nu(z)=\frac{1}{\sqrt{\pi}}\,G_{2,4}^{2,1}\!\left(z,\frac{1}{2}\,\middle|\,\begin{matrix}0,\frac{1}{2}\\ 1,\nu+1,-1,-\nu-1\end{matrix}\right)$$

03.01.26.0082.01

$$J_\mu(z)\,Y_\nu(z)=-\frac{1}{\sqrt{\pi}}\,G_{3,5}^{2,2}\!\left(z,\frac{1}{2}\,\middle|\,\begin{matrix}0,\frac{1}{2},\frac{\mu-\nu+1}{2}\\ \frac{\mu+\nu}{2},\frac{\mu-\nu}{2},\frac{\mu-\nu+1}{2},-\frac{\mu+\nu}{2},\frac{\nu-\mu}{2}\end{matrix}\right)/;\;-\mu-\nu-1\notin\mathbb{N}$$

03.01.26.0153.01

$$J_\nu(z)\,Y_{-n-\nu-1}(z)=\frac{(-1)^n}{\sqrt{\pi}}\,G_{3,5}^{2,2}\!\left(z,\frac{1}{2}\,\middle|\,\begin{matrix}0,\frac{1}{2},\frac{n}{2}+\nu+1\\ \frac{n+1}{2},\frac{n+1}{2}+\nu,-\frac{1}{2}(n+1)-\nu,-\frac{1}{2}(n+1),\frac{n}{2}+\nu+1\end{matrix}\right)-$$

$$\frac{2\cot(\nu\pi)}{\sqrt{\pi}}\sum_{k=0}^{\left\lfloor\frac{n}{2}\right\rfloor}\frac{(-1)^{k+\left\lfloor\frac{n+1}{2}\right\rfloor}z^{2k-n-1}\,\Gamma\left(k-n+\left\lfloor\frac{n}{2}\right\rfloor+\frac{1}{2}\right)\left(1-k+\left\lfloor\frac{n}{2}\right\rfloor\right)_{n-\left\lfloor\frac{n}{2}\right\rfloor}}{k!\,\Gamma(k-n-\nu)\,\Gamma(k+\nu+1)}\;/;\;n\in\mathbb{N}$$

03.01.26.0154.01

$$J_\nu(z)\,Y_{-\nu-1}(z) = \frac{2\cos(\pi\,\nu)}{\pi\,z} + \frac{1}{\sqrt{\pi}}\,G_{3,5}^{2,2}\!\left(z,\ \frac{1}{2}\,\middle|\, \begin{matrix} 0,\ \frac{1}{2},\ \nu+1 \\ \frac{1}{2},\ \nu+\frac{1}{2},\ -\frac{1}{2},\ -\nu-\frac{1}{2},\ \nu+1 \end{matrix}\right)$$

03.01.26.0155.01

$$J_\nu(z)\,Y_{-\nu-2}(z) = -\frac{4\,(\nu+1)\cos(\pi\,\nu)}{\pi\,z^2} - \frac{1}{\sqrt{\pi}}\,G_{3,5}^{2,2}\!\left(z,\ \frac{1}{2}\,\middle|\, \begin{matrix} 0,\ \frac{1}{2},\ \nu+\frac{3}{2} \\ 1,\ \nu+1,\ -1,\ -\nu-1,\ \nu+\frac{3}{2} \end{matrix}\right)$$

03.01.26.0083.01

$$J_\nu(z)\,Y_{-\nu}(z) + J_{-\nu}(z)\,Y_\nu(z) = -\frac{2}{\sqrt{\pi}}\,G_{1,3}^{2,0}\!\left(z,\ \frac{1}{2}\,\middle|\, \begin{matrix} \frac{1}{2} \\ \nu,\ -\nu,\ 0 \end{matrix}\right)$$

03.01.26.0084.01

$$J_\nu(z)\,Y_{-\nu}(z) - J_{-\nu}(z)\,Y_\nu(z) = \frac{\sin(2\,\pi\,\nu)}{\pi^{5/2}}\,G_{1,3}^{3,1}\!\left(z,\ \frac{1}{2}\,\middle|\, \begin{matrix} \frac{1}{2} \\ 0,\ \nu,\ -\nu \end{matrix}\right)$$

03.01.26.0085.01

$$J_\nu(z)\,Y_\mu(z) + J_\mu(z)\,Y_\nu(z) = -\frac{2}{\sqrt{\pi}}\,G_{2,4}^{3,0}\!\left(z,\ \frac{1}{2}\,\middle|\, \begin{matrix} 0,\ \frac{1}{2} \\ \frac{\mu+\nu}{2},\ \frac{\mu-\nu}{2},\ \frac{\nu-\mu}{2},\ -\frac{\mu+\nu}{2} \end{matrix}\right)$$

03.01.26.0086.01

$$J_\nu(z)\,Y_\mu(z) - J_\mu(z)\,Y_\nu(z) = \frac{\sin(\pi\,(\nu-\mu))}{\pi^{5/2}}\,G_{2,4}^{3,2}\!\left(z,\ \frac{1}{2}\,\middle|\, \begin{matrix} 0,\ \frac{1}{2} \\ \frac{\mu-\nu}{2},\ \frac{\nu-\mu}{2},\ \frac{\mu+\nu}{2},\ -\frac{\mu+\nu}{2} \end{matrix}\right) /;\ -\mu-\nu-1 \notin \mathbb{N}$$

03.01.26.0087.01

$$J_\nu(z)^2 + Y_\nu(z)^2 = \frac{2\cos(\pi\,\nu)}{\pi^{5/2}}\,G_{1,3}^{3,1}\!\left(z,\ \frac{1}{2}\,\middle|\, \begin{matrix} \frac{1}{2} \\ 0,\ -\nu,\ \nu \end{matrix}\right)$$

03.01.26.0088.01

$$J_\nu(z)^2 - Y_\nu(z)^2 = -\frac{2}{\sqrt{\pi}}\,G_{2,4}^{3,0}\!\left(z,\ \frac{1}{2}\,\middle|\, \begin{matrix} \frac{1}{2},\ \frac{1}{2}-\nu \\ 0,\ -\nu,\ \nu,\ \frac{1}{2}-\nu \end{matrix}\right)$$

03.01.26.0106.01

$$J_{-\nu}(z)\,J_\nu(z) + Y_{-\nu}(z)\,Y_\nu(z) = \frac{2\cos^2(\pi\,\nu)}{\pi^{5/2}}\,G_{1,3}^{3,1}\!\left(z,\ \frac{1}{2}\,\middle|\, \begin{matrix} \frac{1}{2} \\ 0,\ -\nu,\ \nu \end{matrix}\right)$$

03.01.26.0089.01

$$J_\nu(z)\,J_{-\nu}(z) - Y_\nu(z)\,Y_{-\nu}(z) = -\frac{2}{\sqrt{\pi}}\,G_{2,4}^{3,0}\!\left(z,\ \frac{1}{2}\,\middle|\, \begin{matrix} \frac{1}{2},\ \frac{1}{2} \\ 0,\ -\nu,\ \nu,\ \frac{1}{2} \end{matrix}\right)$$

03.01.26.0090.01

$$J_\mu(z)\,J_\nu(z) - Y_\mu(z)\,Y_\nu(z) = -\frac{2}{\sqrt{\pi}}\,G_{3,5}^{4,0}\!\left(z,\ \frac{1}{2}\,\middle|\, \begin{matrix} 0,\ \frac{1}{2},\ \frac{1-\mu-\nu}{2} \\ \frac{\mu+\nu}{2},\ \frac{\mu-\nu}{2},\ \frac{\nu-\mu}{2},\ -\frac{\mu+\nu}{2},\ \frac{1-\mu-\nu}{2} \end{matrix}\right)$$

03.01.26.0091.01

$$J_\mu(z)\,J_\nu(z)+Y_\mu(z)\,Y_\nu(z)=\frac{\cos(\pi\mu)}{\pi^{5/2}}\,G_{2,4}^{3,2}\!\left(z,\ \frac{1}{2}\,\middle|\,\begin{matrix}0,\ \frac{1}{2}\\ \frac{\mu+\nu}{2},\ \frac{\mu-\nu}{2},\ -\frac{\mu+\nu}{2},\ \frac{\nu-\mu}{2}\end{matrix}\right)+\frac{\cos(\pi\nu)}{\pi^{5/2}}\,G_{2,4}^{3,2}\!\left(z,\ \frac{1}{2}\,\middle|\,\begin{matrix}0,\ \frac{1}{2}\\ \frac{\mu+\nu}{2},\ -\frac{\mu+\nu}{2},\ \frac{\nu-\mu}{2},\ \frac{\mu-\nu}{2}\end{matrix}\right)\ /;\ \mu+\nu\notin\mathbb{Z}$$

Generalized cases involving cos, sin, Y

03.01.26.0092.01

$$\sin(z)\,J_\nu(z)+\cos(z)\,Y_\nu(z)=-\sqrt{2}\,G_{2,4}^{3,0}\!\left(z,\ \frac{1}{2}\,\middle|\,\begin{matrix}\frac{1}{4},\ \frac{3}{4}\\ -\frac{\nu}{2},\ \frac{\nu}{2},\ \frac{\nu+1}{2},\ \frac{1-\nu}{2}\end{matrix}\right)$$

03.01.26.0093.01

$$\cos(z)\,J_\nu(z)-\sin(z)\,Y_\nu(z)=\sqrt{2}\,G_{2,4}^{3,0}\!\left(z,\ \frac{1}{2}\,\middle|\,\begin{matrix}\frac{1}{4},\ \frac{3}{4}\\ \frac{1-\nu}{2},\ \frac{\nu}{2},\ \frac{\nu+1}{2},\ -\frac{\nu}{2}\end{matrix}\right)$$

03.01.26.0094.01

$$\sin(z)\,J_\nu(z)-\cos(z)\,Y_\nu(z)=\frac{\cos(\nu\pi)}{\pi^2\sqrt{2}}\,G_{2,4}^{3,2}\!\left(z,\ \frac{1}{2}\,\middle|\,\begin{matrix}\frac{1}{4},\ \frac{3}{4}\\ -\frac{\nu}{2},\ \frac{\nu}{2},\ \frac{\nu+1}{2},\ \frac{1-\nu}{2}\end{matrix}\right)$$

03.01.26.0095.01

$$\cos(z)\,J_\nu(z)+\sin(z)\,Y_\nu(z)=\frac{\cos(\nu\pi)}{\pi^2\sqrt{2}}\,G_{2,4}^{3,2}\!\left(z,\ \frac{1}{2}\,\middle|\,\begin{matrix}\frac{1}{4},\ \frac{3}{4}\\ \frac{1-\nu}{2},\ \frac{\nu}{2},\ \frac{\nu+1}{2},\ -\frac{\nu}{2}\end{matrix}\right)$$

03.01.26.0096.01

$$\sin(a+z)\,J_\nu(z)-\cos(a+z)\,Y_\nu(z)=\frac{\cos(\pi\nu)}{\pi^2\sqrt{2}}\,G_{3,5}^{4,2}\!\left(z,\ \frac{1}{2}\,\middle|\,\begin{matrix}\frac{1}{4},\ \frac{3}{4},\ \frac{1-\nu}{2}-\frac{a}{\pi}\\ -\frac{\nu}{2},\ \frac{1-\nu}{2},\ \frac{\nu}{2},\ \frac{\nu+1}{2},\ \frac{1-\nu}{2}-\frac{a}{\pi}\end{matrix}\right)$$

03.01.26.0156.01

$$\cos(a+z)\,J_\nu(z)+\sin(a+z)\,Y_\nu(z)=\frac{\cos(\pi\nu)}{\sqrt{2}\,\pi^2}\,G_{3,5}^{4,2}\!\left(z,\ \frac{1}{2}\,\middle|\,\begin{matrix}\frac{1}{4},\ \frac{3}{4},\ -\frac{a}{\pi}-\frac{\nu}{2}\\ \frac{1-\nu}{2},\ -\frac{\nu}{2},\ \frac{\nu}{2},\ \frac{\nu+1}{2},\ -\frac{a}{\pi}-\frac{\nu}{2}\end{matrix}\right)$$

Generalized cases involving ${}_0F_1$

03.01.26.0097.01

$$J_\nu(z)\,{}_0F_1\!\left(;b;-\frac{z^2}{4}\right)=\frac{\Gamma(b)\,2^{b-1}}{\sqrt{\pi}}\,G_{2,4}^{1,2}\!\left(z,\ \frac{1}{2}\,\middle|\,\begin{matrix}\frac{1-b}{2},\ 1-\frac{b}{2}\\ \frac{\nu}{2},\ -\frac{\nu}{2},\ 1-b+\frac{\nu}{2},\ 1-b-\frac{\nu}{2}\end{matrix}\right)\ /;\ -b-\nu\notin\mathbb{N}$$

03.01.26.0157.01

$$J_\nu(z)\,{}_0F_1\!\left(;-n-\nu;-\frac{z^2}{4}\right)=\frac{2^{-n-\nu-1}\,\Gamma(-n-\nu)}{\sqrt{\pi}}\left((-1)^{n-1}\,G_{2,4}^{1,2}\!\left(z,\ \frac{1}{2}\,\middle|\,\begin{matrix}\frac{1}{2}(n+\nu+1),\ \frac{n+\nu}{2}+1\\ n+\frac{\nu}{2}+1,\ -\frac{\nu}{2},\ \frac{\nu}{2},\ n+\frac{3\nu}{2}+1\end{matrix}\right)+2\sum_{k=0}^{\left\lfloor\frac{n}{2}\right\rfloor}\frac{(-1)^{k+\left\lfloor\frac{n+1}{2}\right\rfloor}z^{2k+\nu}\,\Gamma\!\left(k-n+\left\lfloor\frac{n}{2}\right\rfloor+\frac{1}{2}\right)\left(1-k+\left\lfloor\frac{n}{2}\right\rfloor\right)_{n-\left\lfloor\frac{n}{2}\right\rfloor}}{k!\,\Gamma(k-n-\nu)\,\Gamma(k+\nu+1)}\right)\ /;\ n\in\mathbb{N}$$

03.01.26.0158.01

$$J_\nu(z)\,{}_0F_1\left(;-\nu;-\frac{z^2}{4}\right)=-\frac{2^{-\nu-1}\,\Gamma(-\nu)}{\pi}\left(2\sin(\pi\,\nu)\,z^\nu+\sqrt{\pi}\;G_{2,4}^{1,2}\left(z,\frac{1}{2}\,\middle|\,\begin{matrix}\frac{\nu+1}{2},\frac{\nu+2}{2}\\ \frac{\nu+2}{2},-\frac{\nu}{2},\frac{\nu}{2},\frac{3\nu}{2}+1\end{matrix}\right)\right)$$

03.01.26.0159.01

$$J_\nu(z)\,{}_0F_1\left(;-\nu-1;-\frac{z^2}{4}\right)=2^{-\nu-2}\left(\frac{4\,z^\nu}{\Gamma(\nu+1)}+\frac{\Gamma(-\nu-1)}{\sqrt{\pi}}\,G_{2,4}^{1,2}\left(z,\frac{1}{2}\,\middle|\,\begin{matrix}\frac{\nu}{2}+1,\frac{\nu+3}{2}\\ \frac{\nu}{2}+2,-\frac{\nu}{2},\frac{\nu}{2},\frac{3\nu}{2}+2\end{matrix}\right)\right)$$

03.01.26.0160.01

$$J_\nu(z)\,{}_0F_1\left(;\nu;-\frac{z^2}{4}\right)=\frac{2^{\nu-1}\,\Gamma(\nu)}{\sqrt{\pi}}\,G_{1,3}^{1,1}\left(z,\frac{1}{2}\,\middle|\,\begin{matrix}\frac{1-\nu}{2}\\ \frac{\nu}{2},-\frac{\nu}{2},1-\frac{3\nu}{2}\end{matrix}\right)$$

03.01.26.0161.01

$$J_\nu(z)\,{}_0F_1\left(;\nu+1;-\frac{z^2}{4}\right)=\frac{2^\nu\,\Gamma(\nu+1)}{\sqrt{\pi}}\,G_{1,3}^{1,1}\left(z,\frac{1}{2}\,\middle|\,\begin{matrix}\frac{1-\nu}{2}\\ \frac{\nu}{2},-\frac{\nu}{2},-\frac{3\nu}{2}\end{matrix}\right)$$

03.01.26.0162.01

$$J_\nu(z)\,{}_0F_1\left(;1-\nu;-\frac{z^2}{4}\right)=\frac{2^{-\nu}\,\Gamma(1-\nu)}{\sqrt{\pi}}\,G_{1,3}^{1,1}\left(z,\frac{1}{2}\,\middle|\,\begin{matrix}\frac{\nu+1}{2}\\ \frac{\nu}{2},-\frac{\nu}{2},\frac{3\nu}{2}\end{matrix}\right)$$

03.01.26.0098.01

$$J_\nu\,(z)\,{}_0F_1\left(;\nu+1;\frac{z^2}{4}\right)==\sqrt{\pi}\;2^{-\frac{\nu}{2}}\,\Gamma\,(\nu+1)\,G_{0,4}^{1,0}\left(\frac{z}{2\sqrt{2}},\frac{1}{4}\,\middle|\,\frac{\nu}{4},-\frac{\nu}{4},\frac{2-\nu}{4},-\frac{3\nu}{4}\right)$$

03.01.26.0099.01

$$J_\nu\,(z)\,{}_0F_1\left(;1-\nu;\frac{z^2}{4}\right)==2^{\nu/2}\,\sqrt{\pi}\;\Gamma\,(1-\nu)\,G_{1,5}^{2,0}\left(\frac{z}{2\sqrt{2}},\frac{1}{4}\,\middle|\,\begin{matrix}\frac{2-\nu}{4}\\ \frac{\nu}{4},\frac{\nu+2}{4},\frac{3\nu}{4},-\frac{\nu}{4},\frac{2-\nu}{4}\end{matrix}\right)$$

03.01.26.0163.01

$$J_\nu\left(2\sqrt{z}\right)\,{}_0F_1(;\nu+1;z)=2^{-\frac{\nu}{2}}\,\sqrt{\pi}\;\Gamma(\nu+1)\,G_{0,4}^{1,0}\left(\frac{z}{2},\frac{1}{2}\,\middle|\,\frac{\nu}{4},-\frac{\nu}{4},\frac{2-\nu}{4},-\frac{1}{4}\,(3\,\nu)\right)$$

03.01.26.0164.01

$$J_\nu\left(2\sqrt{z}\right)\,{}_0F_1(;1-\nu;z)=2^{\nu/2}\,\sqrt{\pi}\;\Gamma(1-\nu)\,G_{1,5}^{2,0}\left(\frac{z}{2},\frac{1}{2}\,\middle|\,\begin{matrix}\frac{2-\nu}{4}\\ \frac{\nu}{4},\frac{\nu+2}{4},\frac{3\nu}{4},-\frac{\nu}{4},\frac{2-\nu}{4}\end{matrix}\right)$$

Generalized cases involving ${}_0\tilde{F}_1$

03.01.26.0100.01

$$J_\nu\,(z)\,{}_0\tilde{F}_1\left(;b;-\frac{z^2}{4}\right)==\frac{2^{b-1}}{\sqrt{\pi}}\,G_{2,4}^{1,2}\left(z,\frac{1}{2}\,\middle|\,\begin{matrix}\frac{1-b}{2},1-\frac{b}{2}\\ \frac{\nu}{2},-\frac{\nu}{2},1-b+\frac{\nu}{2},1-b-\frac{\nu}{2}\end{matrix}\right)/;\,-b-\nu\notin\mathbb{N}$$

03.01.26.0165.01

$$J_\nu(z)\,{}_0\tilde{F}_1\left(;-n-\nu;-\frac{z^2}{4}\right)=\frac{2^{-n-\nu-1}}{\sqrt{\pi}}\left((-1)^{n-1}\,G_{2,4}^{1,2}\left(z,\frac{1}{2}\,\middle|\,\begin{matrix}\frac{1}{2}(n+\nu+1),\frac{n+\nu}{2}+1\\ n+\frac{\nu}{2}+1,-\frac{\nu}{2},\frac{\nu}{2},n+\frac{3\nu}{2}+1\end{matrix}\right)+2\sum_{k=0}^{\left\lfloor\frac{n}{2}\right\rfloor}\frac{(-1)^{k+\left\lfloor\frac{n+1}{2}\right\rfloor}z^{2k+\nu}\,\Gamma\left(k-n+\left\lfloor\frac{n}{2}\right\rfloor+\frac{1}{2}\right)\left(1-k+\left\lfloor\frac{n}{2}\right\rfloor\right)_{n-\left\lfloor\frac{n}{2}\right\rfloor}}{k!\,\Gamma(k-n-\nu)\,\Gamma(k+\nu+1)}\right)\;/;\;n\in\mathbb{N}$$

03.01.26.0166.01

$$J_\nu(z)\,{}_0\tilde{F}_1\left(;-\nu;-\frac{z^2}{4}\right)=-\frac{2^{-\nu-1}}{\pi}\left(2\sin(\pi\nu)\,z^\nu+\sqrt{\pi}\,G_{2,4}^{1,2}\left(z,\frac{1}{2}\,\middle|\,\begin{matrix}\frac{\nu+1}{2},\frac{\nu+2}{2}\\ \frac{\nu+2}{2},-\frac{\nu}{2},\frac{\nu}{2},\frac{3\nu}{2}+1\end{matrix}\right)\right)$$

03.01.26.0167.01

$$J_\nu(z)\,{}_0\tilde{F}_1\left(;-\nu-1;-\frac{z^2}{4}\right)=2^{-\nu-2}\left(\frac{4z^\nu(\nu+1)\sin(\pi\nu)}{\pi}+\frac{1}{\sqrt{\pi}}\,G_{2,4}^{1,2}\left(z,\frac{1}{2}\,\middle|\,\begin{matrix}\frac{\nu}{2}+1,\frac{\nu+3}{2}\\ \frac{\nu}{2}+2,-\frac{\nu}{2},\frac{\nu}{2},\frac{3\nu}{2}+2\end{matrix}\right)\right)$$

03.01.26.0168.01

$$J_\nu(z)\,{}_0\tilde{F}_1\left(;\nu;-\frac{z^2}{4}\right)=\frac{2^{\nu-1}}{\sqrt{\pi}}\,G_{1,3}^{1,1}\left(z,\frac{1}{2}\,\middle|\,\begin{matrix}\frac{1-\nu}{2}\\ \frac{\nu}{2},-\frac{\nu}{2},1-\frac{3\nu}{2}\end{matrix}\right)$$

03.01.26.0169.01

$$J_\nu(z)\,{}_0\tilde{F}_1\left(;\nu+1;-\frac{z^2}{4}\right)=\frac{2^{\nu}}{\sqrt{\pi}}\,G_{1,3}^{1,1}\left(z,\frac{1}{2}\,\middle|\,\begin{matrix}\frac{1-\nu}{2}\\ \frac{\nu}{2},-\frac{\nu}{2},-\frac{3\nu}{2}\end{matrix}\right)$$

03.01.26.0170.01

$$J_\nu(z)\,{}_0\tilde{F}_1\left(;1-\nu;-\frac{z^2}{4}\right)=\frac{2^{-\nu}}{\sqrt{\pi}}\,G_{1,3}^{1,1}\left(z,\frac{1}{2}\,\middle|\,\begin{matrix}\frac{\nu+1}{2}\\ \frac{\nu}{2},-\frac{\nu}{2},\frac{3\nu}{2}\end{matrix}\right)$$

03.01.26.0101.01

$$J_\nu(z)\,{}_0\tilde{F}_1\left(;\nu+1;\frac{z^2}{4}\right)==\sqrt{\pi}\,2^{-\frac{\nu}{2}}\,G_{0,4}^{1,0}\left(\frac{z}{2\sqrt{2}},\frac{1}{4}\,\middle|\,\frac{\nu}{4},-\frac{\nu}{4},\frac{2-\nu}{4},-\frac{3\nu}{4}\right)$$

03.01.26.0102.01

$$J_\nu(z)\,{}_0\tilde{F}_1\left(;1-\nu;\frac{z^2}{4}\right)==2^{\nu/2}\sqrt{\pi}\,G_{1,5}^{2,0}\left(\frac{z}{2\sqrt{2}},\frac{1}{4}\,\middle|\,\begin{matrix}\frac{2-\nu}{4}\\ \frac{\nu}{4},\frac{\nu+2}{4},\frac{3\nu}{4},-\frac{\nu}{4},\frac{2-\nu}{4}\end{matrix}\right)$$

03.01.26.0171.01

$$J_\nu\left(2\sqrt{z}\right){}_0\tilde{F}_1(;\nu+1;z)=2^{-\frac{\nu}{2}}\sqrt{\pi}\,G_{0,4}^{1,0}\left(\frac{z}{2},\frac{1}{2}\,\middle|\,\frac{\nu}{4},-\frac{\nu}{4},\frac{2-\nu}{4},-\frac{1}{4}(3\nu)\right)$$

03.01.26.0172.01

$$J_\nu\left(2\sqrt{z}\right){}_0\tilde{F}_1(;1-\nu;z)=2^{\nu/2}\sqrt{\pi}\,G_{1,5}^{2,0}\left(\frac{z}{2},\frac{1}{2}\,\middle|\,\begin{matrix}\frac{2-\nu}{4}\\ \frac{\nu}{4},\frac{\nu+2}{4},\frac{3\nu}{4},-\frac{\nu}{4},\frac{2-\nu}{4}\end{matrix}\right)$$

Through other functions

03.01.26.0103.01

$$J_\nu(z) = (-1)^{\nu-\frac{1}{2}} \boldsymbol{H}_{-\nu}(z) /; \nu - \frac{1}{2} \in \mathbb{N}$$

Representations through equivalent functions

With related functions

03.01.27.0001.01

$$J_\nu(z) = \frac{z^\nu}{(i\, z)^\nu} I_\nu(i\, z)$$

03.01.27.0002.01

$$J_\nu(i\, z) = \frac{(i\, z)^\nu}{z^\nu} I_\nu(z)$$

03.01.27.0003.02

$$J_\nu(z) = \csc(\pi\, \nu)\, Y_{-\nu}(z) - \cot(\pi\, \nu)\, Y_\nu(z)$$

03.01.27.0004.01

$$J_\nu(z)\, Y_{\nu+1}(z) - J_{\nu+1}(z)\, Y_\nu(z) = -\frac{2}{\pi\, z}$$

03.01.27.0005.01

$$J_\nu(z) = \left(\frac{z}{2}\right)^\nu {}_0\tilde{F}_1\left(; \nu+1; -\frac{z^2}{4}\right)$$

03.01.27.0006.01

$$J_\nu(z) = \frac{1}{\Gamma(\nu+1)} \left(\frac{z}{2}\right)^\nu {}_0F_1\left(; \nu+1; -\frac{z^2}{4}\right) /; -\nu \notin \mathbb{N}^+$$

03.01.27.0007.01

$$J_\nu(z) = e^{\frac{3 i \pi \nu}{4}} z^\nu \left(-(-1)^{3/4} z\right)^{-\nu} \left(\operatorname{ber}_\nu\left(-(-1)^{3/4} z\right) - i \operatorname{bei}_\nu\left(-(-1)^{3/4} z\right)\right)$$

03.01.27.0008.01

$$J_\nu\left(\sqrt[4]{-1}\; z\right) = e^{\frac{3 i \pi \nu}{4}} z^{-\nu} \left(\sqrt[4]{-1}\; z\right)^\nu (\operatorname{ber}_\nu(z) - i \operatorname{bei}_\nu(z))$$

03.01.27.0009.01

$$J_\nu\left(\sqrt[4]{z}\right) I_\nu\left(\sqrt[4]{z}\right) = (-z)^{-\frac{\nu}{2}} z^{\nu/2} \left(\operatorname{bei}_\nu\left(\sqrt[4]{-z}\right)^2 + \operatorname{ber}_\nu\left(\sqrt[4]{-z}\right)^2\right)$$

Inequalities

03.01.29.0001.01

$$|J_\nu(z)| \leq 1 /; \nu \geq 0$$

03.01.29.0002.01

$$|J_\nu(z)| \leq \frac{1}{\sqrt{2}} /; \nu \geq 1$$

03.01.29.0003.01

$$0 < J_\nu(\nu) < \frac{\sqrt[3]{2}\,\sqrt[3]{\nu}}{3^{2/3}}\,\Gamma\!\left(\frac{2}{3}\right) /;\ \nu > 0$$

03.01.29.0004.01

$$|J_\nu(x)| \leq \frac{1}{\Gamma(\nu+1)}\left(\frac{x}{2}\right)^\nu e^{|\mathrm{Im}(x)|} /;\ \nu \geq -\frac{1}{2} \bigwedge x > 0$$

03.01.29.0005.01

$$|J_\nu(\nu\, x)| \leq \frac{x^\nu\, e^{\nu \sqrt{1-x^2}}}{\left(\sqrt{1-x^2}+1\right)^\nu} /;\ \nu \in \mathbb{N} \land x > 0$$

Zeros

When ν is real, the functions $J_\nu(z)$and $\left(\frac{\partial J_\nu(z)}{\partial z}\right)$ each have an infinite number of real zeros, all of which are simple with the possible exception of $z = 0$.

03.01.30.0001.01

$$J_\nu(z) == 0 /;\ z == z_k \bigwedge k \in \mathbb{N} \bigwedge \nu \in \mathbb{R} \bigwedge \mathrm{Re}(z_k) == z_k$$

03.01.30.0002.02

$$\frac{\partial J_\nu(z)}{\partial z} == 0 /;\ z == z_k \bigwedge k \in \mathbb{N} \bigwedge \nu \in \mathbb{R} \bigwedge \mathrm{Re}\,(z_k) == z_k$$

When $\nu \geq -1$, the zeros of $J_\nu(z)$ are all real. If $\nu < -1$ and ν is not an integer, the number of complex zeros of $J_\nu(z)$ is $2\lfloor -\nu \rfloor$; if $\lfloor -\nu \rfloor$ is odd, two of these zeros lie on the imaginary axis.

If $\nu > 0$, all zeros of $\left(\frac{\partial J_\nu(z)}{\partial z}\right)$ are real.

Theorems

Hankel transformation

$$\hat{f}_\nu(y) == \int_0^\infty f(x)\sqrt{x\,y}\; J_\nu(x\,y)\,dx \Leftrightarrow\ f(x) == \int_0^\infty \hat{f}_\nu(y)\sqrt{x\,y}\; J_\nu(x\,y)\,dy /;\ \mathrm{Re}(\nu) \geq -\frac{1}{2}.$$

Green's function for the Klein-Gordon equation

Green's function for the Klein-Gordon equation $\left(\Box + m^2\right) G == \delta(\mathbf{x}-\mathbf{0})\,\delta(t)$ is given by

$$G(\mathbf{x},\, t,\, \mathbf{0},\, 0) == \frac{\theta(t)}{2\pi}\,\delta\!\left(t^2 - |\mathbf{x}|^2\right) - \frac{m}{2\pi}\,\theta(t - |\mathbf{x}|)\,\frac{J_1\!\left(m\sqrt{t^2-|\mathbf{x}|^2}\right)}{m\sqrt{t^2-|\mathbf{x}|^2}}.$$

The initial value problem for a periodic chain of coupled harmonic oscillators

The initial value problem for a periodic chain of coupled harmonic oscillators with Hamiltonian $H == \sum_{k=1}^{n} \frac{1}{2} \left(p_j^2 + \left(q_j - q_{j+1}\right)^2\right)$ with periodic boundary conditions can be expressed as $\xi_n(t) == \sum_{k=-\infty}^{\infty} \xi_k(0) J_{k-n}(t)$ where $\xi_{2n}(t) == p_n(t)$ and $\xi_{2n+1}(t) == q_{n+1}(t) - q_n(t)$.

The solution of the Kepler equation

The solution of the Kepler equation $w(z) - \varepsilon \sin(w(z)) == z$ is $w(z) == z + 2 \sum_{k=1}^{\infty} \frac{\sin(k z)}{k} J_k(k \varepsilon) /; 0 < \varepsilon < 1$.

The potential of a flat circular disk

The potential $V(r, z)$ of a flat circular disk of radius 1 in cylindrical coordinates is given by $V(r, z) \propto \frac{2}{\pi} \int_o^{\infty} \frac{\sin(\xi)}{\xi} J_0(r \xi) e^{-\xi |z|} d\xi$.

The pair correlation function for two electrons

The pair correlation function $g(r)$ for two electrons in the plane wave approximation is given by $g(r) == 1 - \frac{9 \pi}{4} \frac{J_{3/2}(k_F r)^2}{(k_F r)^3}$ where k_F is the Fermi wave length.

A solution to the Toda equation

A solution to the Toda equation $\partial^2 \frac{\Theta(x,n,t)}{\partial t^2} + \partial^2 \frac{\Theta(x,n,t)}{\partial x^2} == 2 \left(e^{2 (\Theta(x,n+1,t) - \Theta(x,n,t))} - e^{2 (\Theta(x,n,t) - \Theta(x,n-1,t))}\right)$ is given by

$$\Theta(x, n, t) == \frac{1}{2} \log\left(1 + \frac{a^2 J_{n+1}^2\left(2 \sqrt{t^2 + x^2}\right)}{1 - a^2 \left(1 + \sum_{k=n+1}^{\infty} J_k^2\left(2 \sqrt{t^2 + x^2}\right)\right)}\right) /; a^2 < \frac{1}{4}.$$

The components of the magnetic field in a cylindrical coordinate

Let B_z , B_ϕ, B_r the components of the magnetic field in a cylindrical coordinate system $\{r, \phi, z\}$ such that $\mathbf{B}$ is a force-free magnetic field (that is, div $\mathbf{B} = 0$ and curl $\mathbf{B} = a \, \mathbf{B}$ hold simultaneously). One such solution is

$B_z = \xi e^{-l z} J_0(r \xi)$, $B_\phi = a e^{-l z} J_1(r \xi)$, $B_r = l e^{-l z} J_1(r \xi) /; \xi^2 == a^2 + l^2$.

The propagator for the inverse square potential

The function $G(x, t; y) == \frac{\sqrt{x y}}{t} \exp\left(i \frac{x^2 + y^2}{2 t}\right) J_{\lambda + \frac{1}{2}}\left(\frac{x y}{t}\right)$ is the propagator for the singular potential $V(x) == \frac{\lambda(\lambda + 1)}{2} \frac{1}{x^2}$ and satisfies

$$i \frac{\partial}{\partial t} G(x, t; y) == -\frac{1}{2} \frac{\partial^2}{\partial x^2} G(x, t; y) + V(x) G(x, t; y) \bigwedge G(x, 0; y) == \delta(x - y).$$

The solution of the 1+1 dimensional Dirac equation

The spinor $\psi(x,\ t) = \left\{ i\ \frac{(t+x)}{\sqrt{t^2-x^2}}\ J_1\!\left(\frac{m\,t}{\gamma}\right),\ J_0\!\left(\frac{m\,t}{\gamma}\right) \right\}$, $\gamma = 1 \Big/ \sqrt{1 - \frac{x^2}{t^2}}$ is a solution of the 1+1 dimensional Dirac equation

$$i\ \frac{\partial\psi\,(x,\ t)}{\partial t} = -i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \frac{\partial\psi\,(x,\ t)}{\partial x} - m \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \psi(x,\ t).$$

Vibrating circular membrane

Eigenfunctions

The displacement $u(\mathbf{r},\ t)$ of a vibrating membrane, at each point $\mathbf{r} \in \mathbb{R}^2$ as a function of t, satisfies the wave equation

$$\Delta\, u(\mathbf{r},\ t) - \frac{1}{c_{ph}^2}\ \frac{\partial^2 u\,(\mathbf{r},\ t)}{\partial t^2} = 0.$$

Here, Δ is the Laplace operator and c_{ph}^2 (= tension per unit surface density) is the phase velocity. For a circular membrane of radius r_0 that is fixed at the boundary and assumimg harmonic time dependence $u(\mathbf{r},\ t) = \cos(\omega\, t + \alpha)\ u(\mathbf{r})$yields in a polar coordinate system

$$u_{m,k}(\mathbf{r}) = \cos(m\,\varphi)\ J_{|m|}\!\left(\sqrt{\lambda_{m,k}}\ r\right)$$

$$u_{m,k}(\mathbf{r}) = \sin(m\,\varphi)\ J_{|m|}\!\left(\sqrt{\lambda_{m,k}}\ r\right)$$

$\lambda_{m,k}=\mu_{m,k}^2 \big/ r_0^2$ where $\mu_{m,k}^2$ is the kth zero of the Bessel function $J_m(z)$.

History

– J. Bernoulli (1703); D. Bernoulli (1733);
– F. W. Bessel (1816,1824)
- O. Schlömilch (1857) used the name "Bessel functions"
– E. Lommel (1868) used arbitrary real indices
– H. Hankel (1869) used arbitrary complex indices
– I. Todhunter (1875); N. J. Sonin (1880) derived the recurrence relations

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