Abs

Notations

Traditional name

Absolute value function

Traditional notation

$|z|$

Mathematica StandardForm notation

Abs[z]

Primary definition

12.01.02.0001.01

$$|x| = x \;/;\; x \in \mathbb{R} \land x \geq 0$$

12.01.02.0002.01

$$|x| = -x \;/;\; x \in \mathbb{R} \land x < 0$$

12.01.02.0003.01

$$|z| = \sqrt{\operatorname{Re}(z)^2 + \operatorname{Im}(z)^2}$$

$|z|$ is the absolute value of z. The absolute value (or modulus) of a complex number z is the Euclidean distance from z to the origin.

Specific values

Specialized values

12.01.03.0001.01

$$|x| = \operatorname{sgn}(x)\, x \;/;\; x \in \mathbb{R}$$

12.01.03.0002.01

$$|i\, x| = x \;/;\; x \in \mathbb{R} \land x \geq 0$$

12.01.03.0003.01

$$|i\, x| = -x \;/;\; x \in \mathbb{R} \land x < 0$$

12.01.03.0004.01

$$|x + i\, y| = \sqrt{x^2 + y^2} \;/;\; x \in \mathbb{R} \bigwedge y \in \mathbb{R}$$

Values at fixed points

12.01.03.0005.01

$|0| == 0$

12.01.03.0006.01

$|1| == 1$

12.01.03.0007.01

$|-1| == 1$

12.01.03.0008.01

$|i| == 1$

12.01.03.0009.01

$|-i| == 1$

12.01.03.0021.01

$|1+i| == \sqrt{2}$

12.01.03.0022.01

$|-1+i| == \sqrt{2}$

12.01.03.0023.01

$|-1-i| == \sqrt{2}$

12.01.03.0024.01

$|1-i| == \sqrt{2}$

12.01.03.0025.01

$\left|\sqrt{3}+i\right| == 2$

12.01.03.0026.01

$\left|1+i\sqrt{3}\right| == 2$

12.01.03.0027.01

$\left|-1+i\sqrt{3}\right| == 2$

12.01.03.0028.01

$\left|-\sqrt{3}+i\right| == 2$

12.01.03.0029.01

$\left|-\sqrt{3}-i\right| == 2$

12.01.03.0030.01

$\left|-1-i\sqrt{3}\right| == 2$

12.01.03.0031.01

$\left|1-i\sqrt{3}\right| == 2$

12.01.03.0032.01

$\left|\sqrt{3}-i\right| == 2$

12.01.03.0010.01

$|2| == 2$

12.01.03.0011.01

$|-2| == 2$

12.01.03.0012.01

$|\pi| == \pi$

12.01.03.0013.01

$|3\, i| == 3$

12.01.03.0014.01

$|-2\, i| == 2$

12.01.03.0015.01

$|2 + i| == \sqrt{5}$

Values at infinities

12.01.03.0016.01

$|\infty| == \infty$

12.01.03.0017.01

$|-\infty| == \infty$

12.01.03.0018.01

$|i\, \infty| == \infty$

12.01.03.0019.01

$|-i\, \infty| == \infty$

12.01.03.0020.01

$|\tilde{\infty}| == \infty$

General characteristics

Domain and analyticity

$|z|$ is nonanalytical function; it is a real-analytic function of the variable z for $z \neq 0$.

12.01.04.0001.01

$z \longrightarrow |z| :: \mathbb{C} \longrightarrow \mathbb{R}$

Symmetries and periodicities

Parity

$|z|$ is an even function.

12.01.04.0002.01

$|-z| == |z|$

Mirror symmetry

12.01.04.0003.01

$|\bar{z}| == |z|$

Periodicity

No periodicity

Homogeneity

12.01.04.0004.01

$|a\,z| == |a|\,|z|$

Scale symmetry

12.01.04.0005.01

$|z^a| == |z|^a \;/;\; a \in \mathbb{R}$

Sets of discontinuity

The function $|z|$ is continuous function in $\mathbb{C}$.

12.01.04.0006.01

$\mathcal{DS}_z(|z|) == \{\}$

Series representations

Other series representations

12.01.06.0001.01

$$|x| == \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{4\,k^2 - 1} T_{2\,k}(x) + \frac{2}{\pi} \;/;\; x \in \mathbb{R} \land -1 < x < 1$$

12.01.06.0002.01

$$|x| == \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)!} \left(2\,k + \frac{1}{2}\right) \left(-\frac{1}{2}\right)_k P_{2\,k}(x) \;/;\; x \in \mathbb{R} \land -1 < x < 1$$

12.01.06.0003.01

$$|x| == \frac{1}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2\,k)!} \left(-\frac{1}{2}\right)_k H_{2\,k}(x) \;/;\; x \in \mathbb{R} \land -1 < x < 1$$

Limit representations

12.01.09.0001.01

$$|x| == \lim_{n \to \infty} x \frac{p_n(x) - p_n(-x)}{p_n(-x) + p_n(x)} \;/;\; n \in \mathbb{N} \bigwedge -1 < x < 1 \bigwedge p_n(x) == \prod_{k=0}^{n-1} \left(x + e^{-\frac{k}{\sqrt{n}}}\right)$$

Differential equations

Ordinary linear differential equations and wronskians

In a distributional sense:

12.01.13.0001.01

$w'(x) == \theta(x) - \theta(-x) \;/;\; w(x) == |x|$

Transformations

Transformations and argument simplifications

Argument involving basic arithmetic operations

12.01.16.0001.01

$$|-z| = |z|$$

12.01.16.0002.01

$$|a\,z| = a\,|z| \;/;\; a \in \mathbb{R} \land a > 0$$

12.01.16.0003.01

$$|i\,z| = |z|$$

12.01.16.0004.01

$$|-i\,z| = |z|$$

12.01.16.0005.01

$$|\bar{z}| = |z|$$

12.01.16.0006.01

$$\left|\frac{1}{z}\right| = \frac{1}{|z|}$$

Addition formulas

12.01.16.0007.01

$$|x + i\,y| = \sqrt{x^2 + y^2} \;/;\; x \in \mathbb{R} \bigwedge y \in \mathbb{R}$$

12.01.16.0008.01

$$|z_1 + z_2| = ||z_1| - |z_2|| + |z_1| + |z_2| - |z_1 - z_2|$$

Multiple arguments

12.01.16.0010.01

$$|a\,z| = a\,|z| \;/;\; a \in \mathbb{R} \land a > 0$$

12.01.16.0011.01

$$|i\,z| = |z|$$

12.01.16.0012.01

$$|-i\,z| = |z|$$

12.01.16.0013.01

$$\left|\prod_{k=1}^{n} z_k\right| = \prod_{k=1}^{n} |z_k|$$

12.01.16.0014.01

$$|z_1\,z_2| = |z_1|\,|z_2|$$

12.01.16.0015.01

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$$

Power of arguments

12.01.16.0016.01

$$|x^a| = x^{\operatorname{Re}(a)} \;/;\; x \in \mathbb{R} \land x > 0$$

12.01.16.0017.01

$|z^a| == |z|^a \;/;\; a \in \mathbb{R}$

12.01.16.0018.01

$|z^a| == \exp(i\, a\, \mathrm{Im}(\log(z))) \;/;\; i\, a \in \mathbb{R}$

12.01.16.0019.01

$|z^a| == \exp(i\, a \arg(z)) \;/;\; i\, a \in \mathbb{R}$

12.01.16.0020.01

$|z^a| == \exp(\mathrm{Re}(a \log(z)))$

12.01.16.0021.01

$|z^a| == \exp(\mathrm{Re}(a) \log(|z|) - \mathrm{Im}(a) \arg(z))$

12.01.16.0022.01

$|z^a| == |z|^{\mathrm{Re}(a)} \exp\left(-\mathrm{Im}(a) \tan^{-1}(\mathrm{Re}(z), \mathrm{Im}(z))\right)$

12.01.16.0023.01

$|z^a| == |z|^{\mathrm{Re}(a)} \exp(-\mathrm{Im}(a) \arg(z))$

Exponent of arguments

12.01.16.0027.01

$\left|e^{x+i\, y}\right| == e^x$

12.01.16.0028.01

$|e^z| == e^{\mathrm{Re}(z)}$

12.01.16.0029.01

$\left|e^{i\, z}\right| == e^{-\mathrm{Im}(z)}$

Products, sums, and powers of the direct function

Products of the direct function

12.01.16.0024.01

$|z_1|\, |z_2| == |z_1\, z_2|$

Powers of the direct function

12.01.16.0025.01

$|z|^a == |z^a| \;/;\; a \in \mathbb{R}$

Sums of powers of the direct function

12.01.16.0026.01

$$|z_1|^2 + |z_2|^2 == \frac{1}{2}\left(|z_1 - z_2|^2 + |z_1 + z_2|^2\right)$$

Complex characteristics

Real part

12.01.19.0001.01

$\mathrm{Re}(|x + i\, y|) == \sqrt{x^2 + y^2}$

12.01.19.0002.01

$\operatorname{Re}(|z|) = |z|$

Imaginary part

12.01.19.0003.01

$\operatorname{Im}(|x + i\,y|) = 0$

12.01.19.0004.01

$\operatorname{Im}(|z|) = 0$

Absolute value

12.01.19.0005.01

$||x + i\,y|| = \sqrt{x^2 + y^2}$

12.01.19.0006.01

$||z|| = |z|$

Argument

12.01.19.0007.01

$\arg(|x + i\,y|) = 0$

12.01.19.0008.01

$\arg(|z|) = 0$

Conjugate value

12.01.19.0009.01

$\overline{|x + i\,y|} = \sqrt{x^2 + y^2}$

12.01.19.0010.01

$\overline{|z|} = |z|$

Signum value

12.01.19.0011.01

$\operatorname{sgn}(|x + i\,y|) = 1 \;/;\; x + i\,y \neq 0$

12.01.19.0012.01

$\operatorname{sgn}(|z|) = 1 \;/;\; z \neq 0$

Differentiation

Low-order differentiation

In a distributional sense, for $x \in \mathbb{R}$:

12.01.20.0001.01

$$\frac{\partial |x|}{\partial x} = \operatorname{sgn}(x)$$

12.01.20.0002.01

$$\frac{\partial^2 |x|}{\partial x^2} == 2\,\delta(x)$$

Fractional integro-differentiation

12.01.20.0003.01

$$\frac{\partial^\alpha |x|}{\partial x^\alpha} == \frac{|x|\,x^{-\alpha}}{\Gamma(2-\alpha)}$$

Integration

Indefinite integration

Involving only one direct function

For $x \in \mathbb{R}$:

12.01.21.0001.01

$$\int |x|\,dx == \frac{x\,|x|}{2}$$

Definite integration

For the direct function itself

12.01.21.0002.01

$$\int_{-1}^{1} |t|\,dt == 1$$

12.01.21.0003.01

$$\int_{-a}^{a} |t|\,dt == a\sqrt{\operatorname{Im}(a)^2 + \operatorname{Re}(a)^2}$$

12.01.21.0004.01

$$\int_{-a}^{a} t^k\,|t|\,dt == \frac{\left(1+(-1)^k\right)a^{k+2}}{k+2} \;/;\; a \in \mathbb{R} \bigwedge a > 0 \bigwedge \operatorname{Re}(k) > -2$$

Involving the direct function

12.01.21.0005.01

$$\int_{-\infty}^{\infty} e^{-|t|}\,dt == 2$$

12.01.21.0006.01

$$\int_{-\infty}^{\infty} \frac{\cos(t)}{\sqrt{|t|}}\,dt == \sqrt{2\,\pi}$$

12.01.21.0007.01

$$\int_{-\infty}^{\infty} \frac{\sin(t)}{\sqrt{|t|}}\,dt == 0$$

Contour integration

12.01.21.0008.01

$$\int_C \frac{z^{m-1}}{|z-w|^{2(n+1)}}\, dz = 2\pi i \left(\theta(|w|-\rho)\left(\sum_{k=0}^{n} \binom{k+n}{k}\binom{m+n}{n-k} \frac{\rho^{2(k+m)}}{\overline{w}^m \left(|w|^2-\rho^2\right)^{k+n+1}}\right) + \theta(\rho-|w|)\left(\sum_{k=0}^{n}\binom{k+n}{k}\binom{m+n}{n-k}\frac{w^m |w|^{2k}}{\left(\rho^2-|w|^2\right)^{k+n+1}}\right)\right) /; n \in \mathbb{N} \bigwedge m \in \mathbb{N} \bigwedge w \neq 0 \bigwedge |w| \neq \rho$$

In the last formula C is a positively oriented circle around the origin with radius ρ.

Integral transforms

Fourier exp transforms

12.01.22.0001.01

$$\mathcal{F}_t[|t|](x) = -\sqrt{\frac{2}{\pi}}\frac{1}{x^2}$$

12.01.22.0006.01

$$\mathcal{F}_t\left[\frac{1}{\sqrt{|t|}}\right](x) = \frac{1}{\sqrt{|x|}}$$

12.01.22.0007.01

$$\mathcal{F}_t[|t|^\alpha](x) = -\sqrt{\frac{2}{\pi}}\, |x|^{-\alpha-1}\, \Gamma(\alpha+1)\sin\left(\frac{\pi\alpha}{2}\right) /; \operatorname{Re}(\alpha) > -1$$

12.01.22.0008.01

$$\mathcal{F}_t[|t|^\alpha \operatorname{sgn}(t)](x) = i\sqrt{\frac{2}{\pi}}\, |x|^{-\alpha-1}\cos\left(\frac{\pi\alpha}{2}\right)\Gamma(\alpha+1)\operatorname{sgn}(x) /; \operatorname{Re}(\alpha) > -1$$

Inverse Fourier exp transforms

12.01.22.0002.01

$$\mathcal{F}_t^{-1}[|t|](z) = -\sqrt{\frac{2}{\pi}}\frac{1}{z^2}$$

Fourier cos transforms

12.01.22.0003.01

$$\mathcal{F}c_t[|t|](z) = -\sqrt{\frac{2}{\pi}}\frac{1}{z^2}$$

Fourier sin transforms

12.01.22.0004.01

$$\mathcal{F}s_t[|t|](z) = -\sqrt{\frac{\pi}{2}}\,\delta'(z)$$

Laplace transforms

12.01.22.0005.01

$$\mathcal{L}_t[|t|](z) \equiv \frac{1}{z^2}$$

Representations through more general functions

Through Meijer *G*

Classical cases involving cosh

12.01.26.0001.01

$$|1-x|^\nu \cosh\left(\nu \tanh^{-1}\left(\frac{2\sqrt{x}}{x+1}\right)\right) \equiv \frac{\sqrt{\pi}}{\Gamma(-\nu)} \Gamma\left(\nu+\frac{1}{2}\right) G_{2,2}^{1,1}\left(x \,\middle|\, \begin{matrix} \nu+1, \nu+\frac{1}{2} \\ 0, \frac{1}{2} \end{matrix}\right) /; x>0$$

12.01.26.0002.01

$$|1-x|^\nu \cosh\left(\nu \coth^{-1}\left(\frac{1+x}{\sqrt{x}}\right)\right) \equiv \frac{\sqrt{\pi}}{\Gamma(-\nu)} \Gamma\left(\nu+\frac{1}{2}\right) G_{2,2}^{1,1}\left(x \,\middle|\, \begin{matrix} \nu+1, \nu+\frac{1}{2} \\ 0, \frac{1}{2} \end{matrix}\right) /; x>0$$

Classical cases involving sinh

12.01.26.0003.01

$$|1-x|^\nu \sinh\left(\nu \tanh^{-1}\left(\frac{2\sqrt{x}}{x+1}\right)\right) \equiv -\frac{\sqrt{\pi}}{\Gamma(-\nu)} \Gamma\left(\nu+\frac{1}{2}\right) G_{2,2}^{1,1}\left(x \,\middle|\, \begin{matrix} \nu+\frac{1}{2}, \nu+1 \\ \frac{1}{2}, 0 \end{matrix}\right) /; x>0$$

12.01.26.0004.01

$$|1-x|^\nu \sinh\left(\nu \coth^{-1}\left(\frac{1+x}{\sqrt{x}}\right)\right) \equiv -\frac{\sqrt{\pi}}{\Gamma(-\nu)} \Gamma\left(\nu+\frac{1}{2}\right) G_{2,2}^{1,1}\left(x \,\middle|\, \begin{matrix} \nu+\frac{1}{2}, \nu+1 \\ \frac{1}{2}, 0 \end{matrix}\right) /; x>0$$

Generalized cases for powers of Abs

12.01.26.0005.01

$$|1-x|^\nu \equiv \frac{\pi}{\Gamma(-\nu)} \sec\left(\frac{\nu\pi}{2}\right) G_{2,2}^{1,1}\left(x \,\middle|\, \begin{matrix} \nu+1, \frac{\nu+1}{2} \\ 0, \frac{\nu+1}{2} \end{matrix}\right) /; x>0$$

Representations through equivalent functions

With related functions

With Re

12.01.27.0008.01

$$|z| \equiv \sqrt{2\,z\,\mathrm{Re}(z)-z^2}$$

With Im

12.01.27.0009.01

$$|z| \equiv \sqrt{z^2-2\,i\,z\,\mathrm{Im}(z)}$$

12.01.27.0007.01

$$|z| = \sqrt{\text{Re}(z)^2 + \text{Im}(z)^2}$$

With Arg

12.01.27.0001.01

$$|z| = z\, e^{-i \arg(z)}$$

12.01.27.0002.01

$$|z| = z\,(\cos(\arg(z)) - i \sin(\arg(z)))$$

12.01.27.0003.01

$$|z| = \frac{\text{Re}(z)}{\cos(\arg(z))}$$

12.01.27.0004.01

$$|z| = \frac{\text{Im}(z)}{\sin(\arg(z))}$$

With Conjugate

12.01.27.0005.01

$$|z| = \sqrt{z\, \bar{z}}$$

With Sign

12.01.27.0006.01

$$|z| = \frac{z}{\text{sgn}(z)} \;/;\; z \neq 0$$

Inequalities

12.01.29.0001.01

$$|z_1 + z_2| \leq |z_1| + |z_2|$$

12.01.29.0002.01

$$|z_1 - z_2| \geq ||z_1| - |z_2||$$

12.01.29.0003.01

$$|\text{Re}(z)| \leq |z|$$

12.01.29.0004.01

$$|\text{Im}(z)| \leq |z|$$

12.01.29.0005.01

$$|\arg(z)| \leq \pi$$

12.01.29.0006.01

$$|\text{sgn}(z)| \leq 1$$

12.01.29.0007.01

$$|z_1| - |z_2| \leq |z_1 + z_2| \leq |z_1| + |z_2|$$

Triangle inequality

12.01.29.0008.01

$$\left|\sum_{k=1}^{n} z_k\right| \leq \sum_{k=1}^{n} |z_k|$$

Triangle inequality

Zeros

12.01.30.0001.01

$|z| == 0$ /; $z == 0$

History

– J. R. Argand (1806, 1814) introduced the word "module" for absolute value

– K. Weierstrass (1841) introduced the notation $|x|$

Abs is encountered in mathematics and the natural sciences.

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